J. Math. Soc. Japan
Vol. 21, No. 2, 1969

On the Cauchy problem for equations with multiple characteristic roots

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(Received July 17, 1968)

§ 1. Introduction.

Let us consider a kowalewskian

$$(1.1) \qquad Lu=\left(\frac{\partial}{\partial t}\right)^m u+\sum_{\substack{|\nu|+j\leqq m\\ j\leqq m-1}} a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu\left(\frac{\partial}{\partial t}\right)^j u=f$$

where

$$x=(x_1, x_2, \cdots, x_k)\in R^k,$$

$$\left(\frac{\partial}{\partial x}\right)^\nu=\left(\frac{\partial}{\partial x_1}\right)^{\nu_1}\left(\frac{\partial}{\partial x_2}\right)^{\nu_2}\cdots\left(\frac{\partial}{\partial x_k}\right)^{\nu_k},$$

$$|\nu|=\nu_1+\nu_2+\cdots+\nu_k$$

and

$$a_{\nu,j}(x,t)\in \mathcal{B}(R^{k+1})^{1)}.$$

We denote the principal part of L by L_0:

$$(1.2) \qquad L_0=\left(\frac{\partial}{\partial t}\right)^m+\sum_{\substack{|\nu|+j=m\\ j\leqq m-1}} a_{\nu,j}(x,t)\left(\frac{\partial}{\partial x}\right)^\nu\left(\frac{\partial}{\partial t}\right)^j.$$

We look for a necessary condition in order that the Cauchy problem for (1.1) is well posed when (1.2) has multiple characteristic roots with constant multiplicity.

First we give

DEFINITION 1.1. The Cauchy problem for (1.1) is said to be well posed in L^2 sense in an interval $[0, T]$ if the following two conditions are satisfied.

(1) For any prescribed initial data Ψ

$$(1.3) \qquad \Psi=\left\{\left(\frac{\partial}{\partial t}\right)^j u|_{t=0}=u_j\in \mathcal{D}_{L^2}^{m-j-1},\ j=0,1,2,\cdots,m-1\right\}$$

---

1) $\mathcal{B}(R^k)$ is the class of functions $f(x)=f(x_1,\cdots,x_k)$ such that their derivatives $\left(\frac{\partial}{\partial x}\right)^\nu f$ are bounded and continuous for $|\nu|=0, 1, 2, \cdots$.

and $f=f(x, t)\in \mathcal{E}_t^0(L^2)$ [2], there exists a unique solution

$$u(x, t)\in \mathcal{E}_t^0(\mathcal{D}_{L^2}^{m-1})\cap \mathcal{E}_t^1(\mathcal{D}_{L^2}^{m-2})\cap \cdots \cap \mathcal{E}_t^{m-1}(L^2)\ ^{2)} \tag{1.4}$$

which takes the given initial data at $t=0$.

(2) The energy inequality holds, that is

$$E(t:u)\leqq C_T\Big(E(0:u)+\int_0^t \|f(s)\|ds\Big) \tag{1.5}$$

$$0\leqq t\leqq T$$

holds for a constant C_T which depends only on T, where

$$E(t:u)=\sum_{j=0}^{m-1}\Big\|\Big(\frac{\partial}{\partial t}\Big)^j u(t)\Big\|_{m-j-1}. \tag{1.6}$$

Our purpose of this paper is to prove

THEOREM 1.1. *Assume that the characteristic equation*

$$L_0(x, t\,;\,\xi, \lambda)=\lambda^m+\sum_{\substack{|\nu|+j=m\\ j\leqq m-1}} a_{\nu,j}(x, t)\xi^\nu\lambda^j=0 \tag{1.7}$$

corresponding to (1.2) *has real roots with constant*[3] *multiplicity. Moreover we assume that there exists at least one root of* (1.7) *whose multiplicity is greater than unity. Then the Cauchy problem for*

$$(L_0+B)u=f \tag{1.8}$$

is not well posed in L^2 sense for any lower order operator B.

REMARK. In Theorem 1.1, we assume that (1.7) has real roots for any real $\xi\neq 0$. We notice here that if the Cauchy problem for (1.1) is well posed in L^2 sense, then the roots of (1.7) have to be real for any real $\xi\neq 0$. S. Mizohata [1] (also see P. D. Lax [2] and I. G. Petrowsky [3]) has already proved this fact in the case of C^∞-topology.

Using a successive approximation, we can prove that if the Cauchy problem for (1.1) is well posed in L^2 sense, then it is also the case for $(L+B)u=f$ for any lower order operator B. Then to get our Theorem, we only have to prove that there exists at least one lower order operator B such that the Cauchy problem for $(L_0+B)u=f$ is not well posed in L^2 sense.

2) $\mathcal{D}_{L^2}^m$ is the space of all functions $u(x)$ such that $\Big(\frac{\partial}{\partial x}\Big)^\nu u(x)\in L^2$ for $|\nu|\leqq m$, with the norm $\|u\|_m^2=\sum_{|\nu|\leqq m}\Big\|\Big(\frac{\partial}{\partial x}\Big)^\nu u\Big\|_{L^2}^2$. In addition, $\mathcal{E}_t^j(\mathcal{D}_{L^2}^{m-j})$ is a class of functions $v(x, t)$ such that $\frac{\partial^\alpha}{\partial t^\alpha}v(x, t)$ is continuous with the topology of $\mathcal{D}_{L^2}^{m-j}$ for $\alpha=0, 1, 2, \cdots, j$.

3) This condition can be slightly weakened. See condition (II) in the following text.

To make clear our reasoning, we consider rather simplified formulation of the problem as follows. First we formulate the following two conditions on (1.2).

(I) All roots of (1.7) are real for any real $\xi \neq 0$.

(II) There exist a neighbourhood Ω_0 of $(x, t) = (0, 0)$ and a neighbourhood Ω_1 of $\xi_0' = \xi_0 / |\xi_0|$ on the unit sphere $|\xi| = 1$ such that (1.7) can be written as

$$L_0(x, t : \lambda, \xi) = (\lambda - \lambda_1)^p \prod_{j \neq 1} (\lambda - \lambda_j) \tag{1.9}$$

for $(x, t, \xi) \in \Omega_0 \times \Omega_1$, where $\{\lambda_j\}_{j \neq 1}$ are distinct roots of (1.7).

Now let us prove the following

THEOREM 1.2. *Assume that* (1.2) *satisfies* (I) *and* (II), *then there exists a lower order operator B such that the Cauchy problem for $(L_0 + B)u = 0$ is not well posed in L^2 sense.*

Here we give a simple but suggestive example (also see [4] and [5]).

EXAMPLE. Let us consider the Cauchy problem for

$$L_0 u = \frac{\partial^2}{\partial t^2} u = 0 . \tag{1.10}$$

It is well posed in the sense of Petrowsky [3]. In fact for given initial data $u(0) = u_0(x) \in C_0^\infty$ and $\frac{\partial}{\partial t} u(0) = u_1(x) \in C_0^\infty$, $u(x, t) = u_0(x) + t u_1(x) \in C_{x,t}^\infty$ is a unique solution of (1.10) and this is of course continuous with respect to the given initial data. But the forward Cauchy problem for (1.10) is not well posed in L^2 sense. Because for given initial data $u(0) = u_0(x) \in \mathcal{D}_{L^2}^1$ and $\frac{\partial}{\partial t} u(0) = u_1(x) \in L^2$, the solution $u(x, t) = u_0(x) + t u_1(x)$ is not in $\mathcal{D}_{L^2}^1$ for $t > 0$.

We shall prove Theorem 1.2 in §§ 3–4. The method of our proof is quite similar to that of [1]. But we are concerned there with some particular pseudo-differential operators whose symbols have such a form as

$$h(x, \xi) = h_0(x, \xi) + h_1(x, \xi) |\xi|^{-\frac{1}{p}} + h_2(x, \xi) |\xi|^{-\frac{2}{p}} + \cdots \tag{1.11}$$

where $h_j(x, \xi)$, $j = 0, 1, 2, 3, \cdots$, are symbols of Calderón-Zygmund and p is a positive integer. In § 2 we treat the class of pseudo-differential operators attached with such symbols.

ACKNOWLEDGEMENT. The author wishes to express his sincere thanks to Professor S. Mizohata and Professor M. Yamaguti for their kind advice and constant encouragement, and also to Professor H. Kumano-go and Professor M. Matsumura for their helpful suggestions. The résumé of this article was published in [8].

§ 2. Pseudo-differential operators.

2.1. An algebra $\tilde{P}$ of pseudo-differential operators.

In this paragraph we shall define an algebra of pseudo-differential operators which are similar to those of Kohn-Nirenberg [**10**], Yamaguti [**5**].

2.1.1. Pseudo-differential operators of type P.

We consider a class of symbols:

$$h(x, \xi)=h_0(x, \xi)+h_1(x, \xi)|\xi|^{-\frac{1}{p}}+h_2(x, \xi)|\xi|^{-\frac{2}{p}}+\cdots \tag{2.1}$$

which satisfy the following two conditions (P.1) and (P.2):

(P.1) Using the notation of A. P. Calderón-A. Zygmund [**9**], $h_j(x, \xi)$ is of class C_β^{2k}, $\beta=+\infty$ for $j=0, 1, 2, 3, \cdots$.

(P.2) Setting

$$M_{H_j}=\sum_{|\nu|\leqq 2k}\sup_{x\in R^k,|\xi|\geqq 1}\left|\left(\frac{\partial}{\partial\xi}\right)^{\nu}h_j(x, \xi)\right|, \tag{2.2}$$

a power series

$$\sum_{j=0}^{\infty} M_{H_j}\varepsilon^{\frac{j}{p}} \tag{2.3}$$

has a non-zero radius of convergence. We denote it by $\varepsilon_0=\varepsilon_0(H)$. In (2.1) and (2.3), p is a positive integer.

We shall write (2.1) for convenience as

$$H\sim H_0+H_1\Lambda^{-\frac{1}{p}}+H_2\Lambda^{-\frac{2}{p}}+\cdots$$

where Λ is Fourier inverse image of $|\xi|$ and H_j is a singular integral operator of Calderón-Zygmund corresponding to the symbol $h_j(x, \xi)$.

Let us define a C^∞ function $\hat{\gamma}(\xi)$ by

$$\begin{aligned} &\hat{\gamma}(\xi)=\begin{cases} 1 & |\xi|\geqq R+1 \\ 0 & |\xi|\leqq R \end{cases} \\ &1\geqq\hat{\gamma}(\xi)\geqq 0 \qquad R\leqq|\xi|\leqq R+1, \end{aligned} \tag{2.4}$$

where $R>\varepsilon_0(H)^{-1}$. We denote by γ a pseudo-differential operator defined by $(\gamma u)^\wedge=\hat{\gamma}(\xi)\hat{u}$ for $u\in L^2$, where $\hat{u}$ is Fourier image of u.

Consider the class of all functions $\hat{\gamma}(\xi)$ defined as above for $R>\varepsilon_0(H)^{-1}$ and denote it by $\Gamma(H)$.

Now let us define a pseudo-differential operator H_γ by

$$H_\gamma=\sum_{j=0}^{\infty} H_j\gamma\Lambda^{-\frac{j}{p}}, \qquad \gamma\in\Gamma(H), \tag{2.5}$$

and for $u\in L^2$

$$H_\gamma u=\int e^{2\pi i x\cdot\xi}h(x, \xi)\hat{\gamma}(\xi)\hat{u}(\xi)d\xi. \tag{2.6}$$

We denote the symbol of H_γ by $\sigma(H_\gamma)$, namely $\sigma(H_\gamma)=h(x, \xi)\hat{\gamma}(\xi)$.

DEFINITION 2.1. Let $h(x, \xi)$ be a symbol satisfying the conditions (P.1) and (P.2). We call a pseudo-differential operator H_γ defined by (2.5) and (2.6) "a pseudo-differential operator of type P".

First we get

LEMMA 2.1. *A pseudo-differential operator H_γ of type P is a bounded operator in L^2.*

PROOF. Taking $u\in L^2(R^k)$, we have

$$\|H_\gamma u\|_{L^2}\leqq\sum_{j=0}^{\infty}\|H_j(\gamma\Lambda^{-\frac{j}{p}})u\|_{L^2}$$

$$\leqq c\sum_{j=0}^{\infty}M_{H_j}\|\gamma\Lambda^{-\frac{j}{p}}u\|_{L^2}\leqq c\sum_{j=0}^{\infty}M_{H_j}R^{-\frac{j}{p}}\|u\|_{L^2}\,.$$

Here we give an important example of a pseudo-differential operator of type P.

EXAMPLE 2.1. $H_\gamma=\gamma(1+\Lambda)^{-1}$ is of type P. In fact, we get by Taylor's expansion formula

$$(1+|\xi|)^{-1}=|\xi|^{-1}(1-|\xi|^{-1}+|\xi|^{-2}+\cdots)$$

and thus $\sigma(H_\gamma)=(1+|\xi|)^{-1}\hat{\gamma}(\xi)$ satisfies the conditions (P.1) and (P.2).

2.1.2. Orders of pseudo-differential operators.

Now let us define the orders of pseudo-differential operators generally.

DEFINITION 2.2. We say that H is a pseudo-differential operator of order $-r$, if for any $u\in L^2$, H satisfies the following:

$$\|H\Lambda^r u\|_{L^2}\leqq C_r\|u\|_{L^2}\,, \tag{2.7}$$

where C_r is a constant independent of u.

REMARK 1. By this definition we may say that

$$H^{(n)}=\sum_{j=n}^{\infty}H_j\gamma\Lambda^{-\frac{j}{p}}$$

is a pseudo-differential operator of type P of order $-n/p$. Then for sufficiently large n, $H^{(n)}$ is a bounded operator in L^2.

REMARK 2. Let $a(x, D)$ be a pseudo-differential operator of type P of order $-r$. Then we have for any $u\in L^2$,

$$\|a(x, D)\alpha_n u\|\leqq \text{const.}\, n^{-r}\|u\|\text{[4]}\,.$$

LEMMA 2.2. *For any $\gamma_1\in\Gamma(H)$ and $\gamma_2\in\Gamma(H)$, $(H_{\gamma_1}-H_{\gamma_2})$ is a pseudo-differential operator of order $-\infty$, then of course is a bounded operator in L^2.*

Proof is easy and is omitted.

4) The definition of α_n is given in 2.2.2. See page 176.

Now we can define a pseudo-differential operator $(H\circ K)_\gamma$ of type P which is attached to a symbol $h(x,\xi)k(x,\xi)$, where $\sigma(H_\gamma)=h(x,\xi)\hat{\gamma}(\xi)$ and $\sigma(K_\gamma)=k(x,\xi)\hat{\gamma}(\xi)$. In fact, a product of two Puiseux series is also a Puiseux series and has a radius of convergence which is equal to the minimum of two radius of convergence corresponding to

$$\sum_{j=0}^{\infty} M_{H_j}\varepsilon^{\frac{j}{p}} \quad\text{and}\quad \sum_{j=0}^{\infty} M_{K_j}\varepsilon^{\frac{j}{p}}.$$

By this remark, we have the following

LEMMA 2.3.[5] *Let H_γ and K_γ are two pseudo-differential operators of type P, then $(H_\gamma K_\gamma-(H\circ K)_\gamma)\Lambda$*[6] *is of order zero.*

PROOF. Let us denote the equality modulo bounded operators in L^2 by $\equiv$. Taking the Remark 1 into account, we have

$$\begin{aligned}
H_\gamma K_\gamma \Lambda &\equiv H_\gamma(K_0+K_1(\gamma\Lambda^{-\frac{1}{p}})+\cdots+K_{p-1}(\gamma\Lambda^{-\frac{p-1}{p}}))\Lambda\\
&\equiv H_\gamma\Lambda(K_0\gamma+\cdots+K_{p-1}(\gamma\Lambda^{-\frac{p-1}{p}}))\\
&\equiv (H_0\gamma+\cdots+H_{p-1}(\gamma\Lambda^{-\frac{p-1}{p}}))\Lambda(K_0\gamma+\cdots+K_{p-1}(\gamma\Lambda^{-\frac{p-1}{p}}))\\
&\equiv \sum_{i+j=0}^{2(p-1)} H_i(\gamma\Lambda^{-\frac{i}{p}})K_j(\gamma\Lambda^{-\frac{j}{p}})\Lambda\\
&\equiv \sum_{i+j=0}^{2(p-1)} H_iK_j(\gamma^2\Lambda^{-\frac{i+j}{p}})\Lambda\\
&\equiv \sum_{i+j=0}^{2(p-1)} (H_i\circ K_j)\Lambda(\gamma^2\Lambda^{-\frac{i+j}{p}})\\
&\equiv \sum_{i+j=0}^{2(p-1)} (H_i\circ K_j)\Lambda(\gamma\Lambda^{-\frac{i+j}{p}})\equiv (H\circ K)_\gamma\Lambda.
\end{aligned}$$

2.1.3. Now let us prove a fundamental Lemma with respect to pseudo-differential operators of type P.

LEMMA 2.4. (1) *Suppose that* $\inf_{x,\xi}|h_0(x,\xi)|=\delta>0$. *If the support of $\hat{u}(\xi)\in L^2$ lies outside of the ball $|\xi|\leqq R$ $(R>\varepsilon_0(H)^{-1})$, then we have*

$$\|H_\gamma u\|_{L^2}\geqq\Big(\frac{\delta}{2}-c_1R^{-1}-c_2\sum_{j=1}^{\infty}M_{H_j}R^{-\frac{j}{p}}\Big)\|u\|_{L^2}, \tag{2.7}$$

where c_1 and c_2 are constants independent of R and u.

(2) *In the case $p\geqq 2$, suppose* $\mathrm{Re}\,h_0(x,\xi)=0$ *and* $\inf_{x,\xi}\mathrm{Re}\,h_1(x,\xi)=\delta>0$. *Let*

5) In general, we should have to prove for H_{γ_1} and H_{γ_2}, but in view of Lemma 2.2, without loss of generality, we may assume that $\gamma_1=\gamma_2$.

6) Of course $H_\gamma K_\gamma$ is no longer a pseudo-differential operator of type P. Also see Lemma 2.8.

$\{\hat{u}_n(\xi)\}$ be an arbitrary sequence of L^2 whose supports lie between two concentric spheres $|\xi|=c_1 n$ and $|\xi|=c_2 n$ ($c_1<c_2$, $n=1, 2, 3, \cdots$). Then we have

$$\operatorname{Re}(H_\gamma \Lambda u_n, u_n) \geqq c_3 n^{1-\frac{1}{p}} \|u_n\|_{L^2}^2, \tag{2.8}$$

for $n \geqq n_0$. Where (,) means the inner product of L^2, and c_3 is a constant independent of n. $\operatorname{Re} f$ means a real part of f.

PROOF. (1) First we get

$$\|H_\gamma u\| \geqq \|H_0 \gamma u\| - \sum_{j=1}^{\infty} \|H_j(\gamma \Lambda^{-\frac{j}{p}}) u\|.$$

On the other hand, using Lemma 2.1 of [**6**], we get

$$\begin{aligned} \|H_0 \gamma u\| &= \|H_0 \Lambda(\gamma \Lambda^{-1} u)\| \\ &\geqq \frac{\delta}{2} \|\Lambda(\gamma \Lambda^{-1} u)\| - C\|\gamma \Lambda^{-1} u\| \\ &\geqq \frac{\delta}{2} \|u\| - CR^{-1}\|u\|, \end{aligned}$$

and thus we get (1).

(2) For any integer n, we get

$$\operatorname{Re}(H_\gamma \Lambda u_n, u_n) \geqq \operatorname{Re}(H_1(\gamma \Lambda^{-\frac{1}{p}}) \Lambda u_n, u_n) - \sum_{j=2}^{\infty} |(H_j(\gamma \Lambda^{-\frac{j}{p}}) \Lambda u_n, u_n)|.$$

Now we shall show that the first term of the right-hand side is greater than

$$C n^{1-\frac{1}{p}} \|u_n\|_{L^2}^2$$

for some positive constant C which is independent of n[7].

$$\begin{aligned} (H_1 \gamma \Lambda^{1-\frac{1}{p}} u_n, u_n) = &([H_1, \Lambda^{\frac{1}{2}(1-\frac{1}{p})}] \gamma \Lambda^{\frac{1}{2}(1-\frac{1}{p})} u_n, u_n) \\ &+ (H_1 \gamma \Lambda^{\frac{1}{2}(1-\frac{1}{p})} u_n, \Lambda^{\frac{1}{2}(1-\frac{1}{p})} u_n)^{8)}. \end{aligned}$$

A pseudo-differential operator $[H_1, \Lambda^{\frac{1}{2}(1-\frac{1}{p})}] \Lambda^{\frac{1}{2}(1-\frac{1}{p})}$ is of order $-1/p$, that is a bounded operator in L^2. On the other hand, we get by the assumption

$$\operatorname{Re}(H_1 \gamma \Lambda^{\frac{1}{2}(1-\frac{1}{p})} u_n, \Lambda^{\frac{1}{2}(1-\frac{1}{p})} u_n) \geqq \frac{\delta}{2} \|\Lambda^{\frac{1}{2}(1-\frac{1}{p})} u_n\|_{L^2}^2,$$

thus our assertion is clarified. By this fact we have

$$\operatorname{Re}(H_\gamma \Lambda u_n, u_n) \geqq C n^{1-\frac{1}{p}} \|u_n\|^2 - \sum_{j=2}^{\infty} |(H_j(\gamma \Lambda^{-\frac{j}{p}}) \Lambda u_n, u_n)|.$$

7) We refer readers to Lemma 2.1 of [**1**].

8) Let A and B be two operators. We denote the commutator $AB-BA$ by $[A, B]$.

Finally we can easily see that the last term of the right-hand side is dominated by

$$C'n^{1-\frac{2}{p}}\|u_n\|_{L^2}^2$$

where C' is a constant which is independent of n. Thus we get (2) for sufficiently large n.

2.1.4. An algebra $\tilde{P}$.

In this paragraph we define an algebra $\tilde{P}$ of pseudo-differential operators which contains pseudo-differential operators of type P.

DEFINITION 2.3. We say that a pseudo-differential operator H belongs to a class $\tilde{P}$ if for any s there exists a pseudo-differential operator $H_\gamma^{(s)}$ of type P such that $(H-H_\gamma^{(s)})$ is of order $-s$, that is for any $u\in L^2$

$$\|(H-H_\gamma^{(s)})\Lambda^s u\|_{L^2}\leqq c\|u\|_{L^2}$$

with a constant c independent of u.

REMARK 3. By this definition we easily see that

(1) If H belongs to the class $\tilde{P}$, then for $a(x)\in\mathscr{B}(R^k)$, $a(x)H$ also belongs to $\tilde{P}$.

(2) Let h be a distribution such that whose Fourier image $\hat{h}(\xi)$ is infinitely differentiable in $R_\xi^n-\{0\}$ and homogeneous of degree zero. For any H of the class $\tilde{P}$, Hh also belongs to $\tilde{P}$.

Now we shall show that the class $\tilde{P}$ is closed with respect to the commutator operation with Λ. First we get

LEMMA 2.5.[9] *Let $a(x)$ be a function of $\mathscr{B}(R^k)$, then the commutator $[a(x), \Lambda]$ belongs to the class $\tilde{P}$.*

PROOF. First we decompose Λ:

$$\Lambda=\Lambda_1+\Lambda_2=(1-\gamma)\Lambda+\gamma\Lambda,\ \gamma\in\Gamma(H)^{10)}$$

then we get $[a(x), \Lambda]=[a(x), \Lambda_1]+[a(x), \Lambda_2]$. Evidently $[a(x), \Lambda_1]$ is of order $-\infty$.

Now let us show that

$$[a(x), \Lambda_2]=\sum_{1\leqq|\nu|<q}\frac{(-1)^{|\nu|+1}a^{(\nu)}(x)}{\nu\,!}(x^\nu\Lambda_2)+B_q\,, \tag{2.10}$$

where B_q is of order $-(q-k-1)$.

From this fact our lemma is immediate. In fact, for a given integer s, if we take q as $q=k+1+s$, B_q is of order $-s$ and $[a(x), \Lambda_2]-B_q$ is a pseudo-differential operator of type P.

Take $u\in\mathscr{D}_{L^2}^\infty$, then we have

9) We refer readers to Lemma 2.4 of [6].

10) For convenience we use $\gamma\in\Gamma(H)$. Of course γ may be replaced by any function with the similar property.

$$B_q u = \sum_{|\nu|=q} \frac{(-1)^{|\nu|+1}}{\nu!} \int a_\nu(x, y) \Lambda_2(x-y)(x-y)^\nu u(y) dy .$$

By Hausdorff-Young's inequality, we get

$$\|B_q u\|_{L^2} \leqq c\|u\|_{L^2} \|(x^\nu \Lambda_2)\|_{L^1} .$$

Now let us examine $\|x^\nu \Lambda_2\|_{L^1}$ more precisely. By Fourier transform we get

$$|x^\nu \Lambda_2| \leqq c\int \left|\left(\frac{\partial}{\partial\xi}\right)^\nu (\hat{\gamma}(\xi)|\xi|)\right| d\xi, \qquad |\nu| = q .$$

Now we have

$$D_\xi^\nu(\hat{\gamma}(\xi)|\xi|) = \sum_{0<|\alpha|\leqq q} \binom{\nu}{\alpha}(D_\xi^\alpha \hat{\gamma}(\xi))(D_\xi^{\nu-\alpha}|\xi|) + \sum_{|\nu|=q} \hat{\gamma}(\xi)(D_\xi^\nu|\xi|) .$$

By the definition of $\hat{\gamma}(\xi)$, the first term of the right-hand side defines a pseudo-differential operator of order $-\infty$. Next the last term of the right-hand side defines a pseudo-differential operator of order $-(q-k-1)$. In fact, since we have

$$\left|\sum_{|\nu|=q} \hat{\gamma}(\xi)|\xi|^{q-k-1} D^\nu |\xi|\right| \leqq c\hat{\gamma}(\xi)|\xi|^{-k-1} ,$$

we get

$$|\Lambda^{q-k-1} x^\nu \Lambda_2| \leqq c\int \hat{\gamma}(\xi)|\xi|^{-k-1} d\xi < +\infty .$$

In the same way we get

$$\left| |x|^{2s+1}(\Lambda^{q-k-1}(x^\nu \Lambda_2)) \right| \leqq c' \int \hat{\gamma}(\xi)|\xi|^{-k-2s-3} d\xi < +\infty .$$

In the sequel, we get

$$\int |\Lambda^{q-k-1}(x^\nu \Lambda_2)| dx = \int_{|x|\leqq 1} \cdots dx + \int_{|x|\geqq 1} \cdots dx$$

$$\leqq c\int \hat{\gamma}(\xi)|\xi|^{-k-1} d\xi \int_{|x|\leqq 1} dx + c' \int \hat{\gamma}(\xi)|\xi|^{-k-2s-3} d\xi \int_{|x|\geqq 1} \frac{dx}{|x|^{2s+2}} .$$

Thus $\|\Lambda^{q-k-1}(x^\nu \Lambda_2)\|_{L^1} < +\infty$, and this proves that B_q is a pseudo-differential operator of order $-(q-k-1)$.

Now we return to our lemma. In (2.9), the first term is obviously of type P. Finally, for any s given if we take q as $q-k-1=s$, then B_q is of order $-s$. Thus our proof is complete.

COROLLARY 1. *Let h be a distribution mentioned in Remark* 3. *Then* $[a(x), h]$ *belongs to a class* $\tilde{P}$, *where* $a(x)$ *is a function of class* $\mathscr{B}$.

COROLLARY 2. *Let H be a singular integral operator of Calderón-Zygmund whose symbol is of class* C_β^∞, $\beta = +\infty$. *Then* $[H, \Lambda]$ *belongs to the class* $\tilde{P}$.

PROOF. First we develop the symbol $\sigma(H) = h(x, \xi)$ as follows:

$$h(x, \xi)=a_0(x)+\sum_{n=0}^{\infty}\sum_{m=1}^{n(m)}a_{n,m}(x)\hat{Y}_{n,m}(\xi)\,,$$

where $\{\hat{Y}_{n,m}(\xi)\}$ are spherical harmonics of order n. Now applying Lemma 2.5 term by term, we get Corollary 2.

LEMMA 2.6. *If H_γ is of type P, then $[H_\gamma, \Lambda]$ belongs to the class $\tilde{P}$.*

PROOF. For any given number s, we decompose H_γ into

$$H_\gamma=H_{\gamma,0}+H^{(s-1)}\,,$$

where $H^{(s-1)}$ is of order $-(s-1)$ and $H_{\gamma,0}$ is a finite sum of $\{H_j(\gamma\Lambda^{-\frac{j}{p}})\}$ (see Remark 1).

Now we get $[H_\gamma, \Lambda]=[H_{\gamma,0}, \Lambda]+[H^{(s-1)}, \Lambda]$, and evidently $[H^{(s-1)}, \Lambda]$ is of order $-s$.

Finally we see that

$$[H_{\gamma,0}, \Lambda]=\sum_{\text{finite}}[H_j, \Lambda](\gamma\Lambda^{-\frac{j}{p}})$$

belongs to the class $\tilde{P}$. In fact, by Corollary 2 to Lemma 2.5 $[H_j, \Lambda]$ belongs to the class $\tilde{P}$ for $j=0, 1, 2, \cdots, j_0$. Thus $[H_{\gamma,0}, \Lambda]$ belongs to the class $\tilde{P}$.

By the following theorem, we know that a class $\tilde{P}$ is closed with respect to the operation to take commutator with Λ.

THEOREM 2.1. *If a pseudo-differential operator H belongs to $\tilde{P}$, then $[H, \Lambda]$ also belongs to $\tilde{P}$.*

PROOF. For any given number s, there exists a pseudo-differential operator $H^{(s-1)}$ of type P such that $(H-H^{(s-1)})$ is of order $-(s-1)$.

Now we get $[H, \Lambda]=[H-H^{(s-1)}, \Lambda]+[H^{(s-1)}, \Lambda]$, and evidently $[H-H^{(s-1)}, \Lambda]$ is of order $-s$. Finally, by Lemma 2.6 $[H^{(s-1)}, \Lambda]$ belongs to $\tilde{P}$.

Now let us prove that the class $\tilde{P}$ is an algebra. First we prove the following

LEMMA 2.7. *Let H and K are singular integral operators of Calderón-Zygmund. Then $(HK-H\circ K)$ belongs to the class $\tilde{P}$.*

PROOF. Using a development of symbols by spherical harmonics, we only have to prove our lemma in the simplest case such that $\sigma(H)=a_1(x)h$ and $\sigma(K)=a_2(x)k$, where h and k are distributions mentioned in Remark 3. Taking Corollary 1 to Lemma 2.5 and Remark 3 into account, the proof is easy.

LEMMA 2.8. *Let H_γ and K_γ are pseudo-differential operators of type P, then $H_\gamma K_\gamma$ belongs to the class $\tilde{P}$.*

PROOF. For a given s, we can decompose H_γ and K_γ into $H_\gamma=H_{\gamma,0}+H^{(s)}$ and $K_\gamma=K_{\gamma,0}+K^{(s)}$, respectively, where $H^{(s)}$ and $K^{(s)}$ are of order $-s$, and $H_{\gamma,0}$ and $K_{\gamma,0}$ are finite sum of $\{H_j\gamma\Lambda^{-\frac{j}{p}}\}$ and $\{K_j\gamma\Lambda^{-\frac{j}{p}}\}$, respectively.

Now we see

$$H_\gamma K_\gamma=H_{\gamma,0}K_{\gamma,0}+H^{(s)}K_{\gamma,0}+H_{\gamma,0}K^{(s)}+H^{(s)}K^{(s)}\,.$$

Evidently the last three terms of the right-hand side are of order $-s$. Finally let us prove that the first term of the right-hand side belongs to the class $\tilde{P}$. In fact, we see

$$H_{\gamma,0}K_{\gamma,0}=\sum_{i+j=0}^{\text{finite}}(H_i\gamma\Lambda^{-\frac{i}{p}})(K_j\gamma\Lambda^{-\frac{j}{p}})$$

$$=\sum H_i(\gamma\Lambda^{-\frac{i}{p}}K_j-K_j\gamma\Lambda^{-\frac{i}{p}})\gamma\Lambda^{-\frac{j}{p}}$$

$$+\sum(H_iK_j-H_i\circ K_j)\gamma^2\Lambda^{-\frac{i+j}{p}}$$

$$+\sum(H_i\circ K_j)\gamma^2\Lambda^{-\frac{i+j}{p}}.$$

By Lemma 2.7 the second and the last terms of the right-hand side belong to the class $\tilde{P}$. Finally by Corollary 2 to Lemma 2.5, the first term of the right-hand side belongs to a class $\tilde{P}$.

In the sequel, we get the following theorem and thus we see that a class $\tilde{P}$ of pseudo-differential operators is an algebra.

THEOREM 2.2. *Let H and K are pseudo-differential operators of a class $\tilde{P}$, then HK also belongs to a class $\tilde{P}$.*

PROOF. For a given s, there exist $H_\gamma^{(s)}$ and $K_\gamma^{(s)}$ of type P such that $H-H_\gamma^{(s)}$ and $K-K_\gamma^{(s)}$ are of order $-s$. Now we have

$$HK=H(K-K_\gamma^{(s)})+(H-H_\gamma^{(s)})K_\gamma^{(s)}+H_\gamma^{(s)}K_\gamma^{(s)}$$
$$=I+II+III.$$

First it is evident that I is of order $-s$. Next by Lemma 2.8, III belongs to the class $\tilde{P}$. Finally we have

$$II\ \Lambda^s=(H-H_\gamma^{(s)})K_\gamma^{(s)}\Lambda^s=(H-H_\gamma^{(s)})\Lambda^s\Lambda^{-s}K_\gamma^{(s)}\Lambda^s$$

and $\Lambda^{-s}K_\gamma^{(s)}\Lambda^s$ is a bounded operator in L^2. Thus II is of order $-s$, and we get our theorem.

2.2. Class (H) of pseudo-differential operators.

2.2.1. We treat some elementary pseudo-differential operators of the form

$$a(x, D)=\sum_{\text{finite}}a_i(x)\phi_i(D) \tag{2.11}$$

where $a_i(x)\in\mathcal{B}(R^k)$ and $\phi_i(x)$ are distributions belonging to $\mathcal{D}'_{L^2}$ [11].

DEFINITION 2.4. The Fourier image $\hat{\phi}(\xi)$ of $\phi(x)\in\mathcal{D}'_{L^2}$ is said to be a hypoelliptic symbol if it satisfies the following two conditions:

(H_1) $\hat{\phi}(\xi)$ is locally bounded and infinitely differentiable outside of some compact set, say $|\xi|\leqq R$.

11) $\mathcal{D}'_{L^2}$ is a dual space of $\mathcal{D}^\infty_{L^2}=\bigcap_{m=1}^{\infty}\mathcal{D}^m_{L^2}$.

(H_2) There exists a positive integer s such that $D_\xi^\nu \hat{\phi}(\xi)$ is summable for $|\xi| \geqq R$ if $|\nu| \geqq s$.

DEFINITION 2.5. The pseudo-differential operator $a(x, D)$ defined by (2.11) is said to be of class (H) if $\hat{\phi}_i(\xi)$ is a hypoelliptic symbol for all i.

Let $\phi(D)$ be a pseudo-differential operator of class (H), and $b(x) \in \mathscr{B}(R^k)$. Now we consider the commutator $[b(x), \phi(D)]$. Using $\gamma(\xi)$ previously defined[12], we decompose $\hat{\phi}(\xi)$ as

$$\hat{\phi}(\xi) = (1-\gamma(\xi))\hat{\phi}(\xi) + \gamma(\xi)\hat{\phi}(\xi),$$

which gives the corresponding decomposition

$$\phi(x) = \phi_0(x) + \phi_1(x), \quad \hat{\phi}_1(\xi) = \gamma(\xi)\hat{\phi}(\xi):$$

Then we have

$$[b(x), \phi(D)] = [b(x), \phi_0(D)] + [b(x), \phi_1(D)].$$

Evidently $[b(x), \phi_0(D)]$ is a bounded operator.

LEMMA 2.9.

$$[b(x), \phi_1(D)] = \sum_{|\nu|=1}^{s-1} (-1)^{|\nu|+1} \frac{D^\nu b(x)}{\nu!} (x^\nu \phi_1(x))(D) + B_0, \tag{2.12}$$

where B_0 is a bounded operator in L^2 and its operator norm is estimated by

$$c\,|b(x)|_{\mathscr{B}^s}(\|D^s \hat{\phi}_1(\xi)\|_{L^1} + \|D^{s+2k}\hat{\phi}_1(\xi)\|_{L^1})^{13)}$$

where c is a constant independent of b and ϕ_1.

PROOF. Take $u(x) \in \mathscr{D}_{L^2}^\infty$, then we have

$$[b(x), \phi_1]u = \int (b(x)-b(y))\phi_1(x-y)u(y)dy.$$

Now by Taylor's formula

$$b(x)-b(y) = -\Big(\sum_{|\nu|=1}^{s-1} \frac{(-1)^{|\nu|}}{\nu!} D^\nu b(x)(x-y)^\nu + \sum_{|\nu|=s} \frac{(-1)^{|\nu|}}{\nu!} b_\nu(x, y)(x-y)^\nu\Big),$$

then we have

$$[b(x), \phi_1]u = \sum_{|\nu|=1}^{s-1} \frac{(-1)^{|\nu|+1}}{\nu!} D^\nu b(x) \int (x-y)^\nu \phi_1(x-y)u(y)dy$$
$$+ \sum_{|\nu|=s} \frac{(-1)^{|\nu|+1}}{\nu!} \int \phi_1(x-y)(x-y)^\nu b_\nu(x, y)u(y)dy.$$

Denoting the operator of the last term of the right-hand side by B_0, we get (2.12).

Now we consider one of the last terms. Using Hausdorff-Young's in-

12) See the footnote 10).

13) $|b(x)|_{\mathscr{B}^s}$ is defined by $|b(x)|_{\mathscr{B}^s} = \sup_{|\nu| \leqq s, x \in R^k} |D^\nu b(x)|$.

equality, its operator norm is estimated by

$$\sup |b_\nu| \, \|x^\nu \phi_1\|_{L^1}, \qquad |\nu| = s\,.$$

By Fourier transform we get

$$|x^\nu \phi_1| \leqq c \int |D^\nu \hat{\phi}_1(\xi)| \, d\xi = c \|D^\nu \hat{\phi}_1\|_{L^1}\,.$$

Then

$$\begin{aligned}
\|x^\nu \phi_1\|_{L^1} &= \int |x^\nu \phi_1| \, dx \\
&= \int_{|x|\leqq 1} |x^\nu \phi_1| \, dx + \int_{|x|\geqq 1} |x^\nu \phi_1| \, dx \\
&= c' \|D^\nu \hat{\phi}_1(\xi)\|_{L^1} + c'' \|D^{2k+\nu} \hat{\phi}_1(\xi)\|_{L^1} \int_{|x|\geqq 1} \frac{dx}{|x|^{2k}}
\end{aligned}$$

thus we get our lemma.

We define the orders of pseudo-differential operators of class (H) and give some remarks concerned with them.

DEFINITION 2.6. A pseudo-differential operator $\phi(D)$ of class (H) is said to be of order $\sigma(>0)$ if for any $u \in L^2$

$$\|\phi(D)u\|_{-\sigma} \leqq \text{const.} \, \|u\|_0\,.$$

By this definition, we see easily that $\phi(D)$ is of order one if

$$|\hat{\phi}(\xi)| \leqq \text{const.} \, (1+|\xi|)\,.$$

Moreover if $|D_\xi^\nu \hat{\phi}(\xi)| \leqq \text{const.}$ for $|\nu| \geqq 1$, then the commutator $[b(x), \phi(D)]$ is of order 0.

2.2.2. A pseudo-differential operator α_n of class (H).

Here we shall give an important example of pseudo-differential operator of class (H). This is originally considered by S. Mizohata (see [7], [1], etc.).

Let $\hat{\alpha}(\xi)$ be a infinitely differentiable function with a compact support which lies in a neighbourhood of $\xi_0 \neq 0$ not contains $\xi = 0$. Moreover let $\hat{\alpha}(\xi)$ takes the value 1 in a neighbourhood of ξ_0 and $0 \leqq \hat{\alpha}(\xi) \leqq 1$[14]. We define $\hat{\alpha}_n(\xi)$ by $\hat{\alpha}_n(\xi) = \hat{\alpha}\left(\frac{\xi}{n}\right)$, then $\hat{\alpha}_n(\xi)$ is a hypoelliptic symbol whose support lies in a neighbourhood of $n\xi_0$.

If we define a pseudo-differential operator $\alpha_n(D)$ by

$$\alpha_n(D)u \xrightarrow{\mathcal{F}} \hat{\alpha}_n(\xi)\hat{u}(\xi)\,, \qquad u(x) \in L^2\,, \tag{2.13}$$

then $\alpha_n(D)$ is of class (H).

14) Of course this condition is not necessary for $\hat{\alpha}(\xi)$ to be a hypoelliptic symbol. But we assume this for a convenience of an actual use in the following paragraphs.

Finally we notice that

$$|D_\xi^\nu \hat{\alpha}_n(\xi)| \leqq C\Big(\frac{1}{n}\Big)^{|\nu|} \tag{2.14}$$

where C is a constant which is independent of n.

§ 3. Fundamental inequality.

Before the proof of Theorem 1.2 we shall give a differential inequality.

First we consider an approximation to the operator L_0+B. In the assumption (*II*), we assumed $\xi_0 \neq 0$. Then without losing generality, we may assume that the first component $\xi_0^{(1)}$ of ξ_0 is non-zero: for example we consider the case $\xi_0=(1, \cdots, *)$.

Let us take

$$B=b\Big(\frac{\partial}{\partial x_1}\Big)^{m-1} \tag{3.1}$$

as a lower order operator where b is a non-zero real constant. We define its size later.

Setting $U={}^t\Big(u(x, t), \dfrac{\partial}{\partial t}u(x, t), \cdots, \dfrac{\partial^{m-1}}{\partial t^{m-1}}u(x, t)\Big)$, we consider an equivalent system to $(L_0+B)u=0$:

$$\frac{\partial}{\partial t}U=A\Big(x, t; \frac{\partial}{\partial x}\Big)U \tag{3.2}$$

where

$$A\Big(x, t; \frac{\partial}{\partial x}\Big)=\begin{bmatrix} 0, & 1, & \cdots, & 0, & 0 \\ & \ddots & \ddots & & \vdots \\ & & & 0, & 1 \\ -a_m-b\Big(\dfrac{\partial}{\partial x_1}\Big)^{m-1}, & -a_{m-1}, & \cdots, & & -a_1 \end{bmatrix}$$

$$a_j=\sum_{|\nu|=j} a_{\nu,m-j}(x, t)\Big(\frac{\partial}{\partial x}\Big)^\nu .$$

Now we consider a localization of (3.2). We take an infinitely differentiable function $\beta(x)$ which has a compact support contained in Ω_0 and takes the value 1 in a neighbourhood of $x=0$.

Apply $\beta(x)$ to (3.2), then

$$\frac{\partial}{\partial t}(\beta U)=A(\beta U)+[\beta, A]U \tag{3.3}$$

where

$$[\beta, A]U=\beta(x)(AU)-A(\beta(x)U).$$

This is the first localization.

Next we apply α_n defined in § 2 (2.2.2) to (3.3):

$$\frac{\partial}{\partial t}\alpha_n(\beta U)=A\alpha_n(\beta U)+[\alpha_n, A](\beta U)+\alpha_n([\beta, A]U) \tag{3.4}$$

where

$$[\alpha_n, A]U=\alpha_n(AU)-A(\alpha_n u).$$

Now we apply a pseudo-differential operator $E_m(\Lambda)$ of class (H):

$$E_m(\Lambda)=\begin{bmatrix} \{i(\Lambda+1)\}^{m-1} & & & 0 \\ & \{i(\Lambda+1)\}^{m-2} & & \\ & & \ddots & \\ 0, & & & ,1 \end{bmatrix}$$

to (3.4). Then we get[15] a first order system

$$\frac{\partial}{\partial t}E_m\alpha_n(\beta U)=E_mAE_m^{-1}(E_m\alpha_n(\beta U)) \tag{3.5}$$

$$+[\alpha_n, AE_m^{-1}]E_m(\beta U)+\alpha_n([\beta, AE_m^{-1}]E_mU).$$

Here we observe matrices $[\alpha_n, AE_m^{-1}]$ and $[\beta, AE_m^{-1}]$ precisely. All the entries except the m^{th} row are zero. On the other hand each entry of the m^{th} row of AE_m^{-1} is a pseudo-differential operator of class (H). In view of Lemma 2.9, we can see easily

LEMMA 3.1. *$[\alpha_n, AE_m^{-1}]$ and $[\beta, AE_m^{-1}]$ are both bounded operators in $(L^2(R^k))^m$. Moreover, there exists a constant c which is independent of n such that*

$$\|[\alpha_n, AE_m^{-1}]\|\leqq c.$$

Now we can express (3.5) by pseudo-differential operators:

$$\frac{d}{dt}V_n=(\mathcal{H}_0+\mathcal{H}_1)\Lambda V_n+B_1V_n+F_n \tag{3.7}$$

where

$$\mathcal{H}_0=\begin{bmatrix} 0, & i, & 0, \cdots, 0 \\ 0, & \cdots\cdots\cdots, & 0, i \\ h_m, & h_{m-1}, & \cdots, h_1 \end{bmatrix} \quad \textit{and} \quad \mathcal{H}_1=\begin{bmatrix} & 0 & \\ & & \\ b_0, & 0, \cdots, & 0 \end{bmatrix}$$

$$\sigma(h_j)=-ia_j(x, t; \xi)|\xi|^{-j}, \qquad \sigma(b_0)=-b\xi_1^{m-1}|\xi|^{-m}$$

and B_1 is a pseudo-differential operator of order zero of class (H), and finally

15) Since all entries of $[\alpha_n, A]$ and $[\beta, A]$ except the m^{th} row are zero, and moreover the operator α_n is commutative with Λ, then we can easily see that $E_m[\alpha_n, A]U=[\alpha_n, AE_m^{-1}]E_mU$, and $E_m\alpha_n([\beta, A]U)=\alpha_n([\beta, AE_m^{-1}]E_mU)$.

$$V_n = E_m \alpha_n(\beta U)$$
$$F_n = [\alpha_n, AE_m^{-1}]E_m(\beta U) + \alpha_n([\beta, AE_m^{-1}]E_m U).$$

Denoting $\mathcal{H}_0 + \mathcal{H}_1$ by $\mathcal{H}$, $\mathcal{H}$ is obviously a pseudo-differential operator of type P.

By Lemma 3.1, we get the following

LEMMA 3.2. *There exists a positive constant c independent of U and n, such that*

$$\|B_1 V_n + F_n\| \leqq c\|E_m U\|.$$

Now we consider to diagonalize the matrix $\sigma(\mathcal{H})$. Consider

$$P(x, t; i\xi, \lambda) = \det(\lambda I - \sigma(\mathcal{H})) \tag{3.8}$$
$$= \lambda^m + a_1(x, t; i\xi')\lambda^{m-1} + \cdots + a_m(x, t; i\xi') + b(i\xi_1)^{m-1}|\xi|^{-m}.$$

where $\xi' = \xi/|\xi|$. Let us notice that the last term is $b(i\xi_1/|\xi|)^{m-1}|\xi|^{-1}$. In other words, this is a function of homogeneous degree zero multiplied by $|\xi|^{-1}$. Since b is non-zero, we can get roots of $P(x, t; i\xi) = 0$ in Puiseux series for $|\xi|$ sufficiently large and $(x, t, \xi') \in \Omega_0 \times \Omega_1$:

$$\lambda_{r,\varepsilon} = \lambda_1(x, t; i\xi') + \sum_{j=1}^{\infty} e^{\frac{2\pi i}{p}(r-1)} c_j(x, t; \xi')\varepsilon^{\frac{j}{p}} \tag{3.9}$$

where $\varepsilon = |\xi|^{-1}$, $r = 1, 2, \cdots, p$. Let us notice here that there exists a constant δ such that

$$|c_1(x, t; \xi')| \geqq \delta > 0. \tag{3.10}$$

In the same way, perturbed roots $\lambda_{q,\varepsilon}$ corresponding to the distinct roots $\{\lambda_{q+1-p}\}$ of $\det(\lambda I - \sigma(\mathcal{H}_0)) = 0$ are expressed by Taylor series in $\varepsilon = |\xi|^{-1}$:

$$\lambda_{q,\varepsilon} = \lambda_{q-p+1} + \sum_{n=1}^{\infty} c_n^{(q)}(x, t; \xi')\varepsilon^n \tag{3.11}$$
$$q = p+1, p+2, \cdots, m.$$

For $(x, t; \xi') \in \Omega_0 \times \Omega_1$, (3.9) and (3.11) are symbols which satisfy (P.1) and (P.2) in § 2. We shall show it in Appendix.

As previously mentioned, (3.9) and (3.11) are defined only for $(x, t; \xi') \in \Omega_0 \times \Omega_1$. We shall extend the definition of (3.9) and (3.11) in the whole space $R_x^k \times \{|\xi| = 1\} \times [0, T'], [0, T'] \subset \Omega_0 \cap R_t^1$.

Since $\{\lambda_j(x, t; \xi)\}$ are smooth in (x, t, ξ), we may assume the condition (II) in § 1 for $(x, t; \xi') \in \Omega_0' \times \Omega_1'$ where $\Omega_0' \Supset \Omega_0$ and $\Omega_1' \Supset \Omega_1$.

Let $\rho(x) \in C^\infty$ be a function defined in the whole R_x^k which keep Ω_0 invariant, moreover $\rho(x)$ is contained in Ω_0' for any x. Next let $s(\xi') \in C^\infty$ be a transformation on $|\xi| = 1$ into Ω_1' which keeps Ω_1 invariant.

Now we replace x and ξ' in (3.9)-(3.11) by $\rho(x)$ and $s(\xi')$ respectively.

Then

$$c_n^{(j)}(x, t\,;\xi')=c_n^{(j)}(\rho(x), t\,; s(\xi')) \tag{3.12}$$

is defined in the whole space $R_x^k\times\{|\xi|=1\}\times[0, T']$, for $j=1, 2, \cdots, m\,;\ n=1, 2, 3, \cdots$.

We denote Vandermode matrix

$$\begin{bmatrix} 1 & \cdots, & 1 \\ \lambda_{1,\varepsilon} & & \lambda_{m,\varepsilon} \\ \lambda_{1,\varepsilon}^2 & & \lambda_{m,\varepsilon}^2 \\ \vdots & & \vdots \\ \lambda_{1,\varepsilon}^{m-1}, & \cdots, & \lambda_{m,\varepsilon}^{m-1} \end{bmatrix}$$

by $\sigma(\mathcal{N}_1)$. Now let us condider $\sigma(\mathcal{N}_1)^{-1}=\left(\dfrac{\varDelta_{ji}}{\det\sigma(\mathcal{N}_1)}\right)$ where $\varDelta_{ji}$ is (ji)-cofactor of $\sigma(\mathcal{N}_1)$. Since $\det\sigma(\mathcal{N}_1)=\prod\limits_{i>j}(\lambda_{i,\varepsilon}-\lambda_{j,\varepsilon})$, we can write (i, j)-entry of $\sigma(\mathcal{N}_1)^{-1}$ as

$$|\xi|^{r_{ij}}(c_0^{(i,j)}(x, t\,;\xi')+c_1^{(i,j)}(x, t\,;\xi')\varepsilon^{\frac{1}{p}}+\cdots)\,, \tag{3.13}$$

where $\varepsilon=|\xi|^{-1}$ and $r_{ij}\geqq 0$, and

$$c_0^{(i,j)}(x, t\,;\xi')+c_1^{(i,j)}(x, t\,;\xi')\varepsilon^{\frac{1}{p}}+\cdots \tag{3.14}$$

is a symbol which satisfies the conditions (P.1) and (P.2) in § 2. We see easily that $\max\limits_{i,j} r_{i,j}=\dfrac{p-1}{p}$.

Now we define $\sigma(\mathcal{N})$ by

$$\sigma(\mathcal{N})=|\xi|^{\frac{1-p}{p}}\cdot\sigma(\mathcal{N}_1)^{-1}\,, \tag{3.15}$$

then each entry of $\sigma(\mathcal{N})$ satisfies the conditions (P.1) and (P.2) of symbol in § 2. Some entries of $\sigma(\mathcal{N})$, however, begin with positive power of $\varepsilon=|\xi|^{-1}$. Especially we notice here that there exists a positive constant δ such that

$$|c_0^{(i,m)}(x, t\,;\xi')|\geqq\delta\,,\qquad i=1, 2, \cdots, m\,. \tag{3.16}$$

Let us denote by $\mathcal{D}_\gamma$ a pseudo-differential operator associated with

$$\sigma(\mathcal{D})=\begin{bmatrix} \lambda_{1,\varepsilon} & & 0 \\ & \ddots & \\ 0 & & \lambda_{m,\varepsilon} \end{bmatrix}.$$

Thus introducing γ defined in § 2, we see by Lemma 2.4

$$\begin{aligned} \mathcal{N}_\gamma\mathcal{H}_\gamma\varLambda &\equiv(\mathcal{N}_\gamma\circ\mathcal{H}_\gamma)\varLambda\equiv(\mathcal{N}\circ\mathcal{H})_\gamma\varLambda \\ &=(\mathcal{D}\circ\mathcal{N})_\gamma\varLambda\equiv\mathcal{D}_\gamma\mathcal{N}_\gamma\varLambda\equiv\mathcal{D}_\gamma\varLambda\mathcal{N}_\gamma\,, \end{aligned} \tag{3.17}$$

here we mean by $\equiv$ the equality modulo bounded operators in L^2. Defining $R_{j,\varepsilon}$ by $\sigma(R_{j,\varepsilon})=\lambda_{j,\varepsilon}(x, t\,;\xi)$, we can write $\mathcal{D}_\gamma$ as

$$\mathcal{D}_\gamma=\begin{bmatrix} R_{1,\varepsilon}\gamma & & 0 \\ & \ddots & \\ 0 & & R_{m,\varepsilon}\gamma \end{bmatrix}.$$

In the sequel, we get from (3.7) and (3.17)

$$\frac{d}{dt}\mathcal{N}_\gamma V_n=\mathcal{D}_\gamma \Lambda \mathcal{N}_\gamma V_n+\mathcal{N}'_\gamma V_n+B_2 V_n+\mathcal{N}_\gamma B_1 V_n+\mathcal{N}_\gamma F_n\,, \tag{3.18}$$

where B_2 is a bounded operator in L^2 and $\mathcal{N}'_\gamma$ is a pseudo-differential operator defined by

$$\sigma(\mathcal{N}'_\gamma)=-\frac{d}{dt}\sigma(\mathcal{N}_\gamma)\,.$$

By Lemma 2.1 $\mathcal{N}_\gamma$ and $\mathcal{N}'_\gamma$ are bounded operators in L^2.

Setting $\mathcal{N}_\gamma V_n\equiv W_n$, we get from (3.18)

$$\frac{d}{dt}W_n=\mathcal{D}_\gamma \Lambda W_n+(\mathcal{N}'_\gamma+B_2+\mathcal{N}_\gamma B_1)V_n+\mathcal{N}_\gamma F_n\,. \tag{3.19}$$

Now we look at the exact representation of $c_1(x,t;\xi')$ in (3.9):

$$c_1(x,t;\xi')=\left(\frac{i^{p-1}b\left(\frac{\xi_1}{|\xi|}\right)^{m-1}}{\prod_{j=2}^{m-p+1}(\lambda_1-\lambda_j)}\right)^{\frac{1}{p}}, \qquad p\geqq 2\,, \tag{3.20}$$

where $\lambda_1,\cdots,\lambda_{m-p+1}$ are characteristic roots of $\sigma(\mathcal{H}_0)$.

Since p is greater than 2, taking b conveniently there exists at least one integer, we denote it by r_0, in $(1,2,\cdots,p)$ such that

$$\inf_{(x,t,\xi')}\operatorname{Re}\exp\left(\frac{2\pi i}{p}(r_0-1)\right)c_1(x,t;\xi')\geqq\delta_1>0\,, \tag{3.21}$$

for some constant δ_1.

For a convenience of reasoning, we assume that $r_0=1$, that is

$$\inf_{(x,t,\xi')}\operatorname{Re}c_1(x,t,\xi')\geqq\delta_1>0\,. \tag{3.22}$$

Now let us define $S_n(t)$ by

$$S_n(t)=\|W_n^{(1)}(t)\|^2\,, \tag{3.23}$$

where $W_n^{(i)}$ means the i-th component of $W_n(t)$.

We set

$$(\mathcal{N}_\gamma B_1+B_2+\mathcal{N}'_\gamma)V_n+\mathcal{N}_\gamma F_n=G_n(t)\,, \tag{3.24}$$

then we get by Lemma 3.2

$$\|G_n(t)\|\leqq C\|E_m(\Lambda)U\|\,. \tag{3.25}$$

If we define $\widetilde{W}_n(t)$ by

$$\widetilde{W}_n(t) = \alpha_n \mathcal{H}_\gamma E_m(\Lambda)(\beta U)\,, \tag{3.26}$$

then we see by the definition of α_n that

$$\operatorname{supp}\left[\mathcal{F}_x[\widetilde{W}_n(x, t)]\right] \subset \operatorname{supp}[\hat{\alpha}_n(\xi)] \tag{3.27}$$

for any t in $[0, T]$. By (3.25)-(3.27) and by Lemma 2.4, we have that

$$\begin{aligned} &\operatorname{Re}(\mathcal{R}_{1,\varepsilon}\gamma\Lambda \widetilde{W}_n^{(1)}(t) + G_n^{(1)}(t),\ \widetilde{W}_n^{(1)}(t)) \\ &\qquad \geqq c_0 n^{1-\frac{1}{p}} \| \widetilde{W}_n^{(1)}(t)\|^2 - c'\|E_m(\Lambda)U\|^2 \end{aligned} \tag{3.28}$$

for sufficiently large n, where c_0 and c' are constants independent of n. On the other hand, in view of Lemma 2.9 we get

$$\| W_n^{(1)}(t) - \widetilde{W}_n^{(1)}(t)\| \leqq \frac{c}{n} \|E_m(\Lambda)U\|\,, \tag{3.29}$$

of course c is a constant which is independent of n.

Now we get by (3.28) and (3.29) that

$$\frac{d}{dt} S_n(t) \geqq c_0 n^{1-\frac{1}{p}} S_n(t) - c_1 \|E_m(\Lambda)U\|^2 \tag{3.30}$$

for $n > n_0$ and $0 \leqq t \leqq T'$, where c_0 and c_1 are constants which are independent of n.

The inequality (3.30) is fundamental in our proof of Theorem 1.2.

§ 4. Proof of Theorem 1.2.

We shall prove Theorem 1.2 by contradiction. Therefore we assume that the Cauchy problem for $(L_0 + B)u = 0$ is well posed in L^2 sense for any lower order differential operator B. Then the energy inequality (1.5) holds.

Let $\hat{\phi}(\xi)$ be an infinitely differentiable function with a compact support such that $\operatorname{supp}[\hat{\phi}(\xi)] \subset \{\xi\,;\ \hat{\alpha}(\xi) \equiv 1\}$. We define $\hat{\phi}_n(\xi)$ by

$$\hat{\phi}_n(\xi) = \hat{\phi}(\xi - (n-1)\xi_0)\,, \tag{4.1}$$

then $\hat{\phi}_n(\xi)$ has a compact support in a neighbourhood of $n\xi_0$. We denote the Fourier inverse image of $\hat{\phi}_n(\xi)$ by $\phi_n(x)$.

Now let us consider the Cauchy problem

$$\begin{gathered} (L_0 + B)u = 0 \\ u(0) = \cdots = \frac{\partial^{m-2}}{\partial t^{m-2}} u|_{t=0} = 0, \quad \frac{\partial^{m-1}}{\partial t^{m-1}} u|_{t=0} = \phi_n(x) \end{gathered} \tag{4.2}$$

for B defined by (3.1), and $n = 1, 2, 3, \cdots$.

By our assumption the Cauchy problem (4.2) is well posed in L^2 sense, then we have a solution $u_n(x, t)$ such that

$$u_n(x, t) \in \mathcal{E}_t^0(\mathcal{D}_{L^2}^{m-1}) \cap \mathcal{E}_t^1(\mathcal{D}_{L^2}^{m-2}) \cap \cdots \cap \mathcal{E}_t^{m-1}(L^2)$$
$$(0 \leqq t \leqq T')$$

$n = 1, 2, \cdots$.

Taking $U_n = {}^t(u_n(x, t), \frac{\partial}{\partial t} u_n(x, t), \cdots, \frac{\partial^{m-1}}{\partial t^{m-1}} u_n(x, t))$ for U in (3.2), we also get a differential inequality (3.30) for

$$W_n(t) = \mathcal{N}_\gamma V_n = \mathcal{N}_\gamma E_m(\Lambda) \alpha_n(\beta U_n) . \tag{4.3}$$

On the other hand we can prove the following

LEMMA 4.1. *There exists a constant c which is independent of n such that* $\|E_m(\Lambda) U_n\| \leqq c$.

PROOF. By the definition of $E_m(\Lambda)$ we get

$$\|E_m(\Lambda) U_n\| = \sum_{j=0}^{m-1} \left\| \left(\frac{\partial}{\partial t}\right)^j u_n(t) \right\|_{m-j-1}$$
$$\leqq c \sum_{j=0}^{m-1} \left\| \left(\frac{\partial}{\partial t}\right)^j u_n \Big|_{t=0} \right\|_{m-j-1} = c \|\phi_n(x)\|_{L^2} .$$

On the other hand, by the definition $\phi_n(x) = e^{i(n-1)\xi_0 \cdot x} \phi(x)$, so we have

$$\|E_m(\Lambda) U_n\| \leqq c \|\phi_n(x)\| = c \|\phi(x)\| = c' .$$

By this Lemma and (3.30), we get

$$\frac{d}{dt} S_n(t) \geqq c_0 n^{1-\frac{1}{p}} S_n(t) - c \tag{4.4}$$

for $n > n_0$, where c_0 and c are constants which are independent of n. Integrating (4.4) by t, we get

$$S_n(t) \geqq e^{c_0 n^{1-\frac{1}{p}} t} S_n(0) + \frac{c}{c_0 n^{1-\frac{1}{p}}} (1 - e^{c_0 n^{1-\frac{1}{p}} t}) , \quad \text{for } 0 \leqq t \leqq T' . \tag{4.5}$$

In view of Lemma 3.2 and Lemma 4.1, we see easily that

$$S_n(t) \leqq \text{constant independent of } n ; \tag{4.6}$$

Next we can get the following

LEMMA 4.2. *There exists a positive constant δ_0 which is independent of n such that*

$$S_n(0) \geqq \delta_0 . \tag{4.7}$$

If this Lemma is established, our proof of Theorem 1.2 is immediate. In fact, by this Lemma and (4.6), we get from (4.5)

$$C \geqq S_n(t) \geqq \delta_0 e^{c_0 n^{1-\frac{1}{p}} t} + \frac{c}{c_0 n^{1-\frac{1}{p}}} (1 - e^{c_0 n^{1-\frac{1}{p}} t}) \quad \text{for } 0 \leqq t \leqq T' . \tag{4.10}$$

As n tends to infinity, (4.10) is apparently a contradiction unless $t=0$. Thus the proof of Theorem 1.2 is complete.

Finally we give a proof of Lemma 4.2.

First of all we prove that there exists a constant c independent of n such that

$$\|\alpha_n(\beta\psi_n)\| \geqq c - O\left(\frac{1}{n}\right). \tag{4.11}$$

In fact,

$$\|\alpha_n(\beta\psi_n)\| \geqq \|\beta(\alpha_n\psi_n)\| - \|[\alpha_n, \beta]\psi_n\| ,$$

and by Lemma 2.9 the last term of the right-hand side is of order $1/n$. Next by the definition of ψ_n, we see that $\alpha_n\psi_n = \psi_n$, then we get

$$\|\beta(\alpha_n\psi_n)\| = \|\beta\psi_n\| = \|\beta\psi\| = c .$$

Now let us prove (4.7).

Setting $\mathcal{N}_r = (n_{jk})$, $W_n(0)$ can be written as

$$W_n(0) = \mathcal{N}_r \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha_n(\beta\psi_n) \end{bmatrix} = \begin{bmatrix} n_{1m}\alpha_n(\beta\psi_n) \\ \vdots \\ n_{mm}\alpha_n(\beta\psi_n) \end{bmatrix} \tag{4.12}$$

where $\{n_{jk}\}$ are pseudo-differential operators of type P. By Lemma 2.4, we get

$$\|n_{1m}\alpha_n(\beta\psi_n)\| \geqq \left(\frac{\delta}{2} - cR^{-1} - c' \sum_{j=1}^{\infty} M_{C_{j(1,m)}} R^{-\frac{j}{p}}\right) \|\alpha_n(\beta\psi_n)\|$$

where R^{-1} is smaller than the radius of convergence of the symbol of n_{1m}, and c and c' are constants which are independent of n. Thus for sufficiently large R, we get

$$S_n(0) \geqq \frac{\delta^2}{16} \| \alpha_n(\beta\psi_n) \|^2 . \tag{4.13}$$

Now (4.7) follows by (4.11) and (4.13).

Appendix

As mentioned in § 3, here in Appendix we consider a perturbation to the characteristic roots of $\sigma(\mathcal{H}_0)$.

1. We denote the characteristic roots of L_0 by $\lambda_1, \cdots, \lambda_{m-p+1}$, then the characteristic roots of $\sigma(\mathcal{H}_0)$ are given by

$$\tilde{\lambda}_1 = i\lambda_1, \cdots \quad \tilde{\lambda}_{m-p+1} = i\lambda_{m-p+1}, \qquad i = \sqrt{-1}\,.$$

Consider now

$$(3.8) \qquad \begin{aligned} P(x, t\,; i\xi, \lambda) &= \det(\lambda I - \sigma(\mathscr{H})) \\ &= \lambda^m + a_1(x, t\,; i\xi')\lambda^{m-1} + \cdots \\ &\quad + a_m(x, t\,; i\xi') + b(i\xi_1)^{m-1}|\xi|^{-m} \end{aligned}$$

where

$$a_j(x, t\,; i\xi') = \sum_{|\nu|=j} a_{\nu,m-j}(x, t)(i\xi')^{\nu}\,, \qquad \xi' = \xi/|\xi|\,.$$

Under the conditions (I) and (II) we can write (3.8) as follows.

$$(\mathrm{A.1}) \qquad P(i\xi) = (\lambda - \tilde{\lambda}_1)^p \prod_{j\neq 1} (\lambda - \tilde{\lambda}_j) - b(\xi')\varepsilon\,,$$

where

$$b(\xi') = -b\Big(i\frac{\xi_1}{|\xi|}\Big)^{m-1} \quad \text{and } \varepsilon = |\xi|^{-1}\,.$$

Setting $\mu = \lambda - \tilde{\lambda}_1$ we get from (A.1)

$$(\mathrm{A.2}) \qquad \mu^p \prod_{j=2}^{m-n+1} (\mu + (\tilde{\lambda}_1 - \tilde{\lambda}_j)) = b(\xi')\varepsilon\,.$$

If we denote the fundamental symmetric functions with respect to $(\tilde{\lambda}_1 - \tilde{\lambda}_2), \cdots, (\tilde{\lambda}_1 - \tilde{\lambda}_{m-p+1})$ by $\tilde{a}_0, \tilde{a}_1, \cdots, \tilde{a}_{m-p-1}$. Then $\tilde{a}_j(x, t\,; \xi')$ is analytic in ξ except $\xi = 0$ and infinitely differentiable in (x, t) for $j = 0, 1, \cdots, m-p-1$. Finally

$$\tilde{a}_0 = \tilde{a}_0(x, t\,; \xi') \neq 0\,, \qquad (x, t, \xi') \in \Omega_0 \times \Omega_1\,.$$

Now we can write (A.2) in the following form:

$$(\mathrm{A.3}) \qquad \mu^p(\tilde{a}_0 + \tilde{a}_1\mu + \cdots + \tilde{a}_{m-p-1}\mu^{m-p-1} + \mu^{m-p}) = b(\xi')\varepsilon\,.$$

If we put

$$\psi(\mu : x, t, \xi') = \tilde{a}_0 + \tilde{a}_1\mu + \cdots + \tilde{a}_{m-p-1}\mu^{m-p-1} + \mu^{m-p}\,,$$

then for sufficiently small $|\mu|$ we see

$$|\psi(\mu : x, t, \xi')| \geqq \frac{|\tilde{a}_0|}{2} \neq 0\,.$$

Then $\phi(\mu : x, t, \xi') = \psi(\mu : x, t, \xi')^{-1}$ is regular as a function of μ in a neighbourhood of $\mu = 0$.

Setting $b(\xi')\varepsilon = z^p$, we have from (A.3)

$$\mu = {}^p\!\sqrt{\phi}\, z$$

where we mean by ${}^p\!\sqrt{\phi}$ one of its determinations.

Since $z = 0$ yields $\mu = 0$, we get μ in a power series of z in a neighbourhood of $z = 0$.

By Lagrange's inversion formula, we get

(A.4)
$$\mu=\tilde{c}_1 z+\tilde{c}_2 z^2+\cdots$$
$$=c_1\varepsilon^{\frac{1}{p}}+c_2\varepsilon^{\frac{2}{p}}+\cdots$$

where

(A.5)
$$c_n=\frac{1}{n!}[\sqrt[p]{b(\xi')}]^n\Big[\Big(\frac{\partial}{\partial\mu}\Big)^{n-1}\phi(\mu: x, t, \xi')^{\frac{n}{p}}\Big]_{\mu=0}$$
$$=[\sqrt[p]{b(\xi')}]^n\frac{1}{2\pi i}\oint_{|\zeta|=\delta}\frac{\phi(\zeta: x, t, \xi')^{\frac{n}{p}}}{\zeta^n}d\zeta.$$

In (A.5) we take for radical $\sqrt[p]{b(\xi')}$ one of its determinations.

Thus we can express p-perturbed roots corresponding to $\tilde{\lambda}_1$ as follows:

(A.6)
$$\lambda_{j,\varepsilon}=\tilde{\lambda}_1+e^{\frac{2\pi i}{p}(j-1)}\sum_{n=1}^{\infty}c_n(x, t, \xi')\varepsilon^{\frac{n}{p}},$$
$$j=1, 2, \cdots, p.$$

2. Now we show that (A.6) satisfies the conditions (P.1) and (P.2) in § 2.
(1) By the definition, it is evident that $\phi(\mu; x, t, \xi')$ is in $\mathcal{B}_{x,t}$ and infinitely differentiable in $R^k_\xi-\{0\}$. Moreover

$$|\phi(\mu; x, t, \xi')|\geqq\frac{|\tilde{a}_0|}{2}\neq 0, \quad (x, t, \xi')\in\Omega_0\times\Omega_1$$

for sufficiently small μ. Then for sufficiently small $\delta>0$,

$$c_n(x, t, \xi')=\frac{[b(\xi')]^{\frac{n}{p}}}{2\pi i}\oint_{|\zeta|=\delta}\frac{\phi(\zeta; x, t, \xi')^{\frac{n}{p}}}{\zeta^n}d\zeta$$

is infinitely differentiable in (x, t) under the sign of integration. Let us fix this δ. The derivatives of $c_n(x, t, \xi')$ in (x, t) of any order are evidently C^∞ in $R^k_\xi-\{0\}$. Thus for any $n\geqq 0$

$$c_n(x, t, \xi')\in C^{2k}_\beta, \qquad \beta=+\infty.$$

(2) Next we prove that

$$\sum_{n=0}^{\infty}M_{c_n}\varepsilon^n$$

has a positive radius of convergence where

$$M_{c_n}=\sum_{|\nu|\leqq 2k}\sup_{(x,t,\xi')\in\Omega_0\times\Omega_1}|D^\nu_\xi c_n(x, t, \xi')|.$$

First we see that

(A.7)
$$D^\nu_\xi c_n(x, t, \xi')=\frac{1}{2\pi i}\sum_{|\alpha|\leqq|\nu|}\binom{\nu}{\alpha}D^{\nu-\alpha}_\xi b(\xi')^{\frac{n}{p}}\cdot D^\alpha_\xi\oint_{|\zeta|=\delta}\frac{\phi(\zeta; x, t, \xi')^{\frac{n}{p}}}{\zeta^n}d\zeta.$$

We denote the right-hand side as

$$\text{(A.7)}' \qquad \frac{1}{2\pi i}\sum\binom{\nu}{\alpha}Q_{\nu-\alpha}\cdot Q_{\alpha},$$

then we have for $|\xi|\geqq 1$

$$\text{(A.8)} \qquad |Q_{\nu-\alpha}|\leqq C_{\nu-\alpha}\frac{n\cdot|b(\xi')|^{\frac{n}{p}-1}}{p},$$

where $C_{\nu-\alpha}$ is a positive constant.

On the other hand, we have

$$|D_{\xi}^{\alpha}\phi(\zeta;x,t,\xi')|\leqq C_{\alpha}, \quad (x,t,\xi')\in\Omega_0\times\Omega_1, \quad |\xi|\geqq 1$$

where C_{α} is a constant depending on α and $\tilde{a}_0(x,t,\xi')$. Then we get

$$\text{(A.9)} \qquad \begin{aligned} |Q_{\alpha}| &\leqq C_{\alpha}\oint_{|\zeta|=\delta}\frac{n}{p}\frac{|\phi(\zeta,x,t,\xi')|^{\frac{n}{p}-1}}{|\zeta|^{n}}|d\zeta| \\ &\leqq \frac{2\pi C_{\alpha}}{p}\cdot\frac{n}{\delta^{n-1}}\Big(\frac{2}{|\tilde{a}_0|}\Big)^{\frac{n}{p}-1}, \quad (x,t,\xi')\in\Omega_0\times\Omega_1 \\ &\qquad |\xi|\geqq 1. \end{aligned}$$

From (A.7)-(A.9), we get

$$\text{(A.10)} \qquad |D_{\xi}^{\nu}c_n(x,t,\xi')|\leqq C_{\nu}\Big(\frac{n}{p}\Big)^2\Big(\frac{2|b(\xi')|}{|\tilde{a}_0|}\Big)^{\frac{n}{p}-1}\cdot\frac{1}{\delta^{n-1}}$$
$$(x,t,\xi')\in\Omega_0\times\Omega_1, \qquad |\xi|\geqq 1,$$

where C_{ν} is a constant depending on ν and $\tilde{a}_0(x,t,\xi')$.

In the sequel, we have

$$\text{(A.11)} \qquad M_{c_n}\leqq K\Big(\frac{n}{p}\Big)^2\Big(\frac{2|b(\xi')|}{|\tilde{a}_0|}\Big)^{\frac{n}{p}-1}\cdot\frac{1}{\delta^{n-1}}$$

for some positive constant K.

Denoting the right-hand side of (A.11) by N_n, the power series

$$\sum_{n=0}^{\infty}N_n\varepsilon^n$$

converges for

$$\text{(A.12)} \qquad |\varepsilon|<\Big(\frac{|\tilde{a}_0|}{2|b(\xi')|}\Big)^{\frac{1}{p}}\cdot\delta.$$

Thus

$$\sum_{n=0}^{\infty}M_{c_n}\varepsilon^n$$

converges for (A.12).

3. By the same reasoning as in the preceding paragraph

$$\text{(3.11)} \qquad \lambda_{q,\varepsilon}=\tilde{\lambda}_{q-p+1}+\sum_{n=1}^{\infty}c_n^{(q)}(x,t;\xi')\varepsilon^n, \qquad (q=p+1,\cdots,m),$$

also satisfy the conditions (P.1) and (P.2) in § 2.

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