J. Math. Soc. Japan
Vol. 21, No. 3, 1969

Higher dimensional *PL* knots and knot manifolds

Dedicated to Professor Hitoshi Hombu on his sixtieth birthday

By Mitsuyoshi KATO*

(Received Aug. 23, 1968)

§ 1. Introduction.

In this paper, we study *PL* embeddings of spheres with codimension two in terms of regular neighborhoods and exteriors.

In § 2 we list and prove several fundamental facts about *PL* embeddings with codimension two.

We define an *n-knot manifold M* to be a closed $(n+2)$-manifold such that $H_*(M) \cong H_*(S^{n+1} \times S^1)$ and $\pi_1(M)$ has an element whose normal closure equals $\pi_1(M)$ ([**12**], p. 229).

We clarify in § 3 a connection among n-knots, n-knot manifolds and (abstract) regular neighborhoods of $(n+1)$-spheres with codimension two. In particular, it is shown that *PL* n-knot manifolds always bound some regular neighborhoods of $(n+1)$-spheres and that there are at most two distinct *PL* homeomorphism classes of regular neighborhoods of $(n+1)$-spheres with codimension two having *PL* homeomorphic boundaries, provided $n \geqq 3$ (see Theorem 3.11).

In § 4, we investigate the local flatness of a *PL* embedding of S^n with codimension two by means of the homotopy type of the boundary of the regular neighborhood. In fact, we show that if it is 1-flat and if the boundary of the regular neighborhood is homotopically equivalent to $S^n \times S^1$, then it is actually locally flat, provided $n \geqq 4$ (see Corollary 4.6). For each $n \geqq 5$, there is, however, a 2-flat locally knotted embedding of an n-sphere with codimension two whose regular neighborhood has the boundary which is homeomorphic to $S^n \times S^1$ (see Corollary 4.10). As by-products of the argument, we can prove that if a manifold pair (N^{n+2}, bN^{n+2}) is homotopically equivalent to $(S^n \times D^2, S^n \times S^1)$ for $n \geqq 4$, then N is homeomorphic to $S^n \times D^2$. Thus Hauptvermutung for $S^n \times D^2$ $(n \geqq 4)$ is true. (Note that $\pi_1(S^n \times D^2) = \{1\}$, $\pi_1(S^n \times S^1) \cong Z$.) We find two inequivalent n-disk knots having homeomorphic exteriors and knotted $(n+2, n)$-ball pairs in the standard $(n+2, n)$-sphere pair for $n \geqq 4$. (See Corollaries 4.5, 4.9 and 4.11). We also state the unknotting Theorem for n-disk

* Work supported by the Sakkokai Foundation.

knots for $n \geqq 4$.

In § 5, we give a necessary and sufficient condition for that regular neighborhoods of n-spheres with codimension two can be embedded in the $(n+2)$-sphere in terms of slice knots (see Theorem 5.1). We show that two smooth n-knots are diffeomorphic if and only if they are PL homeomorphic. Thus the smooth n-knot cobordism group C^n and the PL n-knot cobordism group C_{PL}^n are connected in the following exact sequence:

$$0 \longrightarrow C^n \longrightarrow C_{PL}^n \longrightarrow \theta_n(\partial\pi) \longrightarrow 0 \qquad \text{for} \quad n \geqq 7$$

$$\text{and} \qquad C^n \cong C_{PL}^n \qquad \text{for} \quad n \leqq 6,$$

(see Corollary 5.3). Finally, we deduce that regular neighborhoods of a $(2m-1)$-sphere with codimension two can be embedded in the $(2m+1)$-sphere and that for each $m \geqq 1$ there is a regular neighborhood of a $2m$-sphere with codimension two which cannot be embedded in the $(2m+2)$-sphere (see Theorem 5.5).

§ 2. Preliminaries.

We refer the reader to the notes of Zeeman [**30**] and Noguchi [**23**] for basic facts and tools about PL manifolds and embeddings with codimension two. We restrict ourselves in the category of polyhedra covered by rectilinear locally finite simplicial complexes and piecewise linear (PL) maps. Thus all maps are to be PL and all manifolds are to be compact, oriented and PL, unless otherwise mentioned. In particular, homeomorphisms between manifolds are to be orientation preserving, and open subsets of manifolds which turn out to be open submanifolds are to be of the induced orientation.

For a manifold M, by $\mathrm{Int}\, M$ and bM we shall denote the interior and the boundary of M, respectively.

Let $f: M \rightarrow W$ and $g: M \rightarrow W'$ be embeddings from a manifold M into manifolds W and W', respectively. We shall say that f and g are *equivalent* if there is a homeomorphism $h: W \rightarrow W'$ such that $h \circ f = g$. The equivalence of embeddings is clearly an equivalence relation. The equivalence class of f will be denoted by $\{f\}$. We shall say that f and g are *micro-equivalent* if there exist derived neighborhoods N of $f(M)$ in W and N' of $g(M)$ in W' so that $f: M \rightarrow N$ and $g: M \rightarrow N'$ are equivalent (for derived neighborhoods, see [**30**]). By the uniqueness of derived neighborhoods, the micro-equivalence of embeddings is an equivalence relation, and the equivalence implies the micro-equivalence. The micro-equivalence class of f will be denoted by $\mu\{f\}$. Following Gluck [4], by an *exterior* of an embedding $f: M \rightarrow W$ we shall mean the closure of the complement of a derived neighborhood of $f(M)$ in W. Again by the uniqueness of derived neighborhoods, exteriors of equivalent embeddings are homeomorphic. It is to be noted that if E is an exterior of

f, then E is a deformation retract of $W-f(M)$.

Let $f: M \to W$ be an embedding from an n-manifold M into an $(n+p)$-manifold W. We shall say that f is *flat at a point* $x \in M$, if there are open neighborhoods U of x in M and V of $f(x)$ in W so that $f|U: U \to V$ and $\times 0^p$: $U \to U \times R^p$ are equivalent, where R^p is the euclidean p-space, 0^p is the origin, $\times 0^p$ stands for the embedding $x \mapsto (x, 0^p)$ and $U \times R^p$ has the product orientation. If f is flat at every point of M, then we shall call f to be *locally flat*. In case $p \geqq 3$, it follows from Zeeman's unknotting Theorem [**29**] that the embedding f is locally flat if and only if it is proper; that is to say, $f(bM) \subset bW$ and $f(\text{Int}\, M) \subset \text{Int}\, W$. However, we shall mainly concern ourselves with the case of codimension $p=2$, in which the embedding might fail to be locally flat.

The following existence and uniqueness Theorem of normal 2-disk bundles for locally flat embeddings with codimension two guarantees us that we can treat them in the same way as smooth ones.

PROPOSITION 2.1. *Locally flat embeddings of manifolds with codimension two have unique (PL) normal 2-disk bundles which triangulate vector bundles.* (For the proof, see [**27**].)

By D^n and S^n we shall denote the standard PL n-disk that is the n-fold cartesian product of the closed interval $D=[-1, 1]$, and the standard PL n-sphere bD^{n+1}, respectively. Following Kervaire [**12**], by an *n-knot* and *n-disk knot* we shall mean locally flat embeddings $f: S^n \to S^{n+2}$ and $g: D^n \to D^{n+2}$, respectively. Note that from definition of local flatness, if g is locally flat, then g is proper. The equivalence classes of f and g will be called the *knot* and *disk knot types*, respectively. The following Corollary, which follows from Proposition 2.1, ensures that the knot and disk knot have collars (see [**27**] and [**10**]).

COROLLARY 2.2. *For any knot $f: S^n \to S^{n+2}$, there is an embedding $F: S^n \times D^2 \to S^{n+2}$ such that $F(x, 0^2)=f(x)$ for $x \in S^n$. For any disk knot $g: D^n \to D^{n+2}$, there is an embedding $G: D^n \times D^2 \to D^{n+2}$ such that $G(x, 0^2)=g(x)$ for $x \in D^n$ and $G(D^n \times D^2) \cap bD^{n+2} = G(bD^n \times D^2)$.*

By K_{PL}^n and D_{PL}^n we shall denote the sets of n-knot and n-disk knot types, respectively. By $K_{PL}'^n$ and $D_{PL}'^n$ we shall denote the sets of homeomorphism classes of locally flat $(n+2, n)$-sphere and -ball pairs respectively. Then maps $K_{PL}^n \to K_{PL}'^n$ and $D_{PL}^n \to D_{PL}'^n$ are defined by

$$\{f: S^n \to S^{n+2}\} \mapsto \{(S^{n+2}, f(S^n))\} \quad \text{and} \quad \{g: D^n \to D^{n+2}\} \mapsto \{(D^{n+2}, g(D^n))\},$$

where $\{X\}$ stands for the class of X. These maps are clearly surjective. Moreover, by Gugenheim's Theorem, if two sphere (or ball) pairs $(S^{n+2}, f(S^n))$ and $(S^{n+2}, f(S^n))$ (or $(D^{n+2}, g(D^n))$ and $(D^{n+2}, g(D^n))$) are homeomorphic, then $f^{-1}f': S^n \to S^n$ (or $g^{-1}g': D^n \to D^n$) is isotopic to the identity. It follows that by Corollary 2.2 this isotopy may be covered by an ambient isotopy of S^{n+2}

(or D^{n+2}) and hence that f and f' (or g and g') are equivalent. Therefore, the maps $K_{PL}^n \to K_{PL}'^n$ and $D_{PL}^n \to D_{PL}'^n$ are bijections. Thus Noguchi's notions of n-knots and n-nodes are essentially the same as ours of n-knots and n-disk knots, respectively. In the following we shall often identify n-knot or n-disk knot types with homeomorphism classes of locally flat $(n+2, n)$-sphere or ball pairs, respectively.

Now we turn to investigate the singularity of a proper embedding $\varphi: M \to W$ from an n-manifold M into an $(n+2)$-manifold W. A point $x \in M$ at which φ fails to be locally flat will be called a *singular point* of φ. By $\mathcal{S}(\varphi)$ we shall denote the *set of all singular points* of φ. Then $\mathcal{S}(\varphi)$ is clearly invariant under the micro-equivalence class of φ. Let K and L be triangulations of M and W respectively such that $\varphi: K \to L$ is simplicial. For each point $x \in M$, let the link $lk(x, K)$ (or $lk(\varphi(x), L)$) be of the orientation coherent with one of $st(x, K)$ (or $st(\varphi(x), L)$) which determines that of M (or W). Thus we have an (oriented) $(n+1, n-1)$-elementary (i.e. sphere or ball) pair $(lk(\varphi(x), L), \varphi(lk(x, K)))$, whose homeomorphism class will be called the *singularity of f at x* and denoted by $\sigma(\varphi, x)$. The pseudo-radial projection argument guarantees us that $\sigma(\varphi, x)$ is determined independently from the choice of triangulations K and L and that $x \in M - \mathcal{S}(\varphi)$ if and only if $\sigma(\varphi, x)$ is the trivial type, that is to say, $\sigma(\varphi, x)$ contains the standard elementary pair $(bD^{n+2}, bD^n \times 0^2)$ or $(D^{n+1}, D^{n-1} \times 0^2)$. Following Noguchi [20], we may describe the singularity in terms of the dual cell pair. For a simplex $\triangle$ of K, by ∇ and $\Box$ we denote the cells dual to $\triangle$ and $\varphi(\triangle)$ in K and L, respectively. Recall that

(1) *the elementary pair $(lk(\varphi(x), L), \varphi(lk(x, K)))$ is homeomorphic to the join pair $b\varphi(\triangle) * (b\Box, b\varphi(\nabla))$, and that*

(2) *if $\triangle'$ is a face of $\triangle$, then $\Box \subset b\Box'$ and $\nabla \subset b\nabla'$.*

It follows from (1) that the singularity $\sigma(\varphi, x)$ of φ at a point x of $\operatorname{Int} \triangle$ is simplicially stable in the sense that $\sigma(\varphi, x) = \sigma(\varphi, y)$ for $y \in \operatorname{Int} \triangle$. This implies that if $\operatorname{Int} \triangle \cap \mathcal{S}(\varphi) \neq \emptyset$, then $\operatorname{Int} \triangle \subset \mathcal{S}(\varphi)$. Moreover, if $\operatorname{Int} \triangle \subset \mathcal{S}(\varphi)$, then by (1) $(b\Box, b\nabla)$ is non-trivial, and hence by (2) $(b\Box', b\nabla')$ must be also non-trivial for any face $\triangle' \subset \triangle$. This implies that if $\operatorname{Int} \triangle \subset \mathcal{S}(\varphi)$, then $\triangle \subset \mathcal{S}(\varphi)$. Therefore, $\mathcal{S}(\varphi)$ is a subpolyhedron of M covered by a subcomplex of K. Now we have the following:

PROPOSITION 2.3 (Noguchi [20] and [23]). *Let $\varphi: M \to W$ be a proper embedding of an n-manifold M into an $(n+2)$-manifold W. If K and L are triangulations of M and W, respectively, such that $\varphi: K \to L$ is simplicial, then the set $\mathcal{S}(\varphi)$ is an underlying set of an at most $(n-2)$-dimensional subcomplex of K.*

In [20], Noguchi defined a 2-dimensional integral cohomology class $\chi(\varphi) \in H^2(M)$ for a proper embedding $\varphi: M \to W$ with codimension two which is invariant under the micro-equivalence class of φ. We shall call the class $\chi(\varphi)$ to be the *Euler class* of φ. In fact, if φ is locally flat, then $\chi(\varphi)$ coincides with the Euler class of a normal bundle for φ. Putting $\sigma(\varphi)=\{(x, \sigma(\varphi, x))\mid x \in \mathcal{S}(\varphi)\}$ we shall call $\sigma(\varphi)$ to be the *singularity of* φ. We shall call an embedding $\varphi: M\to W$ with codimension two to be *k-flat*, if φ is proper and if $\mathcal{S}(\varphi)$ is of dimension $\leqq k-1$. The following was also proved by Noguchi [**21**] (see also [**19**] and [**20**]).

PROPOSITION 2.4 (Noguchi). *Two 1-flat embeddings $\varphi: M\to W$ and $\psi: M\to W'$ are micro-equivalent if and only if $\sigma(\varphi)=\sigma(\psi)$ and $\chi(\varphi)=\chi(\psi)$.*

In the rest of the section we establish a *PL* version of ([**12**], Lemma II. 2) and characterize exteriors of n-knots in algebraic terms. Let G be a group. For a subset A of G, the *normal closure of* A, written (A), will mean the smallest normal subgroup of G containing A. An element ξ of G will be called a *weight element* of G, if the normal closure (ξ) of ξ equals G. Let M be a proper n-submanifold of an $(n+2)$-manifold W. Taking triangulations K and L of M and W, respectively, such that K is a subcomplex of L, let C be the cell dual to an n-simplex of K in L. Then $(C, bC)\subset(W, W-M)$. Let $a: S^1\to W-M$ be an embedding such that $a(S^1)=bC$. Taking a base point $z_0\in S^1$, let $a(z_0)=x_0$ be the base point in $W-M$, and α the homotopy class of the map a. Then we have:

THEOREM 2.5 (Kervaire [**12**]). *Assume that M is connected. Then* Kernel $(i_\#)=(\alpha)$, *where $i: (W-M, x_0)\to(W, x_0)$ is the inclusion map.*

COROLLARY 2.6. *Assume that M is connected and that W is simply connected. Then $\pi_1(W-M, x_0)=(\alpha)$.*

PROOF OF THEOREM 2.5. Since bC bounds a 2-disk C in W, or $i_\#\alpha=1$, it follows that $(\alpha)\subset$ Kernel $(i_\#)$. To see that Kernel $i_\#\subset(\alpha)$, let $b: (S^1, z_0)\to(W-M, x_0)$ be a representative of an element $\beta\in$ Kernel $(i_\#)$. If M is locally flat in W, then by Proposition 2.1 and by the transversal approximation Theorem [**30**], we may prove that $\beta\in(\alpha)$ in a quite similar manner as the proof of ([**12**], Lemma II.2). Now suppose that M is not locally flat. Then from Proposition 2.3 the set $\mathcal{S}(\varphi)$ of singular points of the inclusion map $\varphi: M\to W$ is an underlying set of a proper subcomplex of K. Taking the second barycentric derived neighborhoods U and V of $\mathcal{S}(\varphi)$ in K and L, respectively, we put $W_0=\overline{W-V}$ and $M_0=\overline{M-U}$. Notice that $W_0\supset a(S^1)=bC$ and that by the statements (1) and (2) above, $\varphi|M_0: M_0\to W_0$ is locally flat. Since M is connected, it follows from Proposition 2.3 that $M-\mathcal{S}(\varphi)$ and hence M_0 are connected. Therefore the map $a: (S^1, z_0)\to(W_0-M_0, x_0)$ represents a weight element of $\pi_1(W_0-M_0, x_0)$. On the other hand, $\pi_1(W_0-M_0, x_0)$ is isomorphic with $\pi_1(W-M, x_0)$, since $W_0-M_0=W-(M\cup \operatorname{Int} V)$ is a deformation retract of $W-M$. Therefore, α is a weight element of $\pi_1(W-M, x_0)$. This completes the proof of Theorem 2.5.

THEOREM 2.7. *Let V be a closed $(n+2)$-manifold which is a union of two $(n+2)$-manifolds N and E such that $E\cap N=bE=bN$, and that N is a regular neighborhood of an n-sphere Σ. Assume $n\geqq 2$. Then V is a homotopy $(n+2)$-sphere (necessarily $(n+2)$-sphere for $n\geqq 3$) if and only if*

(1) *$H_*(E)\cong H_*(S^1)$, and*

(2) *there is a weight element $\alpha\in\pi_1(bE)$ such that $(i_\#\alpha)=\pi_1(E)$, where $i: bE\to E$ is the inclusion map.*

Note that $S^n\times D^2$ is a regular neighborhood of $S^n\times 0^2$. Thus we have a characterization of exteriors of n-knots ($n\geqq 3$) in algebraic terms.

COROLLARY 2.8. *Assume $n\geqq 3$. Then an $(n+2)$-manifold E is homeomorphic to an exterior of some n-knot if and only if*

(1) *bE is homeomorphic to $S^n\times S^1$,*

(2) *$H_*(E)\cong H_*(S^1)$, and*

(3) *$(i_*\alpha)=\pi_1(E)$ for a generator α of $\pi_1(bE)\cong Z$.*

REMARK. By [5] and [18], exteriors of n-knots have unique smoothings, since they are $(n+2)$-submanifolds of the $(n+2)$-sphere and since $H_*(E)\cong H_*(S^1)$.

PROOF OF THEOREM 2.7. Since Σ and bN are deformation retracts of N and $N-\Sigma$ respectively, it follows that N is simply connected and hence from Corollary 2.6 that $\pi_1(bN)$ has a weight element α. Hence from Van Kampen Theorem and Corollary 2.6 $\pi_1(V)=1$ if and only if $\pi_1(E)=(i_\#\pi_1(bE))=(i_\#\alpha)$ for some weight element α of $\pi_1(bE)$. Observing the Mayer-Vietoris sequence:

$$\cdots\longrightarrow H_{k+1}(V)\longrightarrow H_k(bE)\longrightarrow H_k(E)+H_k(N)\longrightarrow H_k(V)\longrightarrow\cdots$$

and the homology exact sequence of the pair (E, bE) together with the Poincaré duality $H_k(E, bE)\cong H^{n+2-k}(E)$, we may easily see that $H_*(E)\cong H_*(S^1)$ if and only if V is a homology $(n+2)$-sphere. Now the conclusion follows from the Hurewicz and Whitehead Theorems. In particular, it is to be noted that by *PL* Smale theory V is actually an $(n+2)$-sphere, provided $n\geqq 3$. This completes the proof of Theorem 2.7.

§ 3. Some constructions.

Fixing a point $u\in S^n$, by $\mathcal{K}_u^n$ we shall denote the set of equivalence classes of embeddings of S^n into S^{n+2} which are known to be locally flat at all points of S^n except for the point u. In the following we shall identify S^n with an n-sphere formed from D^n by attaching a cone $u*S^{n-1}$, where $\partial D^n=S^{n-1}$. For an n-disk knot $g: D^n\to D^{n+2}$ we define an embedding $u*g: S^n\to S^{n+2}$ by $u*g|D^n=g$ and $u*g|u*S^{n-1}=u*(g|S^{n-1})$, where $u*(g|S^{n-1})$ is the usual cone extension of $g|S^{n-1}$ from u. Thus if $g|S^{n-1}: S^{n-1}\to S^{n+1}$ is unknotted, then $\{u*g\}\in K_{PL}^n\subset\mathcal{K}_u^n$ and if $g|S^{n-1}: S^{n-1}\to S^{n+1}$ is knotted, then $u*g\in\mathcal{K}_u^n-K_{PL}^n$ and $\sigma(u*g, u)$

$=\{g|S^{n-1}\}$. We define a map $j_n: D^n_{PL}\to\mathcal{K}^n_u$ by $j_n\{g\}=\{u^*g\}$ for $\{g\}\in D^n_{PL}$.

LEMMA 3.1. *The map $j_n: D^n_{PL}\to\mathcal{K}^n_u$ is well-defined and bijective. Moreover, j_n preserves exteriors, that is to say, for any n-disk knot $g: D^n\to D^{n+2}$, the exteriors $E(g)$ and $E(u^*g)$ are homeomorphic.*

PROOF OF LEMMA 3.1. Suppose that we are given a second n-disk knot $g': D^n\to D^{n+2}$ which is equivalent to g. If $G: D^{n+2}\to D^{n+2}$ is an equivalence, then $u^*G: S^{n+2}\to S^{n+2}$ defined by $u^*G|D^{n+2}=G$ and $u^*G|u^*S^{n+1}=u^*(G|S^{n+1})$ is an equivalence between u^*g and u^*g'. Hence j_n is well defined. Let N be a derived neighborhood of $g(D^n)$ in D^{n+2}. Then $u^*S^{n+1}\cup N$ turns out to be a derived neighborhood of $u^*g(S^{n-1})\cup g(D^n)$ in S^{n+2}. Since $\overline{S^{n+2}-(u^*S^{n+1}\cup N)}=\overline{D^{n+2}-N}$, it follows that $E(u^*g)=E(g)$. Thus j_n preserves exteriors. To see the injectivity of j_n, suppose that there is an equivalence $H: S^{n+2}\to S^{n+2}$ from u^*g to u^*g' for $\{g\}, \{g'\}\in D^n_{PL}$. From the invariance of singularities under equivalence and the fact that $(S^{n+1}, g(S^{n-1}))$ has a compatible collar in $(D^{n+2}, g(D^n))$, we may assume that $H(u^*S^{n+1}, u^*g(S^{n-1}))=(u^*S^{n+1}, u^*g'(S^{n-1}))$. Then the equivalence $H: S^{n+2}\to S^{n+2}$ gives rise to an equivalence $H|D^{n+2}: D^{n+2}\to D^{n+2}$ from g to g'. Therefore j_n is injective. Let $\varphi: S^n\to S^{n+2}$ be a representative of an element of $\mathcal{K}^n_u$. Taking a star pair $(st(\varphi(u), S^{n+2}), \varphi(st(u, S^n)))$, we put $(A, B)=(\overline{S^{n+2}-st(\varphi(u), S^{n+2})}, \overline{\varphi(S^n)-\varphi(st(u, S^n))})$. Then (A, B) is a locally flat $(n+2, n)$-disk pair. Since $\varphi(S^n)-u$ is locally flat in $S^{n+2}-u$, we may assume that $\varphi(D^n)=B$. Hence $\varphi|D^n: D^n\to A$ is an n-disk knot and $\varphi|S^{n-1}: S^{n-1}\to bA=lk(\varphi(u), S^{n+2})$ is an $(n-1)$-knot whose type is just $\sigma(\varphi, u)$. Therefore, $j_n\{\varphi|D^n\}=\{\varphi\}$ which implies that j_n is surjective. This completes the proof of Lemma 3.1.

By N^n_u we shall denote the set of micro-equivalence classes of proper embeddings of S^n with codimension two which are known to be locally flat at every point of S^n except for the point u. Then from ([20], Lemma 1), the singularity $\sigma(\varphi, u)$ of a representative $\varphi: S^n\to W$ of an element of N^n_u at u is considered as an element of K^{n-1}_{PL} $(=K'^{n-1}_{PL})$ and invariant under the micro-equivalence class $\mu\{\varphi\}$ of φ. We define a map $\sigma_n: N^n_u\to K^{n-1}_{PL}$ by $\sigma_n(\mu\{\varphi\})=\sigma(\varphi, u)$.

LEMMA 3.2. *The map $\sigma_n: N^n_u\to K^{n-1}_{PL}$ is surjective for $n\geqq 1$ and injective for $n\geqq 3$.*

PROOF OF LEMMA 3.2. Let $f: S^{n-1}\to S^{n+1}$ be an $(n-1)$-knot. From Corollary 2.2 there is an embedding $F: S^{n-1}\times D^2\to S^{n+1}$ such that $F(x, 0^2)=f(x)$ for $x\in S^{n-1}$. Form an $(n+2)$-manifold $N=u^*S^{n+1}\cup(D^n\times D^2)$ from a cone u^*S^{n+1} by attaching $D^n\times D^2$ by the embedding F, and define an embedding $\varphi: S^n\to N$ by $\varphi|u^*S^{n-1}=u^*f$ and $\varphi(x)=(x, 0^2)$ for $x\in D^n$, where u^*f is the cone extension of f from the point u. Then φ is of at most one singularity $\sigma(\varphi, u)=\{f\}$ at u. Hence $\mu\{\varphi\}\in N^n_u$ and $\sigma_n\{\varphi\}=\{f\}$. This implies that σ_n is surjective. From

Corollary 2.4 the micro-equivalence class $\mu\{\varphi\}\in N_u^n$ is completely determined by $\sigma(\varphi, u)=\sigma_n\{\varphi\}$, since $\chi(\varphi)\in H^2(S^n)=0$ for $n\geqq 3$. Therefore, σ_n is injective for $n\geqq 3$. This completes the proof of Lemma 3.2.

For an n-knot $f: S^n\to S^{n+2}$, by a weighted exterior E of f, we shall mean an exterior E of f together with the generator of $H_1(E)$ whose linking number with $f(S^n)$ in S^{n+2} equals 1. In the following, by an exterior of a knot we shall mean its weighted exterior and by a homeomorphism between exteriors of knots a homeomorphism between them preserving the distinguished generators.

By E_{PL}^n, we shall denote the set of homeomorphism classes of exteriors of n-knots. Recall that the homeomorphism class of an exterior $E(f)$ of an embedding f is invariant under the equivalence class of f. Thus we define a map $e_n: K_{PL}^n\to E_{PL}^n$ by $e_n\{f\}=\{E(f)\}$ for $\{f\}\in K_{PL}^n$. Then e_n is obviously surjective. Our oriented version of ([11], Theorem F) is as follows.

PROPOSITION 3.3. *The map $e_n: K_{PL}^n\to E_{PL}^n$ is surjective, and if $n\geqq 2$, then $\sharp e_n^{-1}\{E\}\leqq 2$ for $\{E\}\in E_{PL}^n$.*

Here, for a set X, $\sharp X$ stands for the number of elements of X. As an implication of ([11], Theorem C) we have:

PROPOSITION 3.4. *Every homeomorphism of $S^n\times S^1$ is extendable to a homeomorphism of $D^{n+1}\times S^1$ sending $(0^{n+1}\times S^1, 0^{n+1}\times p)$ onto itself, provided $n\geqq 2$, where 0^{n+1} is the center of D^{n+1} and p is a point of S^1.* (See also [2] and [24].)

Now suppose that we are given a pointed n-knot manifold (M, x_0) and a weight element α of $\pi_1(M, x_0)$. (For the definition of a knot manifold, see Introduction). Since M is orientable, we may take an embedding $G: (D^{n+1}\times S^1, (0^{n+1}, p))\to(M, x_0)$ such that $G|(0^{n+1}\times S^1, 0^{n+1}\times p)$ represents α. By M_α we shall denote a manifold $M_\alpha=M-G(\mathrm{Int}\, D^{n+1}\times S^1)$, together with the generator of $H_1(M_\alpha)$ represented by α.

PROPOSITION 3.5 (M. Kervaire). *Assume $n\geqq 3$. Then M_α is homeomorphic to an exterior of an n-knot.*

For the proof, see ([12], pp. 229-230) and refer to Corollary 2.8. Further we show the following:

LEMMA 3.6. *Let (M, x_0) and (L, y_0) be pointed n-knot manifolds. Assume $n\geqq 2$. Given weight elements $\alpha\in\pi_1(M, x_0)$ and $\beta\in\pi_1(L, y_0)$, then M_α and L_β are homeomorphic if and only if there is a homeomorphism $h: (M, x_0)\to(L, y_0)$ such that $h_\sharp\alpha=\beta$, where $h_\sharp: \pi_1(M, x_0)\to\pi_1(L, y_0)$ is the isomorphism induced from h.*

PROOF OF LEMMA 3.6. Let $G: D^{n+1}\times S^1\to M$ and $H: D^{n+1}\times S^1\to L$ be the embeddings defining M_α and L_β, respectively. Thus $M_\alpha\cup G(D^{n+1}\times S^1)=M$ and $L_\beta\cup H(D^{n+1}\times S^1)=L$. To see the necessity, suppose that there is a homeomorphism $g: M_\alpha\to L_\beta$. From Proposition 3.4, $H^{-1}\circ g\circ G|S^n\times S^1$ is extendable to a homeomorphism of $D^{n+1}\times S^1$ sending $(0^{n+1}\times S^1, 0^{n+1}\times p)$ onto itself. It follows that g is extendable to a homeomorphism $h: (M, x_0)\to(L, y_0)$ so that $h_{\#}\alpha=\beta^{\pm1}$. However, $h_*: H_1(M_\alpha)\to H_1(M_\beta)$ sends the generator represented by α to the generator represented by β. Therefore, we have $h_{\#}\alpha=\beta$. Conversely, suppose that there is such a homeomorphism h. Notice that two embeddings from S^1 into L are isotopic, if they are homotopic, since $n+2\geqq 2\cdot 1+2$. Hence by the uniqueness of regular neighborhoods we may assume that $h\circ G(D^{n+1}\times S^1)=H(D^{n+1}\times S^1)$, and hence $h(M_\alpha)=L_\beta$. Therefore, M_α and L_β are homeomorphic. This completes the proof of Lemma 3.6.

In view of Lemma 3.6, we define a *weighted* n-knot manifold to be a triple (M, x_0, α) consisting of a pointed n-knot manifold (M, x_0) and a weight element α of $\pi_1(M, x_0)$. A second weighted n-knot manifold (L, y_0, β) is *isomorphic* to (M, x_0, α), if there is a homeomorphism $h: (M, x_0)\to(L, y_0)$, called an *isomorphism*, such that $h_{\#}\alpha=\beta$. By M^n_{PL} we shall denote the isomorphism classes of weighted n-knot manifolds. We will define a map $i_n: E^n_{PL}\to M^n_{PL}$ for $n\geqq 2$. To do this, let E be an exterior of an n-knot. Taking a homeomorphism $g: S^n\times S^1\to bE$ such that $g|(0^n, 1)\times S^1$ represents the distinguished generator of $H_1(\partial E)\cong H_1(E)$, form a closed $(n+2)$-manifold $M=E\bigcup_g(D^{n+1}\times S^1)$ from the disjoint union of E and $D^{n+1}\times S^1$ by identifying their boundaries under the homeomorphism g. Letting $G: D^{n+1}\times S^1\to M$ be the natural embedding and $x_0=G(0^{n+1}\times p)$, we denote by α the homotopy class of $G|(0^{n+1}\times S^1, 0^{n+1}\times p)$ in $\pi_1(M, x_0)$. Then we have the following:

LEMMA 3.7. *Assume $n\geqq 2$. Then (M, x_0, α) is a weighted n-knot manifold.*

PROOF OF LEMMA 3.7. First, since $n+2\geqq 4$, by the general position argument we have $\pi_k(M, M-G(0^{n+1}\times S^1))=0$ for $k\leqq 2$, and hence $\pi_1(M)\cong\pi_1(E)$. Secondly, observing the Mayer-Vietoris sequence:

$$\cdots\longrightarrow H_{k+1}(M)\longrightarrow H_k(bE)\longrightarrow H_k(E)+H_k(D^{n+1}\times S^1)\longrightarrow H_k(M)\longrightarrow\cdots$$

together with the isomorphism $H_*(E)\cong H_*(S^1)$ and $H_*(bE)\cong H_*(S^n\times S^1)$, we obtain $H_k(M)\cong H_k(E)=0$ for $2\leqq k\leqq n$. Therefore, from Poincaré duality, we may conclude that M is an n-knot manifold. Moreover, since $\pi_1(M)\cong\pi_1(E)$ and since $\pi_1(bE)=\pi_1(S^n\times S^1)\cong\pi_1(D^{n+1}\times S^1)\cong Z$, it follows from Corollary 2.8 that $G|(0^{n+1}\times S^1, 0^{n+1}\times p)$ represents a weight element of $\pi_1(M, x_0)$. This completes the proof of Lemma 3.7.

Suppose that E' is homeomorphic to E and that (L, y_0, β) is obtained from E' by the construction above. Then $M_\alpha=M-G(\text{Int}\,D^{n+1}\times S^1)=E$ and $L_\beta=L-H(\text{Int}\,D^{n+1}\times S^1)=E'$ are homeomorphic, where $H: D^{n+1}\times S^1\to L$ is the natural embedding. It follows from Lemma 3.6 that (M, x_0, α) and (L, y_0, β) are isomorphic. Therefore, we may define the map $i_n: E^n_{PL}\to M^n_{PL}$ for $n\geqq 2$ by $i_n\{E\}=\{M, x_0, \alpha\}$ for $\{E\}\in E^n_{PL}$, where $\{M, x_0, \alpha\}$ denotes the isomorphism class of

(M, x_0, α). The following is an implication of Proposition 3.5 and Lemma 3.6.

PROPOSITION 3.8. *Assume $n \geqq 2$. The map $i_n: E^n_{PL} \to M^n_{PL}$ is injective for $n \geqq 2$ and bijective for $n \geqq 3$.*

Putting $b_{n+1} = i_n \circ e_n \circ \sigma_{n+1}: N^{n+1}_u \to M^n_{PL}$ for $n \geqq 2$, we have the following implication of Lemma 3.2, Propositions 3.3 and 3.8.

PROPOSITION 3.9. *The map $b_{n+1}: N^{n+1}_u \to M^n_{PL}$ $(n \geqq 2)$ is surjective for $n \geqq 3$ and $\# b^{-1}_{n+1}\{M, x_0, \alpha\} \leqq 2$ for $n \geqq 2$.*

We shall mean by an (abstract) regular neighborhood of an n-sphere a manifold N such that there exists an n-sphere Σ in Int N so that N collapses Σ.

By $\mathcal{N}^n_{PL}$ we shall denote the set of homeomorphism classes of regular neighborhoods of n-spheres with codimension two. Notice that by the uniqueness of regular neighborhoods two proper embeddings $\varphi: S^n \to W$ and $\phi: S^n \to W'$ are micro-equivalent then regular neighborhoods $N(f)$ and $N(g)$ of $\varphi(S^n)$ and $\phi(S^n)$ in W and W', respectively, are homeomorphic. Thus we define a natural map $p_n: N^n_u \to \mathcal{N}^n_{PL}$ by $p_n(\mu\{\varphi\}) = \{N(\varphi)\}$ for $\mu\{\varphi\} \in N^n_u$. Then we prove the following:

LEMMA 3.10. *The map $p_n: N^n_u \to \mathcal{N}^n_{PL}$ is surjective. More precisely, given a regular neighborhood N of the $\hat{n}$-sphere Σ with codimension two, then there is an embedding $\varphi: S^n \to N$ such that $\mu\{\varphi\} \in N^n_u$ and N is a regular neighborhood of $\varphi(S^n)$.*

PROOF OF LEMMA 3.10. Let N be a regular neighborhood of an n-sphere Σ with codimension two. Let K and L be triangulations of Σ and N respectively such that K is a subcomplex of L. Let v be the barycenter of an n-simplex of K. Taking first barycentric subdivision (L', K') of (L, K), let ∇ and $\square$ be the n- and $(n+2)$-cells dual to v in K' and L', respectively, and N' the second barycentric derived neighborhood of Σ in N with respect to (L, K). Put $\overline{N' - \square} = B$ and $\overline{\Sigma - \nabla} = A$. Then (N', Σ) is decomposed into two $(n+2, n)$-disk pairs $(\square, \nabla)$ and (B, A). Since v is the barycenter of an n-simplex of K, and since Σ is flat at each point of the interior of each n-simplex of K, it follows that $(\square, \nabla)$ is a trivial disk pair and hence that (bB, bA) is a locally flat sphere pair. Taking homeomorphism $g: u*S^{n+1} \to B$ and $h: D^n \to \nabla$ and putting $f = g^{-1} \circ h | S^{n-1}: S^{n-1} \to S^{n+1}$, we define an embedding $\varphi: S^n \to N'$ by $\varphi | D^n = j \circ h$ and $\varphi | u*S^{n-1} = g \circ (u*f)$, where $j: \nabla \to \square$ is the inclusion map. Then $\mu\{\varphi\} \in N^n_u$, $\sigma(\varphi, u) = \{bB, bA\}$ and $p_n\mu\{\varphi\} = \{N'\}$. Since N' is a regular neighborhood of Σ in N, it follows that N' is homeomorphic with N. Therefore, $p_n\mu \circ \{\varphi\} = \{N\}$. This completes the proof of Lemma 3.10.

By $\mathcal{M}^n_{PL}$ we shall denote the set of homeomorphism classes of n-knot manifolds. We define a natural map $q_n: M^n_{PL} \to \mathcal{M}^n_{PL}$ and for $n \geqq 2$ a boundary map $b_{n+1}: \mathcal{N}^{n+1}_{PL} \to \mathcal{M}^n_{PL}$ by $q_n\{M, x_0, \alpha\} = \{M\}$ and $b_{n+1}\{N\} = \{bN\}$ for $\{M, x_0, \alpha\} \in M^n_{PL}$ and $\{N\} \in \mathcal{N}^{n+1}_{PL}$, respectively. Finally we define a map $\mu_n: \mathcal{K}^n_u \to N^n_u$ by

$\mu_n\{\varphi\}=\mu\{\varphi\}$ for $\{\varphi\}\in\mathcal{K}^n_u$, where $\mu\{\varphi\}$ denotes the micro-equivalence class of φ.

We must show that the map b_{n+1} is well-defined; that is to say, if N is a regular neighborhood of an $(n+1)$-sphere with codimension two, then the boundary bN is an n-knot manifold. For this, we assume $n\geqq 2$. By Lemma 3.10 we may take an embedding $\varphi: S^{n+1}\to N$ such that N is a regular neighborhood of $\varphi(S^{n+1})$ with codimension two and that $\mu(\varphi)\in N^{n+1}_u$. Moreover, in the proof of Lemma 3.10, we have seen that N is obtained from an $(n+3)$-ball B by attaching a handle $\square=(D^{n+1}\times D^2)$ of index $n+1$ along the n-sphere $\partial A\subset\partial B$. Therefore, the boundary bN is obtained from the exterior of the knot $\sigma_{n+1}\{\mu(\varphi)\}$ and $D^{n+1}\times S^1$ by identifying their boundaries. Thus $b_{n+1}\{N\}=\{bN\}=q_n\circ i_n\circ e_n\circ\sigma_{n+1}\{\mu(\varphi)\}$. From this observation and by definition of maps involved, it is not hard to see that the following diagram commutes.

$$(*)\qquad \begin{array}{ccccc} D^{n+1}_{PL} & \xrightarrow{\partial_{n+1}} & K^n_{PL} & \xrightarrow{e_n} & E^n_{PL} \\ \downarrow{\scriptstyle j_{n+1}} & & \uparrow{\scriptstyle \sigma_{n+1}} & & \downarrow{\scriptstyle i_n}\ (n\geqq 2) \\ \mathcal{K}^{n+1}_u & \xrightarrow{\mu_{n+1}} & N^{n+1}_u & \xrightarrow{b_{n+1}} & M^n_{PL} \\ & & \downarrow{\scriptstyle p_{n+1}} & & \downarrow{\scriptstyle q_n} \\ & & \mathfrak{N}^{n+1}_{PL} & \xrightarrow{b_{n+1}} & \mathcal{M}^n_{PL} \end{array}$$

Consequently, we have the following three theorems:

THEOREM 3.11. *The map* $b_{n+1}:\mathfrak{N}^{n+1}_{PL}\to\mathcal{M}^n_{PL}$ *is well-defined for* $n\geqq 2$, *surjective for* $n\geqq 3$ *and* $\sharp b^{-1}_{n+1}\{M\}\leqq 2$ *for* $n\geqq 2$.

THEOREM 3.12. *Assume* $n\geqq 4$. *A compact* $(n+2)$*-manifold is homeomorphic to a regular neighborhood of an n-sphere* Σ *if and only if*

(1) bN *is an* $(n-1)$*-knot manifold, and*

(2) N *is of the same homotopy type as* S^n.

THEOREM 3.13. *Assume* $n\geqq 4$. *A compact* $(n+2)$*-manifold* E *is homeomorphic to an exterior of some embedding* $\varphi: S^n\to S^{n+2}$ *if and only if*

(1) bE *is an* $(n-1)$*-knot manifold,*

(2) $H_*(E)\cong H_*(S^1)$, *and*

(3) *for some weight element* α *of* $\pi_1(bE)$, $i_\sharp\alpha$ *is a weight element of* $\pi_1(E)$, *where* $i: bE\to E$ *is the inclusion map.*

COROLLARY 3.14. *Assume* $n\geqq 4$. *Then a compact* $(n+2)$*-manifold* E *is homeomorphic to an exterior of some n-disk knot* $g: D^n\to D^{n+2}$ *if and only if* E *satisfies the conditions* (1), (2) *and* (3) *in Theorem* 3.13.

PROOF OF THEOREM 3.11. Since by Proposition 3.9 b_{n+1} is surjective for $n\geqq 3$ and q_n is obviously surjective, it follows from commutativity of the diagram $(*)$ that b_{n+1} is surjective for $n\geqq 3$. To see that $\sharp b^{-1}_{n+1}\{M\}\leqq 2$ for each n-knot manifold M ($n \geqq 2$), let (M, x_0, α) be a weighted n-knot manifold. By Proposition 3.9 and Lemma 3.10, there are at most two embeddings φ_i: $S^{n+1} \to N_i$, $i=1, 2$ such that $\mu\{\varphi_i\} \in N_u^{n+1}$, $i=1, 2$, $b_{n+1}(\mu\{\varphi_1\}) = b_{n+1}(\mu\{\varphi_2\})$ and $\mu\{\varphi_1\} \neq \mu\{\varphi_2\}$, since $n \geqq 2$. Assuming $b_{n+1}^{-1}\{M\} \neq \emptyset$, let N be a representative of an element of $b_{n+1}^{-1}\{M\}$ and $h: bN \to M$ a homeomorphism. By Lemma 3.10 we may take an embedding $\varphi: S^{n+1} \to N$ such that $b_{n+1}(\mu\{\varphi\}) = \{bN, y_0, \beta\}$. If we put $h(y_0) = x_0$ and $h_{\#}\beta = \alpha$, then we have a weighted n-knot manifold (M, x_0, α) such that $\{M, x_0, \alpha\} = \{bN, y_0, \beta\}$. Therefore the embedding φ is micro-equivalent to either φ_1 or φ_2 above, and hence N is homeomorphic with either N_1 or N_2. This completes the proof of Theorem 3.11.

PROOF OF THEOREM 3.12. The necessity follows from Theorem 3.11 and from the fact that N collapses the n-sphere Σ. To see the sufficiency, let N be a compact $(n+2)$-manifold satisfying (1) and (2). Since N is simply connected and since $n+2 > 2 \cdot 2+1$, we may take an embedding $G: (D^n \times D^2, D^n \times S^1) \to (N, bN)$ such that $G|0^n \times S^1$ represents a weight element of $\pi_1(bN)$ and $B \cap bN = T$, where $B = G(D^n \times D^2)$ and $T = G(D^n \times S^1)$. If we put $E = \overline{bN - T}$, $A = \overline{N - B}$ and $U = G(S^{n-1} \times D^2)$, then since $n \geqq 4$ from Proposition 3.5 E is an exterior of an $(n-1)$-knot, and hence that $bA = E \cup U$ is an $(n+1)$-sphere. Since N, B and $A \cap B = U$ are simply connected, it follows from Van Kampen Theorem that A is simply connected. Further, we will show that A is an $(n+2)$-disk. To do this, first, observing the Mayer-Vietoris sequence:

$$\cdots \longrightarrow H_k(U) \longrightarrow H_k(A) + H_k(B) \longrightarrow H_k(N) \xrightarrow{\partial_n} H_{k-1}(U) \longrightarrow \cdots,$$

we have $H_k(A) = 0$ for $k \neq n$, $n-1$, since $H_k(B) = 0$ for $k \geqq 1$, $H_k(N) \cong H_k(S^n)$ and $H_k(U) \cong H_k(S^{n-1})$. Secondly, by Poincaré duality and the universal coefficient theorem, we have

$$H_n(A) \cong H^2(A, \partial A) \cong H^2(A) = 0$$

and

$$H_{n-1}(A) \cong H^3(A, \partial A) \cong H^3(A) = 0,$$

since A is at least 2-connected and by the exact sequence

$$0 \longrightarrow Z \longrightarrow Z \longrightarrow H_{n-1}(A) \longrightarrow 0$$

the free part of $H_{n-1}(A)$ equals zero. Thus A is a compact contractible $(n+2)$-manifold such that bA is an $(n+1)$-sphere. By *PL* Smale theory, we conclude that A is an $(n+2)$-disk, since $n \geqq 4$. Identifying A with the cone $a*(bA)$, we may consider of $N = A \cup B$ as a regular neighborhood of $a*G(S^{n-1} \times 0) \cup G(D^n \times 0)$. This completes the proof of Theorem 3.12.

PROOF OF THEOREM 3.13. The necessity follows from Theorem 2.7. To see the sufficiency, suppose that we are given an $(n+2)$-manifold E satisfying the conditions (1), (2) and (3). From Theorem 3.11 bN bounds a regular neighborhood of an n-sphere Σ such that $bE=bN$. Then by Theorem 2.7 $V=E\cup N$ is an $(n+2)$-sphere and the embedding $S^n\to\Sigma\subset N\subset E\cup N\to S^{n+2}$ is the required one. This completes the proof of Theorem 3.13.

PROOF OF COROLLARY 3.14. Notice that Lemma 3.1 ensures that the set of homeomorphism classes of exteriors of embeddings from S^n into S^{n+2} equals the set of homeomorphism classes of exteriors of n-disk knots. Thus the necessity follows from Theorem 3.13. To see the sufficiency, let $\varphi: S^n\to S^{n+2}$ be an embedding and N a regular neighborhood of $\varphi(S^n)$ in S^{n+2}. Then by Lemma 3.10 we may take an embedding $\phi: S^n\to S^{n+2}$ such that N is a regular neighborhood of $\phi(S^n)$ in S^{n+2} and $\{\phi\}\in\mathcal{K}_u^n$. Hence φ and ϕ have the same exterior $\overline{S^{n+2}-N}$. Therefore, the conclusion again follows from Theorem 3.13. This completes the proof of Corollary 3.14.

§ 4. Singularities and the boundaries of regular neighborhoods.

In [20], Noguchi showed that the relative connected sum operation makes the set $K_{PL}^{\prime n}(=K_{PL}^n)$ into an abelian semi-group. In the quite similar manner we may show that the relative boundary connected sum operation makes the set $D_{PL}^{\prime n}(=D_{PL}^n)$ into an abelian semi-group. These semi-groups K_{PL}^n and D_{PL}^n have the zero-elements that are the trivial knot and disk knot types, respectively. We define a linear map $\partial_{n+1}: D_{PL}^{n+1}\to K_{PL}^n$ by $\partial_{n+1}\{g: D^{n+1}\to D^{n+3}\}=\{g|S^n: S^n\to S^{n+2}\}$. A subset A of an abelian semi-group S with the zero-element 0 is positive if for each non-zero-element $x\in A$, $x+y\neq 0$ for any $y\in S$. From Schubert-Mazur Theorem ([25] and [17]) and Wall's result ([27], p. 6, Remark), we have the following:

PROPOSITION 4.1 (Schubert-Mazur). *K_{PL}^n is positive for $n\neq 2$.*

In order to deduce the analogous result for D_{PL}^n, we must investigate $\partial_n^{-1}(0)$, where 0 is the identity of K_{PL}^{n-1}. It is easily seen that $j_n(\partial_n^{-1}(0))=K_{PL}^n(\subset\mathcal{K}_u^n)$ and $j_n|\partial_n^{-1}(0): \partial_n^{-1}(0)\to K_{PL}^n$ is a linear bijection. Let d and d' be two n-disk knot types such that $d+d'=0$ in D_{PL}^n. Since $\partial_n(d+d')=\partial_n d+\partial_n d'=0$, it follows from Proposition 4.1 that $\partial_n d$ and $\partial_n d'$ equal 0 for $n\neq 3$, and hence that d and d' belong to $\partial_n^{-1}(0)$ for $n\neq 3$. Hence $D_{PL}^n-\partial_n^{-1}(0)$ is positive for $n\neq 3$. Further, if $d\in\partial_n^{-1}(0)$, then $d'\in\partial_n^{-1}(0)$, and $j_n(d)$ and $j_n(d')$ belong to K_{PL}^n. Since $j_n(d)+j_n(d')=j_n(d+d')=0$ in K_{PL}^n, it follows from Proposition 4.1 that $j_n(d)=0$ for $n\neq 2$, and hence that $d=0$ for $n\neq 2$, since j_n is bijective. This implies that $\partial_n^{-1}(0)$ is positive for $n\neq 2$. Thus we conclude the following:

COROLLARY 4.2. *D_{PL}^n is positive for $n\neq 2$, 3 and $D_{PL}^2-\partial_2^{-1}(0)$ and $\partial_3^{-1}(0)$ are positive.*

We prove the following:

THEOREM 4.3. *Let N be a regular neighborhood of an n-sphere Σ with codimension two. Assume that Σ is flat at each point of Σ except for one point $u \in \Sigma$, and that $n \geqq 4$. Then Σ is locally flat if and only if there is a p-connected map $a: S^1 \to bN$ for such an integer p that $n \leqq 2p \leqq 2(n-1)$. In particular, if bN is homotopically equivalent to $S^n \times S^1$, then Σ is locally flat in N.*

PROOF OF THEOREM 4.3. The necessity is obvious by Corollary 2.2. To prove the sufficiency, it suffices to show that the inclusion map $\varphi: \Sigma \to N$ is locally flat, or $\sigma(\varphi, u)$ is trivial. If we put $bN = M$, then $\pi_1(M) \cong \pi_1(S^1) \cong J$, since $p \geqq 2$ and $a: S^1 \to M$ represents a generator α of J, where J is the multiplicative infinite cyclic group. By the unknotting theorem of $(n-1)$-knots due to Levine [**16**], Kervaire [**12**] and Wall [**26**] (in particular, for $n=4$, see [**27**]), it is only necessary to be sure that the exterior M_α of the singularity $\sigma(\varphi, u)$ is of the same homotopy type as S^1. Taking embeddings $G: D^n \times S^1 \to M$ and $H: D^n \times S^1 \to M$ such that $G|0^n \times S^1$ and $H|0^n \times S^1$ are homotopic to the map a, we put $G(D^n \times S^1) = U$, $H(D^n \times S^1) = V$, $E = \overline{M-V}$ and $F = \overline{M-U}$. Then E and F are homeomorphic to M_α and we may assume that $U \cap V = \emptyset$. We will show that U is a deformation retract of E. Since $a: S^1 \to M$ is p-connected, and since $\pi_k(M, E) \cong \pi_k(M, M - H(0^n \times S^1)) = 0$ for $k+1 \leqq n$, it follows that $G|0^n \times S^1 : 0^n \times S^1 \to E$ is p-connected, or $\pi_k(E, U) = 0$ for $k \leqq p$. In the same way we have $\pi_k(F, V) = 0$ for $k \leqq n-p \leqq p$, since $n \leqq 2p$. Let $\hat{M}$ be the universal covering of M. Then the portions $\hat{E}$ and $\hat{U}$ over E and U are also the universal coverings of E and U, respectively, since the inclusion maps $E \to M$ and $U \to M$ induce isomorphisms of the fundamental groups. Thus we may identify $H_k(\hat{E}, \hat{U})$ with $H_k(E, U; Z[J])$, where $Z[J]$ is the integral group ring over J. From excision and Poincaré duality we have

$$H_k(\hat{E}, \hat{U}) \cong H_k(E, U; Z[J]) \cong H_k(W, bU; Z[J])$$
$$\cong H^{n+1-k}(W, bV; Z[J]) \cong H^{n+1-k}(F, V; Z[J]),$$

where $W = E \cap F$, and hence $bW = bU \cup bV$. Since (F, V) is $(n-p)$-connected, it follows that

$$H_k(\hat{E}, \hat{U}) \cong H^{n+1-k}(F, V; Z[J]) = 0 \qquad \text{for } k \geqq p+1.$$

From Hurewicz Theorem, we have $\pi_k(E, U) \cong \pi_k(\hat{E}, \hat{U}) \cong H_k(\hat{E}, \hat{U}) = 0$ for $k \geqq p+1$. Therefore U is a deformation retract of E. This completes the proof of Theorem 4.3.

COROLLARY 4.4. *Let M be a closed m-manifold. Assume $m \geqq 5$. Then M is homeomorphic to $S^{m-1} \times S^1$ if and only if M is homotopically equivalent to $S^{m-1} \times S^1$.* (Refer [1].)

COROLLARY 4.5. *Let N be a compact n-manifold. Assume $n \geqq 6$. Then N is homeomorphic to $S^{n-2} \times D^2$ if and only if N is of the same homotopy type as*

S^{n-2} and bN is homotopically equivalent to $S^{n-2}\times S^1$.

PROOF OF COROLLARIES 4.4 AND 4.5. The necessity of each corollary is trivial. Suppose that M is of the same homotopy type as $S^{m-1}\times S^1$. Since M is an $(m-2)$-knot manifold and $m-2\geqq 3$, it follows from Theorem 3.11 that M bounds a regular neighborhood N of an $(m-1)$-sphere with codimension two. Therefore, in order to prove Corollary 4.4, it suffices to prove Corollary 4.5. Let N be a compact $(m+1)$-manifold such that bN is homotopically equivalent to $S^{m-1}\times S^1$ and N is of the same homotopy type as S^{m-1}. Then from Theorem 3.12, N is a regular neighborhood of an $(m-1)$-sphere. Further, by Lemma 3.10, there is an embedding $\varphi: S^{m-1}\to N$ such that N is a regular neighborhood of $\varphi(S^{n-1})$ and $\mu\{\varphi\}\in N_u^{m-1}$. Since bN is homotopically equivalent to $S^{m-1}\times S^1$, it follows from Theorem 4.3 that φ is locally flat. Therefore, by Proposition 2.3, N is homeomorphic to $S^{m-1}\times D^2$, since $H^2(S^{m-1})=0$ for $m\geqq 4$. This completes the proof of Corollaries 4.4 and 4.5.

COROLLARY 4.6. *Let $\varphi: S^n\to W^{n+2}$ be a proper embedding of S^n with codimension two. Assume that φ is 1-flat and that $n\geqq 4$. If the boundary bN of a regular neighborhood N of $\varphi(S^n)$ in W is homotopically equivalent to $S^n\times S^1$, then φ is locally flat.*

PROOF OF COROLLARY 4.6. By Lemma 3.10, we may take an embedding $\psi: S^n\to N$ such that $\mu\{\psi\}\in N_u^n$ and N is a regular neighborhood of $\psi(S^n)$. Letting $u_1, \cdots, u_m\in S^n$ be the singular points of φ, then we have $\sigma(\psi, u) = \sigma(\varphi, u_1)+\cdots+\sigma(\varphi, u_m)$, see [22].

Since bN is homotopically equivalent to $S^n\times S^1$, it follows from Theorem 4.3 that $\sigma(\psi, u)=0$, and that by Proposition 4.1, $\sigma(\varphi, u_1)=\cdots=\sigma(\varphi, u_m)=0$. Therefore, φ is locally flat. This completes the proof of Corollary 4.6.

Further, we have the following unknotting theorem.

COROLLARY 4.7. *Assume $n\geqq 4$. Then an n-disk knot $g: D^n\to D^{n+2}$ is unknotted, if an exterior E of g is of the homotopy type of S^1 and $\pi_1(bE)\cong\pi_1(E)$.*

PROOF OF COROLLARY 4.7. In order to prove Corollary 4.7, by Lemma 3.1, it suffices to show that $j_n(g)$ is trivial. Since E is homeomorphic to an exterior of $j_n(g)$, if $\pi_k(S^1)\cong\pi_k(bE)$ for all $k\leqq p$, $n\leqq 2p$, then by Theorem 4.3 $j_n(g)\in K_{PL}^n$. In fact, from the assumption $\pi_1(bE)\cong\pi_1(E)$, by taking the universal covering space of (E, bE) and by applying Poincaré duality in the same way as the proof of Theorem 4.3, we have

$$\pi_k(E, bE)=0 \qquad \text{for all } k\leqq n\,.$$

Since E is of the homotopy type of S^1 and $n\geqq 4$, it follows that $\pi_k(S^1)\cong\pi_k(bE)$ for all $k\leqq n-1$, hence $j_n(g)\in K_{PL}^n$ and that by Levine's unknotting theorem [16] $j_n(g)$ is trivial, completing the proof.

In order to ensure that the condition $\pi_1(bE)\cong\pi_1(E)$ is necessary, we construct a remarkable n-disk knot for each $n\geqq 4$.

THEOREM 4.8. *For each $n \geqq 4$, there is an n-disk knot $g: D^n \to D^{n+2}$ such that an exterior E of g is homeomorphic to a product space $S^1 \times W$ of a circle S^1 and a compact contractible manifold W such that $\pi_1(bW)$ is the binary icosahedral group.*

Therefore, there is a non-trivial n-disk knot whose exterior is of the same homotopy type as S^1.

REMARK. In the footnote on page 730 in [8], the unknotting theorem for n-disk knots is incomplete.

PROOF OF THEOREM 4.8. Let G be the binary icosahedral group, that is to say, a group with a presentation $(a, b\,;\, a^4 = bab,\ b^2 = aba)$. According to Newman [28], for each integer $n \geqq 5$, there is a compact contractible n-manifold W^n such that bW^n, say M, is a homology sphere whose fundamental group $\pi_1(M)$ equals G. Letting $E = S^1 \times W^n$, we will show that E is an exterior of an $(n-1)$-disk knot $g: D^{n-1} \to D^{n+1}$ for $n-1 \geqq 4$. To do this, by Corollary 3.14, it is only necessary to be sure that

(1) $bE = S^1 \times M$ is an $(n-2)$-knot manifold,

(2) $H_*(E) \cong H_*(S^1)$ and

(3) $(i_\# \alpha) = \pi_1(E)$ for some weight element α of $\pi_1(bE)$.

Since $S^1 \times W$ is of the same homotopy type as S^1 and M is a homology $(n-1)$-sphere, we have

$$H_*(E) \cong H_*(S^1) \quad \text{and} \quad H_*(bE) = H_*(S^1 \times M) \cong H_*(S^1 \times S^{n-1})\,.$$

From the identity $b = (b^{-1}ab)a$, we conclude that a is a weight element of G. Thus if t is a generator of J, we have a weight element $(t, a) = \alpha \in J \times G$ of $J \times G \cong \pi_1(bE) = \pi_1(S^1 \times M)$. Since $i_\# : \pi_1(bE)(\cong J \times G) \to \pi_1(E)(\cong J)$ is given by the projection onto the first factor $J \times G \to J$, it follows that $i_\# \alpha = i_\#(t, a) = t$ is a weight element of J. Hence E satisfies the conditions (1), (2), and (3). It follows from Corollary 3.14 that E is homeomorphic to an exterior of some $(n-1)$-disk knot $g: D^{n-1} \to D^{n+1}$, since $(n-1) \geqq 4$. This completes the proof of Theorem 4.8.

From Theorem 4.8, we deduce the following three corollaries:

COROLLARY 4.9. *For each integer $n \geqq 4$, there are two inequivalent n-disk knots whose exteriors are homeomorphic.*

REMARK. As is seen from the proof, the exteriors of the disk knots are homeomorphic to $S^1 \times W$ in Theorem 4.8.

COROLLARY 4.10. *For each integer $n \geqq 5$, there exists a 2-flat embedding $\varphi: S^n \to S^{n+2}$ such that a regular neighborhood of $\varphi(S^n)$ in S^{n+2} is homeomorphic to $S^n \times D^2$ and the exterior is homeomorphic to $D^{n+1} \times S^1$.*

This ensures that in general we cannot distinguish local flatness of embeddings by means of the homeomorphy types of the boundaries of their regular neighborhoods. For an n-disk knot $g: D^n \to D^{n+2}$, we have an $(n+1)$-

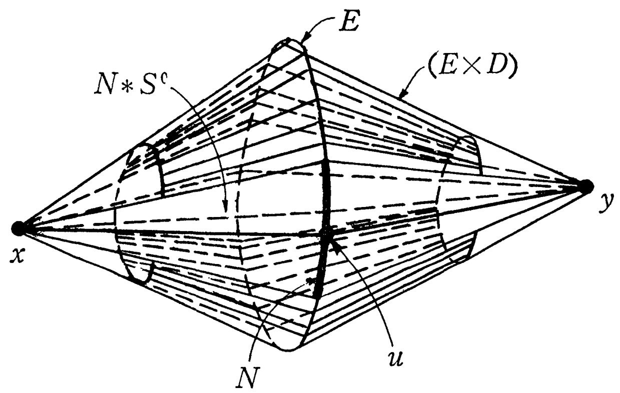
disk knot $g\times D: D^{n+1}\to D^{n+3}$ by $g\times D(x, u)=(g(x), u)$ for $(x, u)\in D^n\times D$. Thus we define a linear map $\times D: D^n_{PL}\to D^{n+1}_{PL}$ by $\times D\{g\}=\{g\times D\}$. Noguchi proposed a problem whether this map $\times D$ is injective or not. The following answers this in the negative:

COROLLARY 4.11. *For each integer $n\geqq 4$, there exists a knotted n-disk knot $g: D^n\to D^{n+2}$ such that $g\times D$ is unknotted.*

In particular, $g\times D|S^n: S^n\to S^{n+2}$ is unknotted. However, the unknotted sphere pair $(S^{n+2}, (g\times D)(S^n))$ is just the double of the locally flat knotted ball pair $(D^{n+2}, g(D^n))$. Thus this also answers ([8], Question 3) in the negative. The first such answer was given by Hudson and Sumners [**9**].

PROOF OF COROLLARY 4.9. For each $n\geqq 4$, let E denote the $(n+2)$-manifold $S^1\times W^{n+1}$ obtained in Theorem 4.8. Taking a base point $x_0\in bE$, we identify $\pi_1(bE, x_0)$ with $J\times G$. By the identity $a=b^{-2}\cdot(aba^{-1})\cdot(a^{-1}ba)\cdot b$, b is a weight element of G. Thus $J\times G$ has at least two weight elements $\alpha=(t, a)$ and $\beta=(t, b)$. Notice that there is no automorphism $\theta: J\times G\to J\times G$ such that $\theta\alpha=\beta$, since $a^5=b^3$, and hence that weighted $(n-1)$-knot manifolds (bE, x_0, α) and (bE, x_0, β) are not isomorphic. On the other hand, as is seen in the proof of Theorem 4.8, E satisfies the conditions (1), (2), and (3) in Theorem 3.13. It follows from Theorem 3.13 and Lemma 4.1 that there exist two n-disk knots $g: D^n\to D^{n+2}$ and $h: D^n\to D^{n+2}$ such that exteriors of g and h are homeomorphic to E and that $b_n\circ\mu_n\circ j_n\{g\}=\{bE, x_0, \alpha\}$ and $b_n\circ\mu_n\circ j_n\{h\}=\{bE, y_0, \beta\}$. Thus g and h should not be equivalent, since (bE, x_0, α) and (bE, y_0, β) are not isomorphic. This completes the proof of Corollary 4.9.

PROOF OF COROLLARY 4.10. For each $n\geqq 5$, let $g: D^{n-1}\to D^{n+1}$ and E denote the $(n-1)$-disk knot obtained in Theorem 4.8 and its exterior. By $\varphi: S^n\to S^{n+2}$ we denote the suspension $(u*g)*S^0: S^n\to S^{n+2}$ of $u*g: S^{n-1}\to S^{n+1}$,



where $S^0=\{x, y\}$. Thus $\mathcal{S}(\varphi)=u*S^0$ and φ is 2-flat, since $g|S^{n-2}: S^{n-2}\to S^n$ is knotted. We take collar neighborhood $(E\times D)$ of E in $E*S^0$ naturally. We will show that $(E\times D)$ is homeomorphic to an exterior of $\varphi: S^n\to S^{n+2}$. For this it suffices to show that $\overline{S^{n+2}-(E\times D)}\searrow\varphi(S^n)$, where $\searrow$ stands for collapsing. Observe that if we put $N=\overline{S^{n+1}-E}$, then

(i) N is a regular neighborhood of $u*g(S^{n-1})$ in S^{n+1},

(ii) $S^{n+2}-\mathrm{Int}\,(E\times D)=N*S^0\cup(E*S^0-(E\times\mathrm{Int}\,D))$
$$=N*S^0\cup(E\times(-1))*x\cup(E\times 1)*y,$$
and

(iii) $N*S^0\cap((E\times(-1))*x\cup(E\times 1)*y)=(bE\times(-1))*x\cup(bE\times 1)*y.$

Since $(bE\times(-1))*x$ and $(bE\times 1)*y$ are subcones of $(E\times(-1))*x$ and $(E\times 1)*y$ respectively and since $N\searrow u*g(S^{n-1})$, it follows that

$$S^{n+2}-\mathrm{Int}\,(E\times D)\searrow N*S^0 \quad (u*g(S^{n-1}))*S^0=\varphi(S^n).$$

Hence N is a regular neighborhood of $\varphi(S^n)$ in S^{n+2}, and $E\times D$ is homeomorphic to an exterior of $\varphi: S^n\to S^{n+2}$. Since $W\times D$ is a compact contractible manifold with simply connected boundary $b(W\times D)$ and since $n+1\geqq 6$, it is an $(n+1)$-ball. Therefore, $E\times D=S^1\times W\times D$ is homeomorphic to $S^1\times D^{n+1}$, and by Corollary 4.5 the regular neighborhood $\overline{S^{n+2}-(E\times D)}$ is homeomorphic to $S^n\times D^2$. Thus φ is the required embedding. This completes the proof of Corollary 4.10.

PROOF OF COROLLARY 4.11. For each $n\geqq 4$, let $g: D^n\to D^{n+2}$ and E be the n-disk knot obtained in Theorem 4.8 and its exterior, respectively. Then $g\times D: D^{n+1}\to D^{n+3}$ has an exterior $E\times D$ which is homeomorphic to $S^1\times D^{n+2}$. By Lemma 3.1, $u*(g\times D): S^{n+1}\to S^{n+3}$ is at most 1-flat and has an exterior $E\times D$, which is homeomorphic to $S^1\times D^{n+2}$. It follows that by Corollary 4.6 $u*(g\times D)$ is locally flat and hence that by unknotting theorem, $u*(g\times D)$ is unknotted. Therefore, $g\times D$ is unknotted. This completes the proof of Corollary 4.11.

§5. Which regular neighborhoods of S^n with codimension two can be embedded in S^{n+2}?

An n-knot will be called a *slice* n-knot, if its knot type belongs to the image of $\partial_{n+1}: D^{n+1}_{PL}\to K^n_{PL}$. By C^n_{PL} we shall denote the n-knot cobordism group defined by Noguchi [22]. Thus we have an exact sequence of abelian semi-groups: $D^{n+1}_{PL}\xrightarrow{\partial_{n+1}}K^n_{PL}\xrightarrow{r_n}C^n_{PL}\longrightarrow 0$, where r_n is the natural linear map. Putting $\gamma_{n+1}=q_n\cdot i_n\cdot e_n\cdot\partial_{n+1}: D^{n+1}_{PL}\to\mathcal{M}^n_{PL}$, we shall say that an n-knot manifold M is *obtained from a slice n-knot*, if $\{M\}\in\mathrm{Image}\,\gamma_{n+1}$. First, we extend Fox-Milnor Theorem ([3] or [22], Theorem 3) in the following form:

TEOREM 5.1. *Let N be a regular neighborhood of an n-sphere with codimension two. Assume $n\geqq 3$. Then there exists an embedding $\Phi: N\to S^{n+2}$ if and only if bN is obtained from a slice $(n-1)$-knot.*

PROOF. Suppose that there is an embedding $\Phi: N \rightarrow S^{n+2}$. We may take an embedding $\varphi: S^n \rightarrow N$ such that $\mu\{\varphi\} \in \mathcal{N}_u^n$ and such that $p_n(\mu\{\varphi\}) = \{N\}$, since $n \geqq 3$. Hence $\{\Phi \circ \varphi\} \in \mathcal{K}_u^n$ and $p_n \circ \mu_n\{\Phi \circ \varphi\} = p_n(\mu\{\varphi\}) = \{N\}$. Thus from the commutative diagram (*) in § 3, we have $\gamma_n(j_n^{-1}\{\Phi \circ \varphi\}) = \{bN\}$. Therefore bN is obtained from a slice $(n-1)$-knot. Conversely, suppose that bN is obtained from a slice $(n-1)$-knot. Then there is an embedding $\psi: S^n \rightarrow S^{n+2}$ such that $\{\psi\} \in \mathcal{K}_u^n$ and $\gamma_n \circ j_n^{-1}\{\psi\} = \{bN\}$. Taking an exterior E of $\psi: S^n \rightarrow S^{n+2}$, we form a closed $(n+2)$-manifold $V = E \cup N$. Then, by Theorem 2.7 and Corollary 2.8, V is an $(n+2)$-sphere. Therefore N can be embedded in an $(n+2)$-sphere. This completes the proof of Theorem 5.1.

For our purpose we must compute the group C_{PL}^n. To do this, in view of Kervaire's result ([12], Theorem III. 6) it is sufficient to clarify the connection between our (PL) n-knots and smooth n-knots. Here a smooth n-knot means a smooth $(n+2, n)$-sphere pair $(S^{n+2}, \tilde{S})$. The diffeomorphism class of a smooth n-knot $(S^{n+2}, \tilde{S})$, written $\{S^{n+2}, \tilde{S}\}$, will be called the *smooth n-knot type*. A smooth n-disk knot means a smooth $(n+2, n)$-disk pair $(D^{n+2}, \tilde{D})$ such that $b\tilde{D} \subset bD^{n+2}$, $\operatorname{Int} \tilde{D} \subset \operatorname{Int} D^{n+2}$ and $\tilde{D}$ intersects transversally to bD^{n+2}. The diffeomorphism class of a smooth n-disk knot $(D^{n+2}, \tilde{D})$, written $\{D^{n+2}, \tilde{D}\}$, will be called the *smooth n-disk knot type*. By K_0^n and D_0^n we shall denote the sets of smooth n-knot and -disk knot types, respectively. Then the relative (boundary) connected sum operation makes the set $K_0^n(D_0^n)$ into an abelian semi-group. We define a linear map $\partial_n: D_0^n \rightarrow K_0^{n-1}$ by $\partial_n\{D^{n+2}, \tilde{D}\} = \{S^{n+1}, b\tilde{D}\}$. It is observed that Kervaire's smooth n-knot cobordism group C^n is defined so that the following sequence of abelian semi-groups is exact: $D_0^{n+1} \rightarrow K_0^n \xrightarrow{r_n} C^n \rightarrow 0$, where r_n is the natural linear map. Notice that given a smooth n-knot $(S^{n+2}, \tilde{S})$ or -disk knot $(D^{n+2}, \tilde{D})$, then we have unique n-knot $t(S^{n+2}, \tilde{S})$ or -disk knot $t(D^{n+2}, \tilde{D})$ up to homeomorphism by triangulating smoothly $(S^{n+2}, \tilde{S})$ or $(D^{n+2}, \tilde{D})$. Thus we define maps $t_n: K_0^n \rightarrow K_{PL}^n$ and $t_n: D_0^n \rightarrow D_{PL}^n$ by $t_n\{S^{n+2}, \tilde{S}\} = \{t(S^{n+2}, \tilde{S})\}$ and $t_n\{D^{n+2}, \tilde{D}\} = \{t(D^{n+2}, \tilde{D})\}$, respectively. Then we have the following theorem:

THEOREM 5.2. (1) *The map* $t_n: K_0^n \rightarrow K_{PL}^n$ *is injective and* (2) *the map* $t_n: D_0^n \rightarrow D_{PL}^n$ *is bijective.*

In other words, two smooth n-knots (or -disk knots) are diffeomorphic if they are homeomorphic. The proof of Theorem 5.2 is postponed at the end of the section. Thus the map $t_n: K_0^n \rightarrow K_{PL}^n$ gives rise to a monomorphism $t_n: C^n \rightarrow C_{PL}^n$, since a diagram

$$\begin{array}{ccccccc} D_0^{n+1} & \longrightarrow & K_0^n & \xrightarrow{r_n} & C^n & \longrightarrow & 0 \\ \downarrow t_{n+1} & & \downarrow t_n & & & & \\ D_{PL}^{n+1} & \longrightarrow & K_{PL}^n & \xrightarrow{r_n} & C_{PL}^n & \longrightarrow & 0 \end{array}$$

commutes.

Further, we will define a map $\mathcal{S}_n : C^n_{PL} \to \theta_n(\partial\pi) \cap \Gamma_n$, where $\theta_n(\partial\pi)$ and Γ_n are the groups of smooth homotopy n-spheres bounding smooth compact parallelizable manifolds and smoothings compatible with S^n. For this, let $f: S^n \to S^{n+2}$ be an n-knot. Since $f(S^n)$ has a collar neighborhood in S^{n+2}, it follows from ([**15**], Theorem 6.3) that there is a smooth manifold pair $(S^{n+2}, \tilde{S})$ compatible with $(S^{n+2}, f(S^n))$. Then by ([**13**], Appendix, Theorem I) $\tilde{S}$ belongs to $\theta_n(\partial\pi) \cap \Gamma_n$. Moreover, from ([**7**], Theorem 7.1) the diffeomorphism class $\{\tilde{S}\}$ of $\tilde{S}$ is uniquely determined by the knot cobordism class $[f]$ of the n-knot $f: S^n \to S^{n+2}$. Thus the map $\mathcal{S}_n : C^n_{PL} \to \theta_n(\partial\pi) \cap \Gamma_n$ is defined by $\mathcal{S}_n[f] = \{\tilde{S}\}$. Again by ([**13**], Appendix, Theorem I) the map $\mathcal{S}_n : C^n_{PL} \to \theta_n(\partial\pi) \cap \Gamma_n$ turns out to be an epimorphism. Therefore, we may conclude the following:

COROLLARY 5.3. *There is an exact sequence:*

$$0 \longrightarrow C^n \xrightarrow{t_n} C^n_{PL} \longrightarrow \theta_n(\partial\pi) \cap \Gamma_n \longrightarrow 0 .$$

Here note that $\Gamma_n = 0$ $(n \leqq 6)$ and $\theta_n(\partial\pi) = \theta_n(\partial\pi) \cap \Gamma_n$ for $n \geqq 7$. From ([**14**], Theorem 5.1), ([**12**], Theorem III. 6) and ([**13**], p. 265), we have the following:

COROLLARY 5.4. (1) *The group C^n is of finite index in C^n_{PL},* (2) $C^{2m}_{PL} = 0$ *for $m \geqq 1$ and* (3) C^{2m-1}_{PL} *has an element of infinite order for each $m \geqq 1$.*

Now we have the following result:

THEOREM 5.5. (1) *Every regular neighborhood of $(2m-1)$-spheres with codimension two can be embedded in S^{2m+1} for $m \geqq 1$.* (2) *For each $m \geqq 1$, there exists a regular neighborhood of S^{2m} with codimension two that cannot be embedded in S^{2m+2}.*

PROOF. By Corollary 5.4, $\partial_{n+1} : D^{n+1}_{PL} \to K^n_{PL}$ is surjective, if $n = 2m$ and not surjective, if $n = 2m-1$. Therefore, by Propositions 3.3 and 3.8 γ_{n+1} is surjective, if $n = 2m \geqq 4$ and not surjective, if $n = 2m-1 \geqq 3$. Thus in case $2m > 2m-1 \geqq 3$, (1) and (2) follow from Theorem 5.1 together with Theorem 3.11. In case $2m-1 = 1$, then a regular neighborhood of S^1 with codimension two is homeomorphic to $S^1 \times D^2$, and hence embeds in S^3. In case $2m = 2$, then we may construct a locally flat embedding $f: S^2 \to N^4$ such that N is a regular neighborhood of $f(S^2)$ and the Euler class $\chi(f) \neq 0$. Since if N^4 embeds in S^4, then $\chi(f) = 0$, it follows that N^4 cannot be embedded in S^4. This completes the proof of Theorem 5.5.

An implication of Theorem 5.5, (1) is the following:

COROLLARY 5.6. *Let N be a regular neighborhood of a $(2m-1)$-sphere Σ with codimension two. Then $(N \times D, \Sigma \times 0)$ is homeomorphic to $(S^{2m-1} \times D^3, S^{2m-1} \times 0^3)$.* (Refer to ([**23**], 3.10, Remark 3)).

PROOF. By Theorem 5.5, we may assume that N is a submanifold of S^{2m+1}, and hence that N is a regular neighborhood of the n-sphere Σ in S^{2m+1}.

If we identify S^{2m+1} with $S^{2m+1}\times 0\subset S^{2m+1}\times D\subset S^{2m+2}$, then $N\times D$ turns out to be a regular neighborhood of $\Sigma\times 0$ in S^{2m+2}. Since Σ is of codimension 3 in S^{2m+2}, it follows from Zeeman's unknotting theorem [**29**] and uniqueness of regular neighborhoods that $(N\times D, \Sigma\times 0)$ is homeomorphic to $(S^{2m-1}\times D^3, S^{2m-1}\times 0^3)$. This completes the proof of Corollary 5.6.

Now we turn to prove Theorem 5.2.

PROOF OF THEOREM 5.2. First, we prove the surjectivity of $t_n: D_0^n \to D_{PL}^n$. Let $g: D^n \to D^{n+2}$ be an n-disk knot type. Then by Corollary 2.2 there is an embedding $G: D^n\times D^2 \to D^{n+2}$ such that $G(D^n\times 0^2)=g(D^n)$ and $G(S^{n-1}\times D^2) = G(D^n\times D^2)\cap S^{n+1}$. By applying Cairns-Hirsch Theorem in the relative case ([**5**], Theorem 2.5 and Remark) twice, we have a smooth n-disk knot $(D^{n+2}, \tilde{D})$ whose smooth triangulation is homeomorphic to $(D^{n+2}, g(D^n))$. Hence $t_n: D_0^n \to D_{PL}^n$ is surjective. Secondly, to see the injectivity of $t_n: K_0^n \to K_{PL}^n$, suppose that we are given two smooth n-knots $(S^{n+2}, \tilde{S}_1)$ and $(S^{n+2}, \tilde{S}_2)$. Then, letting $\tilde{E}_1$ and $\tilde{E}_2$ be the complements of open tubular neighborhoods of $\tilde{S}_1$ and $\tilde{S}_2$ in S^{n+2} respectively, we consider of $(S^{n+2}, \tilde{S}_1)$ and $(S^{n+2}, \tilde{S}_2)$ as to be formed from $\tilde{E}_1$ and $\tilde{E}_2$ by attaching $(S^n\times D^2, S^n\times 0^2)$ under diffeomorphisms $f_1: S^n\times S^1 \to b\tilde{E}_1$ and $f_2: S^n\times S^1\to b\tilde{E}_2$. Thus $(S^{n+2}, \tilde{S}_1)=(\tilde{E}_1 \cup_{f_1} S^n\times D^2, (S^n\times 0^2))$ and $(S^{n+2}, \tilde{S}_2) = (\tilde{E}_2 \cup_{f_2} S^n\times D^2, (S^n\times 0^2))$. Further, suppose that $(S^{n+2}, \tilde{S}_1)$ and $(S^{n+2}, \tilde{S}_2)$ are (PL) homeomorphic. Then by the uniqueness of regular neighborhoods we may take a PR homeomorphism $h: (S^{n+2}, \tilde{S}_1)\to(S^{n+2}, \tilde{S}_2)$ so that $h(\tilde{E}_1)=\tilde{E}_2$ (for PR-homeomorphisms, see [**7**]). Since $H_*(\tilde{E}_1)=H_*(S^1)$ and $H^k(S^1; \Gamma_k)=0$ for $k\geqq 1$, it follows from Munkres-Hirsch obstruction theory ([**18**] and [**6**]) that $h|\tilde{E}_1$ is concordant to a diffeomorphism. Hence we may assume that $h: \tilde{E}_1\to\tilde{E}_2$ is a diffeomorphism. By [**2**] and ([**24**], Theorem C for $n\leqq 4$), the obstructions to extending a diffeomorphism $f_2^{-1}\circ h\circ f_1: S^n\times S^1\to S^n\times S^1$ to one of $(S^n\times D^2, S^n\times 0^2)$ onto itself lie in the groups Γ_{n+2} and $\pi_1(SO_{n+1})=Z_2$. However, as in the proof of ([**2**], Corollary 3), the one corresponding to an element of Γ_{n+2} vanishes. Since $h: (S^{n+2}, \tilde{S}_1)\to(S^{n+2}, \tilde{S}_2)$ is a PR-homeomorphism it follows that $f_2^{-1}\circ h\circ f_1$ is extendable to a PR-homeomorphism of $(S^n\times D^2, S^n\times 0^2)$. Thus another corresponding to an element of $\pi_1(SO_{n+1})$ vanishes, for, otherwise, $f_2^{-1}\circ h\circ f_1: S^n\times S^1\to S^n\times S^1$ cannot be extended to a PR-homeomorphism of $(S^n\times D^2, S^n\times 0^2)$, see [**11**]. Therefore, the diffeomorphism $h|\tilde{E}_1: \tilde{E}_1\to\tilde{E}_2$ extends to a diffeomorphism $h': (S^{n+2}, \tilde{S}_1)\to(S^{n+2}, \tilde{S}_2)$ and hence $(S^{n+2}, \tilde{S}_1)$ and $(S^{n+2}, \tilde{S}_2)$ are diffeomorphic. Thus $t_n: K_0^n\to K_{PL}^n$ is injective. Thirdly, to see the injectivity of $t_n: D_0^n\to D_{PL}^n$, suppose that we are given two smooth n-disk knots $(D^{n+2}, \tilde{D}_1)$ and $(D^{n+2}, \tilde{D}_2)$. Then, letting $\tilde{E}_1$ and $\tilde{E}_2$ be the closures of the complements of tubular neighborhoods of $\tilde{S}_1$ and $\tilde{S}_2$ in S^{n+2} respectively, we think of $(D^{n+2}, \tilde{D}_1)$ and $(D^{n+2}, \tilde{D}_2)$ as formed from $\tilde{E}_1$ and $\tilde{E}_2$ by attaching $(D^n\times D^2, D^n\times 0^2)$ by appropriate smooth embeddings $g_1: D^n\times S^1\to b\tilde{E}_1$ and

$g_2 : D^n \times S^1 \to b\tilde{E}_2$. Thus $(D^{n+2}, \tilde{D}_1) = (E_1 \underset{g_1}{\cup} D^n \times D^2, (D^n \times 0^2))$ and $(D^{n+2}, \tilde{D}_2) = (E_2 \underset{g_2}{\cup} D^n \times D^2, (D^n \times 0^2))$. Further, suppose that $(D^{n+2}, \tilde{D}_1)$ and $(D^{n+2}, \tilde{D}_2)$ are (PL) homeomorphic. Then by the uniqueness of relative regular neighborhoods we may take a PR-homeomorphism $h : (D^{n+2}, \tilde{D}_1) \to (D^{n+2}, \tilde{D}_2)$ so that $h(\tilde{E}_1) = \tilde{E}_2$. Since $h((D^n \times D^2), (D^n \times 0^2)) = ((D^n \times D^2), (D^n \times 0^2))$ and $h|(D^n \times D^2)$ is concordant to the identity keeping $S^{n-1} \times D^2 \cup D^n \times 0^2$ setwise fixed modulo orientation reversing PR-homeomorphisms of $(D^n \times 0^2)$ and $(0^n \times D^2)$, (see [**11**]), we may assume that $h|((D^n \times D^2), (D^n \times 0^2))$ is a diffeomorphism. By Munkres-Hirsch obstruction theory, the obstructions approximating the PR-homeomorphism $h|\tilde{E}_1 : \tilde{E}_1 \to \tilde{E}_2$ by a diffeomorphism $h' : \tilde{E}_1 \to \tilde{E}_2$ relative to $(D^n \times S^1) \subset b\tilde{E}_1$ lie in the cohomology groups $H^k(E_1, (D^n \times S^1); \Gamma_k)$. However, by a short calculation we have $H^k(E_1, (D^n \times S^1)) = 0$ for $k \geqq 0$, and hence the universal coefficient theorem $H^k(E_1, (D^n \times S^1); \Gamma_k) = 0$ for $k \geqq 0$.

It follows that there is a diffeomorphism $h'' : (D^{n+2}, \tilde{D}_1) \to (D^{n+2}, \tilde{D}_2)$ such that $h''|(D^n \times D^2) = h|(D^n \times D^2)$. Therefore, $(D^{n+2}, \tilde{D}_1)$ and $(D^{n+2}, \tilde{D}_2)$ are diffeomorphic. Thus $t_n : D_0^n \to D_{PL}^n$ is injective. This completes the proof of Theorem 5.2.

Tokyo Metropolitan University

References

[1] W. Browder, Manifolds with $\pi_1 = Z$, Bull. Amer. Math. Soc., **72** (1966), 225-231.

[2] W. Browder, Diffeomorphisms of 1-connected manifolds, Trans. Amer. Math. Soc., **128** (1967), 155-163.

[3] R.H. Fox and J.W. Milnor, Singularities of 2-spheres in 4-space and cobordism of knots, Osaka J. Math., **3** (1966), 257-267.

[4] H. Gluck, The embedding of two spheres in the four sphere, Trans. Amer. Math. Soc., **104** (1964), 303-333.

[5] M.W. Hirsch, On combinatorial submanifolds of differentiable manifolds, Comment. Math. Helv., **36** (1961), 103-111.

[6] M.W. Hirsch, Obstruction theories for smoothing manifolds and maps, Bull. Amer. Math. Soc., **69** (1963), 352-356.

[7] M.W. Hirsch and L.P. Neuwirth, On piecewise regular n-knots, Ann. of Math., **80** (1964), 594-612.

[8] J.F.P. Hudson and E.C. Zeeman, On regular neighborhoods, Proc. London Math. Soc., **14** (1964), 719-745.

[9] J.F.P. Hudson and D.W.L. Sumners, Knotted ball pairs in unknotted sphere pairs, J. London Math. Soc., **41** (1966), 717-722.

[10] M. Kato, Combinatorial prebundles, Part II, Osaka J. Math., **4** (1967), 305-311.

[11] M. Kato, A concordance classification of PL homeomorphisms of $S^p \times S^q$, Topology, (to appear).

[12] M. Kervaire, Les noeuds de dimensions superieurs, Bull. Soc. Math. France, **93** (1965), 225-271.

[13] M. Kervaire, On higher dimensional knots, Proceedings of Symposium in Honor

of Marston Morse, Princeton Univ. Press, 1965, 105–119.
[14] M. Kervaire and J. W. Milnor, Groups of homotopy spheres, I, Ann. of Math., **77** (1963), 504–537.
[15] R. K. Lashof and M. Rothenberg, Microbundles and smoothings, Topology, **3** (1965), 357–388.
[16] J. Levine, Unknotting spheres in codimension two, Topology, **3** (1965), 9–16.
[17] B. Mazur, On the structure of certain semi-groups of spherical knot classes, Publs. Math. I. H. E. S., **3** (1959), 19–27.
[18] J. Munkres, Obstructions to the smoothing of piecewise-differentiable homeomorphisms, Ann. of Math., **72** (1962), 521–554.
[19] H. Noguchi, A classification of orientable surfaces in 4-space, Proc. Japan Acad., **39** (1963), 422–423.
[20] H. Noguchi, One flat 3-manifolds in 5-space, Osaka J. Math., **1** (1964), 117–125.
[21] H. Noguchi, One flat submanifolds with codimension two, Illinois J. Math., **13** (1969), 220–223.
[22] H. Noguchi, Obstructions to locally flat embeddings of combinatorial manifolds, Topology, **5** (1966), 203–213.
[23] H. Noguchi, Classical combinatorial topology (mimeographed), University of Illinois, 1967.
[24] H. Sato, Diffeomorphism groups of $S^p \times S^q$ and exotic spheres (to appear).
[25] H. Schubert, Die eindeutige Zerlegbarkeit eines Knotens in Primknoten, S. -B. Heidelberger Akad. Wiss. Math. Nat. Kl. 1949, No. 3, 57–104.
[26] C. T. C. Wall, Unknotting tori in codimension one and spheres in codimension two, Proc. Camb. Phil. Soc., **61** (1965), 659–664.
[27] C. T. C. Wall, Locally flat submanifolds with codimension two, Proc. Camb. Phil. Soc., **63** (1967), 5–8.
[28] M. H. A. Newman, Boundaries of ULC sets in euclidean n-space, Proc. Nat. Acad. Sci. U. S. A., **34** (1948) 193–196.
[29] E. C. Zeeman, Unknotting combinatorial balls, Ann. of Math., **78** (1962), 501–526.
[30] E. C. Zeeman, Seminars on combinatorial topology (mimeographed), Inst. des Hautes Études Sci., Paris (1963–1966).