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Differentiability of nonlinear semigroups

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In the previous paper [4] we discussed the Hille-Yosida theorem in case of nonlinear semigroups in Hilbert spaces: For a maximal dissipative operator A the evolution equation $\frac{d}{dt}u(t) \in A \cdot u(t)$, $u(0) \in D(A)$ has a unique solution in a certain weak sense, and hence such an operator generates uniquely a contraction semigroup, and conversely, if the generator A_0 of a contraction semigroup $\{T_t\}$ is densely defined, a maximal dissipative extension A of A_0 generates the initial semigroup $\{T_t\}$. Thus the following two problems have been left open:

1) whether weak solutions of $\frac{d}{dt}u \in A \cdot u$ for a maximal dissipative operator A are genuine solutions or not,

2) whether the generator of a nonlinear contraction semigroup in Hilbert space is densely defined or not.

In this paper we give positive answers to these problems. Further we study nonlinear holomorphic semigroups: We show a parallel theory with the linear case on such semigroups $\{T_t\}$'s that for fixed $x \in H$, $T_t x$ is holomorphic in t and for a fixed t, T_t is analytic as a mapping $H \to H$. Analytic mapping is a natural generalization of continuous linear operators.

In [3] Kato gave positive answer to the problem 1) in case of single-valued operator A, and extended main part of [4] to the case of Banach spaces with uniformly convex duals. He solved further nonlinear evolution equations in which the generator A depends on t. Some part of our results can be extended to the case of Banach spaces with uniformly convex duals or the case in which the generator A depends on t. For simplicity, however, we restrict ourselves to the case of nonlinear semigroups in Hilbert spaces.

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REMARK. After finishing this work the author was communicated by Professors Crandall, Pazy, Kato and Dorroh their new works [10], [11] and [13] which contain remarkable results. Especially, together with their results we attain to a complete form of the Hille-Yosida Theorem for nonlinear semigroups in Hilbert spaces. A short explanation for this is given in § 5 as additional notes. The author is very grateful for their communication.

§ 1. Genuine solutions of $\frac{d}{dt}u(t) \in A \cdot u(t)$.

In this section we deal with nonlinear evolution equations. Let A be a (multi-valued) maximal dissipative operator, i. e., A satisfies the following:

(1) $$\mathrm{Re}\langle x'-y', x-y\rangle \leqq 0 \quad \text{for } x' \in A \cdot x,\ y' \in A \cdot y,\ x, y \in D(A).$$

(2) $$D((I-\mu A)^{-1}) = H \quad \text{for } \mu > 0.$$

We shall consider the Cauchy problem

(3) $$\frac{d}{dt}u(t) \in A \cdot u(t)$$
$$u(0) = x \in D(A).$$

If $u(t)$ is absolutely continuous, then $u(t)$ is differentiable for a. e. t and is expressed by the indefinite integral of the derivative. We say that $u(t)$ is a *genuine solution* of the equation 3) if $u(t)$ is absolutely continuous, belongs to the domain $D(A)$ of the operator A for a. e. t and satisfies 3) for a. e. t.

In [4] we showed that the equation 3) has a solution in a certain weak sense, and such a solution is unique. More precisely, we constructed an approximating sequence $\{u_n\}$ to the weak solution u such that

(4) $u_n(t)$ is the solution of $\frac{d}{dt}u_n(t) = A_n u_n(t)$, where A_n is a mapping:
$$x - \frac{1}{n}x' \to x' \quad \text{for } x' \in A \cdot x,\ x \in D(A).$$

(5) $u_n(t) \to u(t)$ in the norm topology uniformly in $t \in [0, t_0]$.

(6) $\frac{d}{dt}u_{n_k}(t) \to \frac{d}{dt}u(t)$ in the weak topology $\sigma(L^2_H[0, t_0], L^2_H[0, t_0])$,

where $L^2_H[0, t_0]$ is the Hilbert space of all square integrable H-valued measurable functions on $[0, t_0]$. Our purpose in this section is the following

THEOREM 1. *The Cauchy problem* (3) *has a unique genuine solution.*

For the proof we need several lemmas.

LEMMA 1. *Let $\tilde{A}$ be a mapping $L^2_H[0, t_0] \to L^2_H[0, t_0]$ such that*

(7) $$\dot{f} \in \tilde{A} \cdot f \quad \text{for } \dot{f}, f \in L^2_H[0, t_0] \text{ if and only if } \dot{f}(t) \in A \cdot f(t) \quad \text{for a. e. } t.$$

If A is maximal dissipative, $\tilde{A}$ is also maximal dissipative.

PROOF. Let $\dot{f} \in \tilde{A} \cdot f$, $\dot{g} \in \tilde{A} \cdot g$. From the evident inequality

$$\operatorname{Re}\langle \dot{f}-\dot{g}, f-g\rangle = \operatorname{Re}\int_0^{t_0}\langle \dot{f}(t)-\dot{g}(t), f(t)-g(t)\rangle dt$$
$$\leqq 0$$

it follows that $\tilde{A}$ is dissipative. Hence it suffices to show that $D((I-\tilde{A})^{-1}) = L_H^2[0, t_0]$. For $f\in L_H^2[0, t_0]$ we put $g(t)=(I-A)^{-1}f(t)$. Since $(I-A)^{-1}$ is Lipschitz continuous and $f(t)$ is measurable, the function $g(t)$ is measurable. For a fixed $x\in H$, we have $\|g(t)-(I-A)^{-1}x\|\leqq\|f(t)-x\|$. Hence it holds that

$$\|g(t)\|\leqq\|x\|+\|(I-A)^{-1}x\|+\|f(t)\|\,,$$

which implies $\int_0^{t_0}\|g(t)\|^2dt<\infty$, since $f\in L_H^2[0, t_0]$.

LEMMA 2. *Let B be a dissipative operator. Then the extension*

$$\bar{B}\cdot x=\{\dot{x}: {}^{\exists}x_n\in D(B),\ x_n\to x \text{ strong, and } {}^{\exists}\dot{x}_n\in B\cdot x_n\,,$$
$$\dot{x}_n\to\dot{x} \text{ weak}\}$$

is also dissipative. Hence, if B is maximal dissipative, we have $B=\bar{B}$.

PROOF. If $\dot{x}\in\bar{B}\cdot x$ and $\dot{y}\in\bar{B}\cdot y$ we have evidently

$$\operatorname{Re}\langle\dot{x}-\dot{y},\ x-y\rangle=\lim\operatorname{Re}\langle\dot{x}_n-\dot{y}_n,\ x_n-y_n\rangle\leqq 0\,.$$

PROOF OF THEOREM 1. By Lemma 1, the operator $\tilde{A}$ is maximal dissipative. Hence by Lemma 2, the extension $\bar{\tilde{A}}$ is equal to $\tilde{A}$. Put $v_n(t)=\left(I-\frac{1}{n}A\right)^{-1}u_n(t)$ for an approximating sequence $\{u_n\}$ in (4). Since $\|u_n(t)-v_n(t)\|\to 0$ uniformly in t (see [4]), $v_n\to u$ strongly in $L_H^2[0, t_0]$. The two relations $\frac{d}{dt}u_n(t)=A_nu_n(t)\in A\cdot v_n(t)$ and $\frac{d}{dt}u_{n_k}\to\frac{d}{dt}u$ weakly in $L_H^2[0, t_0]$ by (6) imply $\frac{d}{dt}u\in\bar{\tilde{A}}\cdot u=\tilde{A}\cdot u$. Since $u(t)$ is absolutely continuous, our weak solution $u(t)$ is a genuine solution. The uniqueness of a genuine solution follows from the dissipativity of $\tilde{A}$:

$$\|u(t)-v(t)\|^2=\|u(0)-v(0)\|^2+\int_0^t\frac{d}{ds}\|u(s)-v(s)\|^2ds$$
$$=\|u(0)-v(0)\|^2+2\int_0^t\operatorname{Re}\left\langle\frac{d}{ds}u(s)-\frac{d}{ds}v(s),\ u(s)-v(s)\right\rangle ds$$
$$\leqq\|u(0)-v(0)\|^2,$$

for $\frac{d}{ds}u(s)\in A\cdot u(s)$, $\frac{d}{ds}v(s)\in A\cdot v(s)$ for a. e. s.

§ 2. Domain of generators.

The purpose of this section is to prove the following

Theorem 2. *If the domain $D(T_t)$ of a contraction semigroup $\{T_t\}$ is convex and closed, the domain $D(A_0)$ of the infinitesimal generator A_0 of $\{T_t\}$ is dense in $D(T_t)$.*

In [4] we introduced three notions of infinitesimal generators A_0, A_Φ and $A_{\Phi,\lambda}$. We cite them in some revised form:

(8) $$A_0 x = \lim_{h \downarrow 0} A_h x\,, \qquad \text{where } A_h = \frac{1}{h}(T_h - I)\,.$$

(9) $$A_\Phi x = w\text{-}\lim_{h \in \varphi \in \Phi} A_h x \qquad \text{with } \sup_{h>0} \|A_h x\| < \infty\,,$$

where Φ is an ultra-filter of subsets $\varphi \subset (0, \infty)$ converging to 0.

(10) $$A_{\Phi,\lambda} x = \{x - y : x = w\text{-}\lim_{h \in \varphi \in \Phi} (I - \lambda A_h)^{-1} y\} \qquad \text{for } \lambda > 0\,,$$

where Φ is the same as in (9).

In [4, Theorem 2 and Corollary to Theorem 1] we showed some of their basic properties:

(11) $$A_0 \subset A_\Phi\,, \qquad A_0 \subset A_{\Phi,\lambda}\,,$$

and

(12) $$\overline{D(A_0)} = \overline{D(A_\Phi)}\,,$$

where $\overline{D(A_0)}$ for instance means the closure of $D(A_0)$.

For the proof of Theorem 2, we need more precise properties of these generators: the key to the proof is Lemma 6. Since the generator of a non-contraction semigroup is not necessarily densely defined (see [4, Example 1]), the proof must be based on the contraction property. We use the method of "infinite speed principle" which we shall explain in the following. For a contraction semigroup $\{T_t\}$ and for $x \in D(T_t)$, $T_t x$ is continuous in t by definition. Thus for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

(13) $$\|T_t x - x\| < \varepsilon \qquad \text{for } 0 < t < \delta\,.$$

Roughly speaking, "$T_t x$ is of finite speed at $t = 0$." Let y be another point of $D(T_t)$. If the vector $T_t y - y$ has the opposite direction to $x - y$ (i.e., $T_t y - y = \mu(y - x)$ for some $\mu > 0$), the contraction condition $\|T_t y - T_t x\| \leqq \|x - y\|$ implies

$$\|T_t y - y\| \leqq \|T_t x - x\|\,,$$

that is, "$T_t x$ is no less speedy than $T_t y$". Suppose that for a fixed $x \in H$ there exists a sequence $\{y_n\} \subset D(T_t)$ such that each vector $T_t y_n - y_n$ has the opposite direction to $x - y_n$ and the speed of $T_t y_n$ increases infinitely as $n \to \infty$.

Then, if $x \in D(T_t)$, $T_t x$ must have infinite speed at $t=0$. This is a contradiction, i.e., $x \notin D(T_t)$. More precisely, we have

Infinite speed principle. (i) If for $x \in H$ there exist some sequence $\{y_n\} \subset D(T_t)$, $t_n \downarrow 0$ and a constant $\kappa > 0$ such that

$$\|T_{t_n} y_n - x\| > \|y_n - x\| + \kappa, \qquad \text{for } n = 1, 2, \cdots,$$

then we have $x \notin D(T_t)$.

(ii) If for $x \in H$ there exist some sequence $\{y_n\} \subset D(T_t)$, $h_n \downarrow 0$ and a constant $\kappa > 0$ such that

$$A_{h_n} y_n = \mu_n (y_n - x) \qquad \text{for some } \mu_n > 0 \ \left(A_{h_n} = \frac{1}{h_n}(T_{h_n} - I)\right)$$

and

$$\|A_{h_n} y_n\| \to \infty, \qquad \|y_n - x\| \geqq \kappa > 0,$$

then we have $x \notin D(T_t)$.

Proof of (i). Suppose that $x \in D(T_t)$. Then we have

$$\begin{aligned} \|T_{t_n} x - x\| &= \|T_{t_n} x - T_{t_n} y_n + T_{t_n} y_n - x\| \\ &\geqq \|T_{t_n} y_n - x\| - \|T_{t_n} x - T_{t_n} y_n\| \\ &\geqq \|y_n - x\| + \kappa - \|x - y_n\| = \kappa . \end{aligned}$$

This contradicts (13) since $\kappa > 0$ and $t_n \downarrow 0$.

Proof of (ii). Suppose that $x \in D(T_t)$. Let x' be an element of H such that $\|x' - x\| \leqq \frac{\kappa}{2}$. Let x''_n be the element defined by

$$x''_n - x = \alpha_n (y_n - x), \qquad \text{where } \alpha_n = \frac{\mathrm{Re}\langle x' - x, y_n - x\rangle}{\|y_n - x\|^2}.$$

Then $\mathrm{Re}\langle x' - x''_n, y_n - x\rangle = 0$ and $|\alpha_n| \leqq \frac{1}{2}$. The relation $A_{h_n} y_n = \mu_n (y_n - x)$ means $T_{h_n} y_n - y_n = h_n \mu_n (y_n - x)$. Hence we have

$$\begin{aligned} T_{h_n} y_n - x''_n &= y_n - x + h_n \mu_n (y_n - x) + x - x''_n \\ &= [1 + h_n \mu_n - \alpha_n](y_n - x). \end{aligned} \tag{14}$$

This implies $\mathrm{Re}\langle T_{h_n} y_n - x''_n, x' - x''_n\rangle = [1 + h_n \mu_n - \alpha_n]\, \mathrm{Re}\langle y_n - x, x' - x''_n\rangle = 0$. Thus we have

$$\|T_{h_n} y_n - x'\|^2 = \|T_{h_n} y_n - x''_n\|^2 + \|x' - x''_n\|^2. \tag{15}$$

On the other hand,

$$\begin{aligned} \|y_n - x'\|^2 &= \|y_n - x''_n\|^2 + \|x''_n - x'\|^2 \\ &= [1 - \alpha_n]^2 \|y_n - x\|^2 + \|x''_n - x'\|^2. \end{aligned} \tag{16}$$

By (14), (15) and (16) we have

$$\|T_{h_n}y_n-x'\|^2-\|y_n-x'\|^2 \geqq h_n\mu_n\|y_n-x\|^2,$$

or

$$(17)\qquad \|T_{h_n}y_n-x'\|-\|y_n-x'\| \geqq \frac{h_n\mu_n\|y_n-x\|^2}{\|T_{h_n}y_n-x'\|+\|y_n-x'\|}$$

$$\geqq \frac{h_n\mu_n\|y_n-x\|^2}{\|T_{h_n}y_n-x\|+\|y_n-x\|+2\|x-x'\|}$$

$$\geqq \frac{h_n\kappa}{3\kappa+\varepsilon'}\|A_{h_n}y_n\| = h_n\rho\|A_{h_n}y_n\|,$$

$$\text{for } \rho=\frac{\kappa}{3\kappa+\varepsilon'}$$

since $\|T_{h_n}y_n-x\| \leqq \|y_n-x\|+\varepsilon'$ for $n \geqq n(\varepsilon')$ by (i). Note that the positive constant ρ is independent of $n \geqq n(\varepsilon')$. We put $x_l=T_{lh_n}x$. Suppose that $\|x_l-x\| \leqq \frac{\kappa}{2}$ for $l=1, 2, \cdots, m$. In (17) putting $x'=x_l$ for $l=1, 2, \cdots, m$, we have

$$\|y_n-x_{l-1}\|-\|y_n-x_l\| \geqq \|T_{h_n}y_n-x_l\|-\|y_n-x_l\| \geqq \rho h_n\|A_{h_n}y_n\|,$$

hence

$$(18)\qquad \|y_n-x\|-\|y_n-x_m\| \geqq \rho m h_n\|A_{h_n}y_n\|.$$

Let $m=\left[\frac{\delta}{h_n}\right]$. Then $\|y_n-x\|-\|y_n-x_m\| \leqq \|x-x_m\| < \varepsilon$ by (13). This contradicts (18), since $\rho m h_n\|A_{h_n}y_n\| \to \infty$ as $n \to \infty$.

LEMMA 3. *Let $D(T_t)$ be convex and closed. Then for any $h>0$ and $\lambda \geqq 0$, there exists a $y_{h\lambda} \in D(T_t)$ such that $(I-\lambda A_h)y_{h\lambda}=x$. $y_{h\lambda}$ depends continuously on h for fixed $\lambda \geqq 0$ and also on λ for fixed $h>0$.*

PROOF. We define the mapping $P: z \to \frac{h}{\lambda+h}x+\frac{\lambda}{\lambda+h}T_h z$. Note that the relation $y=(I-\lambda A_h)^{-1}x$ holds for $y \in D(T_t)$ if and only if $y=Py$. For the approximating sequence $\{y_n\}: y_0=x, y_{n+1}=Py_n$, each y_{n+1} is contained in $D(T_t)$ since $x \in D(T_t)$, $y_n \in D(T_t)$. Since P satisfies $\|Pz-Pz'\| \leqq \frac{\lambda}{\lambda+h}\|z-z'\|$ and since $D(T_t)$ is closed, the sequence $\{y_n\}$ converges to an element $y_{h\lambda} \in D(T_t)$. Evidently $y_{h\lambda}$ satisfies the equation $y=Py$. The continuous dependence of $y_{h\lambda}$ on h and λ is evident.

LEMMA 4. *Let $D(T_t)$ be convex and closed. For any point $x \in D(T_t)$ the set $\{y_{h\lambda}=(I-\lambda A_h)^{-1}x: h>0\}$ is bounded. Moreover, for any $\rho>0$, the weak limit $y_\Phi = w\text{-}\lim_{h\in\varphi\in\Phi}(I-\lambda A_h)^{-1}x$ exists in $\{y \in D(T_t): \|y-x\|<\rho\}$ for some $\lambda>0$.*

PROOF. At first we shall prove the second half of our assertion. By Lemma 3, there exists $y_{h\lambda}$ in $D(T_t)$ such that $(I-\lambda A_h)y_h=x$ for $h, \lambda>0$. By Infinite principle ii), for $\kappa=\rho$ there exists a constant $M>0$ such that

$$(19)\qquad \|A_h y_{h\lambda}\|<M, \qquad \text{for } \|y_{h\lambda}-x\| \geqq \rho.$$

Let $\lambda_0 \leqq \frac{\rho}{2M}$. If $\|y_{h\lambda_0}-x\| \geqq \rho$ for some $h>0$, then the relation $(I-\lambda_0 A_h)y_{h\lambda_0} = x$ implies

$$\|A_h y_{h\lambda_0}\| = \frac{1}{\lambda_0}\|y_{h\lambda_0}-x\| \geqq 2M.$$

This contradicts (19). Hence

$$\|y_{h\lambda_0}-x\| \leqq \rho, \qquad \text{for } h>0. \tag{20}$$

Since bounded sets in H are weakly compact, the weak limit $y_\Phi = w\text{-}\lim_{h\in\varphi\in\Phi} y_{h\lambda_0}$ exists in $\|y-x\| \leqq \rho$. Since $D(T_t)$ is convex and closed, it is weakly closed, and so $D(T_t) \ni y_\Phi$.

The relations $\lambda_0 = \frac{\rho}{2M}$ and (20) imply

$$\|y_{h\lambda_0}-x\| \leqq 2\lambda_0 M,$$

which means the boundedness of $\{y_{h\lambda_0}: h>0\}$.

LEMMA 5. *For a constant $h>0$ and a point $z\in D(T_t)$, let F be a real hyperplane which contains z and is orthogonal to $T_h z - z$. If for a point $y\in D(T_t)$, $T_h y$ is in the opposite side to $T_h z$ concerning F, then we have $\|y-z\| \geqq \|T_h y - z\|$.*

PROOF. By assumption, $F=\{x+z: \operatorname{Re}\langle x, T_h z - z\rangle = 0\}$ and $\operatorname{Re}\langle T_h y - z, T_h z - z\rangle < 0$. Hence we have

$$\begin{aligned}
\|T_h y - z\|^2 &= \|T_h y - T_h z + T_h z - z\|^2 \\
&= \|T_h y - T_h z\|^2 + \|T_h z - z\|^2 + 2\operatorname{Re}\langle T_h y - T_h z,\ T_h z - z\rangle \\
&\leqq \|y-z\|^2 - \|T_h z - z\|^2 + 2\operatorname{Re}\langle T_h y - z,\ T_h z - z\rangle \\
&\leqq \|y-z\|^2 .
\end{aligned}$$

LEMMA 6. *Let $D(T_t)$ be convex and closed. For $x\in D(T_t)$ the relation*

$$y = (I-\lambda A_{\Phi\lambda})^{-1}x \qquad (= w\text{-}\lim_{h\in\varphi\in\Phi} (I-\lambda A_h)^{-1}x) \tag{21}$$

implies

$$y = \lim_{h\in\varphi\in\Phi} (I-\lambda A_h)^{-1}x \tag{22}$$

that is, $(I-\lambda A_h)^{-1}x$ converges strongly to y.

PROOF. Note that $y_h = (I-\lambda A_h)^{-1}x$ exists in $D(T_t)$ by Lemma 3. Suppose that y_h does not converge strongly to y. We shall obtain a contradiction, by showing the following steps.

I. There exists a sequence $h_n \downarrow 0$ such that

$$y_n = (I-\lambda A_{h_n})^{-1}x \to y \qquad \text{weakly}, \tag{23}$$

and

$$(24) \qquad \|y_n - y\| \to \rho > 0 .$$

In fact, since the set $\{y_h : h > 0\}$ is separable and bounded by Lemma 4, the weak topology on the set $\{y_h\}$ is metrizable. Hence $w\text{-}\lim_{h\in\varphi\in\Phi} y_h = y$ implies the existence of a sequence $\{h_n > 0\}$ satisfying (23). If every sequence satisfying (23) converges strongly to y, then $\{y_h : h \in \varphi \in \Phi\}$ converges strongly to y. Hence by our assumption there exists a sequence $\{y_n\}$ satisfying (23) and (24) both.

II. For any $\varepsilon > 0$ there exists $n(\varepsilon) > 0$ such that

$$(25) \qquad \|T_{h_n} y_n - y_n\| < \varepsilon, \quad |\langle y_n - y, x - y\rangle| < \varepsilon, \quad |\|y_n - y\|^2 - \rho^2| < \varepsilon \quad \text{for } n \geqq n(\varepsilon) .$$

Moreover, if ε is sufficiently small, for a fixed $n \geqq n(\varepsilon)$ there exist $\kappa_n > 0$ and $m(\varepsilon, n) > n$ such that

$$(26) \qquad \sup_{0<h<h_n} |\langle T_h y_n - y_n, y_m - y\rangle| < \varepsilon, \ |\langle y - y_n, y - y_m\rangle| < \varepsilon, \quad \text{for } m \geqq m(\varepsilon, n),$$

$$(27) \qquad \|y_n - y_m\| + \kappa_n < \|T_{h_n} y_n - y_m\| \qquad \text{for } m \geqq m(\varepsilon, n),$$

$$(28) \qquad \|T_h y_n - T_{h_n} y_n\| < \kappa_n \qquad \text{for } h = \left[\frac{h_n}{h_m}\right] h_m, \ m \geqq m(\varepsilon, n).$$

The first inequality in (25) follows from

$$T_{h_n} y_n - y_n = h_n A_{h_n} y_n = h_n \lambda^{-1}(y_n - x) \to 0 \qquad \text{as } n \to \infty .$$

The second and third inequalities in (25) follow from (23) and (24). The inequality (26) follows from the fact that the sequence $\langle y_n - y, T_h y_n - y\rangle$ converges to 0 uniformly on the compact set $\{T_h y_n - y : 0 \leqq h \leqq h_n\}$. The inequality (27) follows from

$$\begin{aligned} \|T_{h_n} y_n - y_m\|^2 &= \|y_n + h_n \lambda^{-1}(y_n - x) - y_m\|^2 \\ &= \|y_n - y_m\|^2 + \lambda^{-2} h_n^2 \|y_n - x\|^2 + 2\lambda^{-1} h_n \operatorname{Re}\langle y_n - x, y_n - y_m\rangle \\ &\geqq \|y_n - y_m\|^2 + 2\lambda^{-1} h_n \|y_n - y\|^2 - 6\lambda^{-1} h_n \varepsilon \geqq \|y_n - y_m\|^2 + 2\lambda^{-1} h_n(\rho^2 - 4\varepsilon) . \end{aligned}$$

The inequality (28) is evident by the fact $\lim_{m\to\infty} \left[\frac{h_n}{h_m}\right] h_m = h_n$.

III. We shall show that there exists h'_n, $0 < h'_n < h_n$, such that

$$(29) \qquad \operatorname{Re}\langle T_{h'_n} y_n - y, x - y\rangle < -\rho^2 + 3\varepsilon \qquad \text{for } n > n(\varepsilon) .$$

We apply Lemma 5 putting $z = y_m$, $h = h_m$ and $y = T_{kh_m} y_n$ for $k = 0, 1, 2, \cdots, \left[\frac{h_n}{h_m}\right]$. Then if we had $\operatorname{Re}\langle T_{kh_m} y_n - y_m, T_{h_m} y_m - y_m\rangle \leqq 0$ for all k, we had

$$(30) \qquad \|T_{kh_m} y_n - y_m\| \geqq \|T_{(k+1)h_m} y_n - y_m\| .$$

Hence by (28)

$$\|T_{h_n}y_n-y_m\| \leqq \|T_{h_n}y_n-T_{[h_n/h_m]h_m}y_n\|+\|T_{[h_n/h_m]h_m}y_n-y_m\|$$
$$<\kappa_n+\|y_n-y_m\|\,.$$

This contradicts (27). Thus we have for some $h'=kh_m$ $(0<h'<h_n)$

$$\operatorname{Re}\langle T_{h'}y_n-y_m,\ T_{h_m}y_m-y_m\rangle>0\,,$$

or equivalently (using $T_{h_m}y_m-y_m=-h_m\lambda^{-1}(x-y_m)$)

$$\operatorname{Re}\langle T_{h'}y_n-y_m,\ x-y_m\rangle<0\,.$$

Combining this with second and third inequalities in (25) we have

$$\operatorname{Re}\langle T_{h'}y_n-y,\ x-y_m\rangle<-\rho^2+2\varepsilon\,.$$

Using (26) we have $\operatorname{Re}\langle T_{h'}y_n-y,\ x-y\rangle<-\rho^2+3\varepsilon$.

IV. Now we can show $x\notin D(T_t)$, a contradiction. By (29) and (25)

$$\|T_{h'_n}y_n-x\|^2=\|T_{h'_n}y_n-y\|^2+\|x-y\|^2-2\operatorname{Re}\langle T_{h'_n}y_n-y,\ x-y\rangle$$
$$\geqq \|x-y\|^2+2\rho^2-6\varepsilon\geqq \|y_n-x\|^2+\rho^2-8\varepsilon\,.$$

Since the set $\{\|y_n-x\|: n=1, 2, \cdots\}$ is bounded, we have for some $\kappa>0$

$$\|T_{h'_n}y_n-x\|>\|y_n-x\|+\kappa \qquad \forall n>n(\varepsilon),\ 0<\exists h'_n<h_n\,.$$

Thus infinite speed principle i) implies $x\notin D(T_t)$.

LEMMA 7. *Let y be a point of $D(T_t)$. If there exists an $x\in D(T_t)$ such that $y=(I-\lambda A_{\Phi\lambda})^{-1}x$ for some $\lambda>0$, then y is contained in $D(A_\Phi)$.*

PROOF. By Lemma 6, there exists a sequence $h_k\downarrow 0$ such that

$$y_k=(I-\lambda A_{h_k})^{-1}x\to y \qquad \text{strongly as } k\to\infty\,.$$

By the relation $\lambda(T_{h_k}y_k-y_k)=h_k(y_k-x)$ we have for a fixed $h>0$ and for $n_k=[h/h_k]$

$$\lambda\|T_{n_kh_k}y_k-y_k\|\leqq\lambda\sum_{n=1}^{n_k}\|T_{nh_k}y_k-T_{(n-1)h_k}y_k\|$$
$$\leqq\lambda n_k\|T_{h_k}y_k-y_k\|\leqq n_kh_k\|y_k-x\|\,.$$

Since $y_k\to y$, $n_kh_k\to h$ and $T_{n_kh_k}y_k\to T_hy$ as $k\to\infty$, we have

$$\left\|\frac{1}{h}(T_hy-y)\right\|\leqq\frac{1}{\lambda}\|y-x\| \qquad \text{for any } h>0\,.$$

The boundedness of $\{A_hy: h>0\}$ implies the existence of $\underset{h\in\varphi\in\Phi}{w\text{-lim}}\,A_hy$.

PROOF OF THEOREM 2. By Lemma 4, for an arbitrary point $x\in D(T_t)$ there exists $y\in D(T_t)$ such that $y=(I-\lambda A_{\Phi\lambda})^{-1}x$ and $\|x-y\|\leqq\rho$. By Lemma 7 we have $y\in D(A_\Phi)$. Since the constant ρ can be chosen arbitrarily small for a suitable $\lambda=\lambda(\rho)$, the relation (12) $\overline{D(A_0)}=\overline{D(A_\Phi)}$ implies our assertion.

Q. E. D.

We shall give some additional results on the generator $A_{\Phi\lambda}$. In the following we assume that $D(T_t)=H$. By Lemma 6 we see easily that

(31) $$A_{\Phi\lambda}y \ni x \Leftrightarrow {}^{\exists}\{y_h\}: \lim_{h\in\varphi\in\Phi} y_h = y \quad \text{and} \quad \lim_{h\in\varphi\in\Phi} A_h y_h = x\,.$$

In fact, it suffices to put $y_h=(I-\lambda A_h)^{-1}(y-\lambda x)$. By (31) the generator $A_{\Phi\lambda}$ is independent of λ:

$$A_{\Phi\lambda}=A_{\Phi\mu} \qquad \text{for } \lambda,\ \mu>0\,.$$

Further we can prove that $A_{\Phi\lambda}$ is independent of the choice of an ultrafilter Φ. In fact, we put for two ultrafilters Φ and Ψ

$$y=(I-\lambda A_{\Phi\lambda})^{-1}x\,, \qquad z=(I-\lambda A_{\Phi\lambda})^{-1}x\,.$$

Then by Lemma 6 there exist sequences $h_n\downarrow 0$ and $h'_n\downarrow 0$ such that

$$y=\lim_{n\to\infty}(I-\lambda A_{h_n})^{-1}x\,, \qquad z=\lim_{n\to\infty}(I-\lambda A_{h'_n})^{-1}x\,.$$

Assume that $y\neq z$ and $\|y-x\|\geqq\|z-x\|$. Then for some $\kappa>0$,

$$|\operatorname{Re}\langle z-x,\ x-y\rangle|+\kappa\leqq\|x-y\|^2\,.$$

By Lemma 5 and by the same argument as III in the proof of Lemma 6, we obtain

$$\sup_{0<h<h'_n}\operatorname{Re}\langle T_h z_n-x,\ y_m-x\rangle\geqq\|y_m-x\|^2 \qquad \text{for sufficiently large } n,\ m\,.$$

Hence $\sup_{0<h<h'_n}\|T_h z-x\|^2\geqq\|y-x\|^2-\frac{\kappa}{4}$ and

$$\sup_{0<h<h'_n}\operatorname{Re}\langle T_h z-x,\ y-x\rangle\geqq\|y-x\|^2-\frac{\kappa}{4} \qquad \text{for sufficiently large } n\,.$$

Since

$$\begin{aligned}\|z-(2x-y)\|^2&\leqq\|z-x\|^2+\|x-y\|^2-2\operatorname{Re}\langle z-x,\ x-y\rangle\\ &\leqq 4\|x-y\|^2-\kappa\,,\end{aligned}$$

and since for some h with $0<h<h'_n$

$$\begin{aligned}\|T_h z-(2x-y)\|^2&=\|T_h z-x\|^2+\|x-y\|^2+2\operatorname{Re}\langle T_h z-x,\ y-x\rangle\\ &\geqq\|T_h z-x\|^2+3\|x-y\|^2-\frac{\kappa}{2}\\ &\geqq 4\|x-y\|^2-\frac{3\kappa}{4}\,,\end{aligned}$$

we have $2x-y\notin D(T_t)$ by infinite speed principle (i). This contradicts the assumption $D(T_t)=H$.

§ 3. Domain of maximal contraction semigroups.

The Hille-Yosida theorem in [4] connected with the results in §§ 1 and 2 is almost satisfactory for the case that $D(T_t)=H$. For contraction semigroups whose domains are not the whole space H, the situation is a little complicated. However, we must treat such semigroups, especially in the theory of holomorphic semigroups (cf. § 4). We say that $\{T_t\}$ is *a maximal contraction semigroup* if it cannot be extended to a contraction semigroup with larger definition domain. Note that *any contraction semigroup may be extended to a maximal contraction semigroup.*

EXAMPLE. Let H be the one-dimensional Hilbert space C^1. We put

$$\varphi(t)=\begin{cases} 1-\sqrt{1-(1-t)^2} & \text{for } 0\leqq t\leqq 1, \\ 0 & \text{for } 1<t. \end{cases}$$

Then we have $\varphi(\varphi(t))=\min(t, 1)$. We define a contraction semigroup $\{T_t\}$ such that

$$T_t re^{i\theta}=\varphi(\varphi(r)+t)e^{i\theta}, \qquad \text{for } 0\leqq r\leqq 1.$$

The infinitesimal generator A_0 is defined in the unit disc:

$$A_0 re^{i\theta}=\varphi'(\varphi(r))e^{i\theta}=\frac{\varphi(r)-1}{1-r}e^{i\theta} \qquad \text{for } 0\leqq r<1.$$

Since the function $r-\varphi'(\varphi(r))$ is a strictly increasing function and maps $[0, 1)$ onto $[0, \infty)$, the resolvent $(I-A_0)^{-1}$ is defined on C^1. This semigroup $\{T_t\}$ has "infinite speed" on the boundary of the unit disc, hence it is a maximal contraction semigroup.

It should be noted that a maximal dissipative operator does not necessarily generate a maximal contraction semigroup. In fact, let A be the operator such that $A\cdot 0=H$. Then A generates the semigroup $\{T_t\}: T_t 0=0$. Evidently A is maximal dissipative but $\{T_t\}$ is not a maximal contraction semigroup. Thus we are led to the following:

PROBLEM 1. What kind of maximal dissipative operators generate maximal contraction semigroups? Conversely, is a maximal contraction semigroup necessarily generated by a maximal dissipative operator?

PROBLEM 2. Determine the condition on subsets of H in which there exist maximal contraction semigroups (or densely defined maximal dissipative operators).

We shall discuss these problems.

THEOREM 3. i) *The domain of a maximal contraction semigroup $\{T_t\}$ is a closed convex set not contained in any closed hyperplane.*

ii) *The closure of the domain $D(A)$ of a maximal dissipative operator A*

is convex.

For the proof of i) we need some lemmas.

LEMMA 8. *Let $\{S_\alpha : \alpha \in \Gamma\}$ and $\{S'_\alpha : \alpha \in \Gamma\}$ be two systems of spheres in H:*

$$S_\alpha = \{x \in H : \|x - x_\alpha\| \leqq r_\alpha\}, \qquad S'_\alpha = \{x \in H : \|x - x'_\alpha\| \leqq r'_\alpha\}.$$

If $\|x_\alpha - x_\beta\| \geqq \|x'_\alpha - x'_\beta\|$ and $r_\alpha \leqq r'_\alpha$ for every $\alpha, \beta \in \Gamma$, the relation $\bigcap_{\alpha \in \Gamma} S_\alpha \neq \phi$ implies $\bigcap_{\alpha \in \Gamma} S'_\alpha \neq \phi$.

For the proof, see [5].

Put $\Omega =$ the convex closed hull of $D(T_t)$.

LEMMA 9. *For a fixed natural number k, there exists a mapping $U_k : \Omega \to \Omega$ such that*

$$\|U_k x - U_k y\| \leqq \|x - y\| \qquad \text{for any } x, y \in \Omega$$

and

$$U_k x = T_{2^{-k}} x \qquad \text{for any } x \in D(T_t).$$

Moreover, if $x_0, x'_0 \in \Omega$ satisfy

$$\|x'_0 - T_{2^{-k}} x\| \leqq \|x_0 - x\| \qquad \text{for any } x \in D(T_t),$$

then there exists an extension U_k of $T_{2^{-k}}$ such that $U_k x_0 = x'_0$.

PROOF. We let the set $\Omega - D(T_t)$ be well-ordered, as $\{x_\alpha\}$. By transfinite induction we shall construct such a mapping U_k. Assume that the contraction U_k is defined for all x_β with $\beta < \alpha$ and for all $y \in D(T_t)$, satisfying $U_k y = T_{2^{-k}} y$. Let $S(x - z; \tilde{z}) = \{x' : \|x' - \tilde{z}\| \leqq \|x - z\|\}$. We apply Lemma 8 to the two families $\{S(x_\alpha - x_\beta; x_\beta), S(x_\alpha - y; y) : \beta < \alpha, y \in D(T_t)\}$ and $\{S(x_\alpha - x_\beta; U_k x_\beta), S(x_\alpha - y; U_k y) : \beta < \alpha, y \in D(T_t)\}$, then

$$\bigcap_{\beta < \alpha} S(x_\alpha - x_\beta : x_\beta) \cap \bigcap_{y \in D(T_t)} S(x_\alpha - y : y) \neq \phi \qquad (\text{since} \ni x_\alpha)$$

implies

$$\bigcap_{\beta < \alpha} S(x_\alpha - x_\beta; U_k x_\beta) \cap \bigcap_{y \in D(T_t)} S(x_\alpha - y; U_k y) = S^\alpha \neq \phi.$$

We denote by P the projection $H \to \Omega$ i.e. $Px = y$ for $\|y - x\| = \inf_{y' \in \Omega} \|y' - x\|$. Since $\|Pz - x\| \leqq \|z - x\|$ for any $x \in \Omega$, $z \in H$, we have $Pz' \in S^\alpha$ for $z' \in S^\alpha$. Hence $S^\alpha \cap \Omega \neq \phi$. We pick up an element $x'_\alpha \in S^\alpha \cap \Omega$ and let $U_k x_\alpha = x'_\alpha$. Then U_k, defined on $D(T_t) \cup \{x_\beta : \beta \leqq \alpha\}$ is a contraction. By transfinite induction we obtain a required mapping U_k. Q. E. D.

Let $\{U_k^\alpha : \alpha\}$ be the set of all mappings in Lemma 9. For every U_k^α we define $T_{2^{-k}}^\alpha = U_k^\alpha$ and $T_{t+s}^\alpha = T_t^\alpha T_s^\alpha$ for $t = j/2^k$, $s = i/2^k$. Then we have a semi-group $T^\alpha = \{T_t^\alpha : t = j/2^k, j = 0, 1, 2, \cdots\}$. We denote by τ_k the set $\{T^\alpha : \alpha\}$. We define the canonical mapping for $l \geqq k$

$$J_{l,k} : \tau_l \to \tau_k \text{ by } J_{l,k} T^\alpha = T^\beta \text{ for } T_{2^{-k}}^\alpha = T_{2^{-k}}^\beta, \ T^\alpha \in \tau_k, \ T^\beta \in \tau_l.$$

Evidently $D(J_{l,k})=\tau_l$ and $J_{m,l}J_{l,k}=J_{m,k}$. For $T^\alpha\in\tau_k$ we put $A_k^\alpha=2^k(T_{2^{-k}}^\alpha-I)$. By Lemma 3 for any $x\in\Omega$ and any $\lambda>0$ the element $y_k^\alpha(x,\lambda)=(I-\lambda A_k^\alpha)^{-1}x$ exists in Ω.

LEMMA 10. *For a fixed $z\in\Omega$, the family $\{T_t^\alpha z:\alpha\}$ are equicontinuous in t, that is, for any $\varepsilon>0$ there exists some $\delta>0$ such that if $2^{-k}<\delta$ we have*

$$\|T_{2^{-k}}^\alpha z-z\|<\varepsilon \qquad \textit{for any } T^\alpha\in\tau_k .$$

PROOF. First we shall verify our lemma for a z in the convex hull of $D(T_t)$:

$$z=\mu_1x_1+\mu_2x_2+\cdots+\mu_nx_n,\ x_j\in D(T_t),\ 0\leqq\mu_j\leqq 1,\ \sum\mu_j=1 .$$

Without loss of generality we may assume that this representation is unique, i.e. the points $\{x_1, x_2, \cdots, x_n\}$ are linearly independent. For $T^\alpha\in\tau_k$

$$\|T_{2^{-k}}^\alpha z-T_{2^{-k}}^\alpha x_j\|\leqq\|z-x_j\| \qquad j=1,2,\cdots,n .$$

For a sufficiently small $\delta>0$ we have

$$\|T_{2^{-k}}^\alpha x_j-x_j\|<\varepsilon' \qquad \text{for } 2^{-k}<\delta .$$

hence

$$\|T_{2^{-k}}^\alpha z-x_j\|\leqq\|z-x_j\|+\varepsilon' \qquad j=1,2,\cdots,n .$$

Assume that our assertion be verified for $n=n_0$. For $n=n_0+1$, denoting by P_j the orthogonal projection to the linear manifold spanned by $\{x_k:\ k\neq j\}$, we have

$$\|P_jT_{2^{-k}}^\alpha z-T_{2^{-k}}^\alpha z\|\leqq\|P_jz-z\|+\varepsilon'' , \tag{32}$$

since $\|P_jT_{2^{-k}}^\alpha z-x_j\|\leqq\|P_jz-x_j\|+\varepsilon'$. The relation (32) implies

$$\mu_j'\geqq\mu_j-\varepsilon''' \qquad j=1,2,\cdots,n ,$$

where $PT_{2^{-k}}^\alpha z=\sum_{k=1}^n\mu_k'x_k$ and P is the orthogonal projection to the linear manifold spanned by $\{x_k: 1\leqq k\leqq n\}$. Hence

$$\mu_j+n\varepsilon'''\geqq\mu_j'\geqq\mu_j-\varepsilon'''$$

since $1-\sum_{k\neq j}\mu_k'=\mu_j'$. This implies $\|PT_{2^{-k}}^\alpha z-z\|<\varepsilon$, and so we have

$$\|T_{2^{-k}}^\alpha z-z\|\leqq 2\varepsilon , \qquad (\text{since } \|PT_{2^{-k}}^\alpha z-T_{2^{-k}}^\alpha z\|<\varepsilon) .$$

For an arbitrary $z'\in\Omega$ we pick up $z=\sum\mu_jx_j$, $x_j\in D(T_t)$ such that $\|z-z'\|<\varepsilon$. Then

$$\|T_{2^{-k}}^\alpha z'-z'\|\leqq\|T_{2^{-k}}^\alpha z'-T_{2^{-k}}^\alpha z\|+\|T_{2^{-k}}^\alpha z-z\|+\|z-z'\|\leqq 4\varepsilon .$$

LEMMA 11. *Put $y_{k,n}^\alpha=(I-\lambda A_k^\alpha)^{-1}x_n$ for fixed $x_1, x_2, \cdots, x_m\in\Omega$. Then the set $Y_k(\lambda)=\{(y_{k,n}^\alpha)_{n=1}^m\}\subset\underbrace{H\times H\times\cdots\times H}_{m}$ for a fixed $\lambda>0$ is convex and closed. Moreover*

$$\rho(\lambda)=\sup_{(y'_n)\in\cup Y_k(\lambda)}\sqrt{\sum\|y'_n-x_n\|^2}\to 0 \qquad as\ \lambda\downarrow 0.$$

PROOF. Let $(y^{\alpha_j}_{k,n})^m_{n=1}\in Y_k=Y_k(\lambda)$, $y^{\alpha_j}_{k,n}\to y_{k,n}$ as $j\to\infty$. Then we have

$$T^{\alpha_j}_{2^{-k}}y^{\alpha_j}_{k,n}=\Big(I+\frac{1}{\lambda 2^k}\Big)y^{\alpha_j}_{k,n}-\frac{1}{\lambda 2^k}x_n\to\tilde{y}_n=\Big(1+\frac{1}{\lambda 2^k}\Big)y_{k,n}-\frac{1}{\lambda 2^k}x_n.$$

It is clear that the mapping U_k satisfying

$$U_k=T_{2^{-k}} \quad \text{on} \quad D(T_t), \quad \text{and} \quad U_k\cdot y_{k,n}=\tilde{y}_{k,n}$$

is a contraction. Hence by Lemma 9 $(y_{k,n})^m_{n=1}\in Y_k$. Thus Y_k is closed.

Let $(y^\alpha_{k,n})$, $(y^\beta_{k,n})\in Y_k$. Then we have

$$y^\alpha_{k,n}-\lambda 2^k(T^\alpha_{2^{-k}}y^\alpha_{k,n}-y^\alpha_{k,n})=x_n, \qquad y^\beta_{k,n}-\lambda 2^k(T^\beta_{2^{-k}}y^\beta_{k,n}-y^\beta_{k,n})=x_n,$$

and

$$T^\alpha_{2^{-k}}=U^\alpha_k, \qquad T^\beta_{2^{-k}}=U^\beta_k.$$

Hence

$$\lambda 2^k(U^\alpha_k y^\alpha_{k,n}-U^\beta_k y^\beta_{k,n})=(\lambda 2^k+1)(y^\alpha_{k,n}-y^\beta_{k,n}).$$

We put

$$z_n=\frac{1}{2}(y^\alpha_{k,n}+y^\beta_{k,n}),\ \tilde{z}_n=\frac{1}{2}(U^\alpha_k y^\alpha_{k,n}+U^\beta_k y^\beta_{k,n}), \qquad 1\leqq n\leqq m.$$

We shall show that there exists a contraction U^γ_k in Ω satisfying

$$U^\gamma_k z_n=\tilde{z}_n, \qquad U^\gamma_k=T_{2^{-k}} \quad \text{on } D(T_t).$$

Assume that for some n and n'

$$\|\tilde{z}_n-\tilde{z}_{n'}\|>\|z_n-z_{n'}\|.$$

Then, if $\mathrm{Re}\langle y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'},\tilde{z}_n-\tilde{z}_{n'}\rangle\geqq\mathrm{Re}\langle y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'},z_n-z_{n'}\rangle$ we have

$$\Big\|\frac{\lambda 2^k+1}{\lambda 2^k}(y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'})+(\tilde{z}_n-\tilde{z}_{n'})\Big\|$$
$$>\|(y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'})+(z_n-z_{n'})\|,$$

and if $\mathrm{Re}\langle y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'},\tilde{z}_n-\tilde{z}_{n'}\rangle\leqq\mathrm{Re}\langle y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'},z_n-z_{n'}\rangle$ we have

$$\Big\|\frac{\lambda 2^k+1}{\lambda 2^k}(y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'})-(\tilde{z}_n-\tilde{z}_{n'})\Big\|$$
$$>\|(y^\alpha_{k,n}-z_n-y^\alpha_{k,n'}+z_{n'})-(z_n-z_{n'})\|.$$

Thus at least one of the two relations

$$\|U^\alpha_k y^\alpha_{k,n}-U^\alpha_k y^\alpha_{k,n'}\|>\|y^\alpha_{k,n}-y^\alpha_{k,n'}\|$$
$$\|U^\beta_k y^\beta_{k,n}-U^\beta_k y^\beta_{k,n'}\|>\|y^\beta_{k,n}-y^\beta_{k,n'}\|,$$

holds good. This contradicts the contraction property of $U_{k,n}^{\alpha}$ and $U_{k,n}^{\beta}$. Hence we have

$$\|\tilde{z}_n-\tilde{z}_{n'}\| \leqq \|z_n-z_{n'}\|, \qquad 1\leqq n,\ n'\leqq m.$$

Since $\|U_k^\alpha y_{k,n}^\alpha-U_k^\beta y_{k,n}^\beta\| \left(=\frac{\lambda 2^k+1}{\lambda 2^k}\|y_{k,n}^\alpha-y_{k,n}^\beta\|\right)\geqq \|y_{k,n}^\alpha-y_{k,n}^\beta\|$, and since $\|U_k^\alpha y_{k,n}^\alpha-T_{2^{-k}}y\|\leqq\|y_{k,n}^\alpha-y\|$, $\|U_k^\beta y_{k,n}^\beta-T_{2^{-k}}y\|\leqq\|y_{k,n}^\beta-y\|$ for $y\in D(T_t)$, we have

$$\|\tilde{z}_n-T_{2^{-k}}y\|\leqq\|z_n-y\| \qquad \text{for any } y\in D(T_t).$$

Hence by Lemma 9 there exists a contraction U_k^γ in Ω such that

$$U_k^\gamma z_n=\tilde{z}_n \quad \text{for } 1\leqq n\leqq m, \qquad U_k^\gamma=T_{2^{-k}} \quad \text{on } D(T_t).$$

The semigroup $T_t^\gamma(\in\tau_k)$ defined by U_k^γ satisfies

$$z_n-\lambda 2^{-k}(T_{2^{-k}}^\gamma z_n-z_n)=x_n, \qquad \text{for } 1\leqq n\leqq m.$$

This means $(z_n)_{n=1}^m\in Y_k$. Thus Y_k is closed and convex.

Suppose that for some $\varepsilon>0$ there exist sequences $\{\lambda_j\downarrow 0\}$ and $\{(y_n^j)_{n=1}^m \in \bigcup_k Y_k(\lambda_j)\}$ such that

$$\|y_n^j-x\|\geqq\varepsilon.$$

Since $y_n^j-\lambda_j 2^{k_j}(T_{2^{-k_j}}y_n^j-y_n^j)=x_n$ ($(y_n^j)\in Y_{k_j}$), this contradicts the equicontinuity of $\{T_t^\alpha x:\alpha\}$ by infinite speed principle. Q. E. D.

We define $Y_\infty(\lambda)=\bigcap_{n=1}^\infty$ (the weak closure of $\bigcup_{k=n}^\infty Y_k$). By Lemma 11 the set $Y_\infty(\lambda)$ is nonvoid. For fixed $x\in\Omega$, $\lambda>0$ we denote simply $Y_k=Y_k(\lambda)$, $Y_\infty=Y_\infty(\lambda)$.

LEMMA 12. *For any $y\in Y_\infty$ there exists a sequence $\{y_{k_j}^{\alpha j}\in Y_{k_j}:j\}$ such that*

$$y_{k_j}^{\alpha j}\to y \ (strong) \qquad as\ j\to\infty.$$

PROOF. We define the norm $\|\cdot\|$ and the inner product $\langle\,,\rangle$ in $H\times\cdots\times H$ as $\|(x_n)\|=\sqrt{\sum\|x_n\|^2}$ and $\langle(x_n),(y_n)\rangle=\sum_{n=1}^m\langle x_n,y_n\rangle$.

For $T^\alpha\in\tau_l$ and $k\leqq l$, we denote $A_k^\alpha=2^k(T_{2^{-k}}-I)$ and $y_k^\alpha=(I-\lambda A_k^\alpha)^{-1}x$. From the equicontinuity of $\{T_t^\alpha x: T^\alpha\in\cup\tau_l\}$ it follows that for any $\varepsilon>0$ there exists some $k(\varepsilon)$ satisfying

$$\|y_l^\alpha-y_k^\alpha\|<\varepsilon \qquad \text{for } l\geqq k\geqq k(\varepsilon),\ T^\alpha\in\tau_l, \tag{33}$$

by virtue of [12, Lemma 2]. (This can be seen also in a similar but more complicated way to the proof of Lemma 6.)

Let $y_k^{\alpha k}$ be the element of Y_k with

$$\|y-y_k^{\alpha k}\|=\inf_{y'\in Y_k}\|y-y'\|.$$

Such an element $y_k^{\alpha k}$ exists uniquely in Y_k by Lemma 11. Assume that $y_k^{\alpha k}\not\to y$. Then we have $\lim_{k\to\infty}\|y-y_{k_j}^{\alpha k_j}\|=\kappa>0$ for some subsequence $\{k_j\}$. If $\{y_k^{\alpha k}\}$ is not a Cauchy sequence, i. e.,

$$\exists \rho > 0\,,\ \forall k\,,\ \exists l > k: \qquad \| y_k^{\alpha_k} - y_l^{\alpha_l} \| \geqq \rho\,,$$

the relation (33) for $\alpha = \alpha_l$ implies

$$\| y_k^{\alpha_k} - y_k^{\alpha_l} \| \geqq \| y_k^{\alpha_k} - y_l^{\alpha_l} \| - \| y_k^{\alpha_l} - y_l^{\alpha_l} \| \geqq \rho - \varepsilon\,. \tag{34}$$

For sufficiently large k_j, we have

$$|\, \| y - y_{k_j}^{\alpha_{k_j}} \| - \kappa \,| < \varepsilon\,.$$

Since Y_{k_j} is convex and closed, and since $y_{k_j}^{\alpha_{k_j}}$ is the point of Y_{k_j} with minimum distance from y, we have

$$\mathrm{Re}\, \langle y - y_{k_j}^{\alpha_{k_j}}, z - y_{k_j}^{\alpha_{k_j}} \rangle \leqq 0 \qquad \text{for} \quad z \in Y_{k_j}\,. \tag{35}$$

Putting $z = y_{k_j}^{\alpha_l}$ we have

$$\| y - y_{k_j}^{\alpha_{k_j}} \|^2 + \| y_{k_j}^{\alpha_{k_j}} - y_{k_j}^{\alpha_l} \|^2 \leqq \| y - y_{k_j}^{\alpha_l} \|^2 \leqq (\| y - y_l^{\alpha_l} \| + \varepsilon)^2$$

by (33). Let $l = k_m$, $m > j$. Then the relations $\| y - y_{k_j}^{\alpha_{k_j}} \| \geqq \kappa - \varepsilon$ and $\| y - y_{k_m}^{\alpha_{k_m}} \| \leqq \kappa + \varepsilon$ imply

$$\| y_{k_j}^{\alpha_{k_j}} - y_{k_m}^{\alpha_{k_m}} \|^2 \leqq 6\kappa\varepsilon + 2\varepsilon^2\,.$$

This means that $\{ y_{k_j}^{\alpha_{k_j}} \}$ is a Cauchy sequence. Put $y_\infty = \lim_{j \to \infty} y_{k_j}^{\alpha_{k_j}}$. Then $\| y_\infty - y \| = \kappa$. Since $\bigcup_k Y_k$ are bounded, i. e.,

$$\sup_{y' \in \bigcup_m Y_m} \| y' - y_{k_j} \| < M < \infty\,,$$

we have by (35) for $z \in \bigcup_{l \geqq k_j} Y_l$

$$\begin{aligned} \mathrm{Re}\, \langle y - y_\infty, z - y_\infty \rangle &\leqq \mathrm{Re}\, \langle y - y_{k_j}^{\alpha_{k_j}}, z - y_{k_j}^{\alpha_{k_j}} \rangle + \varepsilon(\| y - y_\infty \| + \| z - y_\infty \|) \\ &\leqq \varepsilon(\kappa + M)\,. \end{aligned}$$

By letting z tend weakly to y, we obtain a contradiction for such an ε that $\varepsilon(\kappa + M) < \kappa^2/2$.

LEMMA 13. *Let* $\Phi = \{\varphi = \{(\alpha, k)\}\}$ *be an ultrafilter with* $\lim_{(\alpha,k) \in \varphi \in \Phi} k = \infty$. *Then there exists a filter* $\Psi = \{\varphi = \{(\alpha, k)\}\}$ *such that*

$$\lim_{\Psi} (I - \lambda A_k^\alpha)^{-1} x = w\text{-}\lim_{\Phi} (I - \lambda A_k^\alpha)^{-1} x \qquad \textit{for any} \quad x \in \Omega\,.$$

PROOF. Note that $w\text{-}\lim_{\Phi} (I - \lambda A_k^\alpha)^{-1} x$ exists in Ω. For an arbitrary finite set $\{x_1, x_2, \cdots, x_m\} \subset \Omega$ and for an arbitrary positive $\varepsilon > 0$, we put $\varphi\{x_1, x_2, \cdots, x_m;\ \varepsilon\} = \{(\alpha', k') : \| w\text{-}\lim_{\Phi} (I - \lambda A_k^\alpha)^{-1} x_n - (I - \lambda A_{k'}^{\alpha'})^{-1} x_n \| < \varepsilon,\ 1 \leqq \forall n \leqq m\}$. By Lemma 12, every $\varphi\{x_1, \cdots, x_m;\ \varepsilon\}$ is nonvoid and contains a sequence $\{(\alpha_j, k_j)\}$ with $\lim_{j \to \infty} k_j = \infty$. Thus the filter Ψ generated by $\{\varphi\{x_1, x_2, \cdots, x_m;\ \varepsilon\}\}$ satisfies our requirement.

LEMMA 14. *There exists a filter* Ψ_∞ *such that*

$$y_l(x)=\lim_{\Psi_\infty}(I-2^{-l}A_k^\alpha)^{-1}x$$

exists for every $x\in\Omega$ *and for* $l=0, 1, 2, \cdots$

PROOF. The filter in Lemma 13 for $\lambda=1$ is denoted by

$$\Psi_0=\{\varphi\supset\varphi^0(x_1, \cdots, x_n; \varepsilon): x_1, \cdots, x_n\in\Omega, \varepsilon>0\}.$$

We define a dissipative operator $A^{(0)}$ as

$$A^{(0)}y=\{y-x: y=y_0(x)\}, \qquad y_0(x)=\lim_{\Psi_0}(I-A_k^\alpha)^{-1}x.$$

Let $\tilde{\Psi}_0$ be an ultrafilter containing Ψ_0. Then for every $x\in\Omega$

$$y_1(x)=w\text{-}\lim_{\tilde{\Psi}_0}(I-2^{-1}A_k^\alpha)^{-1}x$$

exists in Ω. Putting $y_{0,k}^\alpha(x)=(I-A_k^\alpha)^{-1}x$, we have

$$y_{0,k}^\alpha(x)=(I-2^{-1}A_k^\alpha)^{-1}(2^{-1}x+2^{-1}y_{0,k}^\alpha(x)), \tag{36}$$

hence $y_0(x)=\lim_{\tilde{\Psi}_0}(I-2^{-1}A_k^\alpha)^{-1}(2^{-1}x+2^{-1}y_0(x))$. Define a dissipative operator $A^{(1)}$:

$$A^{(1)}y=\{2(y-x): y=y_1(x)\}.$$

Since $2^{-1}x+2^{-1}y_0(x)\in\Omega$, we have by (36)

$$A^{(0)}\subset A^{(1)}.$$

The filter in Lemma 13 for $\lambda=2^{-1}$ is denoted by $\Psi_1=\{\varphi\supset\varphi^1(x_1, \cdots, x_n; \varepsilon): x_1, \cdots, x_n\in\Omega, \varepsilon>0\}$. Then for every $x\in\Omega$

$$y_1(x)=\lim_{\Psi_1}(I-2^{-1}A_k)^{-1}x$$

holds good. Note that $\Psi_0\subset\Psi_1$. Repeating this process, we have a sequence $\{A^{(n)}\}$ of dissipative operators and a sequence $\{\Psi_n\}$ of filters of indices (α, k) such that

$$\begin{aligned}&A^{(0)}\subset A^{(1)}\subset A^{(2)}\subset\cdots,\\ &\Psi_0\subset\Psi_1\subset\Psi_2\subset\cdots,\\ &y_n(x)=\lim_{\Psi_n}(I-2^{-n}A_k^\alpha)^{-1}x \quad \text{exists for every} \quad x\in\Omega,\\ &A^{(n)}y=\{2^n(y-x): y_n(x)=y\}.\end{aligned} \tag{37}$$

The filter $\Psi_\infty=\{\varphi\supset{}^\exists\varphi^n: \varphi^n\in{}^\exists\Psi_n\}$ satisfies our requirement.

PROOF OF THEOREM 3. i) The closedness of $D(T_t)$ is clear. Suppose that $D(T_t)$ is a proper subset of the convex closed hull Ω of $D(T_t)$. We define a dissipative operator A defined densely in Ω:

$$A \cdot y = \{2^n(y-x) : {}^{\exists}n,\ y_n(x) = y\}\,,$$

where $y_n(x) = \lim_{\Psi_\infty} (I - 2^{-n}A_k^\alpha)^{-1}x$, and Ψ_∞ is the filter given in Lemma 14. The graph of A is the union of the graph of $A^{(n)}$'s, where $A^{(n)}$ is the operator defined by (37). Since $D((I-2^{-n}A^{(n)})^{-1}) = \Omega$, we have

$$D((I-2^{-n}A)^{-1}) \supset \Omega\,, \qquad n = 0, 1, 2, \cdots.$$

We shall construct a solution of the equation

$$\text{(38)} \qquad \begin{cases} \dfrac{d}{dt}u(t) \in Au(t) \\ u(t) \in \Omega\,, \qquad u(0) = y_0 \in (\Omega - D(T_t)) \cap D(A)\,. \end{cases}$$

For this purpose we construct an approximating sequence $\{u_n(t)\}$ satisfying

$$\text{(39)} \qquad \begin{aligned} &\frac{d}{dt}u_n(t) = A_n u_n(t) \\ &u_n(t) \in \Omega\,, \qquad u_n(0) = y_0 - 2^{-n}y_0',\ y_0' = \lim_{\Psi_\infty} (I - A_k^\alpha)^{-1}x_0 - x_0\,, \end{aligned}$$

where x_0 is an element of $y_0 - Ay_0$ and A_n is the mapping:

$$x - 2^{-n}x' \to x'\,, \qquad x \in D(A)\,, \quad x' \in Ax\,.$$

Let P be a projection $H \to \Omega$ i.e., $Px = y \in \Omega$ with $\|y-x\| = \inf_{y' \in \Omega} \|y'-x\|$. We define $u_n^{(m)}(t)$ by induction:

$$u_n^{(0)}(t) = y_0 - 2^{-n}y_0'\,, \qquad u_n^{(m+1)}(t) = P\Big(y_0 + \int_0^t A_n u_n^{(m)}(s)ds\Big)\,.$$

Then $u_n^{(m)}(t) \in \Omega$, and

$$\begin{aligned} \| u_n^{(m+1)}(t) - u_n^{(m)}(t) \| &= \Big\| P\Big(y_0 + \int_0^t A_n u_n^{(m)}(s)ds\Big) - P\Big(y_0 + \int_0^t A_n u_n^{(m-1)}(s)ds\Big) \Big\| \\ &\leqq \int_0^t \| A_n u_n^{(m)}(s) - A_n u_n^{(m-1)}(s) \| \, ds\,. \end{aligned}$$

Since A_n is Lipschitz continuous, we see that $\Sigma \| u_n^{(m+1)}(t) - u_n^{(m)}(t) \| < \infty$. Hence the limit $u_n(t) = \lim_{m \to \infty} u_n^{(m)}(t)$ exists. The fact that $u_n(t)$ satisfies the equation (39) is verified as in the proof of Theorem 1 in [**14**]. By the same arguement of [**4**, Th. 4], $\{u_n(t)\}$ converges strongly to some function $u(t)$ uniformly in $t \in [0, t_0]$ and the function $u(t)$ is a solution of (38).

Now we have to show that

$$\text{(40)} \qquad \tilde{T}_t x = \begin{cases} T_t x & \text{for } x \in D(T_t) \\ u(t+s) & \text{for } x = u(s)\,,\ s \geqq 0\,, \end{cases}$$

is a contraction semigroup. Since A is dissipative, we see by (38) that

$$\| u(s+t) - u(s'+t) \| \leqq \| u(s) - u(s') \| \qquad \text{for } s, s', t \geqq 0\,.$$

Hence it suffices to show that

(41) $$\| u(s+t)-T_t x \| \leqq \| u(s)-x \| \qquad \text{for} \quad x \in D(T_t),\ s,\ t>0\,.$$

We consider $s, t, s+t \in [0, r]$ for a positive constant r.

We define discrete semigroups $\{T^{\alpha,n}\}$ as follows:

(42) $$\begin{aligned} &T^{\alpha,n}_{2^{-k}} = 2^{-k} A^{\alpha}_k (I-2^{-n}A^{\alpha}_k)^{-1}+I\,,\\ &T^{\alpha,n}_0 = I\,, \quad T^{\alpha,n}_{t+s} = T^{\alpha,n}_t T^{\alpha,n}_s \qquad \text{for} \quad t=j2^{-k},\ s=j'2^{-k}\,. \end{aligned}$$

Since $(I-2^{-n}A^{\alpha}_k)^{-1}$ and $2^{-k}A^{\alpha}_k = T^{\alpha}_{2^{-k}}-I$ are contractions, $\{T^{\alpha,n}_t : t=j2^{-k},\ j=0, 1, 2, \cdots\}$ is evidently a contraction semigroup. Hence $A^{\alpha,n}_k = 2^k(T^{\alpha,n}_{2^{-k}}-I) = A^{\alpha}_k(I-2^{-n}A^{\alpha}_k)^{-1}$ is dissipative and we have

(43) $$\| A^{\alpha,n}_k T^{\alpha,n}_t x \| \leqq \| A^{\alpha,n}_k x \| \qquad \text{for} \quad t=j2^{-k},\ x \in \Omega\,.$$

We shall show for $u^{\alpha}_n = y^{\alpha}_k - 2^{-n} y^{\alpha\prime}_k$, $y^{\alpha}_k = (I-A^{\alpha}_k)^{-1}x_0$, $y^{\alpha\prime}_k = y^{\alpha}_k - x_0$ $(=A^{\alpha}_k y^{\alpha}_k)$ that

(44) $$\| T^{\alpha,n}_t u^{\alpha}_n - T^{\alpha,m}_t u^{\alpha}_m \| \leqq \varepsilon\,, \quad \text{for} \quad n, m \geqq n_0,\ (\alpha, k) \in \varphi_0 \in \Psi_\infty,\ 0 \leqq t=j2^{-k} \leqq r\,.$$

In fact,

$$\begin{aligned} &\| T^{\alpha,n}_{j2^{-k}} u^{\alpha}_n - T^{\alpha,m}_{j2^{-k}} u^{\alpha}_m \|^2 - \| u^{\alpha}_n - u^{\alpha}_m \|^2 \\ &\quad = \sum_{i=1}^{j-1} (\| T^{\alpha,n}_{(i+1)2^{-k}} u^{\alpha}_n - T^{\alpha,m}_{(i+1)2^{-k}} u^{\alpha}_m \|^2 - \| T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n - T^{\alpha,m}_{i2^{-k}} u^{\alpha}_m \|^2) \\ &\quad = \sum_{i=0}^{j-1} 2^{1-k}\, \mathrm{Re} \langle A^{\alpha,n}_k T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n - A^{\alpha,m}_k T^{\alpha,m}_{i2^{-k}} u^{\alpha}_m,\ T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n - T^{\alpha,m}_{i2^{-k}} u^{\alpha}_m \rangle \\ &\qquad + \sum_{i=0}^{j-1} 2^{-2k} \| A^{\alpha,n}_k T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n - A^{\alpha,m}_k T^{\alpha,m}_{i2^{-k}} u^{\alpha}_m \|^2\,. \end{aligned}$$

Since we have

$$\begin{aligned} &\mathrm{Re} \langle A^{\alpha,n}_k T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n - A^{\alpha,m}_k T^{\alpha,m}_{i2^{-k}} u^{\alpha}_m,\ (I-2^{-n}A^{\alpha}_k)^{-1} T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n - (I-2^{-m}A^{\alpha}_k)^{-1} T^{\alpha,m}_{i2^{-k}} u^{\alpha}_m \rangle \\ &\quad \leqq 0 \end{aligned}$$

and

$$\| T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n - (I-2^{-n}A^{\alpha})^{-1} T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n \| \leqq 2^{-n} \| A^{\alpha,n} T^{\alpha,n}_{i2^{-k}} u^{\alpha}_n \|$$

we have by (43)

$$\| T^{\alpha,n}_{j2^{-k}} u^{\alpha}_n - T^{\alpha,m}_{j2^{-k}} u^{\alpha}_m \|^2 \leqq 4r \| y^{\alpha\prime}_k \|^2 (2^{-n}+2^{-m}+2^{-k}) \qquad \text{for} \quad j2^{-k} \leqq r\,.$$

It holds that for any fixed n and t, $0 \leqq t \leqq r$

(45) $$\lim_{\Psi_\infty} T^{\alpha,n}_t u^{\alpha}_n = u_n(t) \qquad (t_k = j_k 2^{-k},\ j_k = [t2^k])\,.$$

In fact, let $\rho_j = \| T^{\alpha,n}_{j2^{-k}} u^{\alpha}_n - u_n(j2^{-k}) \|$. Since

$$\begin{aligned} &T^{\alpha,n}_{(j+1)2^{-k}} u^{\alpha}_n = T^{\alpha,n}_{j2^{-k}} u^{\alpha}_n + \int_0^{2^{-k}} A^{\alpha,n} T^{\alpha,n}_{j2^{-k}} u^{\alpha}_n ds\,, \\ &u_n((j+1)2^{-k}) = u_n(j2^{-k}) + \int_0^{2^{-k}} A_n u_n(j2^{-k}+s) ds\,, \end{aligned}$$

and since $\lim_{\Psi_\infty} A_k^{\alpha,n} u_n(t) = A_n u_n(t)$, we have for $(\alpha, k) \in \varphi$

$$\begin{aligned}\rho_{j+1} &\leqq \| u_n(j2^{-k}) - T^{\alpha,n}_{j2^{-k}} u_n^\alpha \| \\ &\quad + \int_0^{2^{-k}} \| (A_n - A^{\alpha,n}) u_n(j2^{-k}+s) \| ds \\ &\quad + \int_0^{2^{-k}} \| A^{\alpha,n} u_n(j2^{-k}+s) - A^{\alpha,n} T^{\alpha,n}_{j2^{-k}} u_n^\alpha \| ds \\ &\leqq \rho_j + 2^{-k}\varepsilon + 2^{-k} 2^n (\rho_j + 2^{-k} \| y_0' \|) \\ &\leqq \rho_j(1+2^{n-k}) + 2^{1-k}\varepsilon, \text{ for some } \varphi \in \Psi_\infty \\ &\qquad (\text{satisfying } 2^n \| y_0' \| < 2^k \varepsilon).\end{aligned}$$

Hence by induction we have

$$\rho_j \leqq \rho_0 \cdot e^{r2^n} + 2re^{r2^n}\varepsilon \leqq (\| y_k^\alpha - y_0 \| + 2^{-n} \| y_k^{\alpha\prime} - y_0' \|) e^{r2^n} + 2re^{r2^n}\varepsilon \quad \text{for } 0 \leqq j2^{-k} \leqq r,$$

which implies (45). Since $\lim_{n\to\infty} A_k^\alpha (I - 2^{-n} A_k^\alpha)^{-1} y = A_k^\alpha y$ for $y \in D(A_k^\alpha)$ $(= \Omega)$, we have for fixed α

$$\lim_{n\to\infty} \| T^{\alpha,n}_{j2^{-k}} u_n^\alpha - T^\alpha_{j2^{-k}} u_n^\alpha \| = 0, \tag{46}$$

since $\{u_n^\alpha ; n\}$ is relatively compact in Ω.

Now we can show (41) for $s = i2^{-k}$, $t = j2^{-k}$, $s+t = (i+j)2^{-k} \in [0, r]$ by (44), (45) and (46):

$$\begin{aligned}\| u(s+t) - T_t x \| &\leqq \| u(s+t) - u_n(s+t) \| + \| u_n(s+t) - T^{\alpha,n}_{t+s} u_n^\alpha \| \\ &\quad + \| T^{\alpha,n}_{t+s} u_n^\alpha - T^\alpha_{t+s} u_n^\alpha \| + \| T^\alpha_{t+s} u_n^\alpha - T_t x \| \\ &\leqq \| T_s^\alpha u_n^\alpha - x \| + 3\varepsilon \\ &\leqq \| u(s) - x \| + 6\varepsilon,\end{aligned}$$

since

$$\begin{aligned}&\| u(s) - T_s^\alpha u_n^\alpha \| \\ &\quad \leqq \| u(s) - u_n(s) \| + \| u_n(s) - T_s^{\alpha,n} u_n^\alpha \| + \| T_s^{\alpha,n} u_n^\alpha - T_s^\alpha u_n^\alpha \| \\ &\quad \leqq 3\varepsilon.\end{aligned}$$

By the uniform continuity of $T_t x$ and $u(t)$, the relation (41) for any $s, t, s+t \in [0, r]$ holds good. Thus $\{\tilde{T}_t\}$ is a contraction semigroup, which contradicts the maximality of $\{T_t\}$.

It remains to prove that the domain $D(T_t)$ is not contained in any closed hyperplane of H. Suppose that $D(T_t) \subset \{x \in H: \operatorname{Re}\langle x, e\rangle = \alpha\}$ for some $e \in H$ with $\|e\| = 1$. Let $S_t(x+e) = T_t x + e$ for $x \in D(T_t)$. Then $\{S_t\}$ is also a contraction semigroup and its domain $e + D(T_t)$ has the void interesection with $D(T_t)$. Hence

$$\hat{T}_t x = \begin{cases} T_t x & \text{for} \quad x \in D(T_t) \\ S_t x & \text{for} \quad x \in e + D(T_t)\,. \end{cases}$$

is an extension of $\{T_t\}$. $\{\hat{T}_t\}$ is evidently a contraction semigroup. Thus the proof of i) is completed.

LEMMA 15. *Let A be a maximal dissipative operator. Then we have*

$$\lim_{\lambda\downarrow 0} \| x-(I-\lambda A)^{-1}x \| = \inf_{y\in D(A)} \| x-y \| \qquad \textit{for any} \quad x\in H. \tag{47}$$

PROOF. Since $(I-\lambda A)^{-1}x \in D(A)$, we have

$$\| x-(I-\lambda A)^{-1}x \| \geqq \inf_{y\in D(A)} \| x-y \| \qquad \text{for} \quad \lambda > 0\,. \tag{48}$$

Conversely, for $y\in D(A)$ and $y'\in A\cdot y$ we put $y_\lambda = y-\lambda y'$. Then $y_\lambda \to y$ as $\lambda\downarrow 0$. Since $(I-\lambda A)^{-1}$ is a contraction, the operator $I-(I-\lambda A)^{-1}$ is also contraction. Hence we have

$$\| (I-(I-\lambda A)^{-1})x-(I-(I-\lambda A)^{-1})\, y_\lambda \|$$
$$\leqq \| x-y_\lambda \| \to \| x-y \| \qquad \text{as} \quad \lambda \downarrow 0\,.$$

Since $(I-(I-\lambda A)^{-1})\, y_\lambda = y_\lambda - y \to 0$ as $\lambda \downarrow 0$, we have by the above relation

$$\overline{\lim_{\lambda\downarrow 0}} \| (I-(I-\lambda A)^{-1})x \| \leqq \| x-y \|\,. \tag{49}$$

The relations (48) and (49) imply (47).

PROOF OF ii). Let $y, z\in D(A)$ and $x=\mu y-(1-\mu)z$ for $0<\mu<1$. Suppose that $x \notin \overline{D(A)}$. Then, putting $y_\lambda = y-\lambda y'$ for $y'\in A\cdot y$, as in the proof of Lemma 15, we have

$$\| (I-\lambda A)^{-1}x-y \| = \| (I-\lambda A)^{-1}x-(I-\lambda A)^{-1}y_\lambda \|$$
$$\leqq \| x-y_\lambda \| \to \| x-y \| \qquad \text{as} \quad \lambda\downarrow 0\,.$$

Similarly we have $\overline{\lim}_{\lambda\downarrow 0} \| (I-\lambda A)^{-1}x-z \| \leqq \| x-z \|$. Since

$$\| y-z \| \leqq \overline{\lim_{\lambda\downarrow 0}} \left(\| y-(I-\lambda A)^{-1}x \| + \| (I-\lambda A)^{-1}x-z \| \right)$$
$$= \| y-x \| + \| x-z \| = \| y-z \|\,,$$

we see that

$$\lim_{\lambda\downarrow 0} (I-\lambda A)^{-1}x = \nu y-(1-\nu)\cdot z \qquad \text{for} \quad 0\leqq \nu \leqq 1\,,$$

and

$$\lim_{\lambda\downarrow 0} \| (I-\lambda A)^{-1}x-y \| = \| x-y \|\,.$$

Hence we have $\lim_{\lambda\downarrow 0} (I-\lambda A)^{-1}x = x$. But this contradicts the assumption $x \notin \overline{D(A)}$.

THEOREM 4. i) *A maximal contraction semigroup $\{T_t\}$ has a densely defined generator and is generated by a maximal dissipative operator.*

ii) *If a maximal dissipative operator A is single-valued, the semigroup generated by A is a maximal contraction semigroup.*

PROOF OF i). By Theorems 2 and 3 the infinitesimal generator A_0 of $\{T_t\}$ is densely defined in $D(T_t)$. A maximal dissipative extension A of A_0 generates a contraction semigroup $\{S_t\}$. Evidently $\{S_t\}$ is an extension of $\{T_t\}$. The maximality of $\{T_t\}$ implies $\{S_t\}=\{T_t\}$.

PROOF OF ii). Let $\{T_t\}$ be the semigroup generated by A, $\{S_t\}$ a maximal extension of $\{T_t\}$. Suppose that $D(S_t)\supsetneqq D(T_t)$. By virtue of i), the generator B of $\{S_t\}$ is densely defined in $D(S_t)$. Hence there exists a point $x\in D(B)$, $\notin D(T_t)$, since $D(T_t)$ is closed. By the maximal dissipativity of A and by Lemma 7, $y_\lambda=(I-\lambda A)^{-1}x$ exists for $\lambda>0$ and converges to a point $y\in\overline{D(A)}$ as $\lambda\downarrow 0$. Since $y_\lambda\in D(A)$, $T_t y_\lambda$ is weakly differentiable in t by [**3**, Theorem 1], i.e., $w\text{-}\lim_{h\downarrow 0} A_h y_\lambda = A\cdot y_\lambda$. Note that $A\cdot y_\lambda=\frac{1}{\lambda}(y_\lambda-x)$. Hence we have

$$\lim_{h\downarrow 0}\frac{\lambda}{h}\langle T_h y_\lambda-y_\lambda,\ y_\lambda-x\rangle=\|y_\lambda-x\|^2 .$$

Since $\|T_h y_\lambda-S_h x\|=\|S_h y_\lambda-S_h x\|\leqq\|y_\lambda-x\|$, we have

$$\begin{aligned}\mathrm{Re}\,\langle S_h x-x,\ y_\lambda-x\rangle &= \mathrm{Re}\,\langle S_h x-T_h y,\ y_\lambda-x\rangle\\ &\quad+\mathrm{Re}\,\langle T_h y_\lambda-y,\ y_\lambda-x\rangle+\|y_\lambda-x\|^2\\ &\geqq \mathrm{Re}\,\langle T_h y_\lambda-y_\lambda,\ y_\lambda-x\rangle .\end{aligned}$$

Combining above two inequalities we have

$$\lim_{h\downarrow 0}\frac{1}{h}\mathrm{Re}\,\langle S_h x-x,\ y_\lambda-x\rangle\geqq\frac{1}{\lambda}\|y_\lambda-x\|^2 .$$

But this is impossible. In fact, if λ tends to 0, we have

$$\lim_{h\downarrow 0}\mathrm{Re}\left\langle\frac{1}{h}(S_h x-x),\ y_\lambda-x\right\rangle=\mathrm{Re}\,\langle B\cdot x,\ y_\lambda-x\rangle\to\mathrm{Re}\,\langle B\cdot x,\ y-x\rangle ,$$

$$\frac{1}{\lambda}\|y_\lambda-x\|^2\to\infty .$$

COROLLARY. *If the domain $D(T_t)$ of a contraction semigroup $\{T_t\}$ contains an open set Ω, the domain $D(A_0)$ of the infinitesimal generator A_0 is dense in Ω, i.e., $\overline{\Omega\cap D(A_0)}=\bar{\Omega}$.*

PROOF. Let $\{\tilde{T}_t\}$ be a maximal contraction semigroup containing $\{T_t\}$. By Theorem 4, the infinitesimal generator $\tilde{A}_0$ of $\{\tilde{T}_t\}$ is densely defined in $D(\tilde{T}_t)$. Hence $D(\tilde{A}_0)\cap\Omega$ is dense in Ω. Our assertion is now clear, since $\tilde{A}_0=A_0$ in Ω.

§ 4. Holomorphic semigroups.

A continuous (single-valued) mapping $f: H \to H$ is said to be *analytic* if it is Gâteaux differentiable, i.e., $f(x+\lambda y)$ is analytic in λ for fixed $x, y \in H$, if $x+\lambda y \in D(f)$. An analytic mapping f has the Taylor expansion

$$f(x+y) = \sum \frac{1}{n!} \delta^n f(x; y) \qquad x \in D(f),$$

where $\delta^n f(x; \ )$ is a homogeneous mapping of degree n. (See [2].)

To obtain similar results to linear case, a nonlinear holomorphic semigroup $\{T_t\}$ should be not only holomorphic in t, but also analytic as a mapping for each fixed t.

REMARK. Suppose that T_t is a contraction defined on H for a fixed t. If T_t is an analytic mapping, then the function $f(\lambda) = T_t(x+\lambda y)$ for fixed $x, y \in H$ is linear. In fact, $\| f'(\lambda)\| = \| \lim_{\Delta\lambda \to 0} \frac{1}{\Delta\lambda}(f(\lambda+\Delta\lambda) - f(\lambda))\| \leqq \lim_{\Delta\lambda \to 0} \frac{1}{|\Delta\lambda|}\|\Delta\lambda y\| = \|y\|$. Hence by Liouville's theorem $f'(\lambda) =$ a constant y_0. Thus $f(\lambda) = x_0 + \lambda y_0$, where $x_0 = f(0)$. From this fact it follows that $S \cdot x = T_t x - T_t 0$ is a linear operator. The situation for $(I-A)^{-1}$ is the same: If A is maximal dissipative (hence $D((I-A)^{-1}) = H$) and if $(I-A)^{-1}$ is analytic, then $(I-A)^{-1}$ is expressed by the form $x_0 + L$, where L is a linear operator, and $A(L \cdot x + x_0) = (L-I) \cdot x + x_0$. Hence we must consider a generator A which is not maximal dissipative.

LEMMA 16. *Let f be an analytic mapping with the open domain $D(f)$. If the inverse f^{-1} exists and is Lipschitz continuous, it is analytic.*

PROOF. We fix $x_0 \in D(f)$ and $x \in H$. Put $y_0 = f(x_0)$, $y_0 + y_\lambda = f(x_0 + \lambda x)$, where $f(x_0 + \lambda x)$ is defined for λ whose absolute value is sufficiently small. Then we have $y_\lambda = \lambda \delta f \cdot x + o(\lambda)$. We define $y = \lim_{\lambda \to 0} \frac{y_\lambda}{\lambda} = \delta f \cdot x$. It holds that

$$f^{-1}(y_0 + \lambda y) = f^{-1}(y_0 + y_\lambda + o(\lambda)) = x_0 + \lambda x + o(\lambda),$$

since f^{-1} is Lipschitz continuous. Hence $f^{-1}(y_0 + \lambda y)$ is differentiable at $\lambda = 0$. It suffices to show that every element $y \in H$ is expressed by the form $\delta f \cdot x$, i.e. $(\delta f)^{-1}$ is defined on H. Note that δf is a linear mapping. Since f^{-1} is Lipschitz continuous, we have

$$\|\lambda y\| \geqq L \| f^{-1}(y_0 + \lambda y) - f^{-1}(y_0)\| = L \|\lambda x + o(\lambda)\|,$$

and

$$\|\delta f \cdot x\| = \|y\| \geqq L \|x\|.$$

Hence $(\delta f)^{-1}$ is continuous. This implies the closedness of $\delta f \cdot H$. Suppose that $\tilde{y} \notin \delta f \cdot H$. We put $x_0 + x_\lambda = f^{-1}(y_0 + \lambda \tilde{y})$. Then we have

$$y_0 + \lambda \tilde{y} = f(x_0 + x_\lambda) = y_0 + \delta f \cdot x_\lambda + o(x_\lambda),$$

since f is Fréchet differentiable (see [2]). Hence $\lambda\tilde{y}=\delta f\cdot x_\lambda+o(x_\lambda)$. From the relation $\frac{1}{\lambda}\delta f\cdot x_\lambda\in\delta f\cdot H$ it follows that $\frac{1}{\lambda}o(x_\lambda)\neq o(1)$, i.e., $x_\lambda\neq o(\lambda)$. This contradicts the Lipschitz continuity of f^{-1}. Q.E.D.

We consider a sector $\Sigma_\theta=\{t: |\arg t|<\theta\}$ in the complex plane C^1 and a closed set Ω in H. We shall say a nonlinear semigroup $\{T_t\}$ to be *holomorphic in* $\Omega\times\Sigma_\theta$ if it satisfies the following

(50) For each fixed $t\in\Sigma_\theta$, the operator T_t is analytic on a neighbourhood of Ω.

(51) For each fixed $x\in\Omega$, $T_t x$ is holomorphic in $t\in\Sigma_\theta$.

(52) $\{T_t\,;\,t\in\Sigma_\theta\}$ is a contraction semigroup, i.e.,

$$\|T_t x-T_t y\|\leqq\|x-y\| \qquad \text{for } t\in\Sigma_\theta,\ x, y\in\Omega\,.$$

The resolvent $R(\lambda, A)$ of an operator A is defined by:

$$R(\lambda, A)=(\lambda I-A)^{-1}\,.$$

Now we can state the relation of holomorphic semigroups to resolvents of generators as follows.

THEOREM 5. i) *Let* $\{T_t\}$ *be a holomorphic semigroup in* $\Omega\times\Sigma_\theta$. *If a neighbourhood* Ω_1 *of the closed set* Ω *satisfies*

(53) $$\Omega\supset(\lambda I-A_h)^{-1}\Omega_1 \qquad \textit{for } \lambda\in\Sigma_{\theta+\frac{\pi}{2}},\ 0<h<h_0\,,$$

then the resolvent $R(\lambda, A)$ *of* $A=A_{\Phi,\lambda_0}$ $(\lambda_0>0)$ *satisfies*

(54) $R(\lambda, A)$ *is an analytic mapping on a neighbourhood of* Ω.

(55) $$\|R(\lambda, A)x-R(\lambda, A)y\|\leqq\frac{1}{\sup\limits_{|\theta_1|<\theta}\operatorname{Re}(e^{i\theta_1}\lambda)}\|x-y\| \qquad \textit{for } \lambda\in\Sigma_{\theta+\frac{\pi}{2}},\ x, y\in\Omega\,.$$

(56) *For each fixed* $x\in\Omega$, $R(\lambda, A)x$ *is holomorphic in* $\Sigma_{\theta+\frac{\pi}{2}}$.

ii) *Let* Ω *be a closed subset of* H, A *an operator* $\Omega\to H$ *satisfying*

(57) $$D(A_\lambda)\supset\Omega,\ x+\varepsilon(x)A_\lambda x\in\Omega \qquad \textit{for } |\lambda|>C,\ \lambda\in\Sigma_\theta,\ x\in\Omega\,,$$

where $\varepsilon(x)>0$ *and* $A_\lambda\cdot\left(x-\frac{1}{\lambda}x'\right)=x'$ *for* $x'\in A\cdot x$, $x\in D(A)$. *If the resolvent* $R(\lambda, A)$ *satisfies the conditions* (54), (55) *and* (56), *then* A *generates a holomorphic semigroup in* $\Omega\times\Sigma_\theta$.

PROOF OF i). Since $\{T_t\}$ is a contraction semigroup, the operator $A_h=\frac{T_h-I}{h}$ is dissipative. Hence we have for $\lambda=\mu+i\nu$, $\mu>0$,

$$\begin{aligned}\|(\lambda I-A_h)x-(\lambda I-A_h)y\|^2&=\|\lambda(x-y)\|^2+\|A_hx-A_hy\|^2+2\,\mathrm{Re}\,\bar{\lambda}\langle A_hy-A_hx,\,x-y\rangle\\&\geqq\|\lambda(x-y)\|^2+\|A_hx-A_hy\|^2+2\,\mathrm{Re}\,i\nu\langle A_hy-A_hx,\,x-y\rangle\\&\geqq\mu^2\|x-y\|^2+(\|\nu(x-y)\|-\|A_hx-A_hy\|)^2\\&\geqq\mu^2\|x-y\|^2\,.\end{aligned}$$

This means the Lipschitz continuity of $(\lambda I-A_h)^{-1}$ for $\mathrm{Re}\,\lambda>0$:

$$\mu\|(\lambda I-A_h)^{-1}x-(\lambda I-A_h)^{-1}y\|\leqq\|x-y\|\,.$$

By Lemma 16, the mapping $(\lambda I-A_h)^{-1}$ is analytic. We fix $x_0\in\Omega_1$. Since the set $\{(\lambda I-A_h)^{-1}x_0:0<h<h_0\}$ is bounded, the convergence of $w\text{-}\lim_{\Phi}(\lambda I-A_h)^{-1}(x_0+\sigma x)$ is uniform in σ with $|\sigma|<r$, for fixed λ and x and for sufficiently small $r>0$. Thus the resolvent $R(\lambda,A)=w\text{-}\lim_{\Phi}(\lambda I-A_h)^{-1}$ is analytic in Ω_1. By the obvious equality

$$\begin{aligned}R(\lambda+\Delta\lambda,A)x-R(\lambda,A)x&=R(\lambda+\Delta\lambda,A)x-R(\lambda,A)(\lambda I+\Delta\lambda I-A)R(\lambda+\Delta\lambda,A)x\\&=R(\lambda+\Delta\lambda,A)x-R(\lambda,A)(\lambda I-A)R(\lambda+\Delta\lambda,A)x\\&\quad-\Delta\lambda\delta R((\lambda I-A)R(\lambda+\Delta\lambda,A)x\,;\,R(\lambda+\Delta\lambda,A)x)+o(\Delta\lambda)\,,\end{aligned}$$

we have

$$\begin{aligned}\frac{1}{\Delta\lambda}(R(\lambda+\Delta\lambda,A)x-R(\lambda,A)x)&=-\delta R((\lambda I-A)R(\lambda+\Delta\lambda,A)x\,;\,R(\lambda+\Delta\lambda,A)x)+o(1)\\&\to-\delta R(x\,;\,R(\lambda,A)x)\qquad\text{as }\Delta\lambda\to0\,,\end{aligned}$$

since $(\lambda I-A)R(\lambda+\Delta\lambda,A)x\to x$ and $R(\lambda+\Delta\lambda,A)x\to R(\lambda,A)x$. Thus $R(\lambda,A)x$ is holomorphic in λ for $\mathrm{Re}\,\lambda>0$. From this fact and the property that $\{T_t\}$ is extended to the sector $|\arg t|<\theta$, we obtain easily that $R(\lambda,A)x$ is holomorphic in λ for $|\arg\lambda|<\frac{\pi}{2}+\theta$. In fact, put $s=e^{i\theta_1}t$ for $|\theta_1|<\theta$. Then $\{S_t=T_s:t>0\}$ is a semigroup. The generator of $\{S_t\}$ is $e^{i\theta_1}A$. Hence the resolvent of the generator of $\{S_t\}$ is $R(\lambda,e^{i\theta_1}A)=R(e^{-i\theta_1}\lambda,A)e^{-i\theta_1}$. Since $\{S_t\}$ is a holomorphic semigroup in $\Sigma_{\theta-|\theta_1|}$, the resolvent $R(\lambda,e^{i\theta_1}A)x$ is holomorphic in λ for $\mathrm{Re}\,\lambda>0$. Since θ_1 is an arbitrary argument with $|\theta_1|<\theta$, $R(\lambda,A)x$ is holomorphic in $\lambda\in\Sigma_{\frac{\pi}{2}+\theta}$. Moreover we have for $\mu=\mathrm{Re}\,\lambda>0$

$$\|R(e^{-i\theta_1}\lambda,A)e^{-i\theta_1}x-R(e^{-i\theta_1}\lambda,A)e^{-i\theta_1}y\|\leqq\frac{1}{\mu}\|x-y\|\,,$$

and so for $|\arg\lambda|<\frac{\pi}{2}+\theta$, $\mathrm{Re}\,(e^{i\theta_1}\lambda)>0$ and $|\theta_1|<\theta$ we have

$$\|R(\lambda,A)x-R(\lambda,A)y\|\leqq\frac{1}{\mathrm{Re}\,(e^{i\theta_1}\lambda)}\|x-y\|\,.$$

PROOF OF ii). The inequality $\|R(\lambda,A)x-R(\lambda,A)y\|\leqq\frac{1}{\mathrm{Re}\,(e^{i\theta_1}\lambda)}\|x-y\|$ for $\pm\theta_1=|\arg\lambda|<\theta$ implies the dissipativity of $\frac{1}{\lambda}A$ for $|\arg\lambda|<\theta$. We shall show that the operator $A_n : x-\frac{1}{n}x' \to x'$ for $x \in D(A)$, $x' \in Ax$ is analytic. Let $x_t = \left(I-\frac{1}{n}A\right)^{-1}(y_0+ty)$ for fixed $y_0 \in \Omega$, $y \in H$. Obviously x_t is holomorphic in t. We put $x'_t = nx_t - n(y_0+ty)$. Since $x_t - \frac{1}{n}Ax_t \ni y_0+ty$, the element x'_t is contained in Ax_t. Noting that A_n is single-valued, $A_n(y_0+ty) = x'_t$ is holomorphic in t.

We shall construct the solution $u_n(t)$ of the equation

$$\frac{d}{dt}u_n(t) = A_n u_n(t) \qquad t \in \Sigma_\theta,\ |t| \leqq t_1,$$

$$u_n(0) = x \in \Omega .$$

Putting $\tilde{t} = |t|$, $\theta_1 = \arg t$, the approximating sequence

$$u^m(0) = x,\ u^m(\tilde{t}e^{i\theta_1}) = u^m(t_j^m e^{i\theta_1}) + (\tilde{t} - t_j^m)e^{i\theta_1}A_n u^m(t_j^m e^{i\theta_1}) \qquad \text{for } t_j^m \leqq \tilde{t} \leqq t_{j-1}^m ,$$

$$t_0^m = 0,\ t_{j+1}^m = t_j^m + \min\left\{\frac{1}{m},\ \sup\{\tilde{t} : u^m(t_j^m e^{i\theta_1}) + se^{i\theta_1}A_n u^m(t_j^m e^{i\theta_1}) \in \Omega,\ 0 < {}^\forall s < \tilde{t}\}\right\},$$

converges uniformly to a solution $u_n(t)$, since A_n is Lipschitz continuous. Note that for any m there exists some j_m with $t_{j_m}^m > t_1$ by (57), and so $u^m(t) \in \Omega$ for $0 \leqq \tilde{t} \leqq t_1$. Since A_n is analytic, the function $\frac{d}{dt}u_n(t) = A_n u_n(t)$ is p-times differentiable if $u_n(t)$ is so. Thus $u_n(t)$ is infinitely differentiable. By the infinite differentiability of $u_n(t+\tilde{t}e^{i\theta_1})$ by real $\tilde{t}$ for $t \in \Sigma_\theta$, $|\theta_1| < \theta$, the function $u_n(\tilde{t}+is)$ is infinitely differentiable in $\tilde{t}$ and s. Since $\lim_{\tilde{t}\downarrow 0} \frac{1}{e^{i\theta_1}\tilde{t}}(u_n(t+te^{i\theta_1}) - u_n(t))$ has the limit independent of θ_1, the function $u^m(t)$ is holomorphic in $t \in \Sigma_\theta$.

Since $e^{i\theta_1}A$ is dissipative for $|\theta_1| < \theta$, the sequence $\{u^m(t)\}$ is convergent to a function $u(t)$ uniformly in $t \in K$, where K is an arbitrary compact set in the sector Σ_θ. The function $u(t)$ is evidently holomorphic in the sector Σ_θ and satisfies the equation

$$\begin{cases} \frac{d}{dt}u(t) \in Au(t) & t \in \Sigma_\theta , \\ u(0) = x \in \Omega . \end{cases}$$

EXAMPLE. Let H be the one-dimensional complex space C^1. We put

$$Az = z^2, \qquad \Omega = \{z \in C^1 : \mathrm{Re}\, z < -|\mathrm{Im}\, z|\} .$$

Then we can easily see that the operator A and the closed set Ω satisfy the conditions (55), (56) and (57) for $\theta = \frac{\pi}{4}$. Instead of (54), the resolvent $R(\lambda, A)$ is analytic on $\Omega_1 = \{z \in C^1 : \mathrm{Re}\, z < 0\}$. The operator A generates the semigroup $\{T_t\}$ in $\Omega \times \Sigma_{\frac{\pi}{4}}$ such that $T_t z = \left(\frac{1}{z} - t\right)^{-1}$ for $z \in \Omega - \{0\}$. Note that Ω_1 is a neighbourhood of $\Omega-\{0\}$ and $T_t\cdot 0=0$. The semigroup $\{T_t\}$ is evidently holomorphic in $\Omega\times\Sigma_{\frac{\pi}{4}}$.

§ 5. Additional notes.

RESULTS BY CRANDALL-PAZY, KATO and DORROH. We shall explain shortly a part of [**10**], [**11**] and [**13**] closely related to ours. Let A be a maximal (multi-valued) operator. Using the fact that the set Ax is convex and closed for every point $x\in D(A)$, the single-valued restriction A^0 (called the *minimal cross section of* A by Crandall-Pazy and the *canonical restriction of* A by Kato) of A is defined by:

$$A^0x=y\,,\qquad y\in Ax\qquad \text{and}\qquad \|y\|=\inf_{y'\in Ax}\|y'\|\,.$$

The most remarkable fact is:

THEOREM (Crandall-Pazy, Kato and Dorroh). *Let $\{T_t\}$ be the semigroup generated by a maximal dissipative operator A. Then the infinitesimal generator A_0 of $\{T_t\}$ is an extension of A^0.*

More precisely, we obtain the followings:

i) $A_\emptyset=A_0$

ii) *Our solution $u(t)$ of $\frac{d}{dt}u(t)\in Au(t)$ is a strict solution of*

$$D^+u(t)=A^0u(t)\qquad (D^+=\textit{the right differentiation})\,.$$

iii) *The answer to the first half of Problem* 1 *in* § 3.

The result ii) is much better than our Theorem 1.

THE HILLE-YOSIDA THEOREM. We shall begin with explanation of iii) above. If $A^0\subset B^0$ for maximal dissipative operators A and B, then the semigroup $\{T_t\}$ generated by B is an extension of the semigroup $\{S_t\}$ generated by A (Theorem above). Hence, if the semigroup $\{S_t\}$ is maximal contraction semigroup, then A^0 is maximal in the class $\{B^0: B$ is maximal dissipative$\}$. Conversely, if $\{T_t\}$ is not a maximal contraction semigroup, a maximal extension $\{S_t\}$ of $\{T_t\}$ is generated by a maximal dissipative operator A (Theorem 4). The infinitesimal generator of $\{S_t\}$ is A^0 (Theorem above), and so $B^0\subsetneqq A^0$. Hence, if A^0 is maximal in the class $\{B^0: B$ is maximal dissipative$\}$, then the semigroup $\{S_t\}$ generated by A^0 is a maximal contraction semigroup. Thus we obtain

THE HILLE-YOSIDA THEOREM FOR NONLINEAR SEMIGROUPS. *If $\{T_t\}$ is a maximal contraction semigroup, then the infinitesimal generator A_0 is densely defined in $D(T_t)$ and is maximal in the class $\{B^0: B$ is maximal dissipative$\}$. Conversely, if an operator A is a maximal one in the class $\{B^0: B$ is maximal dissipative$\}$, then A generates on $\overline{D(A)}$ uniquely a maximal contraction semi-*

group $\{T_t\}$ *whose infinitesimal generator* A_0 *is* A.

For instance, our Theorem 4 ii) is easily obtained as a special case of this theorem.

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