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On compact complex analytic manifolds of complex dimension 3, II

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The present paper is a continuation of our previous paper [3] and the object is to investigate the structure of compact complex manifolds of dimension 3 with (meromorphic) function fields of dimension 1 or 0. The results are stated in the following two theorems.

THEOREM 1. *Let* $\varphi : M \rightarrow \Delta$ *be a holomorphic mapping of a compact complex manifold of dimension* 3 *onto a compact Riemann surface. If the mapping* φ *induces an isomorphism of the meromorphic function field of* Δ *to the meromorphic function field of* M, *then a general fibre of* φ *must be one of surfaces of the following classes;* (i) $K3$ *surface,* (ii) *surface with first Betti number* $b_1 = 1$, (iii) *complex torus,* (iv) *elliptic surface with a trivial canonical bundle,* (v) *ruled surface with irregularity* $q = 1$, (vi) *rational surface,* (vii) *Enriques surface.*

THEOREM 2. *A compact Kähler manifold of dimension* 3 *which has no non-constant meromorphic functions is bimeromorphically equivalent to* (i) *a complex torus,* (ii) *an elliptic fibre space or a projective line bundle over a complex torus, or* (iii) *a regular manifold with geometric genus* $p_g = 0$ *or* 1.

§ 1. Proof of Theorem 1.

In Kodaira [5] surfaces are classified into the following classes:

I) the class of algebraic surfaces with $p_g = 0$;
II) the class of $K3$ surfaces;
III) the class of complex tori;
IV) the class of elliptic surfaces with $p_g \geqq 1$;
V) the class of algebraic surfaces with $p_g \geqq 1$;
VI) the class of elliptic surfaces with $b_1 \equiv 1(2)$, $p_g \geqq 1$;
VII) the class of surfaces with $b_1 = 1$.

Here p_g and b_1 are the geometric genus and the first Betti number, respectively. Surfaces of class (I) are classified furthermore into (i) rational surfaces, (ii) Enriques surfaces, (iii) elliptic surfaces, (iv) ruled surfaces with irregularity $q = 1$, (v) ruled surface with $q \geqq 2$, and (vi) surfaces of which pluri genera increase infinitely.

In the sequel we shall check whether any surface may appear as a fibre of φ in the theorem.

PROPOSITION 1. *Let $\varphi: M \to \Delta$ be a holomorphic mapping of a compact complex threefold onto a compact Riemann surface. If a general fibre of φ is a surface of class* (IV) *or* (V), *or a surface of class* (VI) *with a non-trivial canonical bundle, then the dimension of the function field of M is greater than* 1.

PROOF. Let $\mathfrak{K}$ be the canonical bundle of M and $\mathcal{O}(m\mathfrak{K})$ the sheaf of holomorphic sections of the bundle $m\mathfrak{K}$. By a fundamental theorem of Grauert [1] the direct image $\varphi_*(\mathcal{O}(m\mathfrak{K}))$ is a coherent sheaf. Hence, by means of Grothendieck [2], the projective space $\lambda: \boldsymbol{P}(\varphi_*(\mathcal{O}(m\mathfrak{K}))) \to \Delta$ is defined. Let U be the set of points of M where the canonical homomorphism $\varphi^*(\varphi_*(\mathcal{O}(m\mathfrak{K}))) \to \mathcal{O}(m\mathfrak{K})$ is surjective, then U is a complement of an analytic set of M, and a morphism Φ of U to $\boldsymbol{P}(\varphi_*(\mathcal{O}(m\mathfrak{K})))$ over Δ is

$$\begin{array}{ccc} \Phi: U \subset M & \to & \boldsymbol{P}(\varphi_*(\mathcal{O}(m\mathfrak{K}))) \\ {\scriptstyle\varphi}\searrow & & \swarrow{\scriptstyle\lambda} \\ & \Delta & \end{array}$$

defined canonically. If u is a general point of Δ, then $\lambda^{-1}(u)$ is a projective space defined by the vector space $H^0(\varphi^{-1}(u), \mathcal{O}(m\mathfrak{K})/\mathfrak{m}_u\mathcal{O}(m\mathfrak{K}))$, where $\mathfrak{m}_u$ is the maximal ideal of the local ring at u, and the restriction Φ_u of Φ to $\varphi^{-1}(u) \cap U$ is a morphism defined by a base of $H^0(\varphi^{-1}(u), \mathcal{O}(m\mathfrak{K})/\mathfrak{m}_u\mathcal{O}(m\mathfrak{K}))$. Now the sheaf $\mathcal{O}(m\mathfrak{K})/\mathfrak{m}_u\mathcal{O}(m\mathfrak{K})$ is isomorphic to the sheaf of sections of multi-canonical bundle mK of $S_u = \varphi^{-1}(u)$, K being the canonical bundle of S_u. By results of Kodaira we know that the dimension of $H^0(S, \mathcal{O}(mK))$ increases infinitely as an integer m increases. Hence for a large m the morphism Φ_u is defined almost everywhere and meromorphic on S_u, and the image by Φ_u is not a point. Therefore U is non-empty, Φ is extensible to a meromorphic map of M, and the image by Φ is an irreducible analytic set of $\boldsymbol{P}(\varphi_*(\mathcal{O}(m\mathfrak{K})))$ of dimension greater than 1. Since the analytic space $\boldsymbol{P}(\varphi_*(\mathcal{O}(m\mathfrak{K})))$ is algebraic, its analytic subset is also algebraic. Consequently dimension of the function field of M is greater than 1.

PROPOSITION 2. *Let $\varphi: M \to \Delta$ be as in the above. If a general fibre S_u over $u \in \Delta$ is a surface with geometric genus $p_g = 0$ and irregularity $q \geqq 1$, then there are a complex space $\lambda: V \to \Delta$ over Δ and a meromorphic mapping Ψ of M to V over Δ such that the restriction Ψ_u of Ψ to S_u is Albanese map of S_u onto the image of Albanese map, that is, a non-singular curve with genus q.*

PROOF. Let $\{a_i\}$ be a finite set of points of Δ such that the restriction φ' of φ to $M' = M - \varphi^{-1}(\{a_i\})$ is a simple morphism of M' onto $\Delta' = \Delta - \{a_i\}$. We may assume that $\Delta' \neq \Delta$. Let $\Omega^1_{M/\Delta}$ be the sheaf of holomorphic 1-differentials along fibres of φ (cf. Grothendieck [2]). By Grauert the sheaf $\varphi_*(\Omega^1_{M/\Delta})$ is a coherent sheaf and for $u \in \Delta' \varphi_*(\Omega^1_{M/\Delta})_u / \mathfrak{m}_u \varphi_*(\Omega^1_{M/\Delta})_u$ is canonically isomorphic to $H^0(S_u, \Omega^1_{M/\Delta}/\mathfrak{m}_u\Omega^1_{M/\Delta})$ which is the space of holomorphic 1-differentials on S_u. Here $\mathfrak{m}_u$ is the maximal ideal of the local ring at u. Since the space Δ' is affine, there are sections $(w^1, \cdots, w^q)$ of $\Gamma(\Delta', \varphi_*(\Omega^1_{M/\Delta}))=\Gamma(M', \Omega^1_{M/\Delta})$ which generate $H^0(S_u, \Omega^1_{M/\Delta}/\mathfrak{m}_u\Omega^1_{M/\Delta})$. Adding some a_i, if necessary, we may assume that $(w^1, \cdots, w^q)$ generate $H^0(S_u, \Omega^1_{M/\Delta}/\mathfrak{m}_u\Omega^1_{M/\Delta})$ for every $u\in\Delta'$. We denote by $(w^1(u), \cdots, w^q(u))$ the elements of $H^0(S_u, \Omega^1_{M/\Delta}/\mathfrak{m}_u\Omega^1_{M/\Delta})$ which correspond to $(w^1, \cdots, w^q)$. We fix a fibre $S_0=S_{u_0}$ and a base $(\gamma_1, \cdots, \gamma_{2q})$ of the first Betti group of S_0. The fibre space $\varphi': M'\to\Delta'$ being topologically locally trivial, we have a base $(\gamma_1(u), \cdots, \gamma_{2q}(u))$ of the first Betti group of S_u such that each $\gamma_j(u)$ depends continuously on u and $\gamma_j(u_0)=\gamma_j$. If we deform $(\gamma_1, \cdots, \gamma_{2q})$ continuously along a path $\beta\in\pi_1(\Delta')=\pi_1(\Delta', u_0)$, $(\gamma_1, \cdots, \gamma_{2q})$ is transformed into $(\sum_j a_{j_1}(\beta)\gamma_j, \cdots, \sum_j a_{j_{2q}}(\beta)\gamma_j)$, where $(a_{jk}(\beta))$ is a unimodular matrix. The integral $\int_{\gamma_j(u)} w^k(u)$ is a multi-valued holomorphic functions of u. Putting $\omega_j(u)=\left(\int_{\gamma_j(u)} w^1(u), \cdots, \int_{\gamma_j(u)} w^q(u)\right)$, the multi-vector-valued function $\omega_j(u)$ is transformed into $\sum_k a_{kj}(\beta)\omega_k(u)$, by the analytic continuation along a path β. Let $\varpi: U'\to\Delta'$ be the universal covering of Δ'. We denote simply by $\omega_j(\tilde{u})$ the single vector-valued holomorphic function $\omega_j(\varpi(\tilde{u}))$ of $\tilde{u}\in U'$. Identifying $\pi_1(\Delta')$ with the covering transformation group with respect to ϖ, we consider the following automorphism of $U'\times C^q$;

$$g(\beta, n): (\tilde{u}, \zeta)\to(\beta\tilde{u}, \zeta+\sum_{k=1}^{2q} n_k\sum_j a_{jk}(\beta)\omega_j(\tilde{u})),$$

where $\beta\in\pi_1(\Delta')$, $n=(n_j)\in Z^{2q}$, $\tilde{u}\in U'$, $\zeta\in \boldsymbol{C}^q$. The set $G=\{g(\beta, n)\,|\,\beta\in\pi_1(\Delta'), n\in Z^{2q}\}$ is a group of automorphisms of $U'\times\boldsymbol{C}^q$ without fixed points. We form the quotient space $U'\times\boldsymbol{C}^q/G$, which is a complex manifold and is denoted by B'. The natural projection of $U'\times\boldsymbol{C}^q$ to U' induces a holomorphic map μ of B' onto Δ' and the map $U'\ni\tilde{u}\to(\tilde{u}, 0)\in U'\times\boldsymbol{C}^q$ induces the holomorphic map ν of Δ' to B' such that $\mu\circ\nu=$ identity. Here 0 is the null-element of vector space $\boldsymbol{C}^q$.

Let $\{U_j\}$ be an open covering of Δ' such that there is a holomorphic section s_j to M' over U_j. The map

$$\phi_j: \varphi^{-1}(U_j)\ni z\to\left(\varphi(z), \left(\int_{s_j(\varphi(z))}^z w^1(\varphi(z)), \cdots, \int_{s_j(\varphi(z))}^z w^q(\varphi(z))\right)\right)\in U'\times\boldsymbol{C}^q/G=B'$$

is a well-defined holomorphic map and it holds $\phi_j\circ s_j=\nu|U_j$. If $U_j\cap U_k$ is non-empty, we have a unique automorphism Λ_{jk} of $B'|U_j\cap U_k=\mu^{-1}(U_j\cap U_k)$, which is a translation along a fibre, such that $\Lambda_{jk}\circ\nu=\phi_j\circ s_k$. We see immediately that $\Lambda_{kl}\cdot\Lambda_{jk}=\Lambda_{jl}$, if $U_j\cap U_k\cap U_l\neq\phi$. Hence we can patch together $B'|U_j=\mu^{-1}(U_j)$ by automorphisms $\{\Lambda_{jk}\}$. The space obtained by patching together is denoted by A'. The map μ may be considered as a map of A' to Δ' and $\{\phi_j\}$ induce a holomorphic map Ψ' of M' to A'. By the theory of surfaces we know the image $\Psi'(S_u)$ of S_u is a non-singular curve with genus q. Hence the image $V'=\Psi'(M')$ of M' is a complex submanifold of A'. We denote by λ' the restriction of μ to V'.

Now we shall extend the complex manifold V' over Δ' to the complex space V over Δ. Take an a_i and a small open disk U_i with center a_i, let m be the least common multiple of the multiplicities of irreducible components of $\varphi^{-1}(a_i)$ and let $\pi: D\rightarrow U_i$ be an m-fold covering with the point $\tilde{a}$ lying over a_i as a branch point of degree $(m-1)$. Considering the normalization M_i^* of the fibre product $(M|U_i)\times_{U_i}D$ and the natural projection φ_i from M_i^* to D, we see immediately that each irreducible component of $\varphi_i^{-1}(\tilde{a})$ has multiplicity 1. Hence for every simple point of $\varphi^{-1}(\tilde{a})$ there exists a section $s_i: D\rightarrow M_i^*$. Since D is a Stein manifold, there are sections $(w^1, \cdots, w^q)$ of $(\varphi_i)_*(\Omega^1_{M_i^*/D})$ over D which generate $(\varphi_i)_*(\Omega^1_{M_i^*/D})/\mathfrak{m}_\sigma(\varphi_i)_*(\Omega^1_{M_i^*/D})$ for every point σ of $D'=D-\tilde{a}$. In the same notation as before $w^j(\sigma)$ is a holomorphic 1-differential in simple points of $S_\sigma=\varphi_i^{-1}(\sigma)$ for $\sigma\in D$. We fix a point σ_0 of D' and a first Betti base $\gamma_1, \cdots, \gamma_{2q}$ of S_{σ_0}. As before we denote by $\gamma_1(\sigma), \cdots, \gamma_{2q}(\sigma)$ the first Betti base such that $\gamma_j(\sigma)$ depends continuously on σ and $\gamma_j(\sigma_0)=\gamma_j$. An element β of the fundamental group $\pi_1(D')$ induces a transformation $(\gamma_j)\rightarrow(\sum_k a_{kj}(\beta)\gamma_k)$. Let $\varpi: U\rightarrow D'$ be the universal covering with the covering transformation group identified with $\pi_1(D')$. Putting

$$\begin{aligned}
&\omega_j(\sigma)=\left(\int_{\gamma_j(\sigma)}w^1(\sigma), \cdots, \int_{\gamma_j(\sigma)}w^q(\sigma)\right),\\
&\omega_j(\tilde{\sigma})=\omega_j(\varpi(\tilde{\sigma})) \qquad \text{for } \tilde{\sigma}\in U,\\
&g(\beta, n): (\tilde{\sigma}, \zeta)\rightarrow(\beta\tilde{\sigma}, \zeta+\textstyle\sum n_k\sum a_{jk}(\beta)\omega_j(\tilde{\sigma})),\\
&G=\{g(\beta, n)\},
\end{aligned}$$

we form the quotient manifold $B_i=U\times C^q/G$. As before we let μ_i be the natural projection from B_i to D' and let ψ_i' be the holomorphic map

$$\psi_i': M_i^*|D'\ni z\rightarrow\left(\varphi_i(z), \left(\int_{s_i(\varphi_i(z))}^z w^1(\varphi_i(z)), \cdots, \int_{s_i(\varphi_i(z))}^z w^q(\varphi_i(z))\right)\right)\in B_i.$$

The analytic set $E'=\psi_i'^{-1}(\psi_i'(s_i(D')))$ is extensible to the analytic set E of M_i^*. In fact $f_j(z)=\int_{s_i(\varphi_i(z))}^z w^j(\varphi_i(z))$ is holomorphic in simple points of $\varphi_i^{-1}(\tilde{a})$. Hence E' is holomorphically extensible to simple points of $\varphi_i^{-1}(\tilde{a})$. Therefore by a theorem of Remmert-Stein E' is holomorphically extensible to M_i^* over all. The analytic set E is an irreducible surface, to which corresponds the complex line bundle $[E]$ on M_i^*. Let $\lambda_i: V_i=P((\varphi_i)_*(\mathcal{O}(mE)))\rightarrow D$ be the projective space defined by the coherent sheaf $(\varphi_i)_*(\mathcal{O}(m[E]))$ on D, where m is a large integer, and let $\phi_i: M_i^*\rightarrow V_i$ be the canonical meromorphic mapping.

Then the fibre $\lambda_i^{-1}(\sigma)$ over $\sigma \in D'$ is isomorphic to the image $\psi_i'(S_\sigma)$ of S_σ in its Albanese variety $\mu_i^{-1}(\sigma) \subset B_i$, and $\psi_i | S_\sigma$ is equivalent to the Albanese map $\psi_i' | S_\sigma$. Now the covering transformation group $\mathcal{G}$ of D with respect to π operates naturally on M_i^* and we have $M_i^*/\mathcal{G} = M | U_i$. It is easily seen that an element of $\mathcal{G}$ transforms the surface E onto itself, and that it induces the automorphism of the line bundle $[E]$, which also induces the automorphism of the complex space V_i. Thus we may consider $\mathcal{G}$ to be a group of automorphisms of V_i. Since $\mathcal{G}$ is a finite group, we can form the quotient space $V_i/\mathcal{G}$ which we denote by the same notation V_i. The mapping ψ_i induces a meromorphic mapping Ψ_i of $M_i | U_i$ to V_i. Seeing the construction it is clear that $V' | U_i'$ is canonically isomorphic to $V_i | U_i'$, and that $\Psi_i | U_i'$ is equivalent to Ψ'. Here $U_i' = U_i - a_i$. Thus we can extend the manifold V' to the complex space V and the holomorphic map Ψ' to the meromorphic map Ψ.

COROLLARY. *If a general fibre S_u of $\varphi : M \rightarrow \Delta$ is a ruled surface with irregularity $q \geqq 2$ or an elliptic surface with canonical bundle K such that $mK \sim 0$ for a positive integer m, the dimension of the function field of M is greater than* 1.

PROOF. In the same notation as above a general fibre C_u of $\lambda : V \rightarrow \Delta$ is a curve with genus q or an elliptic curve. In the latter case it is known by a surface theory of Kodaira that the general fibre S_u is an elliptic surface over C_u with multiple singular fibres. Since the set of points of indeterminacy of the meromorphic map Ψ and points where Ψ is regular and not simple constitute an analytic set of M, the image by Ψ of singular fibres of elliptic surfaces S_u forms a curve Θ of V such that $\lambda(\Theta) = \Delta$. Therefore in both cases by results of Kodaira the surface V is bimeromorphically equivalent to an algebraic surface. The dimension of the function field of M is not less than that of V. This ends the proof.

§ 2. Examples and some propositions.

PROPOSITION. *Let $\varphi : M \rightarrow \Delta$ be a holomorphic map of a compact complex threefold onto a compact Riemann surface. If there lies on M a surface which has no non-constant meromorphic functions, then dimension of the function field of M is* 1.

PROOF. Supposing that dimension of the function field of M is greater than 1, we shall obtain a contradiction. In case the dimension is 3, it is easy. So we suppose that the dimension of the function field is 2. Then there are a compact complex threefold $\tilde{M}$ and a holomorphic map f (resp. ψ) onto M (resp. an algebraic surface V). It suffices to prove that every irreducible surface S on $\tilde{M}$ has infinitely many curves. In case $\psi(S)$ has positive dimension, it is clear. In case $\psi(S)$ is a point a of V, take a general simple point

z_0 of $\psi^{-1}(a)$ which belongs to S and let ξ, ζ, η (resp. x, y) be local coordinates with center z_0 (resp. a). Put $x=x(\psi(z))=g(\xi, \zeta, \eta)$, $y=y(\psi(z))=h(\xi, \zeta, \eta)$ for a neighboring point z of z_0. We may assume that S is defined by the equation $\xi=0$ in a neighborhood of z_0 and that the functions $g(\xi, \zeta, \eta)$, $h(\xi, \zeta, \eta)$ are expressed as

$$g(\xi, \zeta, \eta)=\xi^{\alpha}(a_0(\zeta, \eta)+a_1(\zeta, \eta)\xi+\cdots), \quad h(\xi, \zeta, \eta)=\xi^{\beta}(b_0(\zeta, \eta)+\cdots).$$

Furthermore we may assume that $g(\xi, \zeta, \eta)=\xi^{\alpha}$, the point z_0 being a general point. Now in case $b_0(\zeta, \eta)$ is non-constant, the inverse image $\psi^{-1}(C_\lambda)$ of the curve C_λ defined by the equation $x^{\beta}-\lambda y^{\alpha}=0$ on Δ cuts a variable curve on S, where λ is a variable. If $b_0(\zeta, \eta)$ is a constant b_0 and $b_1(\zeta, \eta)$ is non-constant, the inverse image $\psi^{-1}(C_\lambda)$ of the curve C_λ defined by the equation $(y^{\alpha}-b_0^{\alpha}x^{\beta})^{\alpha}-\lambda x^{\alpha\beta}y=0$ cuts a variable curve on S, and so on. q. e. d.

PROPOSITION. *Let $\varphi: M\to\Delta$ be an analytic fibre bundle of abelian varieties of general type over a compact Riemann surface. If the manifold M is non-algebraic, then dimension of its function field is* 1.

PROOF. Suppose that dimension of the function field of M is greater than 1. Then there is an irreducible hypersurface S on M such that $\varphi(S)=\Delta$. Since each fibre $\varphi^{-1}(u)$ is an abelian variety of general type, $S\cap\varphi^{-1}(u)$ is an ample divisor on $\varphi^{-1}(u)$. Therefore the complex line bundle $m\ [S]$ is very ample for a large integer m by Grothendieck [2].

EXAMPLE 1. Let S be an elliptic surface with a trivial canonical bundle. If S is not a complex torus, then by Kodaira [5] it is isomorphic to a quotient manifold C^2/G, G being an affine transformation group generated by g_1, g_2, g_3, g_4 defined as

$$g_jw_1=w_1+\alpha_j,\ g_jw_2=w_2+\alpha_jw_1+\beta_j,\ j=1, \cdots, 4 \qquad \text{for } (w_1, w_2)\in C^2,$$

where $\alpha_1, \cdots, \alpha_4$, $\beta_1, \cdots, \beta_4$ are constant satisfying the conditions

$$\alpha_1=\alpha_2=0, \qquad \alpha_3\alpha_4-\alpha_4\alpha_3=m\beta_2\neq 0,$$

m being a positive integer.

We put $M=C^3/G'$, G' being an affine transformation group generated by σ, τ, g_1, g_2, g_3, g_4 defined as

$$\sigma z=z+1,\ \sigma w_1=w_1,\ \sigma w_2=w_2$$

$$\tau z=z+\sqrt{-1},\ \tau w_1=w_1+\alpha,\ \tau w_2=w_2,$$

$$g_jz=z,\ g_jw_1=w_1+\alpha_j,\ g_jw_2=\alpha_jw_1+\beta_j \quad \text{for } (z, w_1, w_2)\in C^3,$$

where

$$\alpha=-\frac{3(2+\sqrt{2})}{4}+\frac{-4+3\sqrt{2}}{4}\sqrt{-1}$$

$$\beta_1=-\frac{3(2+\sqrt{2})}{4}+\frac{4+3\sqrt{2}}{4}\sqrt{-1},\ \beta_2=2\sqrt{-1},\ \beta_3=\beta_4=0$$

$$\alpha_1 = \alpha_2 = 0,\ \alpha_3 = 1,\ \alpha_4 = 2+\sqrt{-1}.$$

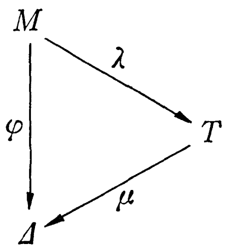
Let Δ be the complex torus $C/\{1, \sqrt{-1}\}$ and let $\varphi: M \to \Delta$ be the natural projection. Then a fibre of φ is the elliptic surface S and dimension of the function field of M is 1. In fact, putting

$$\sigma' z = z+1,\ \sigma' w_1 = w_1,$$

$$\tau' z = z+\sqrt{-1},\ \tau' w_1 = w_1+\alpha,$$

$$g_j z = z,\ g_j w_1 = w_1+\alpha_j,\ j=3, 4 \qquad \text{for } (z, w_1) \in C^2,$$

and letting H be an affine transformation group generated by σ', τ', g_3 and g_4, we obtain an elliptic surface $T = C^2/H$. By results of Kodaira [4] it is easily proved that T is non-algebraic. Let $\lambda: M \to T$ and $\mu: T \to \Delta$ be the natural projections. If dimension of the function field of M were greater than 1, there would be a surface D on M such that $\lambda(D) = T$. Then the elliptic surface S would be algebraic, which yields a contradiction.



EXAMPLE 2. Let P^1 be a projective line with non-homogeneous coordinate x. Putting

$$\sigma z = z+1,\ \sigma x = ax+b,$$

$$\tau z = z+\sqrt{-1},\ \tau x = cx+d,\ ac \neq 0 \qquad \text{for } (z, x) \in C \times P^1,$$

and letting G be a group of automorphisms of $C \times P^1$ generated by σ and τ, we obtain a compact surface $R = C \times P^1/G$. The natural projection induces a holomorphic mapping λ of R onto a 1-dimensional complex torus T with periods $(1, \sqrt{-1})$. The surface R is a ruled surface with irregularity 1. Let S be an analytic principal bundle with group T over a compact Riemann surface Δ which is not an algebraic surface, the existence of which is proved easily. Since translations of C commute with σ and τ, the group T operates naturally on R. Hence an analytic fibre bundle M with fibre R associated to S is defined. Let $\varphi: M \to \Delta$ and $\psi: M \to S$ be the natural projections. Now we shall show that dimension of the function field of M is greater than 1 if a, b, c and d are general complex numbers. Suppose that dimension of the function field is greater than 1. Then we see by [3] that M is elliptic and that almost all irreducible surfaces on M are elliptic. Hence a general fibre of φ, that is, R is an elliptic surface. However R is non-elliptic for general a, b, c and d.

In fact; if R is an elliptic surface, it contains infinitely many elliptic curves C such that $\lambda(C)=T$. It is easily proved by Hurwitz's formula that elliptic curve C is an unramified covering of T. Consequently a projective line bundle induced from the bundle $\lambda: R\to T$ on an appropriate unramified covering of T has infinitely many holomorphic sections. Therefore we infer readily that there are positive integers m, n and infinitely many meromorphic functions $f(z)$ of z such that

$$f(z+m)=a^m f(z)+b(1+a+\cdots+a^{m-1}),$$

$$f(z+n\sqrt{-1})=c^n f(z)+d(1+c+\cdots+c^{n-1}).$$

This yields the relation

$$a^m d\sum_{j=0}^{n-1} c^j+b\sum_{k=0}^{m-1} a^k=c^n b\sum_{k=0}^{m-1} a^k+d\sum_{j=0}^{n-1} c^j.$$

Thus R is not an elliptic surface for general a, b, c, d, e.g., $a=b=1$, $c=d=2$.

REMARK. I have no example of a compact threefold M with a function field of dimension 1 such that there is a holomorphic map φ of M onto a compact Riemann surface Δ and a general fibre of φ is an Enriques surface or a rational surface.

§ 3. Proof of Theorem 2.

Let M be a compact Kähler manifold of dimension 3 on which there is no non-constant meromorphic functions. We denote by h^ν the dimension of the linear space of holomorphic ν-forms on M.

PROPOSITION 3. $h^3\leqq 1$.

PROOF. Suppose that $h^3\geqq 2$. Let ω_1, ω_2 be linearly independent holomorphic 3-forms. By local coordinates (z^1, z^2, z^3) ω_i is expressed as $f_i(z)$ $dz^1\wedge dz^2\wedge dz^3$ and the function $F(z)=f_1(z)/f_2(z)$ is a well-defined meromorphic function on M which is not a constant. This is a contradiction.

PROPOSITION 4. $h^1\leqq 3$.

PROOF. Suppose that $h^1\geqq 4$. Let φ_1, φ_2, φ_3, φ_4 be linearly independent holomorphic 1-forms. By the above proposition there are constants α and β, either of which does not vanish, such that

$$\alpha\varphi_1\wedge\varphi_2\wedge\varphi_3+\beta\varphi_1\wedge\varphi_2\wedge\varphi_4=0.$$

If $\varphi_1\wedge\varphi_2\neq 0$, then there are meromorphic functions f_1, f_2 such that $\alpha\varphi_3+\beta\varphi_4=f_1\varphi_1+f_2\varphi_2$, which yields a contradiction. If $\varphi_1\wedge\varphi_2=0$, then there is a meromorphic function f and we have $\varphi_2=f\varphi_1$. This is a contradiction, too. Q.E.D.

Now we write $h^1=q$ and let $\varphi_1, \cdots, \varphi_q$ be linearly independent holomorphic 1-forms. Since M is Kählerian by assumption, its first Betti number is $2q$. Let $\gamma_1, \cdots, \gamma_{2q}$ be a base of the first Betti group and put

$$\omega_\alpha = \left(\int_{\gamma_\alpha} \varphi_1, \cdots, \int_{\gamma_\alpha} \varphi_q\right),$$

$$\Omega = \{n_1\omega_1 + \cdots + n_{2q}\omega_{2q} \mid n_\alpha \in Z\}.$$

Then Ω is a discrete subgroup of C^q of rank $2q$ and $T^q = C^q/\Omega$ is a complex torus of dimension q. We fix a point z_0 of M and consider the mapping

$$\varphi : M \ni z \to \left(\int_{z_0}^{z} \varphi_1, \cdots, \int_{z_0}^{z} \varphi_q\right) \in C^q/\Omega = T^q.$$

The mapping φ is a well-defined holomorphic map and is called Albanese map.

PROPOSITION 5. *The underlying continuous map of φ is a surjection.*

PROOF. In case $q=3$, we express the 3-form $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ by local coordinates as $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = f(z)dz^1 \wedge dz^2 \wedge dz^3$. Then the points of degeneracy of φ are defined by the equation $f(z)=0$. If $f(z) \neq 0$ at some point (z), it is clear that φ is a surjection. If otherwise, we have $\varphi_1 \wedge \varphi_2 \wedge \varphi_3 = 0$. This produces a contradiction as in the proof of the above proposition. The other case are dealt similarly.

PROPOSITION 6. *In case $q=3$, the mapping φ is one to one almost everywhere. Therefore the threefold M is bimeromorphically equivalent to a complex torus.*

PROOF. There exist no surfaces on T^3. In fact let there be an irreducible surface D on T^3 and let $\theta(x)$ be a reduced theta function defining D, then the function $f(x) = \theta(x+a)\theta(x-a)/\theta(x)^2$ is a non-constant meromorphic function on T^3 for general constant vector a, which contradicts the assumption to the effect that no non-constant meromorphic function exists on M. We denote by A the set of points of degeneracy of φ. The set A is an analytic set and has dimension not greater than 1 by the above. Clearly the map φ induces a surjective homomorphism of the fundamental group $\pi_1(M)$ of M to the fundamental group $\pi_1(T)$ of T. Therefore the restriction $\varphi|(M-A)$ of φ induces a surjective homomorphism of $\pi_1(M-A)$ to $\pi_1(T-\varphi(A))$, for $\varphi(A)$ is an analytic set of codimension greater than 1. The map $\varphi|(M-A)$ is a covering map. Hence the manifold $M-A$ is homeomorphic to the manifold $T-\varphi(A)$.

q. e. d.

PROPOSITION 7. *In case $q=2$, a general fibre of $\varphi : M \to T^2$ is a rational curve or an elliptic curve.*

PROOF. We shall prove at first that every fibre of φ is connected. Let $\varphi : M \xrightarrow{\phi_1} T' \xrightarrow{\phi_2} T$ be the Stein factorization of the map φ. The analytic space T' is by definition a space having connected components of fibres of φ as points. Hence the map $\phi_2 : T' \to T$ is a covering map possibly with ramification. Since T has no curve by the assumption, the covering map ϕ_2 is unramified. The map ϕ_2 induces a surjective homomorphism of the fundamental groups, because $\psi=\psi_2\circ\psi_1$ induces a surjective homomorphism of the fundamental groups. Consequently the covering map $\psi_2: T'\rightarrow T$ is an isomorphism and every fibre is connected. Next we shall prove that genus g of a general fibre of φ is less than 2. Suppose that $g\geqq 2$. Let A be the set of degeneracy points of the map φ. The image $\varphi(A)$ consists of points $\{a_i\}$ of a finite number, for there lies no curve on T because of the fact that it has no non-constant meromorphic functions. Let R_g be the space of moduli of algebraic curves with genus g. It is well-known that R_g is a Zariski-open set of a projective variety. Since a fibre C_t over a point t of $T'=T-\{a_i\}$ is an algebraic curve with genus g, we have a natural map ψ of T' to R_g. By Hartogs' theorem the holomorphic map ψ is meromorphic on T over all. From the fact that T has no non-constant meromorphic function, we infer readily that the image $\psi(T)$ is a single point. Consequently the fibre space $\psi|(M-A)$: $M-A\rightarrow T'$ is an analytic fibre bundle. Its fibre is denoted by C. Taking a polycylinder E_i in a coordinate neighborhood with center a_i, we are to replace $M|E_i=\varphi^{-1}(E_i)$ by $E_i\times C$. Clearly there exists an isomorphism μ of $(E_i-a_i)\times C$ to $M|(E_i-a_i)$. If we prove the existence of many meromorphic functions on $M|E_i$, we see by Hartogs' theorem that μ is extensible to a bimeromorphic map of $E_i\times C$ to $M|E_i$. We denote by $\mathfrak{K}$ the canonical bundle of M. The direct image $\varphi_*(\mathcal{O}(m\mathfrak{K}))$ of the sheaf of holomorphic sections to $m\mathfrak{K}$ is a coherent sheaf on T by a fundamental theorem of Grauert. Since E_i is a Stein manifold, for a large integer m we have sections $f_0, \cdots, f_r$ of $\varphi_*(\mathcal{O}(m\mathfrak{K}))$ over E_i such that the well-defined mapping

$$(M|E_i)\ni z\rightarrow(\varphi(z), (f_0(z), \cdots, f_r(z)))\in E\times P^r$$

is a bimeromorphic map which is an isomorphism on $M|(E_i-a_i)$. Here P^r is a projective space of dimension r. Thus we have proved the existence of many meromorphic functions on $M|E_i$ and we see that μ is extensible to a bimeromorphic map of $E_i\times C$ and $M|E_i$. By means of the isomorphism μ replacing $M|E_i$ by $E_i\times C$, we obtain an analytic fibre bundle $\varphi^*: M^*\rightarrow T$. The threefold M^* is bimeromorphically equivalent to M. There are only a finite number of automorphisms on C and we see immediately the existence of non-constant meromorphic functions on M^*. This is a contradiction.

In case $q=1$, T is an elliptic curve and has non-constant meromorphic functions. Thus we have completed the proof of the theorem.

As a corollary to the theorem we obtain the following proposition.

PROPOSITION. *There are only a finite number of irreducible surfaces on a compact Kähler manifold M of dimension* 3 *which has no non-constant meromorphic function.*

PROOF. In case the irregularity of M is equal to 3, the proof of Proposition 6 shows that there are only a finite number of irreducible surfaces on

M. Let $\varphi: M \to T$ be an elliptic fibre space over a complex torus which has no non-constant meromorphic function and suppose that M has infinitely many irreducible surfaces. There lies an irreducible surface S on M such that $\varphi(S)=T$. Using the same notations as in the proof of Proposition 7, we infer readily that the fibre space $\varphi|(M-A): M-A \to T-\varphi(A)$ is an analytic fibre bundle. Its fibre is denoted by C. We take a small simply connected neighborhood U_i of a point a_i of $\varphi(A)$. We put $M_i=M|U_i$, $S_i=S \cap M_i$, $U_i'=U_i-a_i$, $M_i'=M|U_i'$, $S_i'=S \cap M_i'$. Each connected component of S_i' is an unramified covering of U_i', which is simply connected. Hence each irreducible component of S_i is bimeromorphically equivalent to U_i and consequently we have a holomorphic section to M_i' over U_i' which is extensible to a meromorphic mapping of U_i to M_i. Therefore we can prove in a similar manner to Kodaira [4] that there is a bimeromorphic map of M_i to $U_i \times C$. By means of this bimeromorphic map replacing M_i by $U_i \times C$ for each a_i, we obtain an elliptic fibre space $\varphi^*: M^* \to T$ which is an analytic fibre bundle. The threefold M^* is bimeromorphically equivalent to M. Let S^* be the irreducible surface on M^* corresponding to S. We put $\psi^*=\varphi^*|S^*$ and let $\psi^*: S^* \to T' \to T$ be the Stein factorization. It is easily seen that T' is an unramified covering of T. Clearly the induced bundle from M^* on T' has a holomorphic section and consequently it is isomorphic to a product bundle. Thus we see that M^* has non-constant meromorphic functions. The case where M is a fibre space of projective lines can be dealt with similarly. In case the irregularity of M vanishes, from the exact sequence

$$0 \longrightarrow Z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$$

we have the exact sequence

$$H^1(M, \mathcal{O}) \longrightarrow H(M, \mathcal{O}^*) \longrightarrow H^2(M, Z),$$

in which $H(M, \mathcal{O})=0$. Hence if there were an infinite sequence $S_1, S_2, \cdots, S_i, \cdots$ of irreducible surfaces on M, there would be integers $n_1, \cdots, n_k$, all of which did not vanish, and the divisor $\sum_{i=1}^{k} n_i S_i$ was linearly equivalent to zero. This is a contradiction, q. e. d.

As an example we prove the following

PROPOSITION. *Let $\varphi: M \to S$ be a projective line bundle on a $K3$ surface S which contains no irreducible curve. If M has non-constant meromorphic functions, then it is equivalent to a product bundle.*

PROOF. If M has non-constant meromorphic functions, there are infinitely many irreducible surfaces S_ν such that $\varphi(S_\nu)=S$. We shall prove that each S_ν is isomorphic to S and S_ν does not intersect S_μ for $\mu \neq \nu$. Let A be the set of singular points of S_ν and simple points where the restriction of φ to S_ν degenerates. By the assumption the image $\varphi(A)$ consists of points $\{a_i\}$ of a finite number. Since S is simply connected, $S-\varphi(A)$ is also simply connected and $S_\nu-A$ is isomorphic to $S-\varphi(A)$, for it is clear that $S_\nu-A$ is an unramified covering of $S-\varphi(A)$. Hence there is a meromorphic map s_ν of S to M which is the inverse of the restriction of φ to S_ν. The map s_ν is holomorphic on S over all. In fact if $s_\nu(a_i)$ is a point for a point a_i of $\varphi(A)$, clearly s_ν is holomorphic at a_i. Suppose that $s_\nu(a_i)$ is not a point. Then it must be a projective line and s_ν is a quadratic transformation in a neighborhood of a_i. Taking a small neighborhood U_i of a_i and letting P^1 be a projective line, we identify $M|U_i$ with $U_i\times P^1$. Then the surface $S_\nu|U_i$ is defined by the equation $\lambda^1x-\lambda^0y=0$, where (x, y) are appropriate coordinates with center a_i and (λ^0, λ^1) is a system of homogeneous coordinates of P^1. Now $S_\mu\cap S_\nu$ is a curve for $\mu\neq\nu$ and $\varphi(S_\mu\cap S_\nu)$ consists of a finite number of points to which the point a_i belongs. Therefore taking an appropriate system of coordinates (x_1, y_1) with center a_i, the surface $S_\mu|U_i$ is defined by the equation $\lambda^1x_1-\lambda^0y_1=0$. Consequently $\varphi(S_\mu\cap S_\nu)$ contains a curve defined by the equation $xy_1-x_1y=0$, which is a contradiction. Thus we see S_ν is isomorphic to S and there is a holomorphic section s_ν to M over S. We can prove in a similar manner that S_ν does not intersect S_μ for $\mu\neq\nu$. Once we have infinitely many holomorphic sections to M which do not intersect one another, it is clear that the projective line bundle $\varphi: M\to S$ is equivalent to the product bundle $S\times P^1$.

NOTE. In the following manner we see the existence of a projective line bundle which is not equivalent to a product bundle. From the exact sequence

$$(1)\longrightarrow C^*\longrightarrow GL(2)\longrightarrow PGL(1)\longrightarrow(1),$$

we have the exact sequence

$$H^1(S, \mathcal{O}^*)\longrightarrow H^1(S, GL(2))\xrightarrow{\ \rho\ } H^1(S, PGL(1)).$$

By a result of Kodaira [5] we have $H^1(S, \mathcal{O}^*)=0$ for a general $K3$ surface S. Let τ_S be the tangent bundle of S and let $\varphi: M\to S$ be the associated projective line bundle to $\rho(\tau_S)$. Then it is not equivalent to a product bundle, and M is Kählerian if S is Kählerian.

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