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Abstract homotopy neighborhoods and Hauptvermutung

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1. Introduction and statement of results.

In this note, we shall show some examples of non-simply-connected manifolds for which Hauptvermutung holds. Mitsuyoshi Kato introduced the concept of homotopy neighborhoods and proved the classification theorem ([1]). This concept is the basis for this note.

DEFINITION 1. Let P be a finite connected (simplicial) complex, then an abstract homotopy neighborhood M of P is a compact pl. manifold satisfying the following conditions:

1.) P is a subcomplex of M and contained in $\mathrm{Int}\, M$.
2.) (M, bM) is 2-connected.
3.) P is a deformation retract of M.

In the following, all manifolds are to be (orientable and) oriented and homeomorphisms are to be orientation preserving, we denote by $N(K, X)$ a regular neighborhood of a subcomplex K in a pl. manifold X, $\cong$ represents a pl. homeomorphism, and the Whitehead torsion of a homotopy equivalence $f: P \to Q$ will be denoted by $\tau(f)$ and considered as an element of $\mathrm{Wh}(\pi_1(P))$ as in [1].

Our results are as follows.

THEOREM 1. *Let M^n be an abstract homotopy neighborhood of a finite acyclic complex P^p, and M'^n a pl. manifold. Suppose $n \geqq 6$, $n \geqq 2p+2$ and there exists a homeomorphism $f: M^n \to M'^n$ with $\tau(f)=0$. Then there exists a pl. homeomorphism $g: M^n \to M'^n$ such that g is homotopic to f.*

COROLLARY 1. *Let M^n be an abstract homotopy neighborhood of a finite acyclic complex P^p of which 3-skelton P^3 is r-simple for $3 \leqq r < p$. If $n \geqq 6$, $n \geqq 2p+2$, then Hauptvermutung holds for M^n.*

Let M^n be a compact connected pl. manifold and let $\eta: k_{PL}(M) \to k_{TOP}(M)$ be the natural map.

THEOREM 2. *Let W^{n+k} be an abstract homotopy neighborhood of a connected closed pl. manifold M^n such that $\eta: k_{PL}(M) \to k_{TOP}(M)$ is injective and W'^{n+k} a pl. manifold. Suppose $k \geqq n+2$, $n+k \geqq 6$ and there exists a homeomorphism $f: W \to W'$ with $\tau(f)=0$. Then there exists a pl. homeomorphism $g: W$*

$\to W'$ *such that* g *is homotopic to* f.

COROLLARY 2. *Let* W^{n+k} *be an abstract homotopy neighborhood of a connected closed pl. manifold* M^n. *If* $n+k \geqq 6$, $k \geqq n+2$, *and* $\eta : k_{PL}(M) \to k_{TOP}(M)$ *is injective and* $\mathrm{Wh}(\pi_1(M))=0$, *then Hauptvermutung holds for* W^{n+k}.

COROLLARY 3. *Let* W^n *be an abstract homotopy neighborhood of a connected closed pl. manifold* M^p. *If* $n \geqq 6$, $p \leqq 3$ *and* $n \geqq 2p+2$, *then Hauptvermutung holds for* W^n. *In particular, Hauptvermutung holds for* $M^3 \times D^k$, *for* $k \geqq 5$.

THEOREM 3. *Let* W^{p+k} *be a pl.* π*-manifold such that* $\eta : k_{PL}(W) \to k_{TOP}(W)$ *is injective and an abstract homotopy neighborhood of a finite connected complex* P^p. *Suppose* $p+k \geqq 6$, $k \geqq p+2$ *and* W'^{p+k} *is a pl. manifold and there exists a homeomorphism* $f : W \to W'$ *with* $\tau(f)=0$. *Then there exists a pl. homeomorphism* $g : W \to W'$ *such that* g *is homotopic to* f.

Combining the recent result of D. Sullivan ([**2**]) and the theorem of B. Mazur ([**3**]), we can see that $\eta : k_{PL}(M) \to k_{TOP}(M)$ is injective provided $H_3(M : Z)$ has no 2-torsion. Thus we obtain the following corollaries of Theorems 2 and 3.

COROLLARY 4. *Let* W^{n+k} *be an abstract homotopy neighborhood of a connected closed pl. manifold* M^n. *If* $n+k \geqq 6$, $k \geqq n+2$ *and* $H_3(M : Z)$ *has no two torsion and* $\mathrm{Wh}(\pi_1(M))=0$, *then Hauptvermutung holds for* W^{n+k}.

COROLLARY 5. *Let* W^{p+k} *be a pl.* π*-manifold and an abstract homotopy neighborhood of a finite connected complex* P^p. *If* $p+k \geqq 6$, $k \geqq p+2$ *and* $H_3(W : Z)$ *has no two torsion and* $\mathrm{Wh}(\pi_1(W))=0$, *then Hauptvermutung holds for* W^{p+k}.

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2. Proof of Theorem 1.

We use the following theorem due to Mitsuyoshi Kato.

THEOREM K. *Let* M^n *and* M'^n *be abstract homotopy neighborhoods of* P^p *and* P'^p, *respectively. Let* $i : P^p \to M^n$ *and* $i' : P'^p \to M'^n$ *be natural inclusions. Suppose* $n \geqq 6$, $n-p \geqq 3$ *and there exist* $N(P, M)$ *and* $N(P', M')$ *and a pl. homeomorphism* $f : (N(P, M), P) \to (N(P', M'), P')$. *Then* f *can be extended to a pl. homeomorphism* $g : (M, P) \to (M', P')$ *if and only if* $\tau(i)$ *corresponds to* $\tau(i')$ *by the isomorphism of Whitehead groups induced by* $f | P$.

The proof of this theorem can be done by applying the s-cobordism theorem. For the proof, see [**1**].

PROOF OF THEOREM 1. According to T. Homma ([**4**]), since $n \geqq 2p+2$, f

is homotopic to a homeomorphism which, when restricted to P, is a pl. embedding into Int M'. Thus suppose that f itself has this property, and let $i: P \to M$ and $i': f(P) \to M'$ be inclusions. By Definition 1, M' is also an abstract homotopy neighborhood of $f(P)$. Let $f': P \to f(P)$ be a pl. homeomorphism obtained from $f|P$ by restricting the range to $f(P)$. By the hypothesis, $\tau(f)=0$. Since f' is an onto pl. homeomorphism, $\tau(f')=0$. Hence $f'_*\tau(i)=f'_*(i_*^{-1}\tau(f)+\tau(i))=f'_*\tau(f\circ i)=f'_*\tau(i'\circ f')=f'_*(f'^{-1}_*\tau(i')+\tau(f'))=\tau(i')$.

Since P is acyclic, we can see that bM and bM' are $(n-1)$-homology spheres by computing the homology groups. It is known that every $(n-1)$-homology sphere bounds the unique contractible manifold provided $n \geqq 5$. Let V^n and V'^n be the contractible manifolds bounded by bM and bM', respectively. Glueing V^n to M^n along their boundaries by the identity map, we obtain a closed manifold W^n. Let W'^n be a closed manifold obtained by glueing V'^n to M'^n. Since $n-p \geqq 3$, we can see that $\pi_1(bM)$ is isomorphic to $\pi_1(M)$ by general position argument. Then, it is not hard to see that W^n and W'^n are also n-homology spheres and that W^n and W'^n are simply-connected. Thus W^n and W'^n are n-spheres and M^n and M'^n can be embedded in S^n. Since $n \geqq 2p+2$, applying Gugenheim's theorem ([5]), we can extend f' to a pl. homeomorphism $h: S^n \to S^n$. Let $N(P, M)$ be a small regular neighborhood of P in M such that $h(N(P, M))$ is contained in Int M'. Clearly, $h(N(P, M))$ is a regular neighborhood of $f(P)$ in M'. Put $N(f(P), M')=h(N(P, M))$, then $N(P, M)$ is pl. homeomorphic to $N(f(P), M')$ by a pl. homeomorphism which extends f'.

Thus, by Theorem K, we can extend f' to a pl. homeomorphism $g: (M, P) \to (M', f(P))$. Since P and $f(P)$ are deformation retracts of M and M' respectively, g is homotopic to f. This proves the theorem.

Proof of Corollary 1. Let M' be a pl. manifold and $f: M \to M'$ a homeomorphism. Let $j: P^2 \to P^3$ be the natural inclusion. By the hypothesis, P^p is acyclic and P^3 is r-simple for $3 \leqq r < p$. Then by using the usual obstruction theory we can extend j to a map $\bar{j}: P^p \to P^3$. Let $\bar{r}: M^n \to P^3$ be the composition of the retraction $r: M^n \to P^p$ and $\bar{j}: P^p \to P^3$, then $\bar{r}$ induces the isomorphism of the fundamental groups.

According to Siebenmann [6], when M^n admits a map to a 3-complex inducing an isomorphism of fundamental groups, $\tau(f)$ vanishes. Therefore applying Theorem 1, we get a pl. homeomorphism $g: M \to M'$ such that g is homotopic to f, which completes the proof.

3. Proof of Theorem 2.

Proof of Theorem 2. Since $k \geqq n+2$, we can assume that $f|M: M \to W'$ is a pl. embedding. Let $i: M \to W$ and $i': f(M) \to W'$ be the natural inclusions and f' a pl. homeomorphism obtained from $f|M$ by restricting the range to $f(M)$. Note that W' is also an abstract homotopy neighborhood and that $\tau(i')=f'_*\tau(i)$. Therefore, by Theorem K, we need only show that there exist $N(M, W)$ and $N(f(M), W')$ which are pl. homeomorphic by a pl. homeomorphism which extends f'.

Since $k \geqq n+2$, there are normal pl. microbundles $\nu(i)$ and $\nu(i')$ unique up to isotopy ([7]). By the hypothesis, i and i' are topologically equivalent embeddings, hence $\eta(\{\nu(i)\})=f'^*\eta(\{\nu(i')\})$, where $\eta: k_{PL} \rightarrow k_{TOP}$ is a natural transformation and $\{\nu\}$ represents the stable equivalence class of ν. By the hypothesis η is injective, $\{\nu(i)\}=f'^*\{\nu(i')\}$ holds. Since $k \geqq n+2$, $\nu(i)$ is isomorphic to $\nu(i')$ as pl. microbundles ([7]). This means that there exist neighborhoods N of M in W and N' of $f(M)$ in W' and a pl. homeomorphism $\hat{f}:(N, M) \rightarrow (N', f(M))$ whose restriction to M is f'. This proves the theorem.

Corollary 2 is a direct consequence of Theorem 2.

Recall that B. Mazur proved the following theorem ([3]).

THEOREM M. *Let M^n be a compact pl. manifold. Then $\eta: k_{PL}(M) \rightarrow k_{TOP}(M)$ is injective if and only if the stable Hauptvermutung is true for M^n.*

PROOF OF COROLLARY 3. Since $p \leqq 3$, Hauptvermutung holds for M^p, hence $\eta: k_{PL}(M) \rightarrow k_{TOP}(M)$ is injective. M^p is a deformation retract of W^n, hence the Whitehead torsion of the homeomorphism of W^n vanishes by Siebenmann's result [6]. Applying Theorem 2, we get the result.

4. Proof of Theorem 3 and Corollaries 4, 5.

PROOF OF THEOREM 3. As in Proof of Theorem 2, we can assume that $f|P$ is pl. embedding and we need only show that there exist $N(P, W)$ and $N(f(P), W')$ which are pl. homeomorphic by a pl. homeomorphism which extends $f|P$.

Since $n \geqq 2p+2$ and W is a pl. π-manifold, we can get a p-connected manifold W_s by surgery on W. (For surgery on pl. manifolds, see [8].) By general position argument, we can assume $N(P, W)=N(P, W_s)$. Let D^n be a pl. n-disk in W_s. Since W_s is p-connected, $\pi_r(W_s, \operatorname{Int} D^n)=0$ for $r \leqq p$. Then applying Engulfing Theorem ([9]), we can assume $P \subset \operatorname{Int} D^n$. Therefore $N(P, W)=N(P, W_s) \cong N(P, \operatorname{Int} D^n)$.

Let τ_W and $\tau_{W'}$ are tangent pl. microbundles of W and W', respectively. Since f is a homeomorphism, $\eta(\{\tau_W\})=f^*\eta(\{\tau_{W'}\})=\eta(f^*\{\tau_{W'}\})$. By the hypothesis η is injective, hence $\{\tau_W\}=f^*\{\tau_{W'}\}$ holds. Since f^* is an isomorphism, $\tau_{W'}=0$, i.e. W' is also a pl. π-manifold. Therefore, by the same argument as above, we can see $N(f(P), W') \cong N(f(P), \operatorname{Int} D^n)$. Applying Gugenheim's theorem, we get a pl. homeomorphism $\hat{f}: N(P, W) \rightarrow N(f(P), W')$ which extends $f|P$. This proves the theorem.

PROOF OF COROLLARIES 3 AND 4. According to D. Sullivan [**2**], the stable Hauptvermutung is true for M^n, provided $H_3(M:Z)$ has no two torsion. Therefore $\eta: k_{PL}(M) \to k_{TOP}(M)$ is injective by Theorem M. Then we obtain directly Corollaries 3 and 4 from Theorems 2 and 3.

Added in proof. Recently, Kirby and Siebenmann have obtained a general solution of Hauptvermutung ([**10**]), which indicates most of our results.

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