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On the isometry groups of Sasakian manifolds

By Shûkichi TANNO

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§ 1. Introduction

The dimension of the isometry group of an m-dimensional Riemannian manifold (M, g) is equal to or smaller than $m(m+1)/2$. The maximum is attained if and only if (M, g) is of constant curvature and one of the following manifolds: a sphere S^m, a real projective space P^m, a Euclidean space E^m, and a hyperbolic space H^m (cf. S. Kobayashi and K. Nomizu [**5**], p. 308).

G. Fubini's theorem ([**2**], or [**1**]; p. 229) says that in an m-dimensional Riemannian manifold $(m>2)$ the dimension of the isometry group can not be equal to $m(m+1)/2-1$. Further, by H. C. Wang [**12**] and K. Yano [**13**] it was shown that in an m-dimensional Riemannian manifold $(m \neq 4)$, there exists no group of isometries of order s such that

$$m(m+1)/2 > s > m(m-1)/2+1 . \tag{1.1}$$

Riemannian manifolds admitting isometry groups of dimension $m(m-1)/2+1$ were studied by K. Yano [**13**], and the related subjects were studied by S. Ishihara [**4**], M. Obata [**7**], etc.

We consider similar problems in Sasakian manifolds. For a Sasakian manifold M with structure tensors (ϕ, ξ, η, g) we denote by $I(M)$ and $A(M)$ the group of isometries and the group of automorphisms. By $S^{2n+1}[H]$ for $H>-3$, $E^{2n+1}[-3]$, and $(L, CD^n)[H]$ for $H<-3$, we denote complete and simply connected Sasakian manifolds of $(2n+1)$-dimension with constant ϕ-holomorphic sectional curvature $H>-3$, -3, and $H<-3$, respectively (S. Tanno [**11**]). These Sasakian manifolds admit the automorphism groups of the maximum dimension $(n+1)^2$ (cf. S. Tanno [**10**]). By $F(t)$ we denote the cyclic group generated by $\exp t\xi$ for a real number t. Manifolds are assumed to be connected and structure tensors are assumed to be of class C^∞.

In this paper the main theorem is as follows:

THEOREM A. *Let (M, ϕ, ξ, η, g) be a complete Sasakian manifold of m-dimension, $m=2n+1$.*

(i) *If $\dim I(M)=(n+1)^2$, then (M, ϕ, ξ, η, g) is one of the following manifolds:*

(i—1) *a Sasakian manifold of constant curvature,*

(i—2) $S^m[H]/F(t_1)$ *for* $H>-3$ *and* $H\neq 1$,

(i—3) $E^m[-3]/F(t_2)$,

(i—4) $(L, CD^n)[H]/F(t_3)$ *for* $H<-3$.

(ii) *If* $\dim I(M)>(n+1)^2$, *then* (M, g) *is of constant curvature.*

If M is simply connected, we can give the complete classification of (M, ϕ, ξ, η, g) whose isometry group has dimension $\geqq (n+1)^2=(m+1)^2/4$, (M, g) being complete.

THEOREM A′. *Let* (M, ϕ, ξ, η, g) *be a complete and simply connected Sasakian manifold of m-dimension,* $m=2n+1$. *Then,*

(i) $\dim I(M)=(n+1)^2$, *if and only if* (M, ϕ, ξ, η, g) *is one of the following manifolds:*

(i—1) $S^m[H]$ *for* $H>-3$ *and* $H\neq 1$,

(i—2) $E^m[-3]$,

(i—3) $(L, CD^n)[H]$ *for* $H<-3$.

(ii) $\dim I(M)>(n+1)^2$, *if and only if* $\dim I(M)=(n+1)(2n+1)=m(m+1)/2$ *and* $(M, \phi, \xi, \eta, g)=S^m[1]$.

In Theorem A, Sasakian manifolds (i—2)$\sim$(i—4) have a property $\dim I(M) =\dim A(M)$. More precisely, we have $I(M)=A(M)\cup A'(M)$, where $A'(M)$ is composed of isometries φ satisfying $\varphi\xi=-\xi$.

COROLLARY. *Let* (M, g) *be a complete Riemannian manifold of m-dimension,* $m=2n+1$. *Assume that*

$$\dim I(M)>(n+1)^2=(m+1)^2/4\,.$$

Then (M, g) *is of constant curvature* 1, *if and only if* (M, g) *admits a Sasakian structure* (ϕ, ξ, η, g).

§ 2. Preliminaries

Let (M, g) be a Riemannian manifold with a fixed Riemannian metric g. Then a Sasakian structure (ϕ, ξ, η, g) on (M, g) is characterized by a unit Killing vector field ξ such that

$$\nabla_X(\nabla\xi)\cdot Y=g(\xi, Y)X-g(X, Y)\xi\,, \tag{2.1}$$

or

$$-R(X, \xi)Y=g(\xi, Y)X-g(X, Y)\xi\,, \tag{2.1$'$}$$

where X, Y are vector fields on M, ∇ is the Riemannian connection defined by g, and R is the Riemannian curvature tensor. ϕ and η are defined by $\phi=-\nabla\xi$ and $\eta(X)=g(\xi, X)$ (cf. [3], [8], [10], etc.). So we denote by (M, ξ, g) a Sasakian manifold and by ξ a Sasakian structure on (M, g).

If we have two Sasakian structures $\xi_{(1)}$ and $\xi_{(2)}$, which are orthogonal, namely $g(\xi_{(1)}, \xi_{(2)})=0$ on M, then we have the third Sasakian structure $\xi_{(3)}$:

$$\begin{aligned}\xi_{(3)} &= (1/2)[\xi_{(1)}, \xi_{(2)}] \\ &= \phi_{(1)}\xi_{(2)} = -\phi_{(2)}\xi_{(1)}\end{aligned} \tag{2.2}$$

such that $\xi_{(1)}$, $\xi_{(2)}$ and $\xi_{(3)}$ are mutually orthogonal. A set $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ is called a Sasakian 3-structure and in this case the dimension of M is $4r+3$ for some integer $r \geqq 0$ (Y. Y. Kuo [6]). They satisfy

$$\xi_{(1)} = (1/2)[\xi_{(2)}, \xi_{(3)}]\,, \tag{2.3}$$

$$\xi_{(2)} = (1/2)[\xi_{(3)}, \xi_{(1)}]\,, \tag{2.4}$$

and there is no Sasakian structure $\xi_{(4)}$ on (M, g), which is orthogonal to the above three (S. Tachibana and W. N. Yu [**9**]).

The following lemma is useful in our arguments.

LEMMA 2.1. (S. Tachibana and W. N. Yu [**9**]) *Let (M, g) be a complete and simply connected Riemannian manifold of m-dimension. If (M, g) admits two Sasakian structures ξ and ξ' with non-constant $g(\xi, \xi')$, then (M, g) is isometric with a unit sphere S^m.*

Assume that a Riemannian manifold (M, g) admits a Sasakian structure ξ such that the isometry group $I(M)=I(M, g)$ is different from the automorphism group $A(M)=A(M, \xi, g)$. Then we have an isometry φ which is not an automorphism of the Sasakian structure. If φ preserves ξ, then φ preserves $\nabla\xi=-\phi$ and η, and hence, φ is an automorphism. Therefore, denoting by the same letter φ its differential, we have $\varphi\xi \neq \xi$. We show that this unit Killing vector field $\varphi\xi$ defines a Sasakian structure on (M, g). Let p be an arbitrary point and let X, Y be arbitrary vector fields on M. Since φ is an isometry, it preserves the Riemannian curvature tensor:

$$R_{\varphi p}(\varphi X, \varphi\xi)\varphi Y = \varphi_p(R(X, \xi)Y)_p\,.$$

Consequently, by (2.1)$'$ we have

$$\begin{aligned}-R_{\varphi p}(\varphi X, \varphi\xi)\varphi Y &= \varphi_p(g_p(\xi, Y)X - g_p(X, Y)\xi) \\ &= g_p(\xi, Y)\varphi X - g_p(X, Y)\varphi\xi\,.\end{aligned}$$

Here we have

$$g_p(\xi, Y) = (\varphi^* g)_p(\xi, Y) = g_{\varphi p}(\varphi\xi, \varphi Y)\,,$$

and, therefore, we get

$$-R_{\varphi p}(\varphi X, \varphi\xi)\varphi Y = g_{\varphi p}(\varphi\xi, \varphi Y)\varphi X - g_{\varphi p}(\varphi X, \varphi Y)\varphi\xi\,.$$

This means that $\varphi\xi$ is a Sasakian structure.

In [**10**] we have classified almost contact Riemannian manifolds of $(2n+1)$-

dimension admitting the automorphism groups of the maximum dimension $(n+1)^2$. The classification only for Sasakian manifolds is as follows:

LEMMA 2.2. (S. Tanno [10], [11]) *Let* (M, ξ, g) *be a Sasakian manifold of* $(2n+1)$*-dimension. Then* $\dim A(M) \leqq (n+1)^2$. $\dim A(M)=(n+1)^2$ *holds, if and only if* (M, ξ, g) *has constant* ϕ*-holomorphic sectional curvature* H *and it is one of the followings:*

(1) $S^{2n+1}[H]/F(t_1)$ *for* $H>-3$, *where* $2\pi\cdot 4(H+3)^{-1}/t_1$ *is an integer,*

(2) $E^{2n+1}[-3]/F(t_2)$, *where* t_2 *is a real number,*

(3) $(L, CD^n)[H]/F(t_3)$ *for* $H<-3$, *where* t_3 *is a real number.*

§3. The case $\dim M=3$

First we have

PROPOSITION 3.1. *If a Riemannian manifold of* 3*-dimension admits a Sasakian* 3*-structure, then it is of constant curvature* 1.

PROOF. In a Sasakian manifold, sectional curvature $K(\xi, X)$ for a 2-plane which contains ξ is equal to 1 (cf. (2.1)′, or [3]). If (M, g) has a Sasakian 3-structure $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$, then

$$\xi=a\xi_{(1)}+b\xi_{(2)}+c\xi_{(3)}$$

for constant a, b, c satisfying $a^2+b^2+c^2=1$, is also a Sasakian structure and (M, g) is of constant curvature 1.

THEOREM 3.2. *Let* (M, ξ, g) *be a complete Sasakian manifold of* 3*-dimension.*

(i) *If* $\dim I(M)=4$, *then* (M, ξ, g) *is one of the followings:*

(i—1) *a Sasakian manifold of constant curvature,*

(i—2) $S^3[H]/F(t_1)$ *for* $H>-3$ *and* $H\neq 1$,

(i—3) $E^3[-3]/F(t_2)$,

(i—4) $(L, CD^1)[H]/F(t_3)$ *for* $H<-3$.

(ii) *If* $\dim I(M)>4$, *then* $\dim I(M)=6$ *and* (M, ξ, g) *is either* $S^3[1]$ *or* $P^3[1]=S^3[1]/F(\pi)$.

PROOF. Assume that $\dim I(M)=4$. Since $\dim A(M)\leqq 4$ by Lemma 2.2, we have either $\dim A(M)=\dim I(M)=4$ or $\dim A(M)<\dim I(M)$. The first case implies (i—1)$\sim$(i—4) by Lemma 2.2.

If $\dim A(M)<\dim I(M)$, we see that there is some isometry φ in the identity component of $I(M)$ which satisfies $\varphi\xi \neq \xi$ and $\varphi\xi\neq -\xi$. $\varphi\xi$ defines another Sasakian structure on (M, g). If $g(\xi, \varphi\xi)$ is not constant on M, we consider the universal covering manifold $(^*M, {}^*g)$ of (M, g), and naturally induced Sasakian structures $^*\xi$ and $^*(\varphi\xi)$. Then by Lemma 2.1, $(^*M, {}^*g)$ is isometric with a unit sphere. Hence, (M, g) is of constant curvature 1.

Next we assume that $g(\xi, \varphi\xi)=a$ is constant on M. Since $\varphi\xi\neq\xi$ and

$\varphi\xi \neq -\xi$, we have $|a|<1$. Then we have a Sasakian structure

$$\xi_{(2)} = -a\xi/\sqrt{1-a^2} + (\varphi\xi)/\sqrt{1-a^2}\,, \tag{3.1}$$

which is orthogonal to $\xi=\xi_{(1)}$, and hence (M, g) admits a Sasakian 3-structure by (2.2). By Proposition 3.1, (M, g) is of constant curvature. This is the case (i—1).

Finally assume that $\dim I(M)>4$. By a theorem of G. Fubini we have $\dim I(M)=6$. (M, g) is isometric to either S^3 or P^3. Therefore, we have either $(M, \xi, g)=S^3[1]$ or $(M, \xi, g)=P^3[1]=S^3[1]/F(\pi)$.

§ 4. The difference of $\dim I(M)$ and $\dim A(M)$

PROPOSITION 4.1. *Let (M, ξ, g) be a complete Sasakian manifold. Assume that* $\dim I(M)-\dim A(M)=1$. *Then (M, g) is of constant curvature* 1.

PROOF. Let $(A_1, \cdots, A_{\gamma-1}, A_\gamma=\xi)$ be a basis of the Lie algebra composed of infinitesimal automorphisms, where $\gamma=\dim A(M)$. Since we have some isometry φ satisfying $\varphi\xi\neq\xi$ and $\varphi\xi\neq-\xi$ (cf. proof of Theorem 3.2), we have another Sasakian structure $\varphi\xi$. If $g(\xi, \varphi\xi)$ is not constant on M, (M, g) is of constant curvature. If $g(\xi, \varphi\xi)$ is constant on M, then $\xi_{(2)}$ defined by (3.1) together with $\xi=\xi_{(1)}$, $\xi_{(3)}$ by (2.2), defines a Sasakian 3-structure. Since $\xi_{(2)}$ is not an infinitesimal automorphism of $\xi_{(1)}$, but a Killing vector field, we can consider

$$A_1, \cdots, A_{\gamma-1}, \xi_{(1)}, \xi_{(2)}$$

as a basis of the Lie algebra of Killing vector fields on (M, g). Thus, $\xi_{(3)}$ must be expressed in the form:

$$\xi_{(3)} = a_1A_1 + \cdots + a_{\gamma-1}A_{\gamma-1} + a\xi_{(1)} + b\xi_{(2)} \tag{4.1}$$

for some constant $a_1, \cdots, a_{\gamma-1}, a, b$. However, this implies that $\xi_{(3)}-b\xi_{(2)}$ is an infinitesimal automorphism of $\xi_{(1)}$. On the other hand, by (2.2) and (2.4), we have

$$[\xi_{(3)}-b\xi_{(2)}, \xi_{(1)}] = 2(\xi_{(2)}+b\xi_{(3)})\,,$$

which is a contradiction. Hence, only one possibility is that (M, g) is of constant curvature 1.

LEMMA 4.2. *Let (M, ξ, g) be a complete Sasakian manifold. Assume that* $\dim I(M)-\dim A(M)\geqq 2$. *Then either*

(i) *(M, g) is of constant curvature, or*

(ii) *(M, g) admits a Sasakian 3-structure ξ, $\xi_{(2)}$, $\xi_{(3)}$ and we have a basis of the Lie algebra of Killing vector fields:*

$$A_1, \cdots, A_{\gamma-1}, A_\gamma=\xi, X_1, \cdots, X_\beta, \xi_{(2)}, \xi_{(3)} \tag{4.2}$$

where $\beta=\dim I(M)-\dim A(M)-2$, *and* $(A_1, \cdots, A_\gamma)$ *is a basis of the Lie algebra of infinitesimal automorphisms of* ξ.

PROOF. By $\dim A(M)<\dim I(M)$, we see that either (M, g) is of constant curvature, or (M, g) admits a Sasakian 3-structure. So we consider the latter case. In the proof of the preceding Proposition, we have proved that $\xi_{(3)}$ can not be expressed in the form (4.1). Hence, $\xi_{(3)}$ is taken as one element of (4.2). If $\dim I(M)-\dim A(M)-2>0$, we can add β Killing vector fields $X_1, \cdots, X_\beta$ to get (4.2).

THEOREM 4.3. *Let* (M, ξ, g) *be a complete Sasakian manifold. If* $\dim I(M)$ $-\dim A(M)\geqq 3$, *then* (M, g) *is of constant curvature.*

PROOF. By Lemma 4.2, we may assume that we have a basis (4.2) of the Lie algebra of Killing vector fields on (M, g), where $\beta\geqq 1$. Let $X=X_1$. Since (M, g) is complete, X generates the 1-parameter group $\exp sX$, $-\infty<s<\infty$, of isometries of (M, g). Since $\exp sX\cdot\xi_{(1)}$ is a unit Killing vector field, we have some constant $a_1, \cdots, a_{\gamma-1}, a, b_1, \cdots, b_\beta, b, c$ depending on s such that

$$\begin{aligned}\exp sX\cdot\xi_{(1)}=a_1A_1+\cdots+a_{\gamma-1}A_{\gamma-1}+b_1X_1+\cdots+b_\beta X_\beta\\+a\xi_{(1)}+b\xi_{(2)}+c\xi_{(3)}\,.\end{aligned}$$

We devide our arguments in several steps.

(I) The case where there is some s so that at least one of $a_1, \cdots, a_{\gamma-1}$, $b_1, \cdots, b_\beta$ is non-zero. In this case we have a non-zero Killing vector field Y defined by

$$Y=\exp sX\cdot\xi_{(1)}-a\xi_{(1)}-b\xi_{(2)}-c\xi_{(3)}\,. \tag{4.3}$$

(I—1) First suppose that the inner products of $\exp sX\cdot\xi_{(1)}$ and $\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$ are all constant. Then $g(Y, Y)$ is a non-zero constant on M and $\xi_{[4]}$ defined by

$$\xi_{[4]}=Y/\sqrt{g(Y, Y)} \tag{4.4}$$

is a Sasakian structure on (M, g), because all $\exp sX\cdot\xi_{(1)}$, $\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$ are Sasakian structures and $\xi_{[4]}$ is of unit length. By our construction of $\xi_{[4]}$, the inner products of $\xi_{[4]}$ and $\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$ are constant.

(I—1—i) If $\xi_{[4]}$ belongs to the 3-dimensional distribution defined by $\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$, then $\xi_{[4]}$ or Y is a linear combination of $\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$ with real coefficients. So, $\exp sX\cdot\xi_{(1)}$ is of the form $a'\xi_{(1)}+b'\xi_{(2)}+c'\xi_{(3)}$. This is a contradiction.

(I—1—ii) If $\xi_{[4]}$ does not belong to the 3-dimensional distribution defined by $\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$, then by normalization of the following vector field:

$$\xi_{[4]}-g(\xi_{[4]}, \xi_{(1)})\xi_{(1)}-g(\xi_{[4]}, \xi_{(2)})\xi_{(2)}-g(\xi_{[4]}, \xi_{(3)})\xi_{(3)}$$

we have a Sasakian structure $\xi_{(4)}$, which is orthogonal to all $\xi_{(1)}$, $\xi_{(2)}$, $\xi_{(3)}$.

This is also a contradiction (cf. § 2).

(I—2) Suppose that at least one of the inner products of $\exp sX\cdot\xi_{(1)}$ and $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$ is not constant. Then, by Lemma 2.1 for the universal covering manifold, we see that (M, g) is of constant curvature.

(II) The case where, for any s, $-\infty < s < \infty$, we have

$$\exp sX\cdot\xi_{(1)}=a(s)\xi_{(1)}+b(s)\xi_{(2)}+c(s)\xi_{(3)} \tag{4.5}$$

where a, b, c depend only on s. We differentiate (4.5) with respect to s and get

$$[X, \xi_{(1)}]=A\xi_{(1)}+B\xi_{(2)}+C\xi_{(3)}, \tag{4.6}$$

where A, B, C are constant such that

$$A=-(\partial a(s)/\partial s)_0, \quad B=-(\partial b(s)/\partial s)_0, \quad C=-(\partial c(s)/\partial s)_0.$$

We show that $A=0$. In fact, we have

$$\begin{aligned} A&=g([X, \xi_{(1)}], \xi_{(1)})\\ &=L_X(g(\xi_{(1)}, \xi_{(1)}))-(L_X g)(\xi_{(1)}, \xi_{(1)})-g(\xi_{(1)}, [X, \xi_{(1)}])\\ &=0-0-A, \end{aligned}$$

where L_X is the Lie derivation with respect to X. Hence, we have

$$[X, \xi_{(1)}]=B\xi_{(2)}+C\xi_{(3)}. \tag{4.7}$$

Define a Killing vector field Z by

$$Z=X+(1/2)C\xi_{(2)}-(1/2)B\xi_{(3)}. \tag{4.8}$$

Then we have

$$[Z, \xi_{(1)}]=0 \tag{4.9}$$

by (2.2) and (2.4). Thus, Z is an infinitesimal automorphism of the Sasakian structure $\xi_{(1)}$, and it is written as

$$Z=\alpha_1 A_1+\cdots+\alpha_{r-1}A_{r-1}+\alpha\xi_{(1)} \tag{4.10}$$

for some constant $\alpha_1, \cdots, \alpha_{r-1}, \alpha$. By (4.8) and (4.10), we have

$$X=\alpha_1 A_1+\cdots+\alpha_{r-1}A_{r-1}+\alpha\xi_{(1)}-(1/2)C\xi_{(2)}+(1/2)B\xi_{(3)},$$

which contradicts the choice of the basis (4.2), since $X=X_1$. Thus, only one possibility is that (M, g) is of constant curvature.

THEOREM 4.4. *Let (M, ξ, g) be a complete Sasakian manifold which is not of constant curvature. Then, we have either*

(i) $\dim I(M)=\dim A(M)$ [$\rightleftarrows$ *admitting no Sasakian* 3-*structure*], *or*

(ii) $\dim I(M)=\dim A(M)+2$ [$\rightleftarrows$ *admitting a Sasakian* 3-*structure*].

PROOF. Assume that (M, g) admits no Sasakian 3-structure. If $\dim I(M) > \dim A(M)$, by Proposition 4.1, Lemma 4.2 and Theorem 4.3, (M, g) must be of constant curvature. This is a contradiction. Hence, we have $\dim I(M) = \dim A(M)$.

Assume that (M, g) admits a Sasakian 3-structure $(\xi'_{(1)}, \xi'_{(2)}, \xi'_{(3)})$. Since (M, g) is not of constant curvature, $g(\xi, \xi'_{(1)})$ must be constant. Then we can construct a Sasakian 3-structure $(\xi = \xi_{(1)}, \xi_{(2)}, \xi_{(3)})$. Hence, we have (ii); otherwise (M, g) is of constant curvature.

§ 5. The case $\dim I(M) = \dim A(M) + 2$

In this section we assume that a complete Sasakian manifold (M, ξ, g) is not of constant curvature and $\dim I(M) = \dim A(M) + 2$ holds. Then (M, g) admits a Sasakian 3-structure and $\dim M = 4r + 3$. By Lemma 4.2, we have a basis of Killing vector fields:

$$A_1, \cdots, A_{r-1}, A_r = \xi = \xi_{(1)}, \xi_{(2)}, \xi_{(3)}.$$

LEMMA 5.1. *Let f be an automorphism of ξ. Then we have a constant θ depending on f*

$$f\xi_{(2)} = \sin\theta\,\xi_{(2)} + \cos\theta\,\xi_{(3)}, \tag{5.1}$$

$$f\xi_{(3)} = \mp\cos\theta\,\xi_{(2)} \pm \sin\theta\,\xi_{(3)}. \tag{5.2}$$

PROOF. First we have $f\xi_{(1)} = \xi_{(1)}$. Since $f\xi_{(2)}$ is a Killing vector field, we have

$$f\xi_{(2)} = a_1A_1 + \cdots + a_{r-1}A_{r-1} + a\xi_{(1)} + b\xi_{(2)} + c\xi_{(3)} \tag{5.3}$$

for constant $a_1, \cdots, a_{r-1}, a, b$ and c. If at least one of $a_1, \cdots, a_{r-1}$ is not equal to zero, we see that (M, g) is of constant curvature, as in (I) of proof of Theorem 4.3. This contradicts the assumption. Hence, we get

$$f\xi_{(2)} = a\xi_{(1)} + b\xi_{(2)} + c\xi_{(3)}. \tag{5.4}$$

We show that $a = 0$. This is done by

$$\begin{aligned} a &= g(f\xi_{(2)}, \xi_{(1)}) = g(f\xi_{(2)}, f\xi_{(1)}) \\ &= (f^*g)(\xi_{(2)}, \xi_{(1)}) = g(\xi_{(2)}, \xi_{(1)}) = 0. \end{aligned}$$

Thus, $f\xi_{(2)} = b\xi_{(2)} + c\xi_{(3)}$. Since $f\xi_{(2)}$ is of unit length, b and c are replaced by $\sin\theta$ and $\cos\theta$. Similarly, we have $f\xi_{(3)} = b'\xi_{(2)} + c'\xi_{(3)}$. Then

$$\begin{aligned} g(f\xi_{(2)}, f\xi_{(3)}) &= g(b\xi_{(2)} + c\xi_{(3)}, b'\xi_{(2)} + c'\xi_{(3)}) \\ &= bb' + cc'. \end{aligned}$$

Since f is an isometry, we have $bb'+cc'=0$. Consequently, we have $b'=\mp\cos\theta$ and $c'=\pm\sin\theta$.

LEMMA 5.2. *Let p be a point in M. Then the isotropy group P of the automorphism group $A(M)$ at p is a subgroup of $1\times O(2)\times U(2r)$.*

PROOF. Let $\xi=\xi_{(1)}, \xi_{(2)}, \xi_{(3)}=\phi\xi_{(2)}$,

$$e_1, \cdots, e_{2r}, \qquad \phi e_1, \cdots, \phi e_{2r}$$

be a basis of the tangent space M_p at p. Let f be any element in P. Then we have $f\xi_{(1)}=\xi_{(1)}$ and (5.1), (5.2). This implies that f leaves the three subspaces V^1, V^2 and V^{4r} of M_p invariant, where V^1 is spanned by $\xi_{(1)}$, V^2 is spanned by $\xi_{(2)}$ and $\xi_{(3)}$, and $V^{4r}=V^{4r}(e_1, \cdots, e_{2r}, \phi e_1, \cdots, \phi e_{2r})$ is the orthogonal complement of V^1+V^2 in M_p. The action of f on V^1 is trivial. On V^2 it is an element of the othogonal group $O(2)$. On V^{4r} the action of f is expressed by an element of (the real representation of) the unitary group $U(2r)$.

THEOREM 5.3. *Let (M, ξ, g) be a complete Sasakian manifold which is not of constant curvature. If* $\dim I(M)=\dim A(M)+2$ *(or, equivalently, if (M, g) admits a Sasakian 3-structure), then we have*

$$\dim A(M)\leqq(2r+1)^2+3\,, \qquad m=4r+3\,. \tag{5.5}$$

PROOF. By Lemma 5.2, the dimension of the isotropy group P at p is equal to or smaller than $\dim O(2)+\dim U(2r)=1+(2r)^2$. The dimension of the subspace of M_p spanned by infinitesimal automorphisms is equal to or smaller than $\dim M=4r+3$. Therefore we have $\dim A(M)\leqq(2r)^2+1+(4r+3)$.

§ 6. The case where $I(M)=A(M)\cup A'(M)$

In a Sasakian manifold (M, ξ, g), we denote by $A'(M)$ the set of all isometries φ satisfying $\varphi\xi=-\xi$.

PROPOSITION 6.1. *Let (M, ξ, g) be one of the following Sasakian manifolds:*

$$S^m[H]/F(t_1) \qquad \text{for } H>-3 \text{ and } H\neq 1\,,$$

$$E^m[-3]/F(t_2)\,, \quad (L, CD^n)[H]/F(t_3) \qquad \text{for } H<-3\,.$$

Then we have $\dim I(M)=\dim A(M)$ *and* $I(M)=A(M)\cup A'(M)$.

PROOF. Since (M, g) is not of constant curvature, by Theorem 4.4, we have either $\dim I(M)=\dim A(M)$, or M admits a Sasakian 3-structure, which is assumed to be $(\xi, \xi_{(2)}, \xi_{(3)})$. In the latter case, we have $K(\xi_{(2)}, \xi_{(3)})=1$ by (2.1)$'$. However, $K(\xi_{(2)}, \xi_{(3)})=K(\xi_{(2)}, \phi\xi_{(2)})$; that is, it is ϕ-holomorphic sectional curvature ($=H\neq 1$). This is a contradiction. Hence, $\dim I(M)=\dim A(M)$.

Next we show that there is an isometry h such that $h\xi=-\xi$. Let $(^*M, {}^*\xi, {}^*g)$ be the universal covering manifold of (M, ξ, g) such that (M, ξ, g)

$=(*M, *\xi, *g)/F(t_0)$. Then $(*M, -*\xi, *g)$ is another Sasakian structure on $(*M, *g)$, which has constant $*(-\phi)$-holomorphic sectional curvature H, too. Hence, $(*M, *\xi, *g)$ and $(*M, *(-\xi), *g)$ are isomorphic, and we have an isometry $*h$ such that $*h*\xi=-*\xi$ (cf. Proposition 4.1 of [**11**]). $*h*\xi$ and $-*\xi$ generate the 1-parameter groups $*h^{-1}\cdot \exp t*\xi \cdot *h$ and $\exp(-t*\xi)$, respectively. Therefore we have

$$\exp t*\xi \cdot *h = *h \cdot \exp(-t*\xi)\,.$$

Let $[\ ]: *M \to M$ $(*p \to [*p]=p)$ be the projection. Define h by $hp=[*h*p]$. For $*p'$ such that $[*p']=p$, we have some integer z so that $*p'=\exp t_0 z*\xi \cdot *p$. Then

$$\begin{aligned}[][*h*p'] &= [*h \cdot \exp t_0 z*\xi \cdot *p] \\ &= [\exp(-t_0 z*\xi)\cdot *h \cdot *p] \\ &= [*h*p]\,.\end{aligned}$$

Therefore $*h$ induces a well defined h on (M, ξ, g). Clearly, h is an isometry and satisfies $h\xi=-\xi$.

Now, let φ be any isometry which is not an automorphism of (M, ξ, g). Then $\varphi\xi$ defines a Sasakian structure on (M, g) such that $\varphi\xi \neq \xi$. If $g(\xi, \varphi\xi)$ is not constant, (M, g) must be of constant curvature. This can not happen. Therefore $g(\xi, \varphi\xi)=a$ is constant, and we have either $|a|<1$ or $a=-1$. If $|a|<1$, we can construct a Sasakian 3-structure, and we must have $\dim I(M) \geqq \dim A(M)+2$.

This is a contradiction. Thus, we have $a=-1$ and $\varphi\xi=-\xi$. Then an isometry $h^{-1}\cdot\varphi$ satisfies $h^{-1}\cdot\varphi\xi=\xi$, which implies that $h^{-1}\cdot\varphi$ is an automorphism. Denoting this by f, we have $\varphi=hf$. This means that $I(M)=A(M) \cup A'(M)$, where $A'(M)=h\cdot A(M)=\{hf; f\in A(M)\}$.

§ 7. Theorems and corollaries

THEOREM 7.1. *Let (M, ξ, g) be a complete Sasakian manifold of m-dimension, $m=2n+1$.*

(i) *If* $\dim I(M)=(n+1)^2$, *then (M, ξ, g) is one of the followings:*

(i—1) *a Sasakian manifold of constant curvature,*

(i—2) $S^m[H]/F(t_1)$ *for* $H>-3$ *and* $H\neq 1$,

(i—3) $E^m[-3]/F(t_2)$,

(i—4) $(L, CD^n)[H]/F(t_3)$ *for* $H<-3$.

(ii) *If* $\dim I(M)>(n+1)^2$, *then (M, ξ, g) is of constant curvature* 1.

PROOF. For $m=3$, see Theorem 3.2. Suppose that $m=2n+1\geqq 5$. Assume that $\dim I(M)=(n+1)^2$. By Theorem 4.4, we have (i—1), or $\dim I(M)=\dim A(M)$, or $\dim I(M)=\dim A(M)+2$. If $\dim I(M)=\dim A(M)=(n+1)^2$, we have (i—1)$\sim$(i—4) by Lemma 2.2.

If $\dim I(M)=\dim A(M)+2$ and if (M, g) is not of constant curvature, then $\dim M=4r+3$ and we have $\dim A(M) \leqq (2r+1)^2+3$ by Theorem 6.3. Then

$$\dim I(M)=\dim A(M)+2 \leqq (2r+1)^2+5=(2r+2)^2-(4r-2)\,. \tag{7.1}$$

Since $r \geqq 1$, we have $\dim I(M)<(2r+2)^2=(n+1)^2$, which is a contradiction. This completes the proof of (i).

Next assume that $\dim I(M)>(n+1)^2$. Then (M, g) is of constant curvature, or $\dim I(M)=\dim A(M)+2$ by Theorem 4.4. If $\dim I(M)=\dim A(M)+2$ and if (M, g) is not of constant curvature, we have (7.1) as before. And we have a contradiction.

THEOREM 7.2. *Let (M, ξ, g) be a complete and simply connected Sasakian manifold of m-dimension, $m=2n+1$. Then,*

(i) $\dim I(M)=(n+1)^2$, *if and only if (M, ξ, g) is one of the followings:*

(i—1) $S^m[H]$ *for* $H>-3$ *and* $H \neq 1$,

(i—2) $E^m[-3]$,

(i—3) $(L, CD^n)[H]$ *for* $H<-3$.

(ii) $\dim I(M)>(n+1)^2$, *if and only if* $\dim I(M)=m(m+1)/2$ *and* (M, ξ, g) $=S^m[1]$.

PROOF. This follows from Proposition 6.1 and Theorem 7.1.

COROLLARY 7.3. *Let (M, g) be a complete Riemannian manifold of m-dimension, $m=2n+1$. Assume that*

$$\dim I(M)>(n+1)^2=(m+1)^2/4\,.$$

Then (M, g) is of constant curvature 1, *if and only if (M, g) admits a Sasakian structure (ξ, g).*

PROOF. This follows from Theorem 7.1 and (Proposition 5.1, [**11**]).

COROLLARY 7.4. *Let (M, ξ, g) be a complete Sasakian manifold of* $(4r+1)$-*dimension, which is not of constant curvature. Then* $\dim I(M)=\dim A(M)$.

PROOF. This follows from Theorem 4.4.

REMARK. Let (M, ξ, g) be a complete Sasakian manifold, which is not of constant curvature. Assume that (M, g) admits a Sasakian 3-structure and $\dim M=4r+3$. Then by Theorem 5.3, the dimension of the automorphism group $A(M)$ can not satisfy

$$(4r+3)(4r+4)/2 \geqq \dim A(M)>(2r+2)^2\,, \qquad \text{nor}$$

$$(2r+2)^2>\dim A(M)>(2r+1)^2+3\,.$$

Tôhoku University

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