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The central limit theorem for additive functionals of Markov processes and the weak convergence to Wiener measure

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The first aim of this paper is to disscuss the central limit theorem for additive functionals of conservative strong Feller processes on compact spaces. Secondly, as a refinement of the limit theorem, we shall consider a convergence theorem of measures on $C[0, T]$ formed by certain continuous additive functionals. Then the limit is the Wiener measure, that is, we shall deal with the so-called "invariance principle".

The central limit theorem of this type has been investigated by many authors. Fruitful results were obtained by S. V. Nagaev [**7**], I. S. Volkov [**11**], J. Keilson and D. M. G. Wishart [**5**], and others for discrete time Markov processes. In the case of continuous time Markov processes with finite state spaces, M. Fukushima and M. Hitsuda [**3**] gave the central limit theorem and some applications. Moreover, our central limit theorem is related to other types of limit theorems. In particular, it seems that the limit theorems for a stationary process under quite general conditions (Yu. A. Davydov, I. A. Ibragimov, M.I. Gordin, V.N. Solev [**1**]) are very close to our theorems, where some of our additive functionals can be considered as stationary processes.

The content of this paper:

In § 1, we shall give a basic lemma related to the Fourier transform of the semigroup, and state some results on the spectral theory of operators. In § 2, the central limit theorem (Theorem 2.1) will be established and we shall give the class of the exceptional additive functionals for which the "asymptotic variance" degenerates (Theorem 2.2). § 3 will be devoted to the proof of the invariance principle (Theorem 3.2.). Finally, we shall investigate the central limit theorem for additive vectors in § 4, where the results are analogous to the case of one-dimensional additive functionals.

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§ 1. Preliminaries

Let X be a compact metric space and $\mathfrak{B}_X$ a σ-algebra generated by the open subsets. Let $p(t, x, \Gamma)$, $x \in X$, $\Gamma \in B_X$ be a stochastic transition function. We denote by $C(X)$ (resp. $B(X)$) the space of all complex valued continuous functions on X (resp. all bounded measurable functions on X) with the supremum norm. We set

$$T_t f(x) = \int_X p(t, x, dy) f(y) \qquad \text{for any } f \in B(X).$$

We shall assume the following:

ASSUMPTION 1. (a) $\{T_t, t \geqq 0\}$ is a strongly continuous semigroup on $C(X)$.

(b) For any $f \in B(X)$ and $t > 0$, $T_t f$ belongs to $C(X)$ (Strong Feller property).

(c) $T_t 1 = 1$, for any $t \geqq 0$.

(d) For any $t > 0$, T_t is a completely continuous operator on $C(X)$.

(e) $p(t, x, \Gamma) > 0$, for any $t > 0$, $x \in X$ and non-empty open set Γ.

By Assumption 1, we have

PROPOSITION 1.1. *For any $t > 0$, the eigenvalues of T_t except 1 are less than 1 in absolute value, and the multiplicity of the eigenvalue 1 is one.*

PROOF. Suppose that the equality

$$\int_X p(t, x, dy) f(y) = \lambda f(x) \tag{1.1}$$

holds for some $f \in C(X)$ and $|\lambda| \geqq 1$. Then, we have

$$\begin{aligned} |f(x_0)| \leqq |\lambda|\,|f(x_0)| &\leqq \left| \int_X p(t, x_0, dy) f(y) \right| \\ &\leqq \int_X p(t, x_0, dy) |f(y)|, \end{aligned} \tag{1.2}$$

where $|f(x_0)| = \max_{x \in X} |f(x)|$. Hence, noting that $|f(x_0)| \geqq |f(y)|$, we get

$$\int_X p(t, x_0, dy) \{ |f(x_0)| - |f(y)| \} = 0. \tag{1.3}$$

By Assumption 1 (e), we see that $|f(x_0)| = |f(y)|$ holds for all $y \in X$. Here, assuming that $f(x)$ is not a constant, we easily have

$$|f(x_0)| > \left| \int_X p(t, x_0, dy) f(y) \right|,$$

which contradicts (1.2). Hence we have $f(x) = \text{constant}$. Thus we get $\lambda = 1$ and the multiplicity is one. The proof is now complete.

It is well-known that, under Assumption 1 (a) (c), there exists a conservative strong Feller process $(x_t, \mathfrak{F}_t, P_x, x\in X)$ associated with the transition function $p(t, x, \Gamma)$ (See for example [**2**]). A mapping $\varphi_t(\omega): [0, \infty)\times \Omega \to(-\infty, +\infty)$ is called an *additive functional*[1] of the process $(x_t, \mathfrak{F}_t, P_x, x\in X)$ if it satisfies the following properties:

(1.4) $$\text{for any } t\geqq 0,\ \varphi_t(\omega) \text{ is } \mathfrak{F}_t\text{-measurable;}$$

(1.5) $$P_x[\omega\,;\, \varphi_t(\omega)=\varphi_s(\omega)+\varphi_{t-s}(\theta_s\omega)]=1\,,$$

for any $x\in X$ and $t\geqq s$, where θ_s is the shift operator of $(x_t, \mathfrak{F}_t, P_x, x\in X)$.

In what follows, we shall consider the additive functionals which satisfy the following

ASSUMPTION 2. (a) There exists a positive number $\delta=\delta(t)$ such that

$$\sup_{x\in X} E_x[|\varphi_t(\omega)|^{2+\delta}]<+\infty \qquad \text{for any } t>0\,,$$

and

(b) $$\lim_{t\downarrow 0}\sup_{x\in X} E_x[|\varphi_t(\omega)|^2]=0\,.$$

Here are some typical examples of an additive functional satisfying Assumption 2:

(i) $a(x_t)-a(x_0)$,

(ii) $\int_0^t c(x_s)ds$,

(iii) $\int_0^t c(x_s)dB_s$, where B_s is a Brownian motion independent of x_s[2],

and

(iv) a linear combination of functionals in (i)—(iii), where $a(x)$, $b(x)$ and $c(x)$ are real valued continuous functions on X.

Now, let φ_t be an additive functional satisfying Assumption 2. For each $f\in C(X)$ and each real number z, we define

$$T_t^z f(x)=E_x[f(x_t)e^{iz\varphi_t(\omega)}]\,.$$

Then we get

LEMMA 1.1. (i) *$T_t^z f(x)\in C(X)$, $t\geqq 0$ and $z\in R^1$.*

(ii) *$\{T_t^z, t\geqq 0\}$ is a strongly continuous contraction semigroup on $C(X)$ for each $z\in R^1$.*

(iii) *For any z and $t>0$, T_t^z is a completely continuous operator on $C(X)$.*

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1) In this paper, the terminology *additive functional* means almost additive functional in ordinary sense.

2) If we choose some suitable σ-algebras $\mathfrak{F}_t$ of the process $(x_t, \mathfrak{F}_t, P_x, x\in X)$ and the shift operator θ_t, then we can consider for $\int_0^t c(x_s)dB_s$ to be an additive functional of the process.

(iv) *Define the operators*

$$A_t^z f(x) = E_x[i\varphi_t(\omega) f(x_t) e^{iz\varphi_t}],$$

and

$$B_t^z f(x) = E_x[-\varphi_t(\omega)^2 f(x_t) e^{iz\varphi_t}]$$

for each $f \in C(X)$, z *and* $t > 0$. *Then,* $A_t^z f$ *and* $B_t^z f$ *belong to* $C(X)$.

(v) (*Twice differentiability of* T_t^z *with respect to* z) *For any* $t \geqq 0$ *and* z,

$$\lim_{h \to 0} \left\| \frac{1}{h}(T_t^{z+h} - T_t^z) - A_t^z \right\| = 0$$

and

$$\lim_{h \to 0} \left\| \frac{1}{h}(A_t^{z+h} - A_t^z) - B_t^z \right\| = 0,$$

where $\|\cdot\|$ *denotes the operator norm on* $C(X)$.

PROOF. (i) For any positive $\varepsilon (< t)$ and for $f \in C(X)$, we set

$$T_t^{z,\varepsilon} f(x) = E_x[f(x_t) e^{iz(\varphi_t(\omega) - \varphi_\varepsilon(\omega))}].$$

Then we have

$$T_t^{z,\varepsilon} f(x) = E_x[f(x_{t-\varepsilon}(\theta_\varepsilon \omega)) e^{iz\varphi_{t-\varepsilon}(\theta_\varepsilon \omega)}] \tag{1.6}$$

$$= E_x[E_{x_\varepsilon}[f(x_{t-\varepsilon}) e^{iz\varphi_{t-\varepsilon}}]] = T_\varepsilon T_{t-\varepsilon}^z f(x).$$

Noting that $T_{t-\varepsilon}^z f$ is bounded and measurable, we see that $T_t^{z,\varepsilon} f \in C(X)$ by Assumption 1 (b) and (1.6). Now we have

$$|T_t^{z,\varepsilon} f(x) - T_t^z f(x)| \leqq E_x[|f(x_t)| \cdot |e^{-iz\varphi_\varepsilon(\omega)} - 1|]$$

$$\leqq \|f\| E_x[|z| |\varphi_\varepsilon(\omega)|] \leqq |z| \|f\| (E_x[\varphi_\varepsilon(\omega)^2])^{\frac{1}{2}}.$$

Hence, we get

$$\sup_{x \in X} |T_t^{z,\varepsilon} f(x) - T_t^z f(x)| \leqq |z| \|f\| (\sup_{x \in X} E_x[\varphi_\varepsilon(\omega)^2])^{\frac{1}{2}}. \tag{1.7}$$

Assumption 2 (b) and (1.7) imply

$$\lim_{\varepsilon \downarrow 0} \|T_t^{z,\varepsilon} f(x) - T_t^z f(x)\| = 0.$$

Therefore $T_t^z f(x) \in C(X)$.

(ii) From the inequalities

$$|T_t^z f(x) - f(x)| \leqq |T_t^z f(x) - T_t f(x)| + |T_t f(x) - f(x)|$$

$$\leqq E_x[|f(x_t)| \cdot |e^{iz\varphi_t} - 1|] + |T_t f(x) - f(x)|$$

$$\leqq |z| \|f\| (E_x[\varphi_t(\omega)^2])^{\frac{1}{2}} + |T_t f(x) - f(x)|$$

$$\leqq |z| \|f\| (\sup_x E_x[\varphi_t(\omega)^2])^{\frac{1}{2}} + \|T_t f - f\|,$$

we have

$$\lim_{t \downarrow 0} \|T_t^z f - f\| = 0 .$$

(iii) Because T_ε is completely continuous and $T_{t-\varepsilon}^z$, $t > \varepsilon > 0$, is bounded, we can derive from (1.6) that $T_t^{z,\varepsilon}$ is completely continuous. The inequality (1.7) implies

$$\lim_{\varepsilon \downarrow 0} \| T_t^{z,\varepsilon} - T_t^z \| = 0 .$$

Hence T_t^z is completely continuous.

(iv) Since we have

$$\begin{aligned}(1.8)\qquad & \sup_x \left| \frac{1}{h} \{ T_t^{z+h} f(x) - T_t^z f(x) \} - A_t^z f(x) \right| \\ & \leqq \sup_x E_x \left[|f(x_t) e^{iz\varphi_t}| \left| \frac{e^{ih\varphi_t} - 1}{h} - i\varphi_t \right| \right] \\ & \leqq |h| \|f\| \sup_x E_x(\varphi_t^2) \to 0 \qquad (\text{as } h \to 0),\end{aligned}$$

it is clear that $A_t^z f$ belongs to $C(X)$. Next, noting the inequality

$$\frac{|e^{ihx} - 1 - ihx|}{h^{1+\delta}} \leqq 3|x|^{1+\delta}, \qquad 0 < \delta < 1 ,$$

we have

$$\begin{aligned}(1.9)\qquad & \sup_x \left| \frac{1}{h} \{ A_t^{z+h} f(x) - A_t^z f(x) \} - B_t^z f(x) \right| \\ & = \sup_x \left| E_x \left[i\varphi_t f(x_t) e^{iz\varphi_t} \frac{e^{ih\varphi_t} - 1 - ih\varphi_t}{h} \right] \right| \\ & \leqq |h|^\delta \|f\| \sup_x E_x \left[|\varphi_t| \frac{|e^{ih\varphi_t} - 1 - ih\varphi_t|}{|h|^{1+\delta}} \right] \\ & \leqq 3|h|^\delta \|f\| \sup_x E_x [|\varphi_t|^{2+\delta}] .\end{aligned}$$

Since $A_t^{z+h} f$ and $A_t^z f \in C(X)$, we get $B_t^z f \in C(X)$.

(v) The results are obvious from (1.8) and (1.9).

Our results in the remainder of this section are essentially due to S. V. Nagaev [7] and V. N. Tutubalin [**10**]. So we will give only the outline of the proofs.

Let $\sigma(T_1^z)$ be the spectrum of the operator T_1^z and $\rho(T_1^z)$ the resolvent set of T_1^z. We define

$$R(\lambda, z) = (\lambda I - T_1^z)^{-1}, \qquad \lambda \in \rho(T_1^z),$$

where I is the identity operator.

By Assumption 1, the operator T_1 is completely continuous and has the simple eigenvalue 1. Hence, by Riesz-Shauder's theorem we see that there exists a positive number δ such that

$$\{\lambda;\ \lambda \text{ is complex, } |\lambda| \geqq 1-\delta \text{ and } \lambda \neq 1\} \subset \rho(T_1).$$

Moreover we see by the continuity of T_t^z with respect to z that there exists a neighbourhood (*nbd*) of $z=0$ such that for any z in the *nbd*

$$\{\lambda;\ \lambda \text{ is complex, } |\lambda| \geqq 1-\delta \text{ and } |1-\lambda| \geqq \delta\} \subset \rho(T_1^z).$$

Let I_1 be the circle with the center 1 and the radius δ. Then the image of the operator

$$P(z) = \frac{1}{2\pi i} \oint_{I_1} R(\lambda, z) d\lambda$$

is one-dimensional. With this δ we have

LEMMA 1.2. *For any real z in some nbd of $z=0$, the operator T_1^z has the unique eigenvalue $\lambda(z)$ such that $|\lambda(z)-1| < \delta$. Furthermore, the $\lambda(z)$ is simple and it has the maximum absolute value in $\sigma(T_1^z)$.*

We fix a point x_0 of X. When z belongs to the *nbd*, we denote by e_z the eigenfunction of T_1^z corresponding to $\lambda(z)$ which satisfies $e_z(x_0)=1$. We denote by ν_z the eigenvector of the operator $(T_1^z)^*$ on $C(X)^*$ [3] corresponding to $\overline{\lambda(z)}$ which satisfies $(e_z, \nu_z)=1$. We notice that $e_0(x) \equiv 1$.

THEOREM 1.2. (i) *There exists a nbd of $z=0$ such that for any z in the nbd*

$$T_n^z f(x) = \lambda^n(z)(f, \nu_z)e_z(x) + Q(z)^n f(x), \tag{1.10}$$

holds for any positive integer n and f in $C(X)$, where $Q(z)$ is a bounded operator on $C(X)$ such that $Q(z)e_z = Q(z)^*\nu_z = 0$ and that $\lim_{n\to\infty} \|Q(z)^n\| = 0$ uniformly in z.

(ii) *$\lambda(z)$ is a C^2-class function in the nbd of $z=0$. e_z and ν_z are of C^2-class in the sense of the norm of the space $C(x)$ and $C(X)^*$, respectively. $Q(z)$ is of C^2-class in the sense of the operator norm.*

(iii) *$\lambda'(0)$ is purely imaginary, $-\lambda''(0)+\lambda'(0)^2 \geqq 0$ and*

$$E_\nu(\varphi_n^2) - E_\nu(\varphi_n)^2 = n(-\lambda''(0)+\lambda'(0)^2) + O(1), \tag{1.11}$$

as $n \to \infty$, where $\nu = \nu_0$.

From this theorem, it is easy to see

COROLLARY. *$\nu = \nu_0$ is the unique positive invariant measure for $\{T_t, t \geqq 0\}$ and $\nu(\Gamma) > 0$ holds for any open set $\Gamma \neq \phi$.*

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3) $C(X)^*$ is the dual space of $C(X)$ and T^* is the dual operator of T.

§ 2. The central limit theorem

In this section, we will first prove

THEOREM 2.1. (The central limit theorem) *Set*

$$F(t, x, dy) = P_x\Big[\frac{1}{\sqrt{t}}(\varphi_t(\omega) - \frac{t}{i}\lambda'(0)) \in dy\Big],$$

then the equality,

$$\lim_{t\to\infty}\int_{R^1} e^{iyz}F(t, x, dy) = e^{-\frac{1}{2}(-\lambda''(0)+\lambda'(0)^2)z^2} \tag{2.1}$$

holds, and the convergence is uniform in x.

For the proof, we need the following lemma.

LEMMA 2.1.

$$\lim_{n\to\infty}\int_{R^1} e^{iyz}F(n, x, dy) = e^{-\frac{1}{2}(-\lambda''(0)+\lambda'(0)^2)z^2} \tag{2.2}$$

holds, and the convergence is uniform in x.

PROOF. By virtue of (1.10) and

$$\int_{R^1} e^{iyz}F(t, x, dy) = e^{-\frac{tz\lambda'(0)}{\sqrt{t}}} T_t^{\frac{z}{\sqrt{t}}} 1(x), \tag{2.3}$$

we have

$$\int_{R^1} e^{iyz}F(n, x, dy) = \Big(e^{-\frac{z\lambda'(0)}{\sqrt{n}}}\Big)^n \Big\{\lambda\Big(\frac{z}{\sqrt{n}}\Big)^n \Big(1, \nu_{\frac{z}{\sqrt{n}}}\Big) e_{\frac{z}{\sqrt{n}}} x + Q\Big(\frac{z}{\sqrt{n}}\Big)^n 1(x)\Big\}. \tag{2.4}$$

From Theorem 1.2, we get

$$(1, \nu_{\frac{z}{\sqrt{n}}}) \to 1, \tag{2.5}$$

$$e_{\frac{z}{\sqrt{n}}}(x) \to 1, \quad \text{uniformly in } x, \tag{2.6}$$

and

$$Q\Big(\frac{z}{\sqrt{n}}\Big)^n 1(x) \to 0, \quad \text{uniformly in } x, \text{ as } n\to\infty. \tag{2.7}$$

Noting that $\lambda(z)$ is of C^2-class, we see by a simple calculation

$$\lim_{n\to\infty} (e^{-\frac{z\lambda'(0)}{\sqrt{n}}})^n \lambda\Big(\frac{z}{\sqrt{n}}\Big)^n = e^{-\frac{1}{2}(-\lambda''(0)+\lambda'(0)^2)z^2}. \tag{2.8}$$

On the other hand,

$$(e^{-\frac{z\lambda'(0)}{\sqrt{n}}})^n Q\Big(\frac{z}{\sqrt{n}}\Big)^n 1(x) \to 0, \quad \text{uniformly in } x, \tag{2.9}$$

because of the boundedness of $(e^{-\frac{z\lambda'(0)}{\sqrt{n}}})^n$ and (2.7). Hence, we have the lemma.

PROOF OF THEOREM 2.1. For each $t > 0$, we put $n = n(t)$ is the maximal integer less than t. Then we have,

$$\left|\int_{R^1} e^{iyz}F(t, x, dy) - \int_{R^1} e^{iyz}F(n, x, dy)\right|$$

$$= \left|E_x\left[\exp\left\{iz\frac{1}{\sqrt{t}}\left(\varphi_t - \frac{t}{i}\lambda'(0)\right)\right\} - \exp\left\{iz\frac{1}{\sqrt{n}}\left(\varphi_n - \frac{n}{i}\lambda'(0)\right)\right\}\right]\right|$$

$$\leqq \left|E_x\left[\exp\left\{iz\frac{i}{\sqrt{t}}\left(\varphi_t - \frac{t}{i}\lambda'(0)\right)\right\} - \exp\left\{iz\frac{1}{\sqrt{t}}\left(\varphi_n - \frac{n}{i}\lambda'(0)\right)\right\}\right]\right|$$

$$+ \left|E_x\left[\exp\left\{iz\frac{1}{\sqrt{t}}\left(\varphi_n - \frac{n}{i}\lambda'(0)\right)\right\} - \exp\left\{iz\frac{1}{\sqrt{n}}\left(\varphi_n - \frac{n}{i}\lambda'(0)\right)\right\}\right]\right|$$

$$\leqq E_x\left[\left|\exp\left\{iz\left(\frac{1}{\sqrt{t}}(\varphi_t - \varphi_n) + \frac{n-t}{\sqrt{t}}\lambda'(0)\right)\right\} - 1\right|\right]$$

$$+ E_x\left[\left|\exp\left\{iz\left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{n}}\right)\left(\varphi_n - \frac{n}{i}\lambda'(0)\right)\right\} - 1\right|\right]$$

$$\leqq \frac{1}{\sqrt{t}}|z|(E_x[|\varphi_{t-n}(\theta_n\omega)|] + |\lambda'(0)|)$$

$$+ \frac{\sqrt{t}-\sqrt{n}}{\sqrt{t}}|z|\frac{1}{\sqrt{n}}E_x\left[\left|\varphi_n - \frac{n}{i}\lambda'(0)\right|\right]$$

$$(\text{because } \varphi_t(\omega) = \varphi_n(\omega) + \varphi_{t-n}(\theta_n(\omega)))$$

$$\leqq \frac{1}{\sqrt{t}}|z|(\sup_x E_x[E_{x_n}(|\varphi_{t-n}|)] + \lambda'(0))$$

$$+ \frac{t-n}{\sqrt{t}}|z|\frac{1}{\sqrt{n}}\left[\sup_x E_x\left[\left|\varphi_n - \frac{n}{i}\lambda'(0)\right|^2\right]^{\frac{1}{2}}\right].$$

The last two terms of the above inequalities converge to zero uniformly in x as $t \to \infty$. In fact, noting that

$$\sup_x E_x[E_{x_n}(|\varphi_{t-n}|)] \leqq \sup_{\substack{x \\ 0 \leqq t \leqq 1}} E_x[\varphi_t^2] < +\infty$$

from Assumption 2, we see that the first term converges to zero. For the second term, by differentiating twice the both sides of (1.10) at $z=0$ and putting $f=1$, we have

$$\frac{1}{\sqrt{n}}\sup_x E_x\left[\left|\varphi_n - \frac{n}{i}\lambda'(0)\right|^2\right]^{\frac{1}{2}} < +\infty.$$

Therefore the second term also converges to zero. Thus the proof is complete.

Next, we shall determine the class of additive functionals for which the equality

$$-\lambda''(0)+\lambda'(0)^2=0 \tag{2.11}$$

holds. (2.11) means that the "asymptotic variance" of $\varphi_t/\sqrt{t}$ degenerates.

THEOREM 2.2. *In order that the equality* (2.11) *holds, it is necessary and sufficient that there exist a real valued continuous function $a(x)$ on X and a real number γ such that*

$$P_x[\omega\,;\ \varphi_t(\omega)=a(x_t)-a(x_0)+\gamma t]=1\,,\qquad \text{for any } t\geqq 0 \tag{2.12}$$

and $x\in X$.

First, we prove

LEMMA 2.2. (i) *The equality* (2.11) *is equivalent to*

$$\sup_n \{E_\nu(\varphi_n^2)-E_\nu(\varphi_n)^2\}<+\infty\,. \tag{2.13}$$

(ii) *If* (2.11) *holds, then* $|\lambda(z)|=1$ *in some nbd of* $z=0$.

PROOF. (i) is clear from Theorem 1.2 (iii).

(ii) Set $\psi_t(\omega)=\varphi_t(\omega)-E_\nu[\varphi_t(\omega)]$ and $M=\sup_n E_\nu[\psi_n(\omega)^2]$, then M is finite by (2.13). Since

$$P[|\psi_n|>a]\leqq\frac{1}{a^2}E_\nu[\psi_n(\omega)^2]\leqq\frac{M}{a^2}\,,$$

for any $\varepsilon>0$, we can choose a such that

$$P_\nu[|\psi_n|>a]<\varepsilon \qquad \text{for any } n\,. \tag{2.14}$$

We fix such a for $\varepsilon=\frac{1}{4}$. Then we have

$$\begin{aligned}|1-E_\nu[e^{iz\psi_n}]|&\leqq E_\nu(|1-e^{iz\psi_n}|)\\ &=E_\nu[|1-e^{iz\psi_n}|\,;\ |\psi_n|\leqq a]+E_\nu[|1-e^{iz\psi_n}|\,;\ |\psi_n|>a]\\ &\leqq|z|a+\frac{1}{2}\,.\end{aligned}$$

Hence, if $|z|<\frac{1}{4a}$, we have $|1-E_\nu(e^{iz\psi_n})|<\frac{3}{4}$. Therefore

$$|E_\nu(e^{iz\psi_n})|>\frac{1}{4}\qquad \text{for any } n\,.$$

Hence, if z is near by zero,

$$|E_\nu(e^{iz\varphi_n})|=|E_\nu[e^{iz\psi_n}]e^{izE_\nu(\varphi_n)}|=|E_\nu[e^{iz\psi_n}]|>\frac{1}{4}\qquad \text{for all } n\,. \tag{2.15}$$

On the other hand, assume that there exists z_1 such that $z=z_1$ satisfies (2.15) and $|\lambda(z_1)|<1$. By virtue of (1.10), we have

$$E[e^{iz_1\varphi_n}]=\lambda^n(z_1)(1,\nu(z_1))(e(z_1),\nu)+(Q^n(z_1)1,\nu)\to 0\qquad (\text{as } n\to\infty)\,.$$

This contradicts the inequality (2.15). Thus, the proof is complete.

PROOF OF THEOREM 2.2. The sufficiency is obvious. Hence, we prove that the condition (2.12) is necessary. Suppose the equality (2.11) holds. Then, by Lemma 2.2, $|\lambda(z)|=1$ in some *nbd* of $z=0$. From this fact, we see that there exists a unique real continuous function $\gamma(z)$ such that

$$e^{i\gamma(z)}=\lambda(z) \tag{2.16}$$

and

$$T_t^z e_z=e^{i\gamma(z)t}e_z \tag{2.17}$$

hold for any $t>0$, where e_z is the eigenfunction as in § 1. From (2.17), we have

$$|e_z(x)|=|T_t^z e_z(x)|\leqq E_x(|e_z(x_t)|)=T_t|e_z|(x). \tag{2.18}$$

On the other hand, noting that $\nu=\nu_0$ is the invariant measure, we see that

$$\int_X |e_z(x)|\nu(dx)=\int_X T_t|e_z|(x)\nu(dx). \tag{2.19}$$

From (2.18) and (2.19), we have

$$|e_z(x)|=T_t|e_z|(x) \quad \text{a.e. } \nu(dx). \tag{2.20}$$

By the continuity of both sides of (2.20) and Corollary to Theorem 1.2, we get

$$|e_z(x)|=T_t|e_z|(x) \quad \text{for any } x\in X \text{ and } t>0. \tag{2.21}$$

From Proposition 1.1 and (2.21), it follows that

$$|e_z(x)|=1 \quad \text{for any } x\in X. \tag{2.22}$$

By Theorem 1.2 (ii),

$$\|e_z(x)-1\|<1 \quad \text{in some } nbd \text{ of } z=0. \tag{2.23}$$

Since $e_z(x)$ is a continuous function of (z, x), it follows from (2.22) and (2.23) that there exists a function $a_z(x)$ continuous in (z, x) such that

$$e_z(x)=e^{-ia_z(x)}, \quad a_0(x)\equiv 0 \quad \text{and} \quad |a_z(x)|<\frac{\pi}{3} \tag{2.24}$$

in the *nbd* of $z=0$.
By (2.17) and (2.24), we have

$$E_x[e^{i\{-a_z(x_t)+z\varphi_t(\omega)\}}]=e^{i\{-a_z(x)+\gamma(z)t\}}. \tag{2.25}$$

Since the right-hand side and the integrand of the left-hand side of the equality (2.25) are equal to 1 in absolute value, we have

$$e^{i\{-a_z(x_t)+z\varphi_t(\omega)\}}=e^{i\{-a_z(x)+\gamma(z)t\}} \quad \text{a.e. } P_x, \text{ for any } x\in X. \tag{2.26}$$

The exceptional set does not depend on z because of the continuity of $a_z(x)$ and $\gamma(z)$. Hence we have

$$P_x[\omega;\ -a_z(x_t)+z\varphi_t(\omega)=-a_z(x_0)+\gamma(z)t \text{ in some } nbd \text{ of } z=0]=1, \tag{2.27}$$

for any $t\geqq 0$ and $x\in X$.
If z, z' and $z+z'$ belong to the nbd, we have from (2.26)

$$e^{-i\{a_z(x_t)+a_{z'}(x_t)\}}e^{i(z+z')\varphi_t(\omega)}=e^{-i(a_z(x)+a_{z'}(x))}e^{i\{\gamma(z)+\gamma(z')\}} \quad \text{a. e. } P_x, \text{ for any } x\in X. \tag{2.28}$$

Integrating (2.28), we get

$$T_t^{z+z'}(e^{-ia_z(\cdot)+a_{z'}(\cdot)})(x)=e^{-i\{a_z(x)+a_{z'}(x)\}}e^{i\{\gamma(z)+\gamma(z')\}t}. \tag{2.29}$$

On the other hand, $T_t^{z+z'}$ has a unique eigenvalue which is equal to 1 in absolute value and which is simple (Lemma 1.2), we have

$$e^{-i\{a_z(x)+a_{z'}(x)\}}=e^{-ia_{z+z'}(x)} \quad \text{for any } x\in X \tag{2.30}$$

and

$$e^{i\{\gamma(z)+\gamma(z')\}t}=e^{i\gamma(z+z')t} \quad \text{for any } t\geqq 0. \tag{2.31}$$

Because $|a_z(x)+a_{z'}(x)|<\frac{2}{3}\pi$ and $|a_{z+z'}(x)|<\frac{\pi}{3}$, we obtain

$$a_z(x)+a_{z'}(x)=a_{z+z'}(x) \quad \text{for any } x\in X \tag{2.32}$$

and

$$\gamma(z)+\gamma(z')=\gamma(z+z'). \tag{2.33}$$

Because $a_z(x)$ and $\gamma(z)$ are continuous functions of z, it follows from (2.32) and (2.33) that there exist a real valued continuous function $a(x)$ and a real number γ such that

$$a_z(x)=za(x) \quad \text{for any } x\in X \tag{2.34}$$

and

$$\gamma(z)=\gamma z. \tag{2.35}$$

By (2.27), (2.34) and (2.35), we obtain the theorem.

§ 3. Convergence on the continuous path space $C[0, T]$. (Invariance principle)

In this section, we shall give a more detailed result than in § 2 for additive functionals of a certain type. We consider only the additive functionals φ_t of type (ii) and (iii) in the example of § 1, or their linear combinations. We can assume $E_\nu(\varphi_t)=\frac{t}{i}\lambda'(0)=0$ for each t, whithout loss of generality. Of course, these additive functionals are continuous, so each random process

$$\varphi_t^{(k)}=\frac{\varphi_{kt}}{\sqrt{k\sigma^2}},\quad t\in[0,T],\quad (k=1,2,\cdots),\quad \sigma=(-\lambda''(0))^{\frac{1}{2}}$$

induces the measures μ_x^k from P_x on the space of continuous paths $C[0,T]$.

We will show that the system of measures μ_x^k for each $x\in X$ is relatively compact and μ_x^k converges to Wiener measure μ_W on $C[0,T]$. The next lemma is easy.

LEMMA 3.1. *If the additive functional* $\varphi_t(\omega)$ *has the type* (ii), (iii) *or their linear combinations, then for* $\delta>0$, $\sup_x E_x[\varphi_1^{4+\delta}]<+\infty$, *and* $E_x[\varphi_1^4]$ *belongs to* $C(X)$.

LEMMA 3.2. *Let* $\varphi_t(\omega)$ *be as in Lemma* 3.1 *and* n *be any positive integer. Then* $E_x[\varphi_n^4]\leqq C_2n^2$ *for some constant* $C_2>0$, *where we can choose* C_2 *independent of* $x\in X$.

PROOF. Put $f=1$ in (1.8) then we have

$$E_x[\exp iz\varphi_n]=\lambda(z)^n(1,\nu_z)e_z(x)+Q(z)^n1(x)\,. \tag{3.1}$$

While it is easy to prove that $\lambda(z)$, $Q(z)$, e_z and ν_z which appear in (3.1) are of C^4-class. Thus, differentiating four times the both sides of (3.1) and putting $z=0$, we get

$$\begin{aligned}E_x[\varphi_n^4]=&(\lambda(z)^n)^{(4)}|_{z=0}+(1,\nu_0^{(4)})+e_0^{(4)}(x)+4(1,\nu_0^3)e_0^{(1)}(x)\\&+4(\lambda(z)^n)^{(3)}(1,\nu_0^{(1)})+4(1,\nu_0^{(1)})e_0^{(3)}(x)+4(\lambda(z)^n)^{(3)}e_0^{(1)}(x)\\&+4(1,\nu_0^{(3)})e_0^{(1)}(x)+4(1,\nu_0^{(1)})e_0^{(3)}(x)+6(\lambda(z)^n)^2|_{z=0}(1,\nu_0^{(2)})\\&+6(\lambda(z)^n)^{(2)}|_{z=0}e_0^{(2)}(x)+6(1,\nu_0^{(2)})e_0^{(2)}(x)\\&+12(\lambda(z)^n)^{(2)}|_{z=0}(1,\nu_0^{(1)})e_0^{(1)}(x)+(Q(z)^n1(x))^{(4)}|_{z=0}\,.\end{aligned} \tag{3.2}$$

On the other hand, we have

$$(\lambda(z)^n)^{(2)}|_{z=0}=n\lambda''(0)\,. \tag{3.3}$$

Moreover, we get

$$\begin{aligned}(\lambda(z)^n)^{(3)}=&n\lambda^{(3)}(z)\lambda(z)^{n-1}+3n(n-1)\lambda^{(2)}(z)\lambda^{(1)}(z)\lambda(z)^{n-2}\\&+n(n-1)(n-2)\lambda^{(1)}(z)^3\lambda(z)^{n-2}+6n(n-1)(n-2)\lambda^{(2)}(z)\lambda^{(1)}(z)\lambda(z)^{n-3}\\&+n(n-1)(n-2)(n-3)(\lambda^{(1)}(z))^4\lambda(z)^{n-4}\,.\end{aligned}$$

Then, putting $z=0$,

$$(\lambda(z)^n)^{(3)}|_{z=0}=n\lambda^{(3)}(0)\,, \tag{3.4}$$

$$(\lambda(z)^n)^{(4)}|_{z=0} = n\lambda^{(4)}(0) + 3n(n-1)(\lambda^{(2)}(0))^2, \tag{3.5}$$

because $\lambda'(0) = iE_\nu(\varphi_1) = 0$. Thus, the terms of the order n^3 and n^4 in the right hand of (3.2) vanish. While, in the right hand of (3.2), the derivatives $e_0^i(x)$ $(i = 1, 2, 3, 4)$ at $z = 0$, are continuous and bounded in $x \in X$. Moreover $(Q(z)^n 1(x))^{(4)}|_{z=0}$ tends to zero as $n \to \infty$, because $\lim_{n\to\infty} \|Q(0)^n\| = 0$ (See § 1). Thus Lemma 3.2 is proved.

LEMMA 3.3. *For each* $t \geqq 1$, *the inequality*

$$E_x[\varphi_t^4] \leqq C_3 t^2$$

holds for some constant $C_3 > 0$ *independent of* $x \in X$.

PROOF. Let $n \leqq t < n+1$, then

$$E_x[\varphi_t^4] \leqq 4E_x[\varphi_n^4 + \varphi_{t-n}(\theta_n\omega)^4] \leqq 4\{E_x[\varphi_n^4] + E_x[E_{x_n}(\varphi_{t-n}^4)]\}.$$

The result is easily derived from Lemma 3.2, because φ_t satisfies $\sup_{0\leqq h\leqq 1} \sup_x E_x[\varphi_h^4] < \infty$.

LEMMA 3.4. *For* $0 \leqq t < 1$, *the inequality*

$$E_x[\varphi_h^4] \leqq C_4 t^2 \tag{3.6}$$

holds for some C_4 *independent of* $x \in X$.

PROOF. This inequality is easily derived from the expression of the additive functional φ_t. First, in the case of

$$\varphi_t = \int_0^t a(x_s)ds,$$

we have

$$E_x[\varphi_t^4] \leqq \|a\| t^4 \leqq \|a\| t^2, \qquad \text{for } 0 \leqq t < 1. \tag{3.7}$$

Next, in the case of

$$\varphi_t = \int_0^t b(x_s)dB_s,$$

we have

$$E_x[\varphi_t^4] \leqq E_x^{\frac{1}{2}}\Big[E^2\Big[\int_0^t b(x_s)ds\Big]^4\Big] \leqq 3E_x^{\frac{1}{2}}\Big[\Big(\int_0^t b(x_s)^2 ds\Big)^2\Big] \tag{3.8}$$
$$\leqq 3\|b\| t^2.$$

It is easy to prove the inequality (3.6) for the linear combination

$$\varphi_t = \int_0^t a(x_s)ds + \int_0^t b(x_s)dB_s$$

from (3.7) and (3.8).

Using Lemma 3.4 and Lemma 3.5, we get

THEOREM 3.1. *The inequality*

$$E_x[\varphi_t^4] \leqq Ct^2$$

holds, where C is independent of $x \in X$.

THEOREM 3.2. *Let A be of the σ-algebra* $\mathfrak{A}$ *on* $C[0, T]$, *then the sequence of induced measures,*

$$\mu_x^k(A) = P_x[\varphi^{(k)}_{\cdot} \in A], \qquad (k = 1, 2, \cdots),$$

converges weakly to Wiener measure on $C[0, T]$ *in the Prokhorov's sense.*

PROOF. First, we prove that the finite dimensional distribution of μ_x^k converges to the one of Wiener process. For simplicity, we consider only the case of two time points; $0 < t_1 < t_2 \leqq T$. Then the proof is complete if we prove that the Fourier transform of the distribution μ_x^k's

$$\begin{aligned} &E_x[\exp(iz_1\varphi_{t_1}^{(k)} + iz_2(\varphi_{t_2}^{(k)} - \varphi_{t_1}^{(k)}))] \\ &\quad = E_x[\exp(iz_1\varphi_{t_1}^{(k)}) E_{x_{kt_1}}(\exp(iz_2\varphi_{t_2-t_1}^{(k)}))] \end{aligned}$$

converges to $\exp\left(-\frac{1}{2}(t_1 z_1^2 + (t_2 - t_1)z_2^2)\right)$ uniformly in x as $k \to \infty$. This fact is easily verified by Theorem 2.1. Next, we prove that the system of measures μ_x^k is relatively compact. For this, it is sufficient to prove that Prokhorov's criterion [8] is satisfied. Indeed,

$$\begin{aligned} E_x[(\varphi_t^{(k)} - \varphi_s^{(k)})^4] &= \frac{1}{k^2\sigma^4} E_x[(\varphi_{kt} - \varphi_{ks})^4] \\ &= \frac{1}{k^2\sigma^4} E_x[E_{x_{ks}}(\varphi_{kt-ks}^4)] \\ &\leqq \frac{C}{\sigma^4}(t-s)^2, \text{ (Theorem 3.1)}. \end{aligned}$$

Thus the proof is complete.

§ 4. Remarks on the case of additive vectors

Let $\varphi_t^1, \varphi_t^2, \cdots, \varphi_t^n$ be additive functionals of the Markov process $(x_t, \mathfrak{F}_t, P_x, x \in X)$. The Assumption 1 and 2 are satisfied for the process x_t and the additive functionals φ_t^k; $k = 1, 2, \cdots, n$. We define an additive vector $\Phi_t(\omega)$ by

$$\Phi_t(\omega) = (\varphi_t^1, \varphi_t^2, \cdots, \varphi_t^n).$$

We consider the n-dimensional central limit theorem for the additive vector $\Phi_t(\omega)$. By the same way as in § 1∼§ 3, we can derive the following results.

We set

$$T_t^z f(x) = E_x[e^{i(z_1\varphi_t^1 + \cdots + z_n\varphi_t^n)} f(x_t)]$$

for $t \geqq 0$, $z = (z_1, \cdots, z_n) \in R^n$ and $f \in C(X)$.

Then, for any $t > 0$, T_t^z is a compact operator on $C(X)$, and for sufficiently small z, T_1^z has the unique and simple eigenvalue $\lambda(z)=\lambda(z_1, \cdots, z_n)$ which is maximal in absolute value. The function $\lambda(z)$ is of C^2-class in a *nbd* of $z=0$. We denote derivatives $\frac{\partial\lambda(z)}{\partial z_i}\left(\text{resp. } \frac{\partial^2\lambda(z)}{\partial z_i \partial z_j}\right)$ by $\lambda_i'(z)$ (resp. $\lambda_{ij}''(z)$). We put

$$\mathfrak{M}=\left(\frac{1}{i}\lambda_1'(0), \cdots, \frac{1}{i}\lambda_n'(0)\right)$$

and

$$\mathfrak{C}=(-\lambda_{ij}''(0)+\lambda_i'(0)\lambda_j'(0)).$$

Here, $\mathfrak{M}$ is a vector with real components and $\mathfrak{C}$ is a matrix which is non-negative definite. We get

THEOREM 4.1. *The characteristic function* $\int_{R^n} e^{iyz} m_x^t(dy)$ *of the measure* $m_x^t(A)=P_x\left[\frac{1}{\sqrt{t}}(\Phi_t(\omega)-\mathfrak{M}t)\in A\right]$ *on* R^n *converges to* $e^{-\frac{1}{2}zCz'}$ *as* $t\to\infty$, *where* z' *is the transposed vector of* z.

Moreover, we get the analogous result to Theorem 2.2.

THEOREM 4.2. *The matrix* $\mathfrak{C}$ *degenerates if and only if some linear combination of* φ_t^j,

$$\sum_{j=1}^n y_j\varphi_t^j, \qquad (y_1, \cdots, y_n)\neq(0, \cdots, 0),$$

is expressed in the form (2.12).

COROLLARY. *Let* $a_i(x)$ $(i=1, 2, \cdots, n)$ *be real valued continuous functions on* X. *If* $a_i(x)$ $(i=1, 2, \cdots, n)$ *and* 1 *are linearly independent, then the matrix* $\mathfrak{C}$ *corresponding to the additive vector*

$$\Phi_t(\omega)=\left(\int_0^t a_1(x_s)ds, \int_0^t a_2(x_s)ds, \cdots, \int_0^t a_n(x_s)ds\right)$$

never degenerates.

Next, we consider only the additive vector $\Phi_t=(\varphi_t^1, \cdots, \varphi_t^n)$ where each φ_t^i $(i=1, \cdots, n)$ is of the type (ii) or (iii) or their linear combination. Then we get the following theorem by the same way as in § 3.

THEOREM 4.3. *If the matrix* $\mathfrak{C}$ *corresponding to* Φ_t *does not degenerate, then the system of measures induced by*

$$\Phi_t^{(k)}=\frac{1}{\sqrt{k}}(\Phi_{kt}-t\mathfrak{M})C^{-\frac{1}{2}}$$

on the space of continuous path space $C^n[0, T]=C[0, T]\times\cdots\times C[0, T]$ (*n fold direct product*) *converges weakly to n-dimensional Wiener measure as* $k\to\infty$.

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