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Smooth S^1-action and bordism

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§ 1. Introduction.

In this paper we study smooth actions of the circle group S^1 on smooth manifolds from the view point of bordism theory.

Let G be a fixed compact Lie group and $\mathcal{F}'$ and $\mathcal{F}$ be families of subgroups of G such that $\mathcal{F}' \subset \mathcal{F}$. We assume that both families are closed under inner automorphisms of G. An action of G on a manifold M will be called $(\mathcal{F}, \mathcal{F}')$-free provided that it is effective on each component of M and the isotropy subgroup G_x at each point $x \in M$ belongs to $\mathcal{F}$ and, if $x \in \partial M$, G_x belongs to $\mathcal{F}'$. When $\mathcal{F}' = \emptyset$ then necessarily $\partial M = \emptyset$. In this case we call the action $\mathcal{F}$-free. The n-dimensional bordism group $\Omega_n(G; \mathcal{F}, \mathcal{F}')$ of all orientation preserving $(\mathcal{F}, \mathcal{F}')$-free smooth G-actions on compact oriented smooth n-manifolds is defined in the obvious way. See [3][1]. If $\mathcal{F}' = \emptyset$ then we denote $\Omega_n(G; \mathcal{F}, \emptyset)$ simply by $\Omega_n(G; \mathcal{F})$. These groups are connected by an exact sequence

$$\cdots \longrightarrow \Omega_n(G; \mathcal{F}') \xrightarrow{i_*} \Omega_n(G; \mathcal{F}) \xrightarrow{j_*} \Omega_n(G; \mathcal{F}, \mathcal{F}') \xrightarrow{\partial_*} \Omega_{n-1}(G; \mathcal{F}') \longrightarrow \cdots.$$

In an entirely similar way the U-bordism group $\Omega_n^U(G; \mathcal{F}, \mathcal{F}')$ of all U-structure preserving $(\mathcal{F}, \mathcal{F}')$-free smooth G-actions on compact n-dimensional U-manifolds (weakly complex manifolds) are defined together with natural homomorphisms induced by the inclusion $\mathcal{F}' \subset \mathcal{F}$.

In this paper we consider the case in which $G = S^1$ and $\mathcal{F} = \mathcal{F}_l^+$ where we set

$$\mathcal{F}_l = \{\boldsymbol{Z}_k \mid k \leqq l\}$$

and

$$\mathcal{F}_l^+ = \mathcal{F}_l \cup \{S^1\}.$$

Here $\boldsymbol{Z}_k$ denotes the subgroup of S^1 consisting of k-th roots of unity. Thus $\mathcal{F}_\infty = \cup \mathcal{F}_l$ is the set of all finite subgroups of S^1 and $\mathcal{F}_\infty^+ = \cup \mathcal{F}_l^+$ is the set of all closed subgroups of S^1.

Our main results are the following.

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1) In [3] the assumption of effectiveness in the definition of $(\mathcal{F}, \mathcal{F}')$-free action was not imposed. We add that assumption to simplify the resulting bordism group.

THEOREMS (2.22) and (2.29). *For each integer l, $1<l$, the sequences*

$$0 \longrightarrow \Omega_n^U(S^1; \mathcal{F}_{l-1}^+) \xrightarrow{i_*} \Omega_n^U(S^1; \mathcal{F}_l^+) \xrightarrow{j_*} \Omega_n^U(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow 0$$

and

$$0 \longrightarrow \Omega_n(S^1; \mathcal{F}_{l-1}^+) \xrightarrow{i_*} \Omega_n(S^1; \mathcal{F}_l^+) \xrightarrow{j_*} \Omega_n(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow 0$$

are split exact.

In Section 2 we shall construct splittings

$${}^t\boldsymbol{P}\colon \Omega_n^U(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow \Omega_n^U(S^1; \mathcal{F}_l^+)$$

and

$${}^t\boldsymbol{P}\colon \Omega_n(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow \Omega_n(S^1; \mathcal{F}_l^+)$$

which we call "twisted complex projective space bundle construction". Setting

$${}^tP_n^U(S^1; \mathcal{F}_l^+) = {}^t\boldsymbol{P}\Omega_n^U(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$$

and

$${}^tP_n(S^1; \mathcal{F}_l^+) = {}^t\boldsymbol{P}\Omega_n(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$$

we have immediate corollaries.

COROLLARIES (2.24) and (2.30). *There are canonical isomorphisms*

$$\Omega_n^U(S^1; \mathcal{F}_l^+) \cong \Omega_n^U(S^1; \mathcal{F}_1^+) \oplus \sum_{1<k\leq l} {}^tP_n^U(S^1; \mathcal{F}_k^+)$$

$$\Omega_n^U(S^1; \mathcal{F}_\infty^+) \cong \Omega_n^U(S^1; \mathcal{F}_1^+) \oplus \sum_{1<k} {}^tP_n^U(S^1; \mathcal{F}_k^+)$$

$$\Omega_n(S^1; \mathcal{F}_l^+) \cong \Omega_n(S^1; \mathcal{F}_1^+) \oplus \sum_{1<k\leq l} {}^tP_n(S^1; \mathcal{F}_k^+)$$

and

$$\Omega_n(S^1; \mathcal{F}_\infty^+) \cong \Omega_n(S^1; \mathcal{F}_1^+) \oplus \sum_{1<k} {}^tP_n(S^1; \mathcal{F}_k^+)\,.$$

As was shown in [8] the group $\Omega_n(S^1; \mathcal{F}_1^+)$ is generated by complex projective space bundles. Analogous fact holds for $\Omega_n^U(S^1; \mathcal{F}_l^+)$. We can say that twisted complex projective space bundles are as simple as complex projective space bundles. Thus these corollaries exhibit generators for $\Omega_*^U(S^1; \mathcal{F}_l^+)$ and $\Omega_*(S^1; \mathcal{F}_l^+)$ which are geometrically very simple.

We note here that our methods are applicable to the case of stationary point free actions, i.e. the case of $\mathcal{F}_l$-free actions, with minor modifications in the real case. However that case was already treated by Ossa [7] and indeed our methods are quite similar to his.

In Sections 3 and 4 we shall give an elementary proof of the Kosniowski formula [6] and the Atiyah-Singer formula [1, p. 594] in the framework of bordism theory. These formulae were originally proved by using the Atiyah-Singer G-signature theorem. In the case of semi-free actions proofs in the framework of bordism theory were given by Kawakubo and Uchida [**5**] for the Atiyah-Singer formula and by Takao Matumoto (unpublished) for the Kosniowski formula. Another proof of Atiyah-Singer's formula which uses generalized manifolds was given by Kawakubo and Raymond [**4**].

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§ 2. Twisted complex projective space bundles.

Let $V\to X$ be a vector bundle (real or complex) and let

$$\phi: S^1\times V \longrightarrow V$$

be an effective continuous S^1-action by vector bundle isomorphisms of V. Then ϕ defines an injective homomorphism $S^1\to \mathrm{Isom}(V)$ which we shall also denote by the same letter ϕ where $\mathrm{Isom}(V)$ denotes the group of all vector bundle isomorphisms of V onto itself. Thus, by this convention, we write $\phi(g)v$ for $\phi(g, v)$ for any $g\in S^1$ and $v\in V$. We always indentify X with the zero cross-section image of the bundle V. Set

$$H=\{g\,|\,g\in S^1,\ \phi(g)x=x \text{ for all } x\in X\}\,.$$

Then H is a closed subgroup of S^1. H equals the whole group S^1 if and only if each $\phi(g)$ is an automorphism of the bundle V. If $H\neq S^1$, then H equals $\boldsymbol{Z}_l$, the l-th roots of unity, for some $l\geqq 1$ and it is easy to see that there is a unique S^1-action φ on X such that

$$\phi(g)x=\varphi(g)^l x$$

for all $g\in S^1$ and $x\in X$. In this case we say that *the action ϕ is of order l.*

DEFINITION (2.1). Let l be an integer, $1<l$. An S^1-action ϕ on V is said to be strictly $\mathcal{F}_l^+$-free if the following three conditions are satisfied:

1) ϕ is of order l,
2) the action φ (defined as above) on X is semi-free, i.e. $\mathcal{F}_1^+$-free and
3) the action ϕ restricted on $V-X$ is $\mathcal{F}_{l-1}$-free.

Note that if the action ϕ is strictly $\mathcal{F}_l^+$-free then the fixed point set of ϕ is contained in X as a proper subset. Here by the fixed point set of an action we mean the set of points which are fixed by all elements of the group.

Now let X be a compact U-manifold and V a smooth complex vector bundle on X. Then V, regarded as a smooth manifold, has the obvious induced U-structure. A smooth S^1-action $\phi: S^1\to\mathrm{Isom}(V)$ is called to be U-structure preserving if each $\phi(g)$ preserves the U-structure on the base X. Note that, in that case, each $\phi(g)$ also preserves the induced U-structure on V. Let l be an integer, $1<l$, and let $\mathcal{B}^U_{m,2k}(S^1;\mathcal{F}_l^+)$ denote the totality of

triples (X, V, ϕ) where $V \to X$ is a smooth complex k-vector bundle on a compact m-dimensional U-manifold without boundary X and ϕ is an effective U-structure preserving smooth S^1-action on V which is strictly $\mathcal{F}_l^+$-free. Two triples (X, V, ϕ) and (X', V', ϕ') in $\mathcal{B}_{m,2k}^U(S^1; \mathcal{F}_l^+)$ are called bordant if there is a compact $(m+1)$-dimensional U-manifold Y, a smooth complex k-vector bundle W on Y and a U-structure preserving, strictly $\mathcal{F}_l^+$-free, smooth S^1-action Ψ on W such that

$$\partial Y = X \cup -X'$$

$$W | X = V, \qquad W | X' = V'$$

and

$$\Psi | V = \phi, \qquad \Psi | V' = \phi'$$

where $-X'$ denotes the U-manifold X' with the opposite U-structure as usual. This is clearly an equivalence relation. The set of all equivalence classes of $\mathcal{B}_{m,2k}^U(S^1; \mathcal{F}_l^+)$ will be denoted by $B_{m,2k}^U(S^1; \mathcal{F}_l^+)$ and the class of (X, V, ϕ) will be denoted by $[X, V, \phi]$. $B_{m,2k}^U(S^1; \mathcal{F}_l^+)$ becomes an abelian group where the addition is induced by disjoint union. The verification of the fact is quite routine and is omitted.

Next let X be a compact smooth manifold and V a smooth real vector bundle on X such that $w_1(X)$ equals the first Stiefel-Whitney class of the vector bundle $V \to X$. Then V, regarded as a manifold, is orientable. A triple (X, V, ϕ) in which $V \to X$ is a real vector bundle with the above property and $\phi: S^1 \to \mathrm{Isom}(V)$ is an effective smooth action will be called *oriented* if V, regarded as a manifold, is oriented. For an integer l greater than 1, we shall denote by $\mathcal{B}_{m,k}(S^1; \mathcal{F}_l^+)$ the totality of oriented triples (X, V, ϕ) in which $\dim X = m$, fiber-dim $V = k$ and ϕ is strictly $\mathcal{F}_l^+$-free. The bordism relation between oriented triples and the resulting bordism group $B_{m,k}(S^1; \mathcal{F}_l^+)$ are defined in a similar way as the unitary case.

REMARK (2.2). We shall show later that $\mathcal{B}_{m,k}(S^1; \mathcal{F}_l^+) = \emptyset$ and consequently $B_{m,k}(S^1; \mathcal{F}_l^+) = 0$ for odd k.

Now suppose that a pair (M, ϕ) of a compact smooth manifold M and an $(\mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$-free smooth S^1-action ϕ on M is given. A connected component X of the fixed point set of $\phi(\boldsymbol{Z}_l)$ will be called to be *of the first kind* if it contains a point x whose isotropy subgroup equals precisely $\boldsymbol{Z}_l$.

LEMMA (2.3). *Let ϕ be an $(\mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$-free smooth action on a compact smooth manifold M. If X is a connected component of the first kind of the fixed point set of $\phi(\boldsymbol{Z}_l)$, then X is contained in the interior of M. Consequently X has no boundary. Moreover if V is the normal bundle of X in M then the induced action ϕ on V is strictly $\mathcal{F}_l^+$-free.*

PROOF. Assume that $X \cap \partial M \neq \emptyset$. Then, by the equivariant collar neighborhood theorem, $X \cap \partial M = \partial X$ and the fixed point set F of $\phi(S^1)$ in X contains a neighborbood of ∂X in X. But $F - \partial X$ is a manifold without boundary. Therefore F must coincide with the whole X which is a contradiction. Thus $X \cap \partial M = \emptyset$. The rest of the statement is clear.

LEMMA (2.4). *Let ϕ be an $(\mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$-free smooth S^1-action on a compact smooth n-manifold M. Let $\{X_i\}$ be the totality of connected components of the first kind of the fixed point set of $\phi(\mathbf{Z}_l)$ and let D_i be the ϕ-invariant closed tubular neighborhood of X_i with respect to a ϕ-invariant Riemannian metric on M. Then we have*

$$\sum [D_i, \phi] = [M, \phi]$$

in $\Omega_n(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$.

PROOF. Since the action ϕ restricted on $M - \cup D_i$ is $\mathcal{F}_{l-1}^+$-free, the statement follows from [3, (5.2)]. Similarly we have

LEMMA (2.5). *Let ϕ be a U-structure preserving $(\mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$-free smooth S^1-action on a compact U-manifold M and let X_i and V_i have similar meanings as in* (2.4). *Then*

$$\sum [D_i, \phi] = [M, \phi]$$

in $\Omega_n^U(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$.

We consider the homomorphisms

$$\nu: \Omega_n^U(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow \sum_{m+2k=n} B_{m,2k}^U(S^1; \mathcal{F}_l^+)$$

and

$$\nu: \Omega_n(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow \sum_{m+k=n} B_{m,k}(S^1; \mathcal{F}_l^+)$$

defined by

$$\nu[M, \phi] = \sum [X_i, V_i, \phi]$$

where the summation is taken over the connected components of the first kind of the fixed point set of $\phi(\mathbf{Z}_l)$ and V_i is the normal bundle of X_i in M. In the real case we orient V_i concordantly with M. In the complex case X_i has the natural U-structure and V_i becomes a complex vector bundle on which ϕ acts by U-structure preserving isomorphisms of complex vector bundle. By (2.3) $[X_i, V_i, \phi]$ belongs to $B_{m,2k}(S^1; \mathcal{F}_l^+)$ or $B_{m,k}(S^1; \mathcal{F}_l^+)$ as the case may be.

PROPOSITION (2.6). *The homomorphisms*

$$\nu: \Omega_n^U(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow \sum_{m+2k=n} B_{m,2k}^U(S^1; \mathcal{F}_l^+)$$

and

$$\nu: \Omega_n(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+) \longrightarrow \sum_{m+k=n} B_{m,k}(S^1; \mathcal{F}_l^+)$$

are isomorphisms.

PROOF. Given a triple (X, V, ϕ) in $\mathcal{B}_{m,2k}^U(S^1; \mathcal{F}_l^+)$ or $\mathcal{B}_{m,k}(S^1; \mathcal{F}_l^+)$, let $D(V)$ be the disk bundle of V with respect to a ϕ-invariant metric on the vector bundle V. Then it is a routine matter to verify that the assignment

$$[X, V, \psi] \longmapsto [D(V), \psi]$$

gives a well-defined homomorphism

$$\delta: B_{m,2k}^{U}(S^1; \mathcal{F}_l^+) \longrightarrow \Omega_{m+2k}^{U}(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+)$$

or

$$\delta: B_{m,k}(S^1; \mathcal{F}_l^+) \longrightarrow \Omega_{m+k}(S^1; \mathcal{F}_l^+, \mathcal{F}_{l-1}^+).$$

Then clearly we have

$$\nu \circ \delta = \text{identity}.$$

By (2.5) and (2.4) we also have

$$\delta \circ \nu = \text{identity}.$$

This proves that ν is an isomorphism and $\nu^{-1} = \delta$.

To define twisted complex projective space bundle we need some preliminaries. First we consider the complex case. If (X, V, ψ) is a triple in $\mathcal{B}_{m,2k}^{U}(S^1; \mathcal{F}_l^+)$ then the subgroup $\boldsymbol{Z}_l \subset S^1$ acts on V by automorphisms. We assume that X is connected. This will not destroy the generality of arguments which follow. Then, as is well known, there is a unique eigen-value decomposition of V into a direct sum

$$V = \sum_{0 < l_i < l} V(l_i) \tag{2.7}$$

such that, for all $g \in \boldsymbol{Z}_l$ and $v \in V(l_i)$,

$$\psi(g)v = g^{l_i} v. \tag{2.8}$$

Note that the eigen-values of $\psi(g)$ on V are of the form g^{l_i}, $0 \leqq l_i < l$. But by the condition 3) of (2.1), $1 = g^0$ does not occur in our case. To avoid confusion we denote by $\psi'(g)$ the scalar multiplication by $g \in S^1 \subset \boldsymbol{C}$ in the complex vector bundle V. Thus

$$\psi'(g)v = gv$$

for $g \in S^1$ and $v \in V$. With this notation we have, for $g \in \boldsymbol{Z}_l$,

$$\psi(g) = \psi'(g)^{l_i} \quad \text{on } V(l_i). \tag{2.8$'$}$$

Let F be the fixed point set of the action φ on X (see (2.1)). F is a proper submanifold of X. Let $\{F_j\}$ be the totality of connected components of F. Then the group S^1 acts on $V|F_j$ by automorphisms via ψ and each $V(l_i)|F_j$ is clearly S^1-invariant. Therefore we have eigen-value decomposition

$$V(l_i)|F_j = \sum_{r \in \boldsymbol{Z}} V(l_i, r)$$

where

$$\psi(g)=\psi'(g)^r \qquad \text{on } V(l_i, r)\,.$$

Note that the integers r must satisfy the relation

$$r\equiv l_i \qquad \bmod l\,.$$

Moreover the isotropy subgroup at any point $v\neq 0$ in $V(l_i, r)$ is $\boldsymbol{Z}_{|r|}$. Therefore, in view of the condition 3) in (2.1), the possible ones for which $V(l_i, r)\neq 0$ are l_i and l_i-l. (Recall that $0<l_i<l$.) Setting

$$V_j^+(l_i)=V(l_i, l_i)\,,$$

$$V_j^-(l_i)=V(l_i, l_i-l)\,,$$

we have a ψ-invariant decomposition

$$V(l_i)\,|\,F_j=V_j^+(l_i)\oplus V_j^-(l_i) \tag{2.9}$$

where $\psi(g)=\psi'(g)^{l_i}$ on $V_j^+(l_i)$ and $\psi(g)=\psi'(g)^{l_i-l}$ on $V_j^-(l_i)$.

Next consider the S^1-action on $V(l_i)$ defined by

$$g\longmapsto \psi(g)\psi'(g)^{-l_i}\,.$$

Since ψ and ψ' commute with each other this defines an action of S^1. Moreover since $\psi(g)\psi'(g)^{-l_i}=1$ for $g\in \boldsymbol{Z}_l$ on $V(l_i)$ by (2.8)$'$, there exists a unique S^1-action ψ_i'' on $V(l_i)$ such that

$$\psi_i''(g)^l=\psi(g)\psi'(g)^{-l_i}\,. \tag{2.10}$$

Then ψ_i'' is an action which covers φ. Thus we can form the direct sum action

$$\psi''(g)=\sum \psi_i''(g)$$

on $V=\sum V(l_i)$. It is clear that ψ'' commutes with ψ and ψ'. Furthermore from (2.9) and (2.10) it follows that

$$\psi''(g)=\begin{cases} 1 & \text{on } V_j^+(l_i)\,,\\ \psi'(g)^{-1} & \text{on } V_j^-(l_i)\,.\end{cases} \tag{2.11}$$

Finally we define ψ_1 by

$$\psi_1(g)=\psi''(g)^2\psi'(g)\,. \tag{2.12}$$

Since ψ'' commutes with ψ' this defines an S^1-action $\psi_1: S^1\to \mathrm{Isom}\,(V)$ which commutes with ψ, ψ', and ψ''. Note that the action ψ_1 restricted on X equals φ^2. The behavior of ψ_1 on $V_j^\pm(l_i)$ is given by

$$\psi_1(g)=\begin{cases} \psi'(g) & \text{on } V_j^+(l_i)\,,\\ \psi'(g)^{-1} & \text{on } V_j^-(l_i)\,,\end{cases} \tag{2.13}$$

as is easily seen from (2.9) and (2.11).

We extend the actions ψ and ψ_1 over $V\times \boldsymbol{C}$, Whitney sum of V and the trivial complex line bundle, by putting

$$\phi(g)(v, \alpha)=(\phi(g)v, \alpha) \tag{2.14}$$

and

$$\phi_1(g)(v, \alpha)=(\phi_1(g)v, g\alpha)\,. \tag{2.15}$$

From the above data we readily obtain the following

PROPOSITION (2.16). *Let* $(X, V, \phi)\in \mathcal{B}^U_{m,2k}(S^1; \mathcal{F}^+_l)$ *where* X *is connected. The action* ϕ_1 *on* V *and* $V\times \boldsymbol{C}$ *is strictly* $\mathcal{F}^+_2$*-free. In particular it is free (i.e.,* $\mathcal{F}_1$*-free) on* $V-X$ *and* $V\times\boldsymbol{C}-X$.

Now choose a ϕ-invariant hermitian metric on V and extend it in the obvious way over $V\times\boldsymbol{C}$. Note that the metric is also ϕ''- and ϕ_1-invariant. Let $S(V)$ and $S(V\times\boldsymbol{C})$ be the corresponding unit sphere bundles. The action ϕ_1 keeps $S(V)$ and $S(V\times\boldsymbol{C})$ invariant and it acts freely on them by (2.16). Hence the quotient spaces

$$\boldsymbol{P}_\phi(V)=S(V)/\phi_1$$

and

$$\boldsymbol{P}_\phi(V\times\boldsymbol{C})=S(V\times\boldsymbol{C})/\phi_1$$

are smooth manifolds. We shall call them *twisted projective space bundles of the pairs* (V, ϕ) *and* $(V\times\boldsymbol{C}, \phi)$ respectively, although they are by no means bundles in the usual sense. We denote by $[v]\in\boldsymbol{P}_\phi(V)$ and $[v, \alpha]\in\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ the images of $v\in S(V)$ and $(v, \alpha)\in S(V\times\boldsymbol{C})$ respectively. Since the action ϕ keeps $S(V)$ and $S(V\times\boldsymbol{C})$ invariant and it commutes with ϕ_1, it induces an action on $\boldsymbol{P}_\phi(V)$ and $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ which we shall denote by the same letter ϕ.

Let $W_\phi=W_\phi(V)$ denote the 2-disk bundle associated to the S^1-fibering $S(V)\to\boldsymbol{P}_\phi(V)$. W_ϕ is identified with the quotient space of $S(V)\times D^2$ by the S^1-action ϕ_1 defined by the same formula as (2.15). The class of (v, α) in W_ϕ is denoted by $[v, \alpha]$. We define the map

$$f\colon W_\phi \longrightarrow \boldsymbol{P}_\phi(V\times\boldsymbol{C})$$

by

$$f[v, \alpha]=[v/\sqrt{1+|\alpha|^2},\ \alpha/\sqrt{1+|\alpha|^2}]\,.$$

We also define the map

$$g\colon D(V)\longrightarrow \boldsymbol{P}_\phi(V\times\boldsymbol{C})$$

by

$$g(v)=[v/\sqrt{2},\ \sqrt{1-|v|^2/2}]\,.$$

Then the following lemma is immediate.

LEMMA (2.17). f *and* g *are* ϕ*-equivariant smooth embeddings.* f *and* g *coincide on* $S(V)$. *Moreover we have*

$$g(D(V))\cup f(W_\psi)=\boldsymbol{P}_\psi(V\times\boldsymbol{C})$$

and

$$g(D(V))\cap f(W_\psi)=g(S(V))\,.$$

Lemma (2.17) shows that $\boldsymbol{P}_\psi(V\times\boldsymbol{C})$ is diffeomorphic to the smooth manifold $D(V)\cup W_\psi$ obtained by glueing together $D(V)$ and W_ψ along their common boundary $S(V)$ by the identity automorphism. Henceforth we shall identify $\boldsymbol{P}_\psi(V\times\boldsymbol{C})$ with $D(V)\cup W_\psi$. Then, since the S^1-action ψ_1 preserves the U-structure on V and hence on $S(V)$, it is easy to see that the U-structure can be extended over W_ψ giving a U-structure on $\boldsymbol{P}_\psi(V\times\boldsymbol{C})$. The action ψ clearly preserves this U-structure on $\boldsymbol{P}_\psi(V\times\boldsymbol{C})$.

The manifold $\boldsymbol{P}_\psi(V)$ is contained in the U-manifold W_ψ as a U-submanifold. Namely its normal bundle has the obvious structure of complex line bundle, the one associated to the S^1-bundle $S(V)\to\boldsymbol{P}_\psi(V)$. Thus $\boldsymbol{P}_\psi(V)$ is also a U-manifold.

PROPOSITION (2.18). *Let (X, V, ψ) be in $\mathscr{B}^U_{m,2k}(S^1;\mathscr{F}_l^+)$, $1<l$. Suppose that X is connected. Then the action ψ on $\boldsymbol{P}_\psi(V)$ is $\mathscr{F}_{l-1}^+$-free. The action ψ on $P_\psi(V\times\boldsymbol{C})$ is $\mathscr{F}_l^+$-free and the fixed point set of the first kind of $\psi(\boldsymbol{Z}_l)$ equals precisely X. The normal bundle of X in $\boldsymbol{P}_\psi(V\times\boldsymbol{C})$ is ψ-equivariantly equivalent to V.*

PROOF. Let $v\in S(V)$ and $[v]$ be the image of v in $\boldsymbol{P}_\psi(V)$. Then we have

$$\psi(g)[v]=[v]$$

if and only if

$$\psi(g)v=\psi_1(h)v \qquad \text{for some } h\in S^1\,. \tag{2.19}$$

Let $v\in V_x$, the fiber of V over $x\in X$, and suppose first that $x\notin F$, where F denote the fixed point set of the action φ on X. Write v as

$$v=\sum v_{i_s}\,,\qquad v_{i_s}\neq 0\in V(l_{i_s})$$

according to the decomposition (2.7). Then

$$\psi(g)v=\sum\psi''(g)^l\psi'(g)^{l_{i_s}}v_{i_s}\qquad\text{by (2.10)}$$

and

$$\psi_1(h)v=\sum\psi''(h)^2\psi'(h)v_{i_s}\qquad\text{by (2.12)}\,.$$

Since ψ'' covers φ which is free on $X-F$ and ψ' preserves V_x, the condition (2.19) is equivalent to

$$g^l=h^2\qquad\text{and}\qquad g^{l_{i_s}}=h\,.$$

Such an element h exists if and only if

$$g^{l-2l_{i_s}}=1\qquad\text{for all } s\,. \tag{2.20}$$

Let H be the subgroup of S^1 consisting of all elements satisfying (2.20). H is equal to S^1 if we have only one s and $l_{i_s}=l/2$. Otherwise $H=\boldsymbol{Z}_d$ where d is the greatest common divisor of $\{|l-2l_{i_s}|\}$. Since $0<l_{i_s}<l$, we have $|l-2l_{i_s}|<l$. Hence $d<l$. Thus we have proved that the isotropy subgroup at v belongs to $\mathcal{F}_{l-1}^+$ and $\boldsymbol{P}_\psi(V(l/2))$ is a component of the fixed point set of ψ.

Next suppose that $x\in F_j$, a component of F, and $v\in V_x$. Write v as

$$v=\sum v_{i_s}^+ + \sum v_{k_t}^-$$

where $v_{i_s}^+\in V_j^+(l_{i_s})$ and $v_{k_t}^-\in V_j^-(l_{k_t})$. Then the same reasoning as above using (2.11) and (2.13) shows that (2.19) is equivalent to

$$g^{l_{i_s}}=h \qquad \text{and} \qquad g^{l-l_{k_t}}=h$$

for all s and t. Hence the isotropy subgroup H of ψ at $[v]$ is $\boldsymbol{Z}_d$ when different values occur among l_{i_s} and $l-l_{k_t}$ in which case d is the greatest common divisor of $|l_{i_s}-l_{i_{s'}}|$, $|l_{i_s}-(l-l_{k_t})|$ and $|l_{k_t}-l_{k_{t'}}|$. Since $0<l_i<l$, these numbers are smaller than l. Hence $d<l$ and $H\in\mathcal{F}_{l-1}$.

If there is only one value among l_{i_s} and $l-l_{k_t}$ then H equals S^1. This implies that $\boldsymbol{P}_\psi(V'_j(l_i))$ is a component of the fixed point set of ψ, where $V'_j(l_i)=V_j^+(l_i)\oplus V_j^-(l-l_i)$.

Thus we have proved that ψ is $\mathcal{F}_{l-1}^+$-free on $\boldsymbol{P}_\psi(V)$. Since the open submanifold $\boldsymbol{P}_\psi(V\times\boldsymbol{C})-\boldsymbol{P}_\psi(V)$ is ψ-equivariantly diffeomorphic to V the rest of the statement is clear.

REMARK (2.21). In the above proof we have shown that *the fixed point set of the action ψ on $\boldsymbol{P}_\psi(V)$ is the disjoint union of $\boldsymbol{P}_\psi(V(l/2))=S(V(l/2))/\psi_1$ (when l is even) and $\boldsymbol{P}_\psi(V'(l_i))=S(V'(l_i))/\psi_1$ for $l_i\neq l/2$. In particular, if $l=2$ then any element in $\boldsymbol{P}_\psi(V)$ is fixed by ψ. Indeed in this case the actions ψ and ψ_1 coincide, whence ψ is trivial on $\boldsymbol{P}_\psi(V)$.*

It is again a routine matter to verify that the assignment

$$\mathcal{B}_{m,2k}^U(S^1;\mathcal{F}_l^+)\ni(X,V,\psi)\longmapsto[\boldsymbol{P}_\psi(V\times\boldsymbol{C}),\psi]\in\Omega_{m,2k}^U(S^1;\mathcal{F}_l^+)$$

induces a well-defined homomorphism

$${}^t\boldsymbol{P}\colon B_{m,2k}^U(S^1;\mathcal{F}_l^+)\longrightarrow\Omega_{m,2k}^U(S^1;\mathcal{F}_l^+)\,.$$

THEOREM (2.22). *Let l be an integer, $1<l$. The homomorphism*

$${}^t\boldsymbol{P}\circ\nu\colon\ \Omega_n^U(S^1;\mathcal{F}_l^+,\mathcal{F}_{l-1}^+)\longrightarrow\Omega_n^U(S^1;\mathcal{F}_l^+)$$

is a splitting for

$$j_*\colon\ \Omega_n^U(S^1;\mathcal{F}_l^+)\longrightarrow\Omega_n^U(S^1;\mathcal{F}_l^+,\mathcal{F}_{l-1}^+)\,.$$

PROOF. Consider the composition $\nu\circ j_*\circ{}^t\boldsymbol{P}$. Then, for any (X,V,ψ) with connected X, we have

$$\nu\circ j_*\circ{}^t\boldsymbol{P}[X,V,\psi]=[X,V,\psi]$$

by (2.18). Since such $[X, V, \phi]$ generate $\sum_{m+2k=n} B^U_{m,2k}(S^1; \mathcal{F}^+_l)$, we have

$$\nu \circ j_* \circ {}^t\boldsymbol{P} = \text{identity}.$$

By (2.6), ν is an isomorphism. Hence it follows that

$$j_* \circ {}^t\boldsymbol{P} \circ \nu = \text{identity}.$$

REMARK (2.23). It can be shown easily that, *for any* $[M, \phi] \in \Omega^U_n(S^1; \mathcal{F}^+_l)$, *the element*

$$[M, \phi] - {}^t\boldsymbol{P} \circ \nu \circ j^*[M, \phi] \in i_*\Omega^U_n(S^1; \mathcal{F}^+_{l-1})$$

is represented by $[M_1, \phi]$ *where*

$$M_1 = (M - \bigcup_{X_i} \operatorname{int} D(V_i)) \cup \bigcup_{X_i} - W_\phi(V_i)$$

glued along $\bigcup_{X_i} S(V_i)$.

Here $\{X_i\}$ *is the totality of the connected components of the first kind of the fixed point set of* $\phi(\boldsymbol{Z}_l)$ *and* V_i *is the normal bundle of* X_i *in* M. $-W_\phi$ *denotes the U-manifold* W_ϕ *with the opposite structure.* We may call M_1 twisted blowing up of M along $\cup X_i$.

COROLLARY (2.24). *There are canonical isomorphisms*

$$\Omega^U_n(S^1; \mathcal{F}^+_l) \cong \Omega^U_n(S^1; \mathcal{F}^+_1) \oplus \sum_{1<k\leq l} {}^tP^U_n(S^1; \mathcal{F}^+_k)$$

and

$$\Omega^U_n(S^1; \mathcal{F}^\pm_\infty) \cong \Omega^U_n(S^1; \mathcal{F}^+_1) \oplus \sum_{1<k} {}^tP^U_n(S^1; \mathcal{F}^+_k),$$

where

$${}^tP^U_n(S^1; \mathcal{F}^+_l) = {}^t\boldsymbol{P}(\sum_{m+2k=n} B^U_{m,2k}(S^1; \mathcal{F}^+_l)).$$

PROOF. For $\mathcal{F}^+_l$ it is immediate from (2.23). Since

$$\Omega^U_n(S^1; \mathcal{F}^\pm_\infty) = \varinjlim_l \Omega^U_n(S^1; \mathcal{F}^+_l)$$

the case for $\mathcal{F}^\pm_\infty$ follows from the former.

We turn to the real case. Let $(X, V, \phi) \in \mathcal{B}_{m,k}(S^1; \mathcal{F}^+_l)$ and suppose that X is connected. $\boldsymbol{Z}_l$ acts on V and hence on V^C, the complexification of V, by automorphisms. Decompose V^C into the direct sum of eigensubbundles

$$V^C = \sum_{0<l_i<l} V^C(l_i)$$

where $\phi(g)v = g^{l_i}v$ for $g \in \boldsymbol{Z}_l$ and $v \in V^C(l_i)$. For $0 < l_i < l/2$ we set

$$U(l_i) = V \cap (V^C(l_i) \oplus V^C(l-l_i)).$$

$U(l_i)$ can be given a structure of ϕ-invariant complex vector bundle with a decomposition

$$U(l_i)=V(l_i)\oplus V(l-l_i)$$

such that, for any $g\in \boldsymbol{Z}_l$, we have

$$\psi(g)v=\psi'(g)^{l_i}v \qquad \text{for } v\in V(l_i)$$

and

$$\psi(g)v=\psi'(g)^{l-l_i} \qquad \text{for } v\in V(l-l_i)\,,$$

where $\psi'(g)$ denotes the scalar multiplication in the complex vector bundle $U(l_i)$. For example, the map $\rho\colon V^C(l_i)\to U(l_i)$ given by

$$\rho(v)=(v+\bar{v})/2 \tag{2.25}$$

is a real isomorphism for $l_i\neq l/2$ so that it transports the complex structure of $V^C(l_i)$ onto $U(l_i)$. With this structure we have $U(l_i)=V(l_i)$ and $V(l-l_i)=0$.

If l is even, we set

$$V(l/2)=V\cap V^C(l/2)\,.$$

$\boldsymbol{Z}_l$ acts on $V(l/2)$ by

$$\psi(g)v=g^{l/2}v\,, \qquad g\in \boldsymbol{Z}_l\,, \qquad v\in V(l/2)\,,$$

where it should be noticed that $g^{l/2}=\pm 1$ for $g\in \boldsymbol{Z}_l$. $V(l/2)$ does not have complex vector bundle structure in general. Here we digress to give a proof of Remark (2.2). It clearly suffices to prove that the fiber dimension of $V(l/2)$ is even when l is even. Consider the transformation $\psi(\zeta)$ on $V(l/2)$ where $\zeta=e^{2\pi\sqrt{-1}/l}$. Since it is connected to the identity in $\psi(S^1)$, it preserves the orientation. Since it keeps the base pointwise fixed, it acts on each fiber of $V(l/2)$ preserving orientation. But $\psi(\zeta)=-1$ on each fiber. This implies the dimension of the fiber is even. This proves (2.2).

Now consider the S^1-action on $V(l_i)$, $0<l_i<l$, $l_i\neq l/2$, defined by

$$g\longmapsto \psi(g)\psi'(g)^{-l_i}\,.$$

Since $\psi(g)\psi'(g)^{-l_i}=1$ for $g\in \boldsymbol{Z}_l$ on $V(l_i)$ there exists a unique action ψ_i'' on $V(l_i)$ such that

$$\psi_i''(g)^l=\psi(g)\psi'(g)^{-l_i}\,. \tag{2.26}$$

The action ψ_i'' covers φ. Let ψ'' be the S^1-action on $\sum_{0<l_i<l,\, l_i\neq l/2} V(l_i)$ given by

$$\psi''(g)=\sum \psi_i''(g)\,.$$

We define the S^1-action ψ_1 on $\sum_{0<l_i<l,\, l_i\neq l/2} V(l_i)$ by

$$\psi_1(g)=\psi''(g)^2\psi'(g)\,. \tag{2.27}$$

This action covers φ^2. Next observe that, when l is even, there is a unique S^1-action ψ_1 on $V(l/2)$ such that

(2.28) $$\phi_1(g)^{l/2}=\phi(g)\,.$$

This also covers φ^2. Thus we can form the direct sum action ϕ_1 on V from (2.26) and (2.27).

LEMMA (2.29). *The S^1-action ϕ_1 on V is independent of the choice of ϕ-invariant complex vector bundle structures on $U(l_i)$, $l_i\neq l/2$.*

PROOF. Let $\phi'(g)$ be the scalar multiplication of a ϕ-invariant complex vector bundle structure on $U(l_i)$ and let $\bar{\phi}'(g)$ be the one which is transported by ρ from $V^C(l_i)$ as in (2.25). It is not difficult to see that

$$\phi'(g)=\bar{\phi}'(g) \qquad \text{on } V(l_i)$$

and

$$\phi'(g)=\bar{\phi}'(g)^{-1} \qquad \text{on } V(l-l_i)\,.$$

According to (2.26) we define $\bar{\phi}''$ by

$$\bar{\phi}''(g)^l=\phi(g)\bar{\phi}'(g)^{-l_i} \qquad \text{on } U(l_i)\,.$$

Then we have

$$\bar{\phi}''(g)^l=\phi(g)\phi'(g)^{-l_i}=\phi''(g)^l \qquad \text{on } V(l_i)\,,$$

and

$$\begin{aligned}\bar{\phi}''(g)^l&=\phi(g)\phi'(g)^{l_i}\\&=\phi(g)\phi'(g)^{-(l-l_i)}\phi'(g)^l\\&=\phi''(g)^l\phi'(g)^l \qquad\qquad \text{on } V(l-l_i)\end{aligned}$$

where $0<l_i<l/2$. Hence it follows that

$$\bar{\phi}''(g)=\begin{cases}\phi''(g) & \text{on } V(l_i)\,,\\ \phi''(g)\phi'(g) & \text{on } V(l-l_i)\,.\end{cases}$$

Then

$$\begin{aligned}\bar{\phi}_1(g)&=\bar{\phi}''(g)^2\bar{\phi}'(g)\\&=\begin{cases}\phi_1(g) & \text{on } V(l_i)\,,\\ \phi''(g)^2\phi'(g)^2\phi'(g)^{-1}=\phi''(g)^2\phi'(g)=\phi_1(g) & \text{on } V(l-l_i)\,.\end{cases}\end{aligned}$$

This proves $\bar{\phi}_1=\phi_1$ on $U(l_i)$, $0<l_i<l/2$. Thus $\bar{\phi}_1=\phi_1$ everywhere.

With this ϕ_1 defined we can proceed in an entirely similar way as in the complex case. Note that, in the complex case, $\phi_1(g)$ on $V(l/2)$ satisfied (2.28) too. In particular (2.16) holds for $\mathcal{B}_{m,k}(S^1;\mathcal{F}_l^+)$ instead of $\mathcal{B}^U_{m,2k}(S^1;\mathcal{F}_l^+)$ and we can form the smooth manifolds $\boldsymbol{P}_\phi(V)=S(V)/\phi_1$ and $\boldsymbol{P}_\phi(V\times\boldsymbol{C})=S(V\times\boldsymbol{C})/\phi_1$ which we shall also call *twisted complex projective space bundles.* We orient $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ concordantly with $D(V)\subset\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ as in the complex case. The normal bundle of $\boldsymbol{P}_\phi(V)$ in $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ is oriented by its complex line bundle structure associated to $S(V)\to\boldsymbol{P}_\phi(V)$. Then the above orientations of $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$

and the normal bundle determine the orientation of $\boldsymbol{P}_{\psi}(V)$. Proposition (2.18) holds also for $\mathscr{B}_{m,k}(S^1; \mathscr{F}_l^+)$. We define the homomorphism

$$ {}^t\boldsymbol{P}: B_{m,k}(S^1; \mathscr{F}_l^+) \longrightarrow \Omega_{m+k}(S^1; \mathscr{F}_l^+) $$

by

$$ {}^t\boldsymbol{P}[X, V, \psi] = [\boldsymbol{P}_{\psi}(V \times \boldsymbol{C}), \psi]. $$

Then we obtain

THEOREM (2.29). *Let l be an integer, $1 < l$. The homomorphism*

$$ {}^t\boldsymbol{P} \circ \nu: \Omega_n(S^1, \mathscr{F}_l^+, \mathscr{F}_{l-1}^+) \longrightarrow \Omega_n(S^1; \mathscr{F}_l^+) $$

is a splitting for

$$ j_*: \Omega_n(S^1; \mathscr{F}_l^+) \longrightarrow \Omega_n(S^1; \mathscr{F}_l^+, \mathscr{F}_{l-1}^+). $$

COROLLARY (2.30). *There are canonical isomorphisms*

$$ \Omega_n(S^1; \mathscr{F}_l^+) \cong \Omega_n(S^1; \mathscr{F}_1^+) \oplus \sum_{1<k\leqq l} {}^tP_n(S^1; \mathscr{F}_k^+) $$

and

$$ \Omega_n(S^1; \mathscr{F}_\infty^+) \cong \Omega_n(S^1; \mathscr{F}_1^+) \oplus \sum_{1<k} {}^tP_n(S^1; \mathscr{F}_k^+) $$

where

$$ {}^tP_n(S^1; \mathscr{F}_l^+) = {}^t\boldsymbol{P}_n(\sum_{m+k=n} B_{m,k}(S^1; \mathscr{F}_l^+)). $$

REMARK. Let $(X, V, \psi) \in \mathscr{B}^U_{m,2k}(S^1; \mathscr{F}_l^+)$ and suppose that the action φ induced on X (see (2.1)) is free which implies in particular that the action ψ on V is $\mathscr{F}_l$-free. Even under this assumption *the fixed point set of the action ψ on $\boldsymbol{P}_{\psi}(V \times \boldsymbol{C})$ is not empty in general.* For example *when $l=2$ the submanifold $\boldsymbol{P}_{\psi}(V)$ is the fixed point set by* (2.21). However in this case, i. e. when the fixed point set F of the action φ on X is empty, *the action ψ'' is free so that $S(V \times \boldsymbol{C})/\psi''$ is a smooth manifold. Moreover the action ψ on $S(V \times \boldsymbol{C})/\psi''$ is $\mathscr{F}_{l-1}$-free. This can be used to give a splitting for*

$$ j_*: \Omega_n^U(S^1; \mathscr{F}_l) \longrightarrow \Omega_n^U(S^1; \mathscr{F}_l, \mathscr{F}_{l-1}). $$

Similarly let $(X, V, \psi) \in \mathscr{B}_{m,k}(S^1; \mathscr{F}_l^+)$ and assume that the fixed point set F of φ is empty and V has a structure of ψ-invariant complex vector bundle. Then we can form ψ'' and smooth manifold $S(V \times \boldsymbol{C})/\psi''$ in this case too (but not canonically). This can be used to show that

$$ j_*: \Omega_n(S^1; \mathscr{F}_l) \otimes \boldsymbol{Z}[1/2] \longrightarrow \Omega_n(S^1; \mathscr{F}_l, \mathscr{F}_{l-1}) \otimes \boldsymbol{Z}[1/2] $$

is onto. These constructions were used by Ossa [7].

Finally we remark that in the above constructions *we may replace $\psi''(g)$ by $\psi''(g)\psi'(g)$ which will give another splitting for j_*.*

§ 3. The Kosniowski Formula.

Let M be a closed U-manifold with a smooth S^1-action ϕ preserving the given U-structure. Then each component F_j of the fixed point set F is a U-manifold in a natural way. Moreover the normal bundle V_j of F_j in M decomposes as a direct sum

$$V_j = \sum_s V_{js}$$

of complex vector bundles V_{js} on which the given S^1-action ϕ is expressed by

$$\phi(g)v = g^{k_{js}}v\,, \qquad k_{js} \in \boldsymbol{Z},\ k_{js} \neq 0\,,$$

for $v \in V_{js}$, where $g^{k_{js}}v$ denotes the scalar multiplication in the complex vector bundle V_{js}. We define the integers $d^+(F_j)$ and $d^-(F_j)$ by

$$d^+(F_j) = \sum_{s,\,k_{js}>0} \dim_C V_{js}\,,$$

$$d^-(F_j) = \sum_{s,\,k_{js}<0} \dim_C V_{js}\,.$$

We shall call $d^+(F_j)$ ($d^-(F_j)$) *positive* (negative) *type number* of F_j. With these understood, the Kosniowski formula reads as follows.

THE KOSNIOWSKI FORMULA [6]. *Let M be a closed U-manifold with a smooth S^1-action preserving the U-structure. Then the following relation between the T_y-genera of M and the components of the fixed point set holds.*

$$\begin{aligned} T_y(M) &= \sum_j (-y)^{d^+(F_j)} T_y(F_j) \\ &= \sum_j (-y)^{d^-(F_j)} T_y(F_j)\,, \end{aligned}$$

where T_y is the genus associated to the formal power series in t

$$\frac{t(1+y)}{e^{t(1+y)}-1} + t\,,$$

cf. [2].

In this section we shall give an elementary proof of this formula. In view of Corollary (2.24) it is clearly sufficient to prove the formula for $[M, \phi] \in \Omega_n^U(S^1; \mathcal{F}_1^+)$ and $(M, \phi) = (\boldsymbol{P}_\phi(V \times \boldsymbol{C}), \phi)$ where $(X, V, \phi) \in \mathcal{B}_{m,2k}^U(S^1; \mathcal{F}_l^+)$.

I. Semi-free case. The proof given here is due to Takao Matumoto. We thank him for communicating us his proof.

Let $\phi_{p,q}$ be the S^1-action on $S^{2(p+q)-1}$ defined by

$$\phi_{p,q}(g)(z_1, \cdots, z_p, w_1, \cdots, w_q) = (gz_1, \cdots, gz_p, g^{-1}w_1, \cdots, g^{-1}w_q)$$

where $z_i, w_j \in \boldsymbol{C}$. The action is free so that the quotient space $\boldsymbol{C}P_{p,q} = S^{2(p+q)-1}/\phi_{p,q}$ is a closed smooth manifold. $\boldsymbol{C}P_{p,q}$ is made almost complex manifold by local charts

$$\left(\frac{z_1}{z_i}, \cdots, \frac{z_p}{z_i}, \frac{w_1}{\bar{z}_i}, \cdots, \frac{w_q}{\bar{z}_i}\right) \quad \text{where } z_i \neq 0$$

and

$$\left(\frac{z_1}{\bar{w}_j}, \cdots, \frac{z_p}{\bar{w}_j}, \frac{w_1}{w_j}, \cdots, \frac{w_q}{w_j}\right) \quad \text{where } w_j \neq 0 .$$

LEMMA (3.1). *With the above almost complex structure, we have*

$$T_y(CP_{p,q}) = \frac{1}{1-(-y)}((-y)^q-(-y)^p) .$$

PROOF. Consider the diffeomorphism

$$f\colon CP_{p,q} \longrightarrow CP^{p+q-1}$$

induced by $f\colon S^{2(p+q)-1} \to S^{2(p+q)-1}$ given by

$$f(z_1, \cdots, z_p, w_1, \cdots, w_q) = (z_1, \cdots, z_p, \bar{w}_1, \cdots, \bar{w}_q) .$$

Let CP' denote CP^{p+q-1} with the almost complex structure transported by f. Then it is not difficult to see that

$$\tau(CP') \oplus 1 = p\xi \oplus q\xi^*$$

where τ, 1, ξ and ξ^* denote the complex tangent bundle, the trivial complex line bundle, the canonical line bundle and its dual respectively. It is also clear that the orientation of CP' is $(-1)^q$ times the usual orientation of CP^{p+q-1}. It follows that

$$T_y(CP_{p,q}) = T_y(CP') = \text{coefficient of } (-1)^q x^n \text{ in } h(x)$$

where $n = p+q-1$ and

$$h(x) = \left(\frac{x(y+1)}{e^{x(y+1)}-1}+x\right)^p \left(\frac{-x(y+1)}{e^{-x(y+1)}-1}-x\right)^q .$$

By the Cauchy integral formula

$$T_y(CP_{p,q}) = \frac{(-1)^q}{2\pi i} \oint \frac{h(x)}{x^{n+1}} dx .$$

The substitution $u = e^{x(y+1)}-1$ gives

$$T_y(CP_{p,q}) = \frac{1}{2\pi i} \frac{(-y)^q}{y+1} \oint \frac{(1+u+y)^p\left(1+u+\frac{1}{y}\right)^q}{u^{n+1}(1+u)} du .$$

Hence

$$T_y(CP_{p,q}) = (-y)^q/(y+1)$$
$$\times (\text{coefficient of } u^n \text{ in } (1+u+y)^p(1+u+1/y)^q/(1+u)) .$$

But $(1+u+y)^p(1+u+1/y)^q$ is of the form

$$\sum_{t=0}^{n+1} a_t(1+u)^t$$

with $a_0=y^p/y^q$ and $a_{n+1}=1$. Therefore

$$\text{coefficient of } u^n \text{ in } (1+u+y)^p(1+u+1/y)^q/(1+u)=(-1)^n y^p/y^q+1\,.$$

Hence we obtain

$$T_y(\boldsymbol{CP}_{p,q})=\frac{1}{1-(-y)}((-y)^q-(-y)^p)\,.$$

Now suppose that the S^1-action ϕ is semi-free (i.e. $\mathcal{F}_1^+$-free) on M. We define the S^1-actions ϕ and ϕ_1 on $V_i\times \boldsymbol{C}$ by

$$\phi(g)(v,\alpha)=(\phi(g)v,\alpha)$$

and

$$\phi_1(g)(v,\alpha)=(\phi(g)v,g\alpha)\,.$$

Choose a ϕ-invariant hermitian metric on the complex vector bundle V_j. Let $D(V_j\times \boldsymbol{C})$ and $D(V_j)$ be the associated unit disk bundles and $S(V_j\times \boldsymbol{C})$ and $S(V_j)$ the associated sphere bundles. Since the action ϕ_1 is free on $S(V_j\times \boldsymbol{C})$, the quotient space $\boldsymbol{P}_\phi(V_j\times \boldsymbol{C})=S(V_j\times \boldsymbol{C})/\phi_1$ and $S(V_j)/\phi_1$ are smooth manifolds. Just as in (2.17), $\boldsymbol{P}_\phi(V_j\times \boldsymbol{C})$ is identified with $D(V_j)\cup W_\phi(V_j)$ where $W_\phi(V_j)$ is the disk bundle associated to the S^1-bundle $S(V_j)\to \boldsymbol{P}_\phi(V_j)$. In particular $\boldsymbol{P}_\phi(V_j\times \boldsymbol{C})$ is endowed with a ϕ-invariant U-structure which extends that of $D(V_j)$. Moreover since ϕ_1 acts on V_j by automorphisms, $\boldsymbol{P}_\phi(V_j\times \boldsymbol{C})$ is fibered over F_j with fiber $\boldsymbol{CP}_{d_j^+ +1, d_j^-}$ where $d_j^\pm=d^\pm(F_j)$. Then the U-structure of $\boldsymbol{P}_\phi(V_j\times \boldsymbol{C})$ given above is compatible in the sense of [2, (21.8)]. Similarly $\boldsymbol{P}_\phi(V_j)$ has a ϕ-invariant U-structure and is fibered over F_j with fiber $\boldsymbol{CP}_{d_j^+, d_j^-}$. With these understood,

LEMMA (3.2). *Let ϕ be a U-structure preserving semi-free S^1-action on a closed U-manifold M. Then we have*

$$\sum_j [\boldsymbol{P}_\phi(V_j\times \boldsymbol{C}),\phi]=[M,\phi] \qquad \text{in } \Omega_*^U(S^1;\mathcal{F}_1^+)$$

and

$$\sum_j [\boldsymbol{P}_\phi(V_j)]=0 \qquad \text{in } \Omega_*^U\,.$$

PROOF. Let M_1 be the manifold obtained by glueing together $M-\cup \operatorname{int} D(V_j)$ and $\cup - W_\phi(V_j)$ along their common boundary $\cup S(V_j)$. Then, as in (2.23) we have

$$\Sigma\,[\boldsymbol{P}_\phi(V_j\times \boldsymbol{C}),\phi]+[M_1,\phi]=[M,\phi] \tag{3.3}$$

in $\Omega_*^U(S^1;\mathcal{F}_1^+)$. But the action ϕ restricted on $M_0=M-\cup \operatorname{int} D(V_j)$ is free so that $M_0/\phi=Y$ is a U-manifold. Let N be the 2-disk bundle associated to the S^1-fibering $M_0\to Y$. Then clearly we have

$$\partial N=M_1 \quad \text{and} \quad \partial Y=\cup \boldsymbol{P}_\phi(V_j)\,.$$

Therefore

$$[M_1, \psi]=0 \quad \text{in } \Omega_*^U(S^1;\mathcal{F}_1^+)$$

and

$$\sum [\boldsymbol{P}_\psi(V_j)]=0 \quad \text{in } \Omega_*^U.$$

This together with (3.3) proves Lemma.

PROOF OF THE KOSNIOWSKI FORMULA FOR $[M, \psi]\in\Omega_*^U(S^1;\mathcal{F}_1^+)$. We first remark that the bundle $\boldsymbol{P}_\psi(V_j\times\boldsymbol{C})\to F_j$ has $U(d_j+1)$, $d_j=d_j^+ +d_j^-$, as structure group and the almost complex structure on the fiber $\boldsymbol{CP}_{d_j^+ +1, d_j^-}$ is invariant under the action of $U(d_j+1)$. Therefore, by the strictly multiplicative property of the T_y-genus [2, (22.8)] we get

$$T_y(\boldsymbol{P}_\psi(V_j\times\boldsymbol{C}))=T_y(F_j)T_y(\boldsymbol{CP}_{d_j^+ +1, d_j^-}).$$

Then by (3.1)

$$T_y(\boldsymbol{P}_\psi(V_j\times\boldsymbol{C}))=\frac{1}{1-(-y)}((-y)^{d_j^-}-(-y)^{d_j^+ +1})T_y(F_j).$$

Combining this with (3.2) we obtain

$$T_y(M)=\frac{1}{1-(-y)}\sum_j((-y)^{d_j^-}-(-y)^{d_j^+ +1})T_y(F_j). \tag{3.4}$$

Similarly from the second equality in (3.2) we get

$$0=\frac{1}{(1-(-y))}\sum_j((-y)^{d_j^-}-(-y)^{d_j^+})T_y(F_j). \tag{3.5}$$

Subtracting (3.5) from (3.4) yields

$$T_y(M)=\sum(-y)^{d_j^+}T_y(F_j).$$

This together with (3.5) yields

$$T_y(M)=\sum(-y)^{d_j^-}T_y(F_j).$$

II. Case of $[\boldsymbol{P}_\psi(V\times\boldsymbol{C}), \psi]\in {}^tP_*^U(S^1;\mathcal{F}_t^+)$. Given $(X, V, \psi)\in\mathcal{B}_{m,2k}^U(S^1;\mathcal{F}_t^+)$, let F be the fixed point set of φ in X and let $\{F_j\}$ be the connected components of F. Let U_j be the normal bundle of F_j in X. The action φ decomposes U_j into the direct sum $U_j=\sum U_{jt}$ so that

$$\varphi(g)u=g^{k_{jt}}u \qquad \text{for } g\in S^1 \text{ and } u\in U_{jt}$$

where $k_{jt}\in\boldsymbol{Z}$. Set

$$U_j^+=\sum_{k_{jt}>0}U_{jt}$$

and

$$U_j^-=\sum_{k_{jt}<0}U_{jt}.$$

We also set

$$V_j^+ = \sum_{l_i} V_j^+(l_i)\,,$$
$$V_j^- = \sum_{l_i} V_j^-(l_i)$$

where $V_j^\pm(l_i)$ are as in (2.9). Note that $\{F_j\}$ is a part of connected components of the fixed point set of the action ψ in $\boldsymbol{P}_\psi(V\times \boldsymbol{C})$ and we have

$$\begin{aligned} d^+(F_j) &= \dim U_j^+ + \dim V_j^+\,, \\ d^-(F_j) &= \dim U_j^- + \dim V_j^-\,. \end{aligned} \tag{3.6}$$

Here and throughout this Section dim means the complex dimension.

We consider the action ψ'' defined in (2.10). Since ψ'' commutes with ψ_1 it can be extended to the S^1-action ψ'' on $\boldsymbol{P}_\psi(V\times \boldsymbol{C})$ by the formula

$$\psi''(g)[v, \alpha] = [\psi''(g)v, \alpha]\,.$$

LEMMA (3.7). *Let $(X, V, \psi) \in \mathscr{B}^U_{m,2k}(S^1\,;\, \mathscr{F}_l^+)$ and suppose that X is connected. The action ψ'' on $\boldsymbol{P}_\psi(V\times \boldsymbol{C})$ is semi-free. Its fixed point set consists of components $\boldsymbol{P}_\psi(V_j^+\times \boldsymbol{C})$ and $\boldsymbol{P}_\psi(V_j^-)$. Their type numbers are given by*

$$\begin{aligned} d^+(\boldsymbol{P}_\psi(V_j^+\times \boldsymbol{C})) &= \dim U_j^+\,, \\ d^-(\boldsymbol{P}_\psi(V_j^+\times \boldsymbol{C})) &= \dim U_j^- + \dim V_j^-\,, \\ d^+(\boldsymbol{P}_\psi(V_j^-)) &= \dim U_j^-\,, \\ d^-(\boldsymbol{P}_\psi(V_j^-)) &= \dim U_j^+ + \dim V_j^+ + 1\,. \end{aligned}$$

PROOF. Since ψ'' covers φ, its fixed point set is contained in $\cup \boldsymbol{P}_\psi(V|F_j\times \boldsymbol{C})$. Then, using (2.11), we see that the fixed point set is as stated. As to the type numbers of $\boldsymbol{P}_\psi(V_j^+\times \boldsymbol{C})$, since it contains F_j around which the action ψ'' is equivalent to the given action ψ'' on V the statement follows from the definition of $U_j^\pm$ and $V_j^\pm$.

Next consider $\boldsymbol{P}_\psi(V_j^-)$. Let $\mathring{D}(U_j)$ be a small φ-invariant open tubular neighborhood of F_j in X. Then the bundle $V|\mathring{D}(U_j)$ can be ψ-equivariantly identified with the complex vector bundle $V\oplus U_j$. With this in mind, given a point $(v_0, 0) \in S(V_j^-) \subset S(V_j^- \times \boldsymbol{C})$ any point in $S(V_j^-\times \boldsymbol{C})$ near $(v_0, 0)$ can be expressed in the form (v_0+v, α), $v\in V\oplus U_j$, $\alpha \in \boldsymbol{C}$. Note that the normal vectors to $\boldsymbol{P}_\psi(V_j^-)$ in $\boldsymbol{P}_\psi(V\times \boldsymbol{C})$ at $[v_0, 0]$ are spanned by $[v_0, \alpha]$, $[v_0+v, 0]$ with $v\in V_j^+$ and $[v_0+u, 0]$ with $u \in U_j^\pm$. We compute the effect of $\psi''(g)$ on these generators.

$$\begin{aligned} \psi''(g)[v_0, \alpha] &= [\psi''(g)v_0, \alpha] \\ &= [\psi'(g)^{-1}v_0, \alpha] && \text{by (2.11)} \\ &= [\psi_1(g)v_0, \alpha] && \text{by (2.13)} \\ &= [v_0, g^{-1}\alpha]\,. \end{aligned}$$

$$\begin{aligned}\psi''(g)[v_0+v,\,0]&=[\psi'(g)^{-1}v_0+v,\,0] \qquad \text{for } v\in V_j^- \text{ by (2.11)}\\&=[\psi_1(g)v_0+v,\,0] \qquad \text{by (2.13)}\\&=[v_0+\psi_1^{-1}(g)v,\,0]\\&=[v_0+\psi'(g)^{-1}v,\,0]\,.\end{aligned}$$

$$\begin{aligned}\psi''(g)[v_0+u,\,0]&=[\psi'(g)^{-1}v_0+\varphi(g)u,\,0] \qquad \text{for } u\in U_j \text{ by (2.11)}\\&=[\psi_1(g)v_0+\varphi(g)u,\,0]\\&=[v_0+\varphi(g)^{-2}\varphi(g)u,\,0] \qquad \text{since } \psi_1(g)=\varphi^2(g) \text{ on } U_j\\&=[v_0+\varphi(g)^{-1}u,\,0]\,.\end{aligned}$$

Therefore we have

$$d^+(\boldsymbol{P}_\psi(V_j^-))=\dim U_j^-$$

$$d^-(\boldsymbol{P}_\psi(V_j^-))=\dim U_j^+ +\dim V_j^+ +1\,.$$

In an entirely similar way we obtain

LEMMA (3.8). *Under the same assumption as in* (3.7), *the fixed point set of* ψ'' *in* $\boldsymbol{P}_\psi(V)$ *consists of components* $\boldsymbol{P}_\psi(V_j^+)$ *and* $\boldsymbol{P}_\psi(V_j^-)$ *for which the type numbers are given by*

$$d^+(\boldsymbol{P}_\psi(V_j^+))=\dim U_j^+\,,$$

$$d^-(\boldsymbol{P}_\psi(V_j^+))=\dim U_j^- +\dim V_j^-\,,$$

$$d^+(\boldsymbol{P}_\psi(V_j^-))=\dim U_j^-\,,$$

$$d^-(\boldsymbol{P}_\psi(V_j^-))=\dim U_j^+ +\dim V_j^+\,.$$

The following Corollary (3.10) is a variant of the Kosniowski formula for $(\boldsymbol{P}_\psi(V\times\boldsymbol{C}),\,\psi)$.

PROPOSITION (3.9). *Let* $(X,\,V,\,\psi)\in\mathscr{B}^U_{m,2k}(S^1;\,\mathscr{F}_l^+)$. *Let* $\{F_j\}$ *be the components of the fixed point set of* φ *in* X, *and let* $U_j^\pm$ *and* $V_j^\pm$ *be defined as above. We have*

$$T_y(\boldsymbol{P}_\psi(V\times\boldsymbol{C}))=\frac{1}{1-(-y)}\sum_j\{(-y)^{\dim U_j^-+\dim V_j^-}-(-y)^{\dim U_j^+ +\dim V_j^+ +1}\}T_y(F_j)$$

and

$$T_y(\boldsymbol{P}_\psi(V))=\frac{1}{1-(-y)}\sum_j\{(-y)^{\dim U_j^-+\dim V_j^-}-(-y)^{\dim U_j^+ +\dim V_j^+}\}T_y(F_j)\,.$$

COROLLARY (3.10). *Under the same assumption as in* (3.9) *the following relations hold.*

$$\begin{aligned}T_y(\boldsymbol{P}_\psi(V\times\boldsymbol{C}))&=T_y(\boldsymbol{P}_\psi(V))+\sum_j(-y)^{\dim U_j^+ +\dim V_j^+}T_y(F_j)\\&=(-y)T_y(\boldsymbol{P}_\psi(V))+\sum_j(-y)^{\dim U_j^- +\dim V_j^-}T_y(F_j)\,,\end{aligned}$$

$$0=\sum_j((-y)^{\dim U_j^+}-(-y)^{\dim U_j^-})T_y(F_j).$$

PROOF OF (3.9) AND (3.10). The action φ on X is semi-free so that we can apply the Kosniowski formula proved in I to get the last relation of (3.10). The action ϕ'' on $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ is semi-free. Hence we can apply the Kosniowski formula to this action. By the strictly multiplicative property of T_y-genus and (3.1),

$$T_y(\boldsymbol{P}_\phi(V_j^+\times\boldsymbol{C}))=\frac{1}{1-(-y)}(1-(-y)^{\dim V_j^++1})T_y(F_j),$$

$$T_y(\boldsymbol{P}_\phi(V_j^-))=\frac{1}{1-(-y)}((-y)^{\dim V_j^-}-1)T_y(F_j).$$

Using the data in (3.7) we obtain

$$T_y(\boldsymbol{P}_\phi(V\times\boldsymbol{C}))=\frac{1}{1-(-y)}\sum_j\{(-y)^{\dim U_j^+}(1-(-y)^{\dim V_j^++1})$$
$$+(-y)^{\dim U_j^-}((-y)^{\dim V_j^-}-1)\}T_y(F_j).$$

Using the last relation in (3.10) we obtain

$$T_y(\boldsymbol{P}_\phi(V\times\boldsymbol{C}))=\frac{1}{1-(-y)}\sum_j\{(-y)^{\dim U_j^-+\dim V_j^-}-(-y)^{\dim U_j^++\dim V_j^++1}\}T_y(F_j).$$

The formula for $T_y(\boldsymbol{P}_\phi(V))$ is proved similarly using (3.8). This proves (3.9). The rest of the statement in (3.10) is immediate from (3.9).

Now we shall deduce the Kosniowski formula for $(\boldsymbol{P}_\phi(V\times\boldsymbol{C}),\phi)$ from (3.10). We proceed by induction on l where $(X, V, \phi)\in\mathscr{B}^U_{m,2k}(S^1;\mathscr{F}_l^+)$, $1<l$. Let $\{F_j\}$ be the components of the fixed point set of φ in X. First suppose $l=2$. Then by (2.21) the fixed point set of ϕ is the union of F_j and $\boldsymbol{P}_\phi(V)$. As in the proof of (3.7) we see that the type number of $\boldsymbol{P}_\phi(V)$ is given by

$$d^+(\boldsymbol{P}_\phi(V))=0 \qquad\text{and}\qquad d^-(\boldsymbol{P}_\phi(V))=1.$$

Thus with this and (3.6) the formulae in (3.10) are nothing but Kosniowski's one in this case.

Next suppose $l>2$. Then the components of the fixed point set consists of $\{F_j\}$ and $\{F_s'\}$ where $F_s'\subset\boldsymbol{P}_\phi(V)$. See (2.21). Let $d^\pm(F_s')$ be the type numbers of F_s', and let $d'^\pm(F_s')$ denote the type numbers of F_s' with respect to the action ϕ restricted on $\boldsymbol{P}_\phi(V)$. As in the proof of (3.7) we have

(3.11) $$d^+(F_s')=d'^+(F_s')\qquad\text{and}\qquad d^-(F_s')=d'^-(F_s')+1.$$

By (2.18) the action ϕ on $\boldsymbol{P}_\phi(V)$ is $\mathscr{F}_{l-1}^+$-free. Hence by the induction assumption we can apply the Kosniowski formula to this action to get

$$T_y(\boldsymbol{P}_\varphi(V))=\sum_s(-y)^{d'^+(F'_s)}T_y(F'_s)$$
$$=\sum_s(-y)^{d'^-(F'_s)}T_y(F'_s)\,.$$

Substitute this in the formula (3.10) and use (3.6) and (3.11). We obtain

$$T_y(\boldsymbol{P}_\varphi(V\times \boldsymbol{C}))=\sum_s(-y)^{d^+(F'_s)}T_y(F'_s)+\sum_j(-y)^{d^+(F_j)}T_y(F_j)$$
$$=\sum_s(-y)^{d^-(F'_s)}T_y(F'_s)+\sum_j(-y)^{d^-(F_j)}T_y(F_j)\,.$$

This proves the Kosniowski formula in its full generality.

§ 4. The Atiyah-Singer Formula.

In the case of oriented manifold with a smooth S^1-action, the normal bundle V_j of each component F_j of the fixed point set has still an S^1-invariant complex vector bundle structure with a direct sum decomposition

$$V_j=\sum_s V_{js}$$

such that

$$\varphi(g)v=g^{k_{js}}v$$

for $v\in V_{js}$. Here the complex structure on V_{js} is determined up to sign of k_{js}. We fix it by requiring $k_{js}>0$. Then the normal bundle V_j and the manifold F_j are canonically oriented. We set $d(F_j)=\dim_C V_j$. With the above orientation convention we have

THE ATIYAH-SINGER FORMULA [**1**, p. 594]. *Let M be an oriented closed smooth manifold with a smooth S^1-action. Then*

$$\operatorname{sign} M=\sum_{F_j,d(F_j)\text{ even}}\operatorname{sign}(F_j)\,,$$
$$0=\sum_{F_j,d(F_j)\text{ odd}}\operatorname{sign}(F_j)\,.$$

An elementary proof of this formula will be given in the sequel. By (2.30) it is sufficient to prove it for $[M,\varphi]\in\Omega_*(S^1;\mathcal{F}_1^+)$, and $[M,\varphi]=[\boldsymbol{P}_\varphi(V\times\boldsymbol{C}),\varphi]\in{}^tP_n(S^1;\mathcal{F}_l^+)$, $1<l$. As to the case of $\Omega_*(S^1;\mathcal{F}_1^+)$ we refer to [**5**] where a proof similar to that of I in Section 3 is given. Thus we confine our attention to the case of ${}^tP_n(S^1;\mathcal{F}_l^+)$. Given $(X,V,\varphi)\in\mathcal{B}_{m,2k}(S^1;\mathcal{F}_l^+)$ the real vector bundle V does not necessarily have a structure of complex vector bundle. Consequently we can not in general use auxiliary action φ'' as in the complex case. To remedy this point we first make some cohomological considerations for a special type of (X,V,φ). Let $\{F_j\}$ be the components of the fixed point set F of φ on X as before.

I. We first *assume that each F_j has real codimension 2 in X.*

We shall prove

PROPOSITION (4.1). *Suppose that $(X, V, \psi) \in \mathscr{B}_{m,2k}(S^1; \mathscr{F}_l^+)$ and each F_j has real codimension 2 in X. Then*

$$\operatorname{sign} \boldsymbol{P}_\psi(V \times \boldsymbol{C}) = \begin{cases} \sum_j \operatorname{sign} F_j & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \text{ is even}. \end{cases}$$

$$\operatorname{sign} \boldsymbol{P}_\psi(V) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ \sum_j \operatorname{sign} F_j & \text{if } k \text{ is even}. \end{cases}$$

REMARK. In (4.1), *assume moreover that the manifold X (and hence the vector bundle V too) is orientable. Then the Atiyah-Singer formula applied to the semi-free action φ on X yields $\sum_j \operatorname{sign} F_j = 0$. Therefore* $\operatorname{sign} \boldsymbol{P}_\psi(V \times \boldsymbol{C}) = \operatorname{sign} \boldsymbol{P}_\psi(V) = 0$ *in this case.*

The proof of (4.1) is preceded by several lemmas. We shall only give proof for $\boldsymbol{P}_\psi(V)$, the case for $\boldsymbol{P}_\psi(V \times \boldsymbol{C})$ being entirely similar.

We use the following notations. $P = \boldsymbol{P}_\psi(V)$, $P_0 = \boldsymbol{P}_\psi(V|F) = S(V|F)/\psi_1$, $Y = X/\varphi$. Let $i: P_0 \subset P$, $j: P \subset (P, P_0)$, $i': F \subset Y$ and $j': Y \subset (Y, F)$ be inclusions. Since the projection $S(V) \to X$ is equivariant with respect to ψ_1 and φ^2 it induces a map $\pi: P \to Y$. Let $\pi_0: P_0 \to F$ be the restriction of π and set $\pi_1 = (\pi, \pi_0): (P, P_0) \to (Y, F)$. It is easy to see that *$\pi: P - P_0 \to Y - F$ is a fiber bundle which has $(2k-1)$-dimensional real projective space $\boldsymbol{RP}^{2k-1}$ as fiber and $\pi_0: P_0 \to F$ is a fiber bundle with fiber $\boldsymbol{CP}^{k-1}$ associated to the vector bundle V with the complex structure determined by our orientation convention.* Moreover, since each F_j has real codimension 2 in X, *the quotient space $Y = X/\varphi$ is a compact manifold with boundary F.* Take a collar neighborhood $F \times [0, 1]$ of $F = \partial Y$ in Y and set $Y_1 = Y - F \times [0, 1)$, $Q_0 = \pi^{-1}(F \times [0, 1])$ and $Q_1 = \pi^{-1}(Y_1)$. Note that Q_0 is a tubular neighborhood of P_0 in P.

We shall consider the following commutative diagram.

(4.2)
$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & & {} \arrow[d] & \\
\cdots \arrow[r, "\delta'^*"] & H^q(Y, F) \arrow[r, "j'^*"] \arrow[d, "\pi_1^*"] & H^q(Y) \arrow[r, "i'^*"] \arrow[d, "\pi^*"] & H^q(F) \arrow[r, "\delta'^*"] \arrow[d, "\pi_0^*"] & H^{q+1}(Y, F) \arrow[r] \arrow[d, "\pi_1^*"] & {} \\
\cdots \arrow[r] & H^q(P, P_0) \arrow[r, "j^*"] \arrow[d, "\pi_{1!}"] & H^q(P) \arrow[r, "i^*"] \arrow[d, "\pi_!"] & H^q(P_0) \arrow[r, "\delta^*"] & H^{q+1}(P, P_0) & \\
 & \hat{H}^{q-(2k-1)}(Y, F) \arrow[r, equal] \arrow[d, "\rho_1"] & \hat{H}^{q-(2k-1)}(Y, F) \arrow[d, "\rho"] & & & \\
H^q(F) \arrow[r, "\delta'^*"] & H^{q+1}(Y, F) \arrow[r, "j'^*"] \arrow[d] & H^{q+1}(Y) \arrow[d] & & & \\
 & {} & {} & & &
\end{tikzcd}$$
.

Here H^* denotes the usual rational cohomology and $\hat{H}^*$ denotes the cohomology with coefficients in the rational orientation sheaf of the manifold Y. $\pi_!$ and $\pi_{1!}$ are Gysin homomorphisms; i.e. $\pi_! = \vartheta^{-1}\pi_*\vartheta$ where ϑ denotes the Poincaré-Lefschetz duality and $\pi_{1!}$ is the transform (via excision) of $\vartheta^{-1}\pi_*\vartheta$: $H^q(Q_1, \partial Q_1) \to \hat{H}^{q-(2k-1)}(Y_1, \partial Y_1)$. The homomorphism ρ_1 is given by $\rho_1(y) = y \cdot \bar{\chi}$ where $\bar{\chi} \in \hat{H}^{2k}(Y)$ is the rational characteristic class of $\boldsymbol{R}P^{2k-1}$-bundle π: $Q_1 \to Y_1 \cong Y$, and $\rho = j'^* \circ \rho_1$.

Let $i_!: H^*(P_0) \to H^*(P)$ be the Gysin homomorphism of i. As is well-known, the element $\chi = i^*i_!(1)$ of $H^2(P_0)$ is the Euler class of the normal bundle ν_i of the embedding i. Let $e \in H^2(P_0)$ denote the first Chern class of the canonical line bundle ξ of the complex projective space bundle P_0. Let $c_1 \in H^2(F)$ denote the first Chern class of the normal bundle μ of F in X with the complex structure determined by our orientation convention.

LEMMA (4.3). *With the above notations, we have*

$$\chi = 2e + \pi_0^*(c_1).$$

PROOF. We claim that

$$\nu_i = \xi^2 \otimes \pi_0^*(\mu),$$

which implies Lemma (4.3). For the additivity of the first Chern class with respect to the tensor product of complex line bundle yields

$$\begin{aligned} \chi = c_1(\nu_i) &= 2c_1(\xi) + \pi_0^* c_1(\mu) \\ &= 2e + \pi_0^* c_1. \end{aligned}$$

To prove the claim, note that the normal bundle $\tilde{\nu}$ of $S(V|F)$ in $S(V)$ is equivalent to $\tilde{\pi}^*\mu$ where $\tilde{\pi}: S(V|F) \to F$ is the projection. Moreover we can choose an equivalence equivariantly with respect to ψ_1. Thus we may assume that $\tilde{\nu} = S(V|F) \underset{F}{\times} \mu$ (fiber product) with the action given by

$$\psi_1(g)(v, u) = (\psi'(g)v, \varphi(g)^2 u).$$

$S(V|F)$ is contained in the Hopf bundle $\bar{\xi}$, the conjugate bundle of ξ, as the sphere bundle. With this understanding, it is easy to see that the assignment

$$[v, u] \longmapsto \bar{v} \otimes \bar{v} \otimes u$$

gives a well-defined equivalence

$$\nu_i = \tilde{\nu}/\psi_1 \longrightarrow \xi^2 \otimes \pi_0^*(\mu)$$

where $\bar{v}$ is the conjugation of $v \in \bar{\xi}$ in ξ. This proves (4.3).

To proceed further we recall some fundamental properties of the Gysin homomorphism which we shall use later.

(4.4) $$i^*i_!(\pi_0^*(y)\chi^j) = \pi_0^*(y)\chi^{j+1} \qquad \text{for } y \in H^*(F).$$

(4.5) $$\pi_! i_! = \pm\delta^* \pi_{0!}\,.$$

(4.6) $$\pi_!(\pi^*(y)x) = y\pi_!(x) \qquad \text{for } y \in H^*(Y) \text{ and } x \in H^*(P)\,.$$

Now since P_0 is a complex projective space bundle, $H^*(P_0)$ is a free $H^*(F)$-module (via π_0^*) on generators $1, e, \cdots, e^{k-1}$. In virtue of (4.3) $1, \chi, \cdots, \chi^{k-1}$ also form a system of free $H^*(F)$-module generators.

LEMMA (4.7). *The Gysin homomorphism*

$$\pi_{0!} : H^*(P_0) \longrightarrow H^*(F)$$

is onto. Its kernel equals

$$A = \sum_{j=0}^{k-2} H^*(F)\chi^j\,.$$

PROOF. $\pi_{0!}$ lowers degree by $2(k-1)$. Hence $\pi_{0!}(\chi^j) = 0$ for $j < k-1$. Then

$$\pi_{0!}(\pi_0^*(y)\chi^j) = y\pi_{0!}(\chi^j) = 0 \qquad \text{for } j < k-1\,.$$

Thus $\pi_{0!}(A) = 0$. If we assume $\pi_{0!}(\chi^{k-1}) = 0$, then $\pi_{0!}$ would be trivial. But the Gysin homomorphism maps the top dimensional classes of P_0 into the top dimensional classes of F non-trivially. Hence $\pi_{0!}(\chi^{k-1}) \neq 0$ and $\pi_{0!}(H^*(F)\chi^{k-1}) = H^*(F)$.

LEMMA (4.8). *Let* $A = \sum_{j=0}^{k-2} H^*(F)\cdot\chi^j$ *as above. Then* $i_!|A$ *and* $i^*|i_!(A)$ *are injective.*

PROOF. This follows immediately from (4.4).

LEMMA (4.9). *The rows of* (4.2) *are exact. The columns of* (4.2) *are exact except for the part*

$$H^q(Y) \xrightarrow{\pi^*} H^q(P) \xrightarrow{\pi_!} H^{q-(2k-1)}(Y, F)\,.$$

PROOF. The rows are part of exact sequences of pairs and hence exact. The first column is exact as part of the Gysin exact sequence in the rational cohomology of the $\boldsymbol{RP}^{2k-1}$-bundle $\pi_1 : (Q_1, \partial Q_1) \to (Y_1, \partial Y_1)$.

To prove the exactness of

(4.10) $$H^q(P) \xrightarrow{\pi_!} \hat{H}^{q-(2k-1)}(Y, F) \xrightarrow{\rho} H^{q+1}(Y)$$

we consider the following commutative diagram

$$\begin{tikzcd}
\arrow[r] & H^q(P, Q_1) \arrow[r, "j_1^*"] \arrow[d, equal] & H^q(P) \arrow[r] \arrow[dd, "\pi_!"] & H^q(Q_1) \arrow[r, "\delta_1^*"] \arrow[dd, "\pi_!'"] & H^{q+1}(P, Q_1) \arrow[r, "j_1^*"] \arrow[d, equal] & {} \\
& H^q(Q_0, \partial Q_0) \arrow[d, "\pi_{0!}'"] & & & H^q(Q_0, \partial Q_0) \arrow[d, "\pi_{0!}'"] & \\
\arrow[r] & H^{q-2k}(F) \arrow[r] & \hat{H}^{q-(2k-1)}(Y, F) \arrow[r] \arrow[d, "\rho"] & \hat{H}^{q-(2k-1)}(Y) \arrow[r] \arrow[d, "\rho'"] & H^{q+1-2k}(F) \arrow[r] & {} \\
& & H^{q+1}(Y) \arrow[r, equal] & H^{q+1}(Y) & &
\end{tikzcd}$$

where $\pi'_{0!}$ and $\pi'_!$ are Gysin homomorphisms and $\rho'(y)=y\bar{\chi}$. Let

$$\phi : H^*(P_0) \longrightarrow H^*(Q_0, \partial Q_0)$$

be the Thom isomorphism. Then we have

$$\pi'_{0!}=\pi_{0!}\circ\phi^{-1} \qquad \text{and} \qquad j_1^*\circ\phi=i_! .$$

Therefore from (4.7) and (4.8) it follows that $\pi'_{0!}$ is surjective and $\delta_1^*(H^q(Q_1)) \cap \text{Kernel}\ \pi'_{0!}=0$. Then the exactness of (4.10) follows from a diagram chasing using the exactness of the third column of the above diagram.

PROPOSITION (4.11). *Let* $(X, V, \psi)\in \mathcal{B}_{m,2k}(S^1; \mathcal{F}_l^+)$ *and assume that* $\operatorname{codim}_R F_j=2$ *for all* F_j. *Then*

$$\textit{Kernel of } \pi_! : H^q(P) \longrightarrow \hat{H}^{q-(2k-1)}(Y, F)$$
$$=\pi^*H^q(Y)\oplus i_!(A^{q-2}) \qquad (\textit{direct sum})$$

where

$$A^{q-2}=\sum_{j=0}^{k-2} H^{q-2-2j}(F)\chi^j .$$

PROOF. Since $\pi_!$ lowers degree by $2k-1$ we have $\pi_!(1)=0$. Then, by (4.6),

$$\pi^*H^q(Y)\subset \text{Kernel of } \pi_! .$$

By (4.5) and (4.7) we have, for $j\leqq k-2$,

$$\pi_! i_!(\pi_0^*(y)\chi^j)=\pm\delta^*\pi_{0!}(\pi_0^*(y)\chi^j)=0 .$$

Thus $i_!(A)\subset$ Kernel of $\pi_!$.

Next, using (4.4) we obtain

$$i^*(\pi^*H^q(Y)\cap i_!(A^{q-2}))\subset \pi_0^*H^q(F)\cap A^{q-2}\cdot\chi=0 .$$

But i^* is injective on $i_!(A^{q-2})$ by (4.8). Hence $\pi^*H^q(Y)\cap i_!(A^{q-2})=0$. We have proved that

$$\pi^*H^q(Y)\oplus i_!(A^{q-2})\subset \text{Kernel } \pi_! .$$

To prove the equality it is therefore sufficient to show that

$$\dim \pi^*H^q(Y)+\dim A^{q-2}=\dim \text{Kernel } \pi_! ,$$

or

$$\dim \pi^*H^q(Y)+\dim A^{q-2}+\dim \pi_!H^q(P)=\dim H^q(P) .$$

This follows from a diagram chasing of (4.2) using (4.9). We leave the details to the reader. We only note that

$$i^*H^q(P)=A^{q-2}\cdot\chi\oplus\pi_0^*(\delta'^{*-1}(\rho_1\hat{H}^{q-(2k-1)}(Y, F)))$$

as follows easily from (4.4).

LEMMA (4.12). *Suppose that* $(X, V, \psi)\in \mathcal{B}_{2m,2k}(S^1; \mathcal{F}_l^+)$ *and* $\operatorname{codim}_R F_j=2$

for all F_j. Then the pairing

$$\pi^* H^{m+k-1}(Y) \times \pi_! H^{m+k-1}(P) \longrightarrow \boldsymbol{R}$$

defined by

$$\pi^* y \cdot \pi_!(x) = (\pi^* y \cdot x)[P]$$

is a dual pairing. In particular

$$\dim \pi^* H^{m+k-1}(Y) = \dim \pi_! H^{m+k-1}(P)\,.$$

PROOF. If $\pi^* y \cdot \pi_!(x) = 0$ for any x, then by Poincaré duality in $P, \pi^* y = 0$. Suppose that $\pi^* y \cdot \pi_!(x) = 0$ for any $y \in H^{m+k-1}(Y)$. Then

$$0 = \pi_!(\pi^* y \cdot x)[Y, F] = (y \cdot \pi_!(x))[Y, F]$$

for all y by (4.6). Hence $\pi_!(x) = 0$. This proves (4.12).

We are now ready to prove Proposition (4.1). In the case of $\boldsymbol{P}_\phi(V)$ we may clearly assume that $(X, V, \phi) \in \mathcal{B}_{2m,2k}(S^1; \mathcal{F}_t^+)$ and $m+k-1$ is even. We set

$$B_1 = i_!(A^{m+k-3})\,, \qquad B_2 = \pi^* H^{m+k-1}(Y)$$

and

$$B_3 = \text{a complement of } B_1 \oplus B_2 \text{ in } H^{m+k-1}(P)\,.$$

Then by (4.6), (4.11) and (4.12) the matrix of the cup product

$$H^{m+k-1}(P) \times H^{m+k-1}(P) \longrightarrow \boldsymbol{R}$$

with respect to the decomposition $H^{m+k-1}(P) = B_1 \oplus B_2 \oplus B_3$ is of the following form.

B_1	M_{11}	0	*
B_2	0	0	M_{23}
B_3	*	${}^tM_{23}$	*

It follows easily that

$$\operatorname{sign} \boldsymbol{P}_\phi(V) = \operatorname{sign} M_{11}\,.$$

But using (4.4) we get

$$i_!(y_1 \chi^{j_1}) i_!(y_2 \chi^{j_2}) = i_!(y_1 y_2 \chi^{j_1+j_2+1})$$

and hence

$$i_!(y_1 \chi^{j_1}) i_!(y_2 \chi^{j_2})[P] = y_1 y_2 \chi^{j_1+j_2+1}[P_0]\,.$$

Therefore $\operatorname{sign} M_{11}$ is equal to the signature of the bilinear form Q on $A^{m+k-3} = \sum_{j=0}^{k-2} H^{m+k-3-2j}(F)\chi^j$ defined by

$$Q(y_1\chi^{j_1}, y_2\chi^{j_2}) = y_1 y_2 \chi^{j_1+j_2+1}[P_0]\,.$$

We set

$$C_j = H^{m+k-3-2j}(F)\chi^j\,.$$

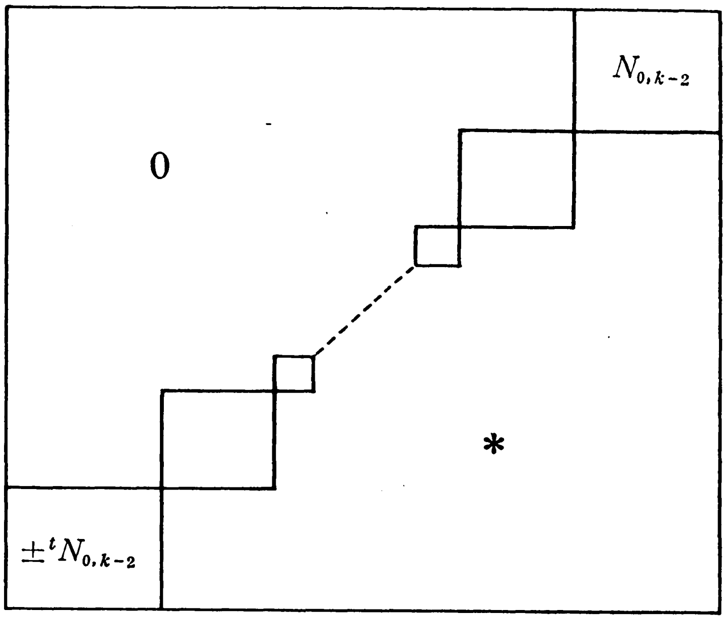
Since the fundamental cohomology class of P_0 is $\mu\chi^{k-1}$ where μ is that of F, we get

$$Q(C_{j_1}, C_{j_2}) = 0 \qquad \text{for } j_1+j_2+1 < k-1$$

and

$$Q(y_1\chi^{j_1}, y_2\chi^{j_2}) = y_1 y_2[F] \qquad \text{for } j_1+j_2+1 = k-1\,.$$

Therefore the matrix of Q with respect to the decomposition $A^{m+k-3} = C_1 \oplus \cdots \oplus C_{k-2}$ is of the form



where $N_{j,k-2-j}$ is the matrix of the cup product $H^{m+k-3-2j}(F) \times H^{m-k+1+2j}(F) \to \boldsymbol{R}$. From this it follows easily that

$$\begin{aligned} \operatorname{sign} \boldsymbol{P}_\varphi(V) &= \operatorname{sign} M_{11} \\ &= \operatorname{sign} Q = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \operatorname{sign} F, & \text{if } k \text{ is even}. \end{cases} \end{aligned}$$

This completes the proof of (4.1) for $\boldsymbol{P}_\varphi(V)$. The case of $\boldsymbol{P}_\varphi(V \times \boldsymbol{C})$ is similarly proved.

II. General case. First we shall prove the following proposition which is a variant of the Atiyah-Singer formula. Cf. (3.9) and (3.10).

PROPOSITION (4.13). *Let $(X, V, \varphi) \in \mathscr{B}_{m,2k}(S^1; \mathscr{F}_t^+)$. Let $\{F_j\}$ be the components of the fixed point set F of φ in X. Then*

$$\operatorname{sign} \boldsymbol{P}_\varphi(V \times \boldsymbol{C}) = \sum_{\operatorname{codim}_{\boldsymbol{C}} F_j \text{ even}} \operatorname{sign} F_j\,,$$

$$\operatorname{sign} \boldsymbol{P}_\phi(V) = \sum_{\operatorname{codim}_C F_j \text{ odd}} \operatorname{sign} F_j ,$$

where codim_C *means the complex codimension in* V.

PROOF. First we shall decompose $[X, V, \phi]$ into a sum of elements with certain simple properties. Take a φ-invariant tubular neighborhood $D(U_j)$ around F_j and let $p_j\colon D(U_j) \to F_j$ be the projection of the normal bundle. Then there is a ϕ-equivariant bundle equivalence

$$\theta_j\colon\ V\,|\,D(U_j) \longrightarrow p^*(V\,|\,F_j) = D(U_j) \underset{F_j}{\times} V\,|\,F_j$$

where the action ϕ on $p^*(V\,|\,F_j)$ is given by

$$\begin{aligned}\phi(g)(u, v) &= (\phi(g)u,\ \phi(g)v)\\ &= (\varphi(g)^l u,\ \phi(g)v)\,.\end{aligned}$$

We identify both bundles through θ_j and consider the S^1-action ϕ'' defined by

$$\phi''(g)(u, v) = (\varphi(g)u, v)\,.$$

Clearly ϕ'' commutes with ϕ. Moreover it is semi-free outside of $V\,|\,F_j$. Therefore the mapping cylinder W_j of the projection $V\,|\,S(U_j) \to V\,|\,S(U_j)/\phi''$ is a vector bundle over the mapping cylinder Y_j of the projection $S(U_j) \to S(U_j)/\varphi$ where $S(U_j) = \partial D(U_j)$. Thus we can form a vector bundle

$$V_j = V\,|\,D(U_j) \cup W_j$$

on the complex projective space bundle $X_j = \boldsymbol{P}(U_j \times \boldsymbol{C}) = D(U_j) \cup Y_j$. The orientation of the manifold V_j is given concordantly with that of $V\,|\,D(U_j)$. The actions ϕ and ϕ'' are extended over V_j in the obvious way. Define

$$V' = (V - \bigcup \operatorname{int} V\,|\,D(U_j)) \cup \bigcup W_j$$

glued along $\bigcup V\,|\,S(V_j)$, and

$$X' = (X - \bigcup \operatorname{int} D(U_j)) \cup \bigcup Y_j$$

glued along $\bigcup S(U_j)$. The action ϕ is also extended on V'. We have

$$[X, V, \phi] = [X', V', \phi] + \sum [X_j, V_j, \phi]\,.$$

It is therefore sufficient to prove (4.13) for (X', V', ϕ) and (X_j, V_j, ϕ) separately. The fixed point set of φ in X' is the union of $L_j = \boldsymbol{P}(U_j)$. Since each L_j has real codimension 2 in X' we can apply (4.1) which is a special case of (4.13).

The fixed point set of φ in X_j is the union of F_j and $-L_j$, where $-L_j$ is $\boldsymbol{P}(U_j)$ with the opposite orientation. The action ϕ'' on $\boldsymbol{P}_\phi(V \times \boldsymbol{C})$ is semi-free and its fixed point set is the union of $\boldsymbol{P}(V_j|F_j\times\boldsymbol{C})$ and $-\boldsymbol{P}(V_j|L_j\times\boldsymbol{C})$. Applying the Atiyah-Singer formula in the semi-free case we obtain

$$\operatorname{sign}\boldsymbol{P}_\phi(V_j\times\boldsymbol{C})=\begin{cases}\operatorname{sign}\boldsymbol{P}(V_j|F_j\times\boldsymbol{C}), & \text{if } \dim_{\boldsymbol{C}} U_j \text{ is even},\\ 0, & \text{if } \dim_{\boldsymbol{C}} U_j \text{ is odd},\end{cases}$$

$$=\begin{cases}\operatorname{sign}F_j, & \text{if } \dim_{\boldsymbol{C}} U_j \text{ and } k \text{ are both even},\\ 0, & \text{otherwise}.\end{cases}$$

When k is even this proves the formula in (4.13) for $\operatorname{sign}\boldsymbol{P}_\phi(V\times\boldsymbol{C})$. When k is odd then

$$\operatorname{sign}F_j-\operatorname{sign}L_j=0$$

since $L_j=\boldsymbol{P}(U_j)$. Thus the formula holds in this case too.

The proof for $\operatorname{sign}\boldsymbol{P}_\phi(V)$ is entirely similar and is left to the reader.

Now the Atiyah-Singer formula for $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ takes the following form.

PROPOSITION (4.14). *Let $(X, V, \phi)\in\mathcal{B}_{m,2k}(S^1;\mathcal{F}_1^+)$. Let $\{F_j\}$ be the components of the fixed point set of φ in X and $\{F'_s\}$ be the components of the fixed point set of ϕ in $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$ which are contained in $\boldsymbol{P}_\phi(V)$, cf.* (2.21). *We orient F_j and F'_s in accordance with the orientation convention with respect to the action ϕ on $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$. Then we have*

$$\operatorname{sign}\boldsymbol{P}_\phi(V\times\boldsymbol{C})=\sum_{\operatorname{codim}F_j\ \text{even}}\operatorname{sign}F_j,$$

$$0=\sum_{\operatorname{codim}F'_s\ \text{even}}\operatorname{sign}F'_s$$

and

$$\sum_{\operatorname{codim}F_j\ \text{odd}}\operatorname{sign}F_j+\sum_{\operatorname{codim}F'_s\ \text{odd}}\operatorname{sign}F'_s=0$$

where codim means the complex codimension in $\boldsymbol{P}_\phi(V\times\boldsymbol{C})$.

The deduction of (4.14) from (4.13) is quite similar to that of the Kosniowski formula from (3.10) and is left to the reader. This finishes our proof of the Atiyah-Singer formula.

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