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# Foliations of total spaces of sphere bundles over spheres

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Recently it was shown that every odd dimensional homotopy sphere has a codimension-one foliation (Tamura [3]). The purpose of this paper is to construct foliations for various differentiable manifolds. The main tool is the following lemma which is a direct consequence of the existence of a codimension-one foliation of the $(2n+1)$-sphere $S^{2n+1}$.

LEMMA. *$S^{2n-1}\times D^2$ has a codimension-one foliation having the boundary $S^{2n-1}\times S^1$ as a compact leaf.*

PROOF. We may assume $n\geqq 2$. Let $\gamma$ be a closed smooth curve in $S^{2n+1}$ which is transverse to leaves of a codimension-one foliation of $S^{2n+1}$ and let $N$ be a sufficiently small tubular neighborhood of $\gamma$ in $S^{2n+1}$. Then, by modifying the foliation in the well known way, we have a codimension-one foliation of $S^{2n+1}-\mathrm{Int}\, N=S^{2n-1}\times D^2$ having $\partial N=S^{2n-1}\times S^1$ as a compact leaf (cf. Lawson [1], Cor. 2).

Let $(E, p, S^m, S^r)$ be a sphere bundle over $m$-sphere $S^m$ having the total space $E$, the fibre $S^r$, the projection $p: E\rightarrow S^m$ and the structural group $\mathrm{Diff}(S^r)$, where $\mathrm{Diff}(S^r)$ denotes the diffeomorphism group of $S^r$. Then $E$ is an $(m+r)$-dimensional differentiable manifold whose differentiable structure is defined by the differentiable structures of $S^m$ and $S^r$.

THEOREM 1. *If $m$ or $r$ is odd, then $E$ has a codimension-one foliation.*

PROOF. Suppose that $m$ is odd. Then $S^m$ has a codimension-one foliation $\mathcal{F}=\{F_\lambda\}$, where $F_\lambda$ is a leaf (Tamura [3]). It is then obvious that $p^*\mathcal{F}=\{p^{-1}(F_\lambda)\}$ is a codimension-one foliation of $E$.

Now suppose that $m$ is even and $r$ is odd. We may assume $m\geqq 2$. Let $S^{m-2}$ be the $(m-2)$-sphere naturally imbedded in $S^m$ and let $S^{m-2}\times D^2$ be a tubular neighborhood of $S^{m-2}$ in $S^m$. Then $S^m$ is decomposed as follows:

$$S^m=(S^{m-2}\times D^2)\cup(D^{m-1}\times S^1).$$

Since $S^{m-2}\times D^2$ and $D^{m-1}\times S^1$ are homotopic to a point in $S^m$, the sphere bundles restricted on $S^{m-2}\times D^2$ and on $D^{m-1}\times S^1$ are both trivial. Thus we have

$$p^{-1}(S^{m-2}\times D^2)=S^{m-2}\times D^2\times S^r, \qquad p^{-1}(D^{m-1}\times S^1)=D^{m-1}\times S^1\times S^r.$$

According to Lemma, $S^r\times D^2$ has a codimension-one foliation $\mathcal{F}'=\{F'_{\lambda'}\}$ having the boundary $S^r\times S^1$ as a compact leaf. Thus $p^{-1}(S^{m-2}\times D^2)$ has a codimension-one foliation $p_1^*\mathcal{F}'=\{p_1^{-1}(F'_{\lambda'})\}$, where $p_1: S^{m-2}\times D^2\times S^r\to D^2\times S^r$ is the projection. On the other hand, it is well known that $D^{m-1}\times S^1$ has a codimension-one foliation $\mathcal{F}''=\{F''_{\lambda''}\}$ having the boundary $S^{m-2}\times S^1$ as a compact leaf. Thus $p^{-1}(D^{m-1}\times S^1)$ has a codimension-one foliation $p^*\mathcal{F}''=\{p^{-1}(F''_{\lambda''})\}$. Since $p^{-1}(S^{m-2}\times S^1)$ is a compact leaf for both of $p_1^*\mathcal{F}'$ and $p^*\mathcal{F}''$, the union of $p_1^*\mathcal{F}'$ and $p^*\mathcal{F}''$ defines a codimension-one foliation of $E$. This completes the proof.

REMARK. If $m$ and $r$ are even, the Euler number of $E$ is 4. Thus $E$ cannot have any codimension-one foliation in this case.

By slicing the leaves of $p_1^*\mathcal{F}'$, we have the following theorem.

THEOREM 2. *If $m$ is even and $r$ is odd, then $E$ has a codimension $m-1$ foliation.*

PROOF. $p^{-1}(S^{m-2}\times D^2)=S^{m-2}\times D^2\times S^r$ has a codimension $m-1$ foliation $\hat{\mathcal{F}}'$ whose leaves are $\{x\}\times F'_{\lambda'}$ $(x\in S^{m-2},\ F'_{\lambda'}\in\mathcal{F}')$. On the other hand, $p^{-1}(D^{m-1}\times S^1)$ $=D^{m-1}\times S^1\times S^r$ has a codimension $m-1$ foliation $\hat{\mathcal{F}}''$ whose leaves are $\{y\}\times S^1\times S^r$ $(y\in D^{m-1})$. Since $\{x\}\times S^1\times S^r$ $(x\in S^{m-2})$ are leaves for both of $\hat{\mathcal{F}}'$ and $\hat{\mathcal{F}}''$, the union of $\hat{\mathcal{F}}'$ and $\hat{\mathcal{F}}''$ defines a codimension $m-1$ foliation of $E$. This completes the proof.

In case $m=r+1$, $E$ is an $(r-1)$-connected $(2r+1)$-dimensional differentiable manifolds. In a subsequent paper (Tamura [4]), codimension-one foliations of such manifolds will be dealt in generalities.

As an application of Theorem 1, we have the following.

THEOREM 3. *Stiefel manifolds $V_{n,k}=O(n)/O(n-k)$, $W_{n,k}=U(n)/U(n-k)$, $X_{n,k}=Sp(n)/Sp(n-k)$ have codimension-one foliations, except $V_{n,1}=S^{n-1}$ ($n$ odd).*

PROOF. First suppose that $n$ is even. Let $\bar{p}: V_{n,k}\to V_{n,1}=S^{n-1}$ be the natural projection. Then $\bar{p}^*\mathcal{F}$ is a codimension-one foliation of $V_{n,k}$, where $\mathcal{F}$ denotes a codimension-one foliation of $S^{n-1}$. By the similar methods, we can construct codimension-one foliations of $W_{n,k}$ and of $X_{n,k}$.

Now suppose that $n$ is odd and $k\neq n-1$. Let $(V_{n,2}, p, S^{n-1}, S^{n-2})$ be the sphere bundle over $S^{n-1}$ having the projection $p: V_{n,2}\to V_{n,1}=S^{n-1}$ and the fibre $SO(n-1)/SO(n-2)=S^{n-2}$. Then, by Theorem 1, $V_{n,2}$ has a codimension-one foliation $\hat{\mathcal{F}}$. Therefore $V_{n,k}$ has a codimension-one foliation $\hat{p}^*\hat{\mathcal{F}}$, where $\hat{p}: V_{n,k}\to V_{n,2}$ is the natural projection. This completes the proof.

By applying Theorem 2 to the fibering $p: S^7\to S^4$ (resp. $p: S^{15}\to S^8$), we have the following. (See Thomas [5], Problem 12.)

THEOREM 4. *$S^7$ (resp. $S^{15}$) has a codimension 3 (resp. 7) foliation.*

Let $\tilde{S}^7$ be an exotic 7-sphere with the Milnor invariant $\lambda'(\tilde{S}^7)=$ $-m(m+1)/2 \bmod 28$ for an integer $m$. Then there exists a fibering $(\tilde{S}^7, p, S^4, S^3)$, (Tamura [2]). Thus Theorem 2 yields the following.

THEOREM 5. *Exotic 7-sphere $\tilde{S}^7$ such that $\lambda'(\tilde{S}^7)=-m(m+1)/2$ has a codimension 3 foliation.*

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## References

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