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On Bazilevič functions of bounded boundary rotation

By Mamoru NUNOKAWA

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§1. Introduction.

Let

$$f(z)=\left\{\frac{\beta}{1+\alpha^{2}}\int_{0}^{z}(h(\zeta)-\alpha i)\zeta^{[-\alpha\beta i/(1+\alpha^{2})]-1}g(\zeta)^{\beta/(1+\alpha^{2})}d\zeta\right\}^{(1+\alpha i)/\beta} \tag{1}$$

where $h(z)=1+\sum_{n=1}^{\infty}c_{n}z^{n}$ satisfies $\operatorname{Re} h(z)>0$ in $|z|<1$, $g(z)$ is starlike in $|z|<1$, α is any real number and $\beta>0$.

Bazilevič [1] introduced the above class of functions and showed that each such function is univalent in $|z|<1$.

Let $\alpha=0$ in (1). On differentiating we get

$$zf'(z)=f(z)^{1-\beta}g(z)^{\beta}h(z) \tag{2}$$

and

$$\operatorname{Re} h(z)=\operatorname{Re}(zf'(z)/f(z)^{1-\beta}g(z)^{\beta})>0 \quad \text{in} \quad |z|<1. \tag{3}$$

Thomas [6] called a function satisfying the condition (3) a Bazilevič function of type β. Let $C(r)$ denote the curve which is the image of the circle $|z|=r<1$ under the mapping $w=f(z)$, $L(r)$ the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. Let $M(r)=\max_{|z|=r}|f(z)|$.

Hayman [2] gave an example of a bounded starlike function satisfying

$$\limsup_{r\to 1}\frac{L(r)}{\log 1/(1-r)}>0.$$

In [7] Thomas gave the following open problems: Does there exist a starlike function for which

$$\lim_{r\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L(r)}{M(r)\log 1/(1-r)}>0$$

or

$$\lim_{r\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L(r)}{\sqrt{A(r)}\log 1/(1-r)}>0.$$

In this paper the author gives some results concerning this and others.

§ 2. On Bazilevič functions of bounded boundary rotation.

LEMMA 1. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *be regular and univalent in* $|z|<1$. *If* $\phi(z)=1+\sum_{n=1}^{\infty}c_nz^n$ *is regular and* $\operatorname{Re}\phi(z)>0$ *in* $|z|<1$, *then we have*

$$\int_0^r\int_0^{2\pi}|f'(z)\phi(z)|\,d\theta\,d\rho\leqq C\int_\delta^r\frac{M(\rho)}{1-\rho}d\rho+C$$

where δ *is fixed* $0<\delta\leqq\rho\leqq r<1$ *and* C *is an absolute constant.*

We can prove this lemma by the same method as in the proof of [5, Theorem 3].

THEOREM 1. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ *be a Bazilevič function of type* β *and* $\arg f(z)$ *be a function of bounded rotation on* $|z|=r<1$. *Let*

$$M(r)=O((1-r)^{-\alpha}(\log 1/(1-r))^{\lambda})\quad as\quad r\to 1\qquad for\quad 0<\alpha\leqq 2.$$

Then we have

$$L(r)=O((1-r)^{-\alpha}(\log 1/(1-r))^{\lambda})\quad as\quad r\to 1\qquad for\quad 0<\alpha\leqq 2.$$

PROOF. Applying the same method as in the proof of [3, Theorem 1], we have also that

$$\begin{aligned}L(r)&=\int_0^{2\pi}|zf'(z)|\,d\theta\\&=\int_0^{2\pi}|f(z)^{1-\beta}g(z)^{\beta}h(z)|\,d\theta\\&\leqq\int_0^r\int_0^{2\pi}|(1-\beta)f'(z)f(z)^{-\beta}g(z)^{\beta}h(z)|\,d\theta\,d\rho\\&\quad+\int_0^r\int_0^{2\pi}|f(z)^{1-\beta}\beta g'(z)g(z)^{\beta-1}h(z)|\,d\theta\,d\rho\\&\quad+\int_0^r\int_0^{2\pi}|f(z)^{1-\beta}g(z)^{\beta}h'(z)|\,d\theta\,d\rho\\&=J_1+J_2+J_3\qquad\text{say}.\end{aligned}$$

Then we have

$$J_1\leqq 2\pi|1-\beta|M(r)=O((1-r)^{-\alpha}(\log 1/(1-r))^{\lambda})\quad\text{as}\quad r\to 1 \tag{4}$$

and

$$J_2=|\beta|\int_0^r\int_0^{2\pi}|f'(z)\phi(z)|\,d\theta\,d\rho$$

where $\phi(z)=1+\sum_{n=1}^{\infty}c_nz^n$ is regular and $\operatorname{Re}\phi(z)>0$ in $|z|<1$.

Hence we have by Lemma 1

$$J_2 \leqq C\int_\delta^r \frac{M(\rho)}{1-\rho}d\rho + C \tag{5}$$

$$\leqq C\int_\delta^r \frac{1}{(1-\rho)^{\alpha+1}}(\log 1/(1-\rho))^\lambda d\rho + C$$

$$\leqq C\frac{1}{\alpha}(1-r)^{-\alpha}(\log 1/(1-r))^\lambda + C$$

where C is an absolute constant, not necessarily the same each time.

Therefore we have

$$J_2 = O((1-r)^{-\alpha}(\log 1/(1-r))^\lambda) \quad \text{as} \quad r\to 1.$$

Now we have also

$$J_3 = 2\pi\{|1-\beta|C + |\beta|\}\int_0^r \frac{M(\rho)}{1-\rho}d\rho \tag{6}$$

$$= O((1-r)^{-\alpha}(\log 1/(1-r))^\lambda) \quad \text{as} \quad r\to 1.$$

From (4), (5) and (6) we obtain

$$L(r) = O((1-r)^{-\alpha}(\log 1/(1-r))^\lambda) \quad \text{as} \quad r\to 1 \qquad \text{for} \quad 0<\alpha\leqq 2.$$

COROLLARY 1. *Let* $f(z) = z + \sum_{n=2}^\infty a_n z^n$ *be a Bazilevič function of type* β *and* $\arg f(z)$ *be a function of bounded boundary rotation on* $|z| = r < 1$. *If*

$$M(r) = O((1-r)^{-\alpha}(\log 1/(1-r))^\lambda) \tag{7}$$

as $r\to 1$ *for* $0<\alpha\leqq 2$ *and* O *in* (7) *can not be replaced by* 0, *then there is not any Bazilevič function satisfying the above conditions and*

$$\lim_{r\to 1}{}^{\sup}_{\inf} \frac{L(r)}{M(r)\log 1/(1-r)} > 0.$$

REMARK. We notice that if $\beta = 0$ in (3) we have the class of starlike functions whose boundary rotation is 2π.

Applying the same method as in the proof of [5, Theorem 2] we can prove the following result:

THEOREM 2. *Let* $f(z) = z + \sum_{n=2}^\infty a_n z^n$ *be regular and close-to-convex in* $|z|<1$. *Let*

$$M(r) = O((1-r)^{-\alpha}(\log 1/(1-r))^\lambda) \quad \text{as} \quad r\to 1 \qquad \text{for} \quad 0<\alpha\leqq 2.$$

Then we have

$$L(r) = O((1-r)^{-\alpha}(\log 1/(1-r))^\lambda) \quad \text{as} \quad r\to 1$$

and therefore

$$\lim_{r\to 1}{}^{\sup}_{\inf} \frac{L(r)}{M(r)\log 1/(1-r)} = 0.$$

In [4, Theorem 1] the author got the following result:

THEOREM 3. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ *be regular and convex in* $|z|<1$. *Then we have*

$$L(r)=O(A(r)\log 1/(1-r))^{1/2} \quad as \quad r\to 1.$$

On the other hand, the author gave a question whether there is a positive constant α and a convex function $f(z)$ for which

$$L(r)\geqq \alpha(A(r)\log 1/(1-r))^{1/2} \quad \text{as} \quad r\to 1. \tag{8}$$

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Gunma University

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