J. Math. Soc. Japan
Vol. 23, No. 4, 1971

# Deformations of compact complex surfaces III

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(Received Dec. 12, 1970)

## § 1. Introduction.

This is a continuation of the papers "Deformations of compact complex surfaces" and "Deformations of compact complex surfaces II" to which we refer as Parts I and II, respectively. We studied topological properties of plurigenera of compact complex surfaces and obtained Theorem II in Part I and Theorem IV in Part II. Moreover, we showed the invariance of plurigenera of compact complex surfaces under deformations in Theorem III in Part II. In this paper, we first investigate the structure of compact complex manifolds of dimension 4, and we prove Theorem V which would be important in the future study of bimeromorphic classification of compact complex manifolds. Furthermore, we shall show that the plurigenera of an elliptic surface are determined by the homotopy invariants of the surface, if its fundamental group is neither a finite abelian group generated by at most two elements nor a dihedral group of order $4k$, $k \geqq 1$. In order to prove the above assertion, we determine the structure of the fundamental groups of elliptic surfaces. We recall that the fundamental group of an algebraic curve determines its genus ($=h^{1,0}$) and conversely the genus determines the abstract structure of the fundamental group. In the case of surfaces, such a connection between topology and genera[1] might be lost. However if we restrict ourselves to elliptic surfaces, we can say that the fundamental group of any elliptic surface determines its plurigenera completely with minor exceptions. Conversely, the essential structure of any elliptic surface of general type can be given by use of its plurigenera, as will be shown later.

## § 2. Statement of the results.

We employ the notation and the terminology of Parts I and II. Thus, by a (compact complex) surface we mean a connected compact complex manifold of dimension 2. In [8], S. Kawai has developed a theory of bimero-

1) Here, by genera we mean the basic discrete complex analytic invariants of compact complex manifolds.

morphic classification of connected compact complex manifolds of dimension 3. By the same way as he did, we can prove the following: Let $M$ be a connected compact complex manifold of dimension 4. We denote by the symbol $a(M)$ the algebraic dimension of $M$ which is defined to be the dimension of the field[2] of all meromorphic functions on $M$. Then, (A) *in the case where* $a(M)=4$, *there exists by the* Мойшезон *theorem a compact complex manifold, admitting a structure of a projective algebraic variety, which is bimeromorphically equivalent to* $M$. (B) *In the case in which* $1 \leqq a(M) \leqq 3$, *there exists a fiber space of compact complex manifolds* $f: M^* \to V$ *such that*

(1) $M^*$ *is bimeromorphically equivalent to* $M$,

(2) $V$ *has the dimension* $a(M)$ *and has a structure of a projective algebraic variety,*

(3) $f$ *is a proper surjective holomorphic map,*

(4) $f$ *induces an isomorphism of the field* $\boldsymbol{C}(V)$ *onto the field* $\boldsymbol{C}(M)=\boldsymbol{C}(M^*)$,

(5) *the general fiber* $F_v$ *is a connected compact complex manifold of dimension* $4-a(M)$ *and of Kodaira dimension* $\leqq 0$.

We call the above $V$ *an algebraic equivalent* to $M$. Note that the notion of the Kodaira dimension $\kappa(M)$ of $M$ has been introduced in [**5**] by the author. *Moreover, when* $a(M)=3$, *we obtain* $\kappa(F_v)=0$, *namely,* $F_v$ *is an elliptic curve.* (C) *In the case in which* $a(M)=0$, *we restrict ourselves to the case where* $M$ *admits a Kähler metric. Then, we have the Albanese map* $\Phi: M \to T$ *where* $T$ *denotes the Albanese torus of* $M$. $T$ *is a complex torus of dimension* $q(M)$ *(by which we denote the irregularity of* $M$*).* By referring to the results of Kawaï, we conclude that

(i) $T$ *has no effective divisors,*

(ii) $\Phi$ *is proper and surjective,*

(iii) *any general fiber* $\Phi^{-1}(t)$ *is connected,*

(iv) *the case in which* $q(M)=1$ *does not occur,*

(v) *if* $q(M)=3$, *any general fiber* $\Phi^{-1}(t)$ *has the Kodaira dimension* $\leqq 0$.

Now in the case in which $q(M)=2$, a new problem arises which we did not meet in classifying compact complex manifolds of dimension 2 or 3. The problem is *to classify surfaces which are general fibers of* $\Phi: M \to T$. It seems to be true that those surfaces have the Kodaira dimension $\leqq 0$. Here, as an application of Theorem III we shall prove

THEOREM V. *Any general fiber of* $\Phi: M \to T$ *could not be an elliptic surface of general type.*

For the sake of simplicity, we call an elliptic surface of which fundamental group is neither a finite abelian group generated by at most two elements nor a dihedral group of order $4k$, $k \geqq 1$, a $G$-surface.

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2) This field is denoted by the symbol $\boldsymbol{C}(M)$.

THEOREM VI. *Any plurigenus of a G-surface is computed in terms of its certain homotopy invariants, namely, its geometric genus, its irregularity, and its fundamental group.*

COROLLARY. *Any plurigenus of a surface in the following three classes is a homotopy invariant:*

(1) *the class of surfaces with* $b_1 \equiv 1 \bmod 2$ *and* $b_1 > 1$,

(2) *the class of surfaces with* $c_1^2 = c_2 = 0$ *and* $b_1 > 1$,

(3) *the class of surfaces each of which has a finite unramified covering manifold belonging to the class* (1) *or* (2).

The following Theorem VII shows that the converse of Theorem VI is almost true:

THEOREM VII. *Let $S$ be an elliptic surface which is not a hyperelliptic surface. By $\Psi: S \to \Delta$ we denote a structure of an elliptic fiber space of $S$. Then $\pi(\Delta)$ and the set of multiplicities of multiple fibers of $\Psi$ can be computed in terms of its plurigenera $P_m(S)$ and its irregularity $q(S)$.*

Thus, we have arrived at the conclusion that the essential structure of the fundamental group of $S$ is determined by the values of $P_m(S)$ and $q(S)$ as will be explained below in the Table of the fundamental groups of elliptic surfaces.

About sixty years ago, F. Enriques and G. Castelnuovo discovered various characterizations of rational surfaces, ruled surfaces, Enriques surfaces, abelian varieties of dimension 2, Kummer surfaces (K3 surfaces), and hyperelliptic surfaces by means of the values of their irregularities and plurigenera. We note that our Theorem VI and VII treat a similar problem for elliptic surfaces of general type. On the other hand, K. Kodaira proved that plurigenera of an algebraic surface of general type are expressed in terms of the values of their Chern numbers. Therefore, we dare say that *the problem of characterization of surfaces by use of their irregularities and plurigenera are completely solved.*

## § 3. Proof of Theorem V.

Let $M_t$ be a general fiber of the Albanese map $\Phi: M \to T$. We shall prove Theorem V by deriving a contradiction under the assumption that $M_t$ is an elliptic surface of general type. Fix an integer $\alpha \geqq 86$. Referring to the theory of A. Grothendieck, we have a meromorphic map $h$ of $M$ into $\boldsymbol{P}(\Phi_*(\mathcal{L}))$ over $T$ where $\mathcal{L}$ denotes the invertible sheaf $\mathcal{O}(\alpha K)$. Let $N$ be the meromorphic transform of $M$ by $h$, and $g$ the structure map of $N$ over $T$. Then $N$ is a closed analytic subset of $\boldsymbol{P}(\Phi_*(\mathcal{L}))$. We denote by $T'$ the set of points $t \in T$ such that the Jacobian matrix of $\Phi$ has the maximal rank at every point over $t$. $T-T'$ is a closed analytic set of $T$ and so the codimension of $T-T'\geqq 2$, for $T$ has no effective divisors by (i). For simplicity, we abbreviate $\Phi^{-1}(T')$, $\Phi|(M')$, $g^{-1}(T')$, and $h|M'$, respectively, by $M'$, $\Phi'$, $N'$, and $h'$. It is clear that $\Phi': M'\to T'$ gives a family of elliptic surfaces of general type and $h': N'\to T'$ a family of base curves of them by the results in § 3 in Part II. Note that this result may be recognized as a direct consequence of Theorem III. We wish to prove that *$N$ has a non-constant meromorphic function so that $M$ has a non-constant meromorphic function.* This contradicts the assumption: $a(M)=0$. If $\pi=\pi(M_t)\geqq 2$, then we have $a(N)>0$ by applying Proposition 7 in [**8**, II] to the fiber space $g: N\to T$. If $\pi=1$ [or $\pi=0$], we make use of the following

LEMMA 1. *An elliptic surface $S$ of general type with $\pi(S)=1$ [or $=0$] has at least one singular fiber [or at least three singular fibers].*

PROOF. Since $S$ is of general type, we have the inequality (the formula (6) in Part II):

$$2\pi-2+1-q+p_g+\sum_{\lambda=1}^{s}\left(1-\frac{1}{m_\lambda}\right)>0, \tag{1}$$

where we write $\pi(S)=\pi$, $q(S)=q$, and $p_g(S)=p_g$. Furthermore, we denote by $\{m_1,\cdots,m_s\}$ the set of multiplicities of multiple fibers of the elliptic fiber space $\Psi: S\to\Delta$ of $S$. If $\pi=1$ and $s=0$, then we have $1-q+p_g>0$ by (1). On the other hand, we have the formula (12.6) in [**9**, III]:

$$\begin{aligned}12(1-q+p_g)=c_2(S)=j+\sum_{b>0}6\nu(I_b^*)+2\nu(II)+10\nu(II^*)\\ +3\nu(III)+9\nu(III^*)+4\nu(IV)+8\nu(IV^*)\,,\end{aligned} \tag{2}$$

where $j$ denotes the order of the functional invariant of $S$, $\nu(\sharp)$ the number of the singular fibers of type $\sharp$ (see [**9**, III] for the definition of type $\sharp$). Hence, we conclude that in this case there exist singular fibers of type other than ${}_mI_0$, $m>1$. If $\pi=1$ and $s\neq 0$, we see that there exist singular fibers of type ${}_mI_0$. If $\pi=0$ and $s=0$, we have $1-q+p_g\geqq 3$ by (1). Hence, from the formula (2) we get

$$j+\sum 6\nu(I_b^*)+\cdots\geqq 36\,. \tag{3}$$

Now, in the case of $j=0$, we see that there exist at least three singular fibers by (3). In the case of $j>0$, we shall derive a contradiction under the assumption that there exist at most two singular fibers. Suppose that there exist two singular fibers $\Psi^*(a_1)$ and $\Psi^*(a_2)$. Then we consider the usual representation $\rho$ of the fundamental group of the Riemann surface $\Delta-\{a_1,a_2\}$ in the 1-homology group $H_1(C_u,\boldsymbol{Z})$ where $C_u$ denotes a general fiber of $\Psi: S\to\Delta$. Set $B_1=\rho(\beta_1)$ and $B_2=\rho(\beta_2)$ where $\beta_1$, $\beta_2$ are small circles around $a_1$, $a_2$, respectively, with positive orientation, then we have $B_1B_2=1$.

On the other hand, by Table I in [7] we see that if $\Psi^*(a_i)$ is of type ${}_1I_b$ [or $I_b^*$], then $B_i$ is conjugate to

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \left[\text{or } -\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right],$$

in $SL_2(\boldsymbol{Z})$, where $b$ is a positive integer. Moreover, if $\Psi^*(a_i)$ is of type other than $I_b$ or $I_b^*$, then we have $B_i^{12}=1$. Therefore, since $\Psi^*(a_1)$ is of type $I_b^*$ or ${}_1I_b$, $B_1B_2=1$ does not hold, whatever the type of $\Psi^*(a_2)$ may be. In the other cases, we can easily show that there exist at least three singular fibers.

Now, we proceed with the proof of Theorem V. We can find a compact complex manifold $M_1$ and a holomorphic bimeromorphic map $\mu: M_1 \to M$ such that the composed map $h_1 = h\cdot\mu$ is holomorphic. Let $A_1$ be a set of points $p$ such that the Jacobian matrix of $h_1$ has not the maximal rank at $p\in M_1$. Then, $B=h_1(A_1)$ is a closed analytic set in $N$. We note that for every point $t'\in T''$ the elliptic fiber space $M_{t'}\to N_{t'}=g^{-1}(t')$ has singular fibers only over points of $B_{t'}=B\cap N_{t'}$. Consider the case of $\pi(N_t)=1$ for $t\in T'$. In view of Lemma 1, we have $g(B)=T$. Hence, the analytic set $B$ has an irreducible component $B_1$ of which image by $g$ is $T$. By the same argument as in the proof of Proposition in p. 613 of [8, II] we obtain $a(N)=1$. Next, consider the case of $\pi(N_t)=0$ for any point $t\in T'$. Let $B_1$ be one of the irreducible components of $B$ such that $g(B_1)=T$. We denote by $g_{B_1}$ the restriction of $g$ to $B_1$. We define three analytic sets as follows:

$\Sigma_1=$ the set of singular points of $B_1$,

$\Sigma_2=$ the set of points $p$ such that the Jacobian matrix of $g_B$ has not the maximal rank,

$\Sigma_3=g(\Sigma_1)\cup g(\Sigma_2)\cup(T-T')$.

We write $T-\Sigma_3=T''$, $g^{-1}(T'')=N''$, $\Phi^{-1}(T'')=M''$, and $B''=B\cap N''$. Then the holomorphic map $g_{B''}=g_{B_1}|B''$ is an unramified covering map. From the fact that $T$ has no effective divisors it follows that the codimension of $\Sigma_3\geqq 2$. Hence, the natural surjection $\pi_1(T'')\to\pi_1(T)$ is isomorphic. This implies that $T$ has an unramified covering map $f: T^*\to T$ and an open immersion $B''\subsetneq T^*$ such that the following diagram

$$\begin{array}{ccc} B'' & \hookrightarrow & T^* \\ {\scriptstyle g_{B''}}\downarrow & & \downarrow{\scriptstyle f} \\ T'' & \hookrightarrow & T \end{array}$$

commutes. Using the map $f$ we extend the base of the fiber space $\Phi: M\to T$. Thus we get a fiber space $\Phi^*: M^*=M\times_T T^*\to T^*$ where $\Phi^*$ is the natural projection of the fiber product. This fiber space has the following properties: (1) $a(M^*)=0$, (2) $T^*$ is a complex torus without effective divisors, (3) $g^*=N^* = N\underset{T}{\times}T^*\to T^*$ gives a family of base curves of the elliptic surfaces $M_t^*$ for any $t\in T'^*$ where $T'^*$ is the full inverse image of $T'$. (4) $B''^*=B''\underset{T}{\times}T^*\subset N''^* = N''\underset{T}{\times}T^*$ is a section of $N''^*\to T''^*$.

In view of Lemma 1 we can perform the similar construction for the fiber space $\Phi^*: M^*\to T^*$. Repeat once more this construction. Then we get fiber spaces $\Phi^{***}: M^{***}\to T^{***}$, $h^{***}: M^{***}\to N^{***}$, and $g^{***}: N^{***}\to T^{***}$ such that $\Phi^{***}=g^{***}\cdot h^{***}$. Furthermore, $N''^{***}\to T''^{***}$ is the projective line bundle with three sections. From this it follows that $\boldsymbol{C}(N''^{***})=\boldsymbol{C}(\boldsymbol{P}^1)\times\boldsymbol{C}(T''^{***}) =\boldsymbol{C}(\boldsymbol{P}^1)$. On the other hand, referring to Theorem II in [8, II] we have a $\boldsymbol{P}^1$-bundle $g^{\#}: N^{\#}\to T^{***}$ which is bimeromorphically equivalent to the fiber space $g^{***}: N^{***}\to T^{***}$. Let $H$ and $H'$ denote analytic sets of $N^{\#}$ and $N^{***}$, respectively, such that the bimeromorphic map induces the isomorphism: $N^{\#}-H\simeq N^{***}-H'$. Moreover, we can assume that $N^{***}-H'\subset N''^{***}$. Since $\operatorname{codim}(g^{\#}(H))\geqq 2$ and $\dim(g^{\#-1}(g^{\#}(H)))=\dim g^{\#}(H)+1$, we get $\operatorname{codim} H\geqq \operatorname{codim} g^{\#-1}(g^{\#}(H))\geqq 2$. Hence it follows that $\boldsymbol{C}(N^{\#}-H)=\boldsymbol{C}(N^{\#})$. Furthermore, we have $\boldsymbol{C}(N^{\#}-H)=\boldsymbol{C}(N^{***}-H')\supset\boldsymbol{C}(N''^{***})=\boldsymbol{C}(P^1)$ and $\dim\boldsymbol{C}(N)= \dim\boldsymbol{C}(N^{***})=\boldsymbol{C}(N^{\#})$. These lead to the inequality $a(N)=\dim\boldsymbol{C}(N)\geqq 1$, q. e. d.

## § 4. Proof of Theorem VI.

Let $S$ be an elliptic surface which is minimal[3] and $\Psi: S\to\Delta$ an elliptic fiber space of $S$. Let $\Psi^*(a_1),\cdots,\Psi^*(a_s)$ be all the multiple fibers of $\Psi$. We denote by $m_i$ the multiplicity of $\Psi^*(a_i)$ for $1\leqq i\leqq s$. Since any elliptic surface can be obtained from an elliptic surface free from multiple fibers by means of logarithmic transformations (see [10, I], [11]), we can find an elliptic surface $\bar{S}$ over $\Delta$ such that its fiber space $\bar{\Psi}: \bar{S}\to\Delta$ has no multiple fibers and $L(m_s)L(m_{s-1})\cdots L(m_1)\bar{S}=S$. Here, we denote by $L(m_i)$ a logarithmic transformation of order $m_i$ at the point $a_i\in\Delta$. Letting $\Psi^*(a_i)=P_i$ and $\bar{\Psi}^*(a_i) =\bar{P}_i$ for $1\leqq i\leqq s$, we have the biholomorphic map:

$$S'=S-\sum_{i=1}^{s}P_i\simeq\bar{S}'=\bar{S}-\sum_{i=1}^{s}\bar{P}_i\,.$$

Then we get the exact sequence of abstract groups:

$$\pi_1(S')=\pi_1(\bar{S}')\longrightarrow\pi_1(S)\longrightarrow\{1\}\,.$$

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3) As we consider birational properties of surfaces, e. g., plurigenera and fundamental groups, it is no loss of generality to assume $S$ to be minimal.

In order to determine the structure of $\pi_1(S)$, we first investigate that of $\pi_1(S')$. We define as usual the loops $\alpha_1, \cdots, \alpha_\pi, \beta_1, \cdots, \beta_\pi$ ($\pi$ denotes the genus of $\Delta$) in $\Delta$ which generate $\pi_1(\Delta)$ and satisfy the following fundamental relation:

$$\alpha_1\beta_1{}^{4)}\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_\pi\beta_\pi\alpha_\pi^{-1}\beta_\pi^{-1}=1 .$$

We can assume that the loops $\alpha_1, \cdots, \alpha_\pi, \beta_1, \cdots, \beta_\pi$ contain none of points $a_1, \cdots, a_s$. Then these loops can be regarded as loops in $\Delta'=\Delta-\{a_1, \cdots, a_s\}$. We denote by $\tau_i$ a small circle around $a_i$ in $\Delta'$. If we denote by the same letter $\alpha$ the homotopy class of the loop $\alpha$ in $\Delta'$, we can give the description of $\pi_1(\Delta')$ as follows:

$$\pi_1(\Delta') \text{ is generated by } \alpha_1, \cdots, \beta_\pi, \tau_1, \cdots, \tau_s ;$$

there exists the fundamental relation:

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_\pi\beta_\pi\alpha_\pi^{-1}\beta_\pi^{-1}\tau_1 \cdots \tau_s=1 .$$

Now we denote by $\alpha_1', \cdots, \beta_1', \cdots, \tau_1', \cdots, \tau_s'$ the loops in $S'$ which lie over $\alpha_1, \cdots, \beta_1, \cdots, \tau_1, \cdots, \tau_s$, respectively. Moreover, we let $\gamma, \delta$ be generators of $\pi_1(\Psi^{-1}(u))$, where $\Psi^{-1}(u)$ is a regular fiber of $S$. Thus we get the following description of $\pi_1(S')$:

$$\pi_1(S') \text{ is generated by } \alpha_1', \cdots, \beta_1', \cdots, \tau_1', \cdots, \gamma, \delta ;$$

there exist the following fundamental relations:

(1) $\gamma\delta=\delta\gamma$,

(2) $\{\gamma, \delta\}$[5] is normal in $\pi_1(S')$,

(3) $\alpha_1'\beta_1'\alpha_1'^{-1}\beta_1'^{-1} \cdots \alpha_\pi'\beta_\pi'\alpha_\pi'^{-1}\beta_\pi'^{-1}\tau_1' \cdots \tau_s' \in \{\gamma, \delta\}$,

(4) $R(\gamma, \delta)$[6] $=1$.

Next we consider a neighbourhood of each multiple fiber. We denote by $U_i$ a small open disc around $a_i$ $(i=1, \cdots, s)$ such that the elliptic fiber space $V_i=\Psi^{-1}(U_i)\to U_i$ has no singular fibers except $\Psi^*(a_i)$. Write $V_i'=V_i-P_i$, $\bar{V}_i=\bar{\Psi}^{-1}(U_i)$, and $\bar{V}_i'=\bar{V}_i-\bar{P}_i$. Then $\pi_1(V_i')$ is generated by $\tau_i', \gamma, \delta$. Moreover, we have the following exact sequence:

$$\pi_1(V_i') \simeq \pi_1(\bar{V}_i') \longrightarrow \pi_1(V_i) \longrightarrow \{1\} .$$

We denote by $p$ the composition of the first and the second arrows in the above sequence.

Choose suitable generators $\gamma^*, \delta^*$ of the group $\{\gamma, \delta\}$, then the fundamental relations among them are

4) For simplicity, we denote by the same letter $\gamma$ the homotopy class of the loop $\gamma$.

5) By the symbol $\{\gamma, \delta\}$ we denote the subgroup generated by $\gamma, \delta$.

6) $R(\gamma, \delta)$ denotes a word consisting of $\gamma, \delta$. Note that $R(\gamma, \delta)$ might be the word 1.

$$\gamma^*\delta^* = \delta^*\gamma^*, \quad \tau_i'\delta^* = \delta^*\tau_i', \quad \tau_i'\gamma^* = \delta^{*h}\gamma^*\tau_i' \qquad \text{(see [11])}.$$

Furthermore, we have $p(\tau_i')^{m_i} \in \{p(\gamma), p(\delta)\}$. In the case in which $h>0$, we get $p(\delta^*)=1$. Thus we conclude that

$$\pi_1(V_i) \text{ is generated by } \tilde{\tau}_i = p(\tau_i'),\ \tilde{\gamma} = p(\gamma'),\ \tilde{\delta} = p(\delta')$$

and there exist the fundamental relations:

$$\tilde{\gamma}\tilde{\delta} = \tilde{\delta}\tilde{\gamma}, \qquad \tilde{\tau}_i^{m_i} \in \{\tilde{\gamma}, \tilde{\delta}\}, \qquad \tilde{\tau}_i\tilde{\gamma} = \tilde{\gamma}\tilde{\tau}_i, \qquad \tilde{\tau}_i\tilde{\delta} = \tilde{\delta}\tilde{\tau}_i.$$

Finally, applying van Kampen's theorem we have the description of $\pi_1(S)$ in the following way:

$$\pi_1(S) \text{ is generated by the letters } \alpha_1, \cdots, \alpha_\pi, \beta_1, \cdots, \beta_\pi, \tau_1, \cdots, \tau_s, \gamma, \delta.$$

The fundamental relations among them are

(1) $\gamma\delta = \delta\gamma$,

(2) $\{\gamma, \delta\}$ is normal in $\pi_1(S)$,

(3) $\tau_i\gamma = \gamma\tau_i$, $\tau_i\delta = \delta\tau_i$, $\tau_i^{m_i} \in \{\gamma, \delta\}$ for $1 \leqq i \leqq s$,

(4) $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_\pi\beta_\pi\alpha_\pi^{-1}\beta_\pi^{-1}\cdot\tau_1 \cdots \tau_s = 1$,

(5) $R(\gamma, \delta) = 1$.

Hence, in the case in which $\pi=0$ and $s \leqq 2$, $\pi_1(S)$ is an abelian group generated by at most three elements. Moreover, if $c_2(S)=0$, then the equality $\kappa(S) = -\infty$ follows from the formula of the canonical bundle (the formula (40) in [**10**, I]). This implies that $S$ is a Hopf surface or a ruled surface of genus 1 or 0. If $c_2(S) \neq 0$, then $\Psi: S \to \Delta$ has singular fibers of type other than ${}_mI_0$. Hence, we may assume that $\pi_1(S)$ is generated by $\tau_1, \tau_2, \delta$ among which there exist the fundamental relations $\tau_1\tau_2 \in \{\delta\}$, $\tau_1^{m_1} \in \{\delta\}$, $\tau_2^{m_2} \in \{\delta\}$, $R(\delta)=1$ (here we admit $m_1, m_2 = 1$). From this we infer that $\pi_1(S)$ is an abelian group generated by at most two elements. Suppose that $\pi_1(S)$ is infinite. Then we have $b_1(S)=1$ and so $S$ is an elliptic surface belonging to the class $\mathrm{VII}_0$ of surfaces by Kodaira (see [**10**], I). Hence it follows that $c_2(S)=0$. This is a contradiction. In the other case, we can construct a universal normally ramified covering manifold $\tilde{\Delta}$ of $\Delta$ with ramification indices $m_1, \cdots, m_s$, respectively, at $a_1, \cdots, a_s$. Then the elliptic fiber space $S \times_{\Delta} \tilde{\Delta} \to \tilde{\Delta}$ has no multiple fibers. Hence, we have the exact sequence:

$$\{1\} \longrightarrow \pi_1(\tilde{S}) = \{\gamma, \delta\} \longrightarrow \pi_1(S) \longrightarrow \pi_1^*(\tilde{\Delta}/\Delta) \longrightarrow \{1\},$$

where $\pi_1^*(\tilde{\Delta}/\Delta)$ denotes the transformation group associated with the normal covering $\tilde{\Delta} \to \Delta$. Now we deal with the case $\pi=0$. Then we have $\pi_1(\tilde{S}) = \{\gamma, \delta\} \subseteqq \mathrm{Cent}\,\pi_1(S)$. Here, we denote by the symbol $\mathrm{Cent}\,G$ the center of the group $G$. Except the case in which $\pi_1^*(\tilde{\Delta}/\Delta)$ is a dihedral group of order $4k$ for an integer $k \geqq 2$, we see that $\mathrm{Cent}\,\pi_1^*(\tilde{\Delta}/\Delta)$ reduces to the trivial group $\{1\}$. Hence, in this case, we have $\pi_1(\tilde{S}) = \mathrm{Cent}\,\pi_1(S)$ and so $\pi_1(S)/\mathrm{Cent}\,\pi_1(S)$

$\simeq \pi_1^*(\tilde{\Delta}/\Delta)$. However, in the case in which $\pi_1^*(\tilde{\Delta}/\Delta)$ is a dihedral group of order $4k$ for $k \geqq 2$, the conclusion would be slightly more complicated. In this case, we have $\tilde{\Delta} = \boldsymbol{P}^1$. If $\pi_1(\tilde{S})$ is not finite, then $\pi_1(S)$ is also not finite. Hence, by the previous consideration, we see that $S$ is a Hopf surface or a ruled surface. If $\pi_1(S)$ is finite, then $\pi_1(\tilde{S})$ is finitely cyclic. In this case, we wish to prove:

A) If $\mathrm{Cent}\,\pi_1(S)$ is cyclic and its order $\neq 2$, then we have $\mathrm{Cent}\,\pi_1(S) = \pi_1(\tilde{S})$ and so $m_1 = 2$, $m_2 = k$, $m_3 = 2$ where $2k$ is the order of the group $\pi_1(S)/\mathrm{Cent}\,\pi_1(S)$.

B) If $\mathrm{Cent}\,\pi_1(S)$ is not cyclic, then we have $\mathrm{Cent}\,\pi_1(S) = \pi_1(\tilde{S})$ and so $m_1 = 2$, $m_2 = 2k$, $m_3 = 2$ where $2k$ is the order of the group $\pi_1(S)/\mathrm{Cent}\,\pi_1(S)$.

C) If $\mathrm{Cent}\,\pi_1(S) \simeq \boldsymbol{Z}_2$, then $\pi_1(S)$ is a dihedral group of order $4k$ for an integer $k \geqq 2$.

For this we note that the dihedral group $\Gamma_k$ has a non-trivial center if and only if $k$ is even and the center of $\Gamma_{2k}$ is $\{1, \beta^k\}$ where $\Gamma_{2k}$ is generated by $\alpha, \beta$ among which the fundamental relations are $\alpha^2 = \beta^{2k} = (\alpha\beta)^2 = 1$. Moreover, $\Gamma_{2k}$ can be regarded as a central extension of $\boldsymbol{Z}_2$ by $\Gamma_k$. Therefore, since $\pi_1(S)$ is an extension of $\pi_1(\tilde{S})$ ($\subset \mathrm{Cent}\,\pi_1(S)$) by $\Gamma_k$, we have $\mathrm{Cent}\,\pi_1(S) = \boldsymbol{Z}_d$ or $= \boldsymbol{Z}_d \oplus \boldsymbol{Z}_2$. Clearly it follows that the isomorphism $\mathrm{Cent}\,\pi_1(S) \simeq \boldsymbol{Z}_d \oplus \boldsymbol{Z}_2$ for $d \geqq 2$ implies $\mathrm{Cent}\,\pi_1(S) = \pi_1(\tilde{S})$ and that the isomorphism $\mathrm{Cent}\,\pi_1(S) = \boldsymbol{Z}_d$ for $d \geqq 1$ implies that $\mathrm{Cent}\,\pi_1(S) = \pi_1(\tilde{S})$ or that $\pi_1(\tilde{S}) = \{1\}$ and $\mathrm{Cent}\,\pi_1(S) = \boldsymbol{Z}_2$. From these assertions A, B, and C follow at once.

Next we deal with the case $\pi = 1$ and $s = 0$. If $c_2(S) = 0$, then we have $\kappa(S) = 0$ and $h^{1,0}(S) \geqq 1$. Hence, $S$ is a complex torus or a hyperelliptic surface or a surface with the trivial canonical bundle and $b_1 = 3$ by virtue of Theorem 19 in [**10**, I]. Any hyperelliptic surface can be regarded as an elliptic surface over $\boldsymbol{P}^1$. This was used by Enriques and Severi and has been rigorously proved by T. Suwa in [**13**]. Hence, we can apply the previous consideration to such a surface. It is clear that the fundamental group of a complex torus is isomorphic to $\boldsymbol{Z}^4$. We denote by $G_1$ the fundamental group of type (1) with $b_1 = 3$. Then we have a realization of $G_1$ as a central extension:

$$\{1\} \longrightarrow \boldsymbol{Z}^2 \longrightarrow G_1 \longrightarrow \boldsymbol{Z}^2 \longrightarrow \{1\}\,.$$

We can easily prove this referring to the fact that $G_1$ can be recognized as an affine transformation group: $G_1$ is generated by four affine transformations $g_1, g_2, g_3, g_4$; the fundamental relations among them are $g_i g_j = g_j g_i$ for any $1 \leqq i \leqq 2$, $1 \leqq j \leqq 4$ and $g_3 g_4 = g_4 g_3 g_2^m$ for an integer $m \geqq 1$. Hence, by an easy computation we have

$$\mathrm{Cent}\,G_1 = \{g_1, g_2\} \simeq \boldsymbol{Z}^2\,, \qquad G_1/\mathrm{Cent}\,G_1 \simeq \boldsymbol{Z}^2\,.$$

Now, if $c_2(S) \neq 0$, we see that $S$ has singular fibers of type other than ${}_mI_0$

for an integer $m>1$. Therefore, letting $\tilde{\Delta}=\boldsymbol{C}$ and $\tilde{S}=S\underset{\Delta}{\times}\tilde{\Delta}$ we conclude that $\pi_1(\tilde{S})$ is cyclic and $\pi_1(S)/\pi_1(\tilde{S})\simeq\pi_1(\Delta)\simeq\boldsymbol{Z}^2$. From this it follows that $\pi_1(S)/\mathrm{Cent}\,\pi_1(S)$ is an abelian group generated by at most two elements. Furthermore, in the case in which $-2+\sum_{\nu=1}^{s}\left(1-\frac{1}{m_\nu}\right)=0$, we make use of the following

LEMMA 2. *Let $\Gamma$ be a group generated by $\tau_1, \cdots, \tau_s$ among which there exist the fundamental relations: $\tau_1^{m_1}=\cdots=\tau_s^{m_s}=\tau_1\cdots\tau_s=1$, where the $m_1, \cdots, m_s$ are positive integers and satisfy the inequality: $-2+\sum_{\nu=1}^{s}\left(1-\frac{1}{m_\nu}\right)=0$. Then $\Gamma$ has the following properties:*

1) *$\Gamma$ can be realized as the extension below:*

$$\{1\}\longrightarrow\boldsymbol{Z}^2\longrightarrow\Gamma\longrightarrow\boldsymbol{Z}_i\longrightarrow\{1\}\qquad \textit{for}\quad i=2, 3, 4, 6.$$

2) $\mathrm{Cent}\,\Gamma=\{1\}$.

3) *the structure of the abelianized group of $\Gamma$ are described as follows:*

Table II*

| class | $i$ | $\{m_1, \cdots, m_s\}$ | $\Gamma/[\Gamma, \Gamma]$ |
|---|---|---|---|
| 1 | 2 | $\{2, 2, 2, 2\}$ | $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$ |
| 2 | 3 | $\{3, 3, 3\}$ | $\boldsymbol{Z}_3\oplus\boldsymbol{Z}_3$ |
| 3 | 4 | $\{2, 4, 4\}$ | $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_4$ |
| 4 | 6 | $\{2, 3, 6\}$ | $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_3$ |

,

4) $[\Gamma, \Gamma]\simeq\boldsymbol{Z}^2$.

PROOF. We can verify this by a routine work.

Finally, let us consider the case in which $2\pi-2+\sum_{\nu=1}^{s}\left(1-\frac{1}{m_\nu}\right)>0$. Then, by a similar argument as in the previous cases, we have a realization of $\pi_1(S)$ as an extension of $N$ by $F$, where $N$ is one of $\boldsymbol{Z}^2$, $\boldsymbol{Z}$, $\boldsymbol{Z}_d$, $1\leqq d$, and $F$ is a fuchsian group without parabolic elements. In this case, we need the following two lemmas:

LEMMA 3 (Reidemeister). *If two fuchsian groups without parabolic elements are isomorphic to each other as abstract groups, then they have the common signature.*

PROOF. A proof of this lemma can be found in [12].

LEMMA 4. *Any fuchsian group contain no normal abelian subgroups.*

PROOF. Considering fixed points of a fuchsian group acting naturally on the complex upper half plane, we can immediately prove this.

Summarizing the above results, we obtain the following classification of elliptic surfaces by the structure of their fundamental groups:

Let $S$ be an elliptic surface. We denote by $G$, $C$, $H$, $K$, and $A$, respectively, the fundamental group of $S$, $\mathrm{Cent}\, G$, $G/C$, $[H, H]$, and $H/K$.

Table III

(I) $G$ is abelian

| class | $G$ | $b_1$ | $\pi$ | $s$ | $p_g$ | structure of $S$ |
|---|---|---|---|---|---|---|
| (I-1) | $\{1\}$ | 0 | 0 | $\leqq 1$ | 0 | rational |
| (I-2) | $\{1\}$ | 0 | 0 | $\leqq 1$ | 0 | of general type |
| (I-3) | $\{1\}$ | 0 | 0 | $\leqq 2$ | 1 | $K3$ |
| (I-4) | $\{1\}$ | 0 | 0 | $\leqq 2$ | 1 | of general type |
| (I-5) | $\boldsymbol{Z}_2$ | 0 | 0 | 2 | 0 | Enriques |
| (I-6) | $\boldsymbol{Z}_2$ | 0 | 0 | 2 | 0 | of general type |
| (I-7) | $\boldsymbol{Z}_{d'} \oplus \boldsymbol{Z}_{d''}$ | 0 | 0 | $\leqq 2$ | 0 | of general type |
| (I-8) | $\boldsymbol{Z} \oplus \boldsymbol{Z}_d$ | 1 | 0 | $\leqq 3$ | 0 | Hopf surfaces |
| (I-9) | $\boldsymbol{Z}^2$ | 2 | 0 | $\leqq 3$ | 0 | ruled surfaces of genus 1 |
| (I-10) | $\boldsymbol{Z}^4$ | 4 | 1 | 0 | 1 | complex tori of dim 2 |
| (I-11) | $B$ | 2 | 1 | 0 | | of general type |

Here, $B$ is an abelian group generated by at most 3 elements.
$d$, $d'$, $d''$ are integers such that $d \geqq 1$, $d' \geqq d'' \geqq 1$, and $(d', d'') = (2, 1)$.

(II) $G$ is not abelian; $H$ is finite.

| class | $G$ | $H$ | $C$ | $\{m_1, \cdots, m_s\}$ | structure of $S$ |
|---|---|---|---|---|---|
| (II-1) | | $\boldsymbol{Z}_d$ | $\boldsymbol{Z}_d \oplus \boldsymbol{Z}_2$ | $\{2, 2, 2k\}$ | |
| (II-2) | | $D_{2k}$ | $\boldsymbol{Z}_d$ | $\{2, 2, k\}$ | |
| (II-3) | | $D_{2k}$ | $\boldsymbol{Z}_2$ | $\{2, 2, k\}$ | |
| (II-4) | finite | $D_{2k}$ | $\boldsymbol{Z}_2$ | $\{2, 2, 2k\}$ | of general type |
| (II-5) | | $A_4$ | $\boldsymbol{Z}_d$ | $\{2, 3, 3\}$ | |
| (II-6) | | $S_4$ | $\boldsymbol{Z}_d$ | $\{2, 3, 4\}$ | |
| (II-7) | | $A_5$ | $\boldsymbol{Z}_d$ | $\{2, 3, 5\}$ | |
| (II-1*) ~ (II-7*) | infinite | the same as above | $\boldsymbol{Z}$ or $\boldsymbol{Z} \oplus \boldsymbol{Z}_2$ | the same as above | Hopf surfaces |

Here, we used the following notation:

$D_{2k}$ = dihedral group of order $2k$,

$A_4$ = alternating group of four letters,

$S_4$ = symmetric group of four letters,

$A_5$ = alternating group of five letters,

$d', d, d''$ are integers satisfying $2 \leqq d'$, $1 \leqq d$, and $d'' = 1, 3, 4, \cdots$.

(III) $G$ is neither abelian nor finite; $K$ is abelian.

| class | $H$ | $A$ | $K$ | $\{m_1, \cdots, m_s\}$ | structure of $S$ |
|---|---|---|---|---|---|
| (III-1) | $\boldsymbol{Z}^2$ | $\boldsymbol{Z}^2$ | $\{1\}$ | $s=0$ | of type (1) and $b_1=3$ |
| (III-2) | $B'$ | $B'$ | $\{1\}$ | $s=0$ | of general type with $\pi=1$, $b_1=2$ |
| (III-3) | | $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$ | $\boldsymbol{Z}^2$ | $\{2, 2, 2, 2\}$ | of type (2) |
| (III-4) | | $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$ | $\boldsymbol{Z}^2$ | $\{3, 3, 3\}$ | of type (3) |
| (III-5) | | $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_4$ | $\boldsymbol{Z}^2$ | $\{2, 4, 4\}$ | of type (4) |
| (III-6) | | $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$ | $\boldsymbol{Z}^2$ | $\{2, 3, 6\}$ | of type (6) |
| (III-3*) ~ (III-6*) | the same as above | | | | of general type |

Here, $B'$ is an abelian group generated by at most 2 elements.

(IV) $G$ can be realized as an extension of $N$ by $F$, where $N$ is one of $\boldsymbol{Z}^2$, $\boldsymbol{Z}$, $\boldsymbol{Z}_d$ $(d \geqq 1)$ and $F$ is a fuchsian group without parabolic elements.

In this case $F$ is uniquely determined by $G$ in view of Lemma 4. Furthermore, $\{\pi; m_1, \cdots, m_s\}$ is the signature of $F$ as a fuchsian group by virtue of Lemma 3. Note that $K$ cannot be abelian.

Therefore, we can conclude that

(i) the genus of the base curve of an elliptic surface $S$ is a homotopy invariant, if and only if $S$ is not hyperelliptic.

(ii) Plurigenera of $S$ are all homotopy invariants, if $S$ is a $G$-surface.

For the proof of the latter, it suffices to note that plurigenera of any elliptic surface of general type can be computed in terms of its $p_g$, $q$, $\pi$ and $m_1, \cdots, m_s$, referring to the formula (3) in Part II.

## § 5. Proof of Theorem VII.

Setting $f(m)=P_m(S)$ and $\lambda=\lim_{m\to\infty} f(m)/m$, we get the invariants $\pi, p_g$ as follows:

$$\lambda=2\pi-2+1+q+p_g+\sum_{\nu=1}^{s}\left(1-\frac{1}{m_\nu}\right),$$

$$1-\pi=\overline{\lim_{m\to\infty}}(f(m)-\lambda m),$$

$$p_g=f(1).$$

Furthermore, we define $f_1(m)$ to be $f(m)-(2\pi-2+1-q+p_g)m+\pi-1$. Then $f_1(m)=\sum_{\nu=1}^{s}\left[m\left(1-\frac{1}{m_\nu}\right)\right]$. Hence, we have $s=f_1(2)$, $s_1+2s_1'=f_1(3)$, where we denote by $s_1$ the number of $m_\nu$ such that $m_\nu=2$ and we define $s_1'$ to be $s-s_1$. Next, setting $f_2(m)=f_1(m)-s_1\left[\frac{m}{2}\right]=\sum_{m_\nu\geqq 3}\left[m\left(1-\frac{1}{m_\nu}\right)\right]$, we have $f_2(4)=2s_2+3s_2'$, where $s_2=$ the number of $m_\nu$ such that $m_\nu=3$, and we write $s_2'=s-s_1-s_2$. Repeating these arguments, we arrive at the conclusion: $\{m_1,\cdots,m_s\}$ can be computed in terms of the values of $f$. Hence, comparing the table of the fundamental groups of elliptic surfaces, we may say that the essential structures of the fundamental groups of elliptic surfaces are determined by the values of their irregularities and plurigenera. We remark that for an elliptic surface of general type, its irregularity is determined by its plurigenera.

## § 6. Proof of Corollary.

Let $S$ be a surface which has the same homotopy type as a surface in the class (2). By Proposition 12 in Part II we see that $S$ is minimal. Hence $S$ is either an elliptic surface or a surface classified in Table I in Part I. Applying Theorem VI we can prove the Corollary. Note that it has been proved in pp. 26-27 in [9] that the class (2) contains the class (1). The proof of Corollary for the class (3) is easy.

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