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On the alternating groups IV

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§ 0. Introduction.

The object of this note is to prove the following result.

THEOREM. *Let $\tilde{z}$ be an arbitrary involution of the alternating group $\mathfrak{A}_m$ of degree m. Let G be a finite group with the following properties:*

(i) *G has no subgroup of index 2,*

(ii) *G contains an involution z such that $C_G(z)$ is isomorphic to $C_{\mathfrak{A}_m}(\tilde{z})$.*

Then if $m \equiv 2$ or $3 \bmod 4$ and $m \geqq 7$, G is isomorphic to $\mathfrak{A}_m$.

Obviously this is a generalization of [**3**; Th. I] in which the author proved the theorem in the case where $\tilde{z}$ is an involution of $\mathfrak{A}_m$ with the longest cycle decomposition. In [**4**; Th. A], the author also proved that, in the case $m \equiv 0$ or $1 \bmod 4$, if $\tilde{z}$ is an involution of $\mathfrak{A}_m$ with the longest cycle decomposition, G is isomorphic to $\mathfrak{A}_m$ with a few exceptions in the case of small degrees. Of course, we can expect a generalization of this result similar to the theorem, but the author has not obtained any such results. The reason lies in the point that we cannot find out any method to examine the fusion of involutions.

The main work of this note is to examine the fusion of involutions of the groups which satisfy the conditions of the theorem. On the basis of these results, we can determine the precise structures of the normalizers of some elementary abelian subgroups. Then it turns out that this knowledge enables us to calculate the centralizers of involutions other than a given one in the exact same way as in [**3**; § 5 and § 6]. (So we shall omit the detailed discussions of this part.) Then, by applying our previous result [**3**; Th. I], we can obtain the theorem. Essentially the method to examine the fusion of involutions is also similar to our previous work [**5**], but, in some points, we need different kinds of arguments from [**5**]. So we shall discuss the examination of the fusion of involutions in full detail.

The notations and the terminologies which were introduced in the introduction of [**3**] or [**4**] will be freely used.

§ 1. Some 2-local subgroups of $\mathfrak{A}_m$.

Let $\mathfrak{A}_m$ be the alternating group on m letters $\{1, 2, \cdots, m\}$. In this first section, we shall introduce some notations and describe some 2-local subgroups of $\mathfrak{A}_m$, where $m \equiv 2$ or $3 \bmod 4$. Throughout the paper, n denotes the largest integer not exceeding $m/4$. Thus we have $m = 4n+2$ or $4n+3$.

1.1. Firstly we shall define some involutions of $\mathfrak{A}_m$ as follows:

$$\begin{aligned}
&\pi_s = (4s-3,\ 4s-2)(4s-1,\ 4s)\,, \\
&\pi'_s = (4s-3,\ 4s-1)(4s-2,\ 4s)\,, \\
&\lambda_s = (4s-3,\ 4s-2)(4n+1,\ 4n+2) \qquad (1 \leqq s \leqq n)\,, \\
&\alpha_s = \pi_1\pi_2 \cdots \pi_s\,, \\
&\sigma'_s = (4s-1,\ 4s+1)(4s,\ 4s+2)\,, \\
&\sigma_t = (\pi'_t\pi'_{t+1})^{\sigma'_t} \qquad (1 \leqq t \leqq n-1)\,.
\end{aligned}$$

It is well known that the involutions $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the representatives of conjugacy classes of involutions of $\mathfrak{A}_m$. Set

$$\begin{aligned}
&S = \langle \pi'_1, \pi_1, \pi'_2, \pi_2, \cdots, \pi'_n, \pi_n \rangle\,, \\
&M = \langle \lambda_1, \pi_1, \lambda_2, \pi_2, \cdots, \lambda_n, \pi_n \rangle\,, \\
&J = S \cdot M\,, \\
&\Sigma = \langle \pi'_1, \sigma'_1, \pi'_2, \sigma'_2, \cdots, \sigma'_{n-1}, \pi'_n \rangle\,, \\
&\Sigma^0 = \langle \Sigma, \sigma'_n \rangle\,, \\
&P = \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \rangle\,.
\end{aligned}$$

We have $(\pi'_s\lambda_s)^2 = \pi_s$ for each s $(1 \leqq s \leqq n)$. So the group $\langle \lambda_s, \pi'_s \rangle$ is isomorphic to a dihedral group of order 8 with the center $\langle \pi_s \rangle$, and, if $s \neq t$, we have

$$[\langle \pi'_s, \lambda_s \rangle, \langle \pi'_t, \lambda_t \rangle] = 1\,.$$

So J is isomorphic to a direct product of n copies of a dihedral group of order 8, and S and M are elementary abelian subgroups of order 2^{2n}. Σ (resp. Σ^0) is isomorphic to the symmetric group of degree $2n$ (resp. $2n+1$) and the ordered set $\{\pi'_1, \sigma'_1, \cdots, \sigma'_{n-1}, \pi'_n\}$ (resp. $\{\pi'_1, \sigma'_1, \cdots, \sigma'_{n-1}, \pi'_n, \sigma'_n\}$) is a canonical set of generators of Σ (resp. Σ^0). P is isomorphic to $\mathfrak{S}_n$ and the ordered set $\{\sigma_1, \cdots, \sigma_{n-1}\}$ is a canonical set of generators of P. Furthermore we easily see that Σ and Σ^0 normalize M and P normalizes J.

Now we shall describe the actions of Σ and Σ^0 on M and show

$$N_{\mathfrak{A}_m}(M) = M \cdot \Sigma^0\,. \tag{1}$$

Before describing them, we remark that, as is easily seen from the order formula of $C_{\mathfrak{A}_m}(\alpha_n)$,

(2) $$C_{\mathfrak{A}_m}(\alpha_n)=\langle\nu\rangle\cdot M\cdot\Sigma\rhd\langle\nu\rangle,$$

where

$$\nu=\begin{cases}1 & \text{if } m\equiv 2 \bmod 4\\ (4n+1,\ 4n+2,\ 4n+3) & \text{if } m\equiv 3 \bmod 4,\end{cases}$$

and

(3) $$C_{\mathfrak{A}_m}(\alpha_n)\cap N_{\mathfrak{A}_m}(M)=M\cdot\Sigma,$$

because ν is inverted by λ_1. Let $\Lambda=\{\lambda_1, \lambda_1\pi_1, \lambda_2, \lambda_2\pi_2, \cdots, \lambda_n, \lambda_n\pi_n\}$ which is a basis of M. Then we see that every element of Σ induces a permutation on Λ. In particular, when Σ is regarded as a permutation group on Λ, π'_s acts on Λ as a transposition which interchanges λ_s and $\lambda_s\pi_s$, and σ'_s $(1\leqq s\leqq n-1)$ does as a transposition which interchanges $\lambda_s\pi_s$ and λ_{s+1}. The action of σ'_n on M, which is not a permutation on Λ, is as follows:

$$\lambda_s^{\sigma'_n}=\lambda_s(\lambda_n\pi_n) \qquad (1\leqq s\leqq n),$$

$$(\lambda_s\pi_s)^{\sigma'_n}=\begin{cases}(\lambda_s\pi_s)(\lambda_n\pi_n) & (1\leqq s\leqq n-1)\\ \lambda_n\pi_n & (s=n).\end{cases}$$

From these observations and the fact that any two involutions of $\mathfrak{A}_m$ are conjugate in $\mathfrak{A}_m$ if and only if they have the same type of cycle decompositions, it turns out that, if we let Π_h $(1\leqq h\leqq n)$ be a set of elements each of which is a product of $2h-1$ or $2h$ elements of Λ, then the Π_h $(1\leqq h\leqq n)$ are the totality of distinct orbits under the action of Σ^0 on $M-\{1\}$. (Observe also that each element of Π_h is a product of $2h$ transpositions without common letters.) In particular, we have $2n+1$ elements which are conjugate in $M\cdot\Sigma^0$ to α_n. Furthermore, we also see that any two involutions of M are conjugate in $\mathfrak{A}_m$ if and only if they are so in $M\cdot\Sigma^0\subseteqq N_{\mathfrak{A}_m}(M)$. This implies $[N_{\mathfrak{A}_m}(M): N_{\mathfrak{A}_m}(M)\cap C_{\mathfrak{A}_m}(\alpha_n)]=2n+1$. On the other hand, since we have $C_{\mathfrak{A}_m}(\alpha_n)\cap N_{\mathfrak{A}_m}(M)=M\cdot\Sigma$ by (3) and $[M\cdot\Sigma^0: M\cdot\Sigma]=2n+1$, we get $N_{\mathfrak{A}_m}(M)=M\cdot\Sigma^0$ which proves (1).

We shall describe the action of P on J. It can be easily checked that

$$\lambda_s^{\sigma_t}=\begin{cases}\lambda_{s+1} & \text{if } t=s\\ \lambda_{s-1} & \text{if } t=s-1\\ \lambda_s & \text{otherwise},\end{cases}$$

and

$$\pi_s'^{\sigma_t}=\begin{cases}\pi'_{s+1} & \text{if } t=s\\ \pi'_{s-1} & \text{if } t=s-1\\ \pi'_s & \text{otherwise}.\end{cases}$$

Namely, P operates on J in such a way that every element of P induces a permutation on the index set of generators π'_s and λ_s $(1 \leqq s \leqq n)$ of J. In other words, $P \cdot J$ is isomorphic to a wreathed product of a dihedral group of order 8 by the symmetric group of degree n. From a property of the automorphism group of J (cf. Proposition 2 in § 2), we see that $N_{\mathfrak{A}_m}(J)/C_{\mathfrak{A}_m}(Z(J)) \cap N_{\mathfrak{A}_m}(J)$ is isomorphic to a subgroup of $\mathfrak{S}_n$. Since we have

$$C_{\mathfrak{A}_m}(Z(J)) \cap N_{\mathfrak{A}_m}(J) = J \tag{4}$$

as is easily checked by direct computations, we must have

$$N_{\mathfrak{A}_m}(J) = P \cdot J . \tag{5}$$

1.2. Let k be a fixed integer such that $1 \leqq k \leqq n$. We shall describe some properties of $C_{\mathfrak{A}_m}(\alpha_k)$ which will be used throughout this paper. We shall write $H_m(k)$ for $C_{\mathfrak{A}_m}(\alpha_k)$. If there is no confusion, we shall write $H = H_m(k)$ more simply.

We easily see

$$H_m(k) = (W_k \times X_k)\langle \lambda_1 \rangle , \tag{6}$$

where W_k is the centralizer of α_k in the alternating group on $4k$ letters $\{1, 2, \cdots, 4k\}$ and X_k is the alternating group on $m-4k$ letters $\{4k+1, 4k+2, \cdots, m\}$.

Set

$$E_k = \langle \lambda_1\lambda_2, \lambda_2\lambda_3, \cdots, \lambda_{k-1}\lambda_k, \pi_1, \pi_2, \cdots, \pi_k \rangle ,$$

$$\Sigma_k^1 = \langle \pi'_1, \sigma'_1, \cdots, \sigma'_{k-1}, \pi'_k \rangle ,$$

$$\Sigma_k^2 = \langle \pi'_{k+1}, \sigma'_{k+1}, \cdots, \sigma'_{n-1}, \pi'_n \rangle ,$$

$$\Sigma_k^3 = \langle \Sigma_k^2, \sigma'_n \rangle ,$$

$$P'_k = \langle \sigma_1, \sigma_2, \cdots, \sigma_{k-1} \rangle ,$$

$$P_k^2 = \langle \sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_{n-1} \rangle .$$

Then we have

$$W_k = E_k \cdot \Sigma_k^1 \triangleright E_k ,$$

$$X_k \supset \Sigma_k^3 \supset \Sigma_k^2 ,$$

$$\Sigma^0 \supset \Sigma_k^1 \times \Sigma_k^3 ,$$

$$P \supset P_k^1 \times P_k^2 .$$

From (1), (3) and (5), we get

$$N_H(M) \cap C_{\mathfrak{A}_m}(\alpha_n) = M \cdot (\Sigma_k^1 \times \Sigma_k^2) , \tag{7}$$

$$N_H(M) = M \cdot (\Sigma_k^1 \times \Sigma_k^3) , \tag{8}$$

(9) $$N_H(J)=J\cdot(P_k^1\times P_k^2).$$

The actions of $\Sigma_k^1\times\Sigma_k^2$ and $\Sigma_k^1\times\Sigma_k^3$ on M (and also the action of $P_k^1\times P_k^2$ on J) have been described in 1.1. We note here that, among all the involutions of $\mathfrak{A}_m$ introduced in 1.1, just two involutions σ_k' and σ_k are not contained in $H=H_m(k)$.

The following lemma on some automorphisms of $H_m(k)$ will be used in our later discussions.

LEMMA 1. (α) *$H_m(k)$ has (outer) automorphisms ε_1 and ε_2 as follows:*

(i) $\varepsilon_1:\varepsilon_1(\pi_s')=\pi_s'\alpha_k$ $(1\leqq s\leqq k)$,
$\varepsilon_1(\sigma_s')=\sigma_s'\alpha_k$ $(1\leqq s\leqq k-1)$,
$\varepsilon_1=$ *the identity on* X_k *and* M,

(ii) $\varepsilon_2:\varepsilon_2(\lambda_s)=\lambda_s\alpha_k$ $(1\leqq s\leqq k)$,
$\varepsilon_2(\pi_s)=\pi_s$ $(1\leqq s\leqq k)$,
$\varepsilon_2=$ *the identity on* X_k *and* Σ_k^1.

(β) *When* $m\equiv 2 \bmod 4$ *and* $k=n-1$, $H_m(n-1)$ *has an (outer) automorphism* ε_0 *as follows:*

$\varepsilon_0=$ *the identity on* W_{n-1} *and* ε_0 *induces an automorphism on* $X_{n-1}\cong\mathfrak{A}_6$ *such that* $\langle\lambda_n,\pi_n\rangle^{\varepsilon_0}\doteq\langle\pi_n',\pi_n\rangle$.

PROOF. The existence of ε_1 and ε_2 is obvious, because it is easily seen from the definitions of ε_1 and ε_2 that ε_1 and ε_2 leave the relations between generators of H invariant. We shall show the existence of ε_0. It is well known that $\mathfrak{A}_6$ has two conjugacy classes of four groups and a well-known outer automorphism of $\mathfrak{S}_6$ interchanges these two classes of four groups. In our present case, $\langle\lambda_n,\pi_n\rangle$ and $\langle\pi_n',\pi_n\rangle$ are representatives of conjugacy classes of four groups of $X_{n-1}\cong\mathfrak{A}_6$. From this we easily see the existence of ε_0.

§ 2. Some properties of $N_G(J)$.

2.1. Let $G_m(k)$ be a finite group with the following properties:

$G_m(k)$ contains an involution $\tilde{\alpha}_k$ such that the centralizer $C_{G_m(k)}(\tilde{\alpha}_k)$ is isomorphic to $H_m(k)$.

Clearly, in order to obtain the theorem in the introduction, it will be sufficient to prove that, under the additional condition that $G_m(k)$ has no subgroup of index 2, $G_m(k)$ is isomorphic to $\mathfrak{A}_m$ for each integer m with $m\equiv 2$ or $3 \bmod 4$ and $m\geqq 7$ and each integer k with $1\leqq k\leqq n$. For simplicity, we shall identify $C_{G_m(k)}(\tilde{\alpha}_k)$ and $\tilde{\alpha}_k$ with $H_m(k)$ and α_k respectively. So the notations which are introduced for involutions and subgroups of $H_m(k)$ in § 1 will be used also for those of $G_m(k)$. If there is no confusion, we shall frequently write $G=G_m(k)$ and $H=H_m(k)$. The theorem of the case $k=n$ was treated in [3] and [5]. So henceforth we shall assume $k<n$ for simplicity, although many parts of our subsequent discussions will work, including the case $k=n$.

In § 1, we observed that

(i) G contains a subgroup J which is isomorphic to a direct product of n copies of a dihedral group of order 8, and

(ii) $N_G(J)\cap C_G(Z(J))=J$.

(ii) follows from (4), because $C_{\mathfrak{A}_m}(Z(J))\subseteqq H_m(k)$. Furthermore, we note that $Z(J)=\langle \pi_1, \pi_2, \cdots, \pi_n\rangle$. The following proposition is very fundamental for our subsequent discussions.

PROPOSITION 2. (i) *Every element of* $N_G(J)$ *induces a permutation on the set* $\{\pi_1, \pi_2, \cdots, \pi_n\}$.

(ii) *Every elementary abelian group of* $N_G(J)$ *of order* 2^{2n} *is contained in* J.

PROOF. Let $\mathfrak{D}$ be an arbitrary direct factor of J which is isomorphic to a dihedral group of order 8. Then it follows from a theorem of Krull-Remak-Schmidt [2; p. 66] that we can find an integer s $(1\leqq s\leqq n)$ such that

$$\mathfrak{D}=\langle \lambda_s z, \pi_s' z'\rangle,$$

where z and z' are some elements of $Z(J)$. Then π_s is an involution of $Z(\mathfrak{D})$. This implies that $\{\pi_1, \pi_2, \cdots, \pi_n\}$ is characterized as the set which consists of the totality of non-identity elements in the centers of all direct factors of J. From this fact the first statement of our proposition follows. We shall prove the second statement. Let E be an elementary abelian subgroup of $N_G(J)$ of order 2^{2n}. Assume by way of contradiction that $E\not\subseteq J$. Set

$$E=E_0\oplus(E\cap J) \qquad \text{(a direct sum)},$$

$$|E_0|=2^l \qquad (l\geqq 1).$$

By the fact that $N_G(J)\cap C_G(Z(J))=J$ and the first statement of the proposition which has been just proved, E_0 operates faithfully on the set $\{\pi_1, \pi_2, \cdots, \pi_n\}$. Now we need a simple lemma the proof of which is left to readers.

LEMMA. *Let* E_0 *be an elementary abelian subgroup of order* 2^l *of the symmetric group on* n *letters, and* h *be the number of distinct orbits of* E_0. *Then we have* $2h+l<2n$.

Returning to the proof of the proposition, let E_1 be an arbitrary elementary abelian 2-subgroup of J which commutes elementwise with E_0. Then in order to complete the proof of our proposition, it will be sufficient to see

$$|E_1|\leqq 2^{2h}, \tag{10}$$

where h is the number of distinct orbits under the action of E_0 on the set $\{\pi_1, \pi_2, \cdots, \pi_n\}$, because, by applying (10) to the case $E_1=E\cap J$ and using the above lemma, we get

$$|E|=|E\cap J|\cdot|E_0|\leqq 2^{2h+l}<2^{2n},$$

which is a contradiction. Let I be one of the orbits under the action of E_0 on $\{\pi_1, \pi_2, \cdots, \pi_n\}$. For convenience of description, we shall identify the set $\{\pi_1, \pi_2, \cdots, \pi_n\}$ with the index set $\{1, 2, \cdots, n\}$. By considering the factor group $J/Z(J)$ on which E_0 operates, we can choose one of λ_i and π_i' for each $i\in I$ in such a way that, if we let it be x_i and the other one be y_i, E_0 operates transitively on the sets $\{\bar{x}_i=x_iZ(J)\,|\,i\in I\}$ and $\{\bar{y}_i=y_iZ(J)\,|\,i\in I\}$, because any indecomposable direct factor of J is of the form $\langle\lambda_s z, \pi_s' z\rangle$ for some s $(1\leqq s\leqq n)$ and some elements z and z' in $Z(J)$ as was seen in the first paragraph of the proof of this proposition. Then set

$$\xi_I=\prod_{i\in I}x_i,$$

$$\eta_I=\prod_{i\in I}y_i,$$

$$z_I=\prod_{i\in I}\pi_i.$$

Since E_1 commutes elementwise with E_0, $\bar{E}_1$ must be contained in the group $\langle\bar{\xi}_I, \bar{\eta}_I\,|\,I$ ranges over all distinct orbits$\rangle$. Noting that ξ_I and η_I do not commute with each other, we can choose one of the groups $\langle\xi_I, z_I\rangle$ and $\langle\eta_I, z_I\rangle$ for each I in such a way that the product of chosen groups, which is of order 2^{2h}, contains E_1. This yields $|E_1|\leqq 2^{2h}$. Thus the proof of our proposition is complete.

LEMMA 3. *A 2-Sylow subgroup of $N_G(J)$ is that of G.*

PROOF. Let D be a 2-Sylow subgroup of $N_G(J)$, and T be a 2-Sylow subgroup of G containing D. If $T>D$, we have $N_T(D)>D$ by a fundamental property of groups of prime power order. On the other hand, we have $N_T(D)\subseteqq N_G(J)$, because J is a characteristic subgroup of D by Proposition 2. This contradicts that D is a 2-Sylow subgroup of $N_G(J)$, q.e.d.

REMARK. In [3, Th. I], we assumed that the given involution lies in the center of a 2-Sylow subgroup of the group $G(n, r)$ $(r=2$ or $3)$. However, this assumption can be dropped. In fact, let $k=n$ and then we have $N_G(J)\subseteqq C_G(\alpha_n)$ by (i) of Proposition 2. So by Lemma 3 α_n must be in the center of 2-Sylow subgroup of G. We note that, in the case $r=1$ of [4, Th. A], the same assumption can be also dropped, though the proof will be omitted here.

Put for each integer s with $1\leqq s\leqq n$,

$$L_s=\{\pi_{i_1}\pi_{i_2}\cdots\pi_{i_s}\,|\,1\leqq i_1<i_2<\cdots<i_s\leqq n\}.$$

If an involution x of G is conjugate in G to one of elements of $Z(J)$, we say that x is *of positive length* and write $l(x)>0$. Otherwise we shall write

$l(x)=0$. More precisely, if x is conjugate in G to an element of L_s, we say that x is *of length* s and write $l(x)=s$.

LEMMA 4. *No two elements with distinct positive length are conjugate in* G.

PROOF. Since J is weakly closed in 2-Sylow subgroup of G with respect to G by Proposition 2 and Lemma 3, two elements of $Z(J)$ are conjugate in G if and only if they are so in $N_G(J)$. Then our lemma follows from (i) of Proposition 2.

LEMMA 5. *Let X be an elementary abelian subgroup of $N_G(J)$ of order 2^{2n}. If X is normal in $N_G(J)$, X is weakly closed in a 2-Sylow subgroup of G with respect to G.*

PROOF. Let D be a 2-Sylow subgroup of $N_G(J)$ containing X. By Lemma 3, D is a 2-Sylow subgroup of G. Suppose $X^x \subseteq D$ for some x of G. By (ii) of Proposition 2, we have $X^x \subseteq J$. Since every elementary abelian subgroup of J of order 2^{2n} is normal in J, we get $J^{x^{-1}}, J \subseteq N_G(X)$. Since J is generated by all elementary abelian subgroups of order 2^{2n} of any 2-Sylow subgroup of G containing J by (ii) of Proposition 2, we can find an element $y \in N_G(X)$ such that $J^{x^{-1}}=J^y$. Then we have $yx \in N_G(J)$ and so $yx \in N_G(X)$, because X is normal in $N_G(J)$. Then we have $X=X^{yx}=X^x$. This completes the proof.

LEMMA 6. (i) *If $N_G(J)>N_H(J)$, we have $l(\lambda_1)>0$ and $l(\pi_1')>0$.*

(ii) *If G has no subgroup of index 2, we have $l(\lambda_1)>0$.*

PROOF. Suppose $N_G(J)>N_H(J)$. Then we can find an element δ of $N_G(J)$ such that $\pi_k^\delta=\pi_{k+1}$, because there are just two orbits $\{\pi_1, \pi_2, \cdots, \pi_k\}$ and $\{\pi_{k+1}, \cdots, \pi_n\}$ under the action of $N_H(J)$ on the set $\{\pi_1, \pi_2, \cdots, \pi_n\}$ as was seen in §1. Then we must have $\langle \pi_k', \lambda_k\rangle^\delta=\langle \pi_{k+1}'z, \lambda_{k+1}z'\rangle$, where z and z' are some elements of $Z(J)$ because any direct factor of J with the center $\langle \pi_{k+1}\rangle$ must be of the form $\langle \pi_{k+1}'z, \lambda_{k+1}z'\rangle$ as was seen in the first paragraph of the proof of Proposition 2. Let β and γ be elements of H corresponding to elements with the cycle decompositions $(4k+1, 4k+2, 4k+3)$ and $(4k+1, 4k+3, 4n+1)(4k+2, 4k+4, 4n+2)$ respectively. Then by transforming by suitable elements of $\langle \beta, \gamma\rangle$ we easily see that all involutions of the group $\langle \pi_{k+1}'z, \lambda_{k+1}z'\rangle$ are of positive length. So we must have $l(\pi_k')>0$ and $l(\lambda_k)>0$, and then $l(\pi_1')>0$ and $l(\lambda_1)>0$ because π_1' and λ_1 are conjugate in $N_H(J)$ to π_k' and λ_k respectively. This proves (i). Next we shall prove (ii).

Assume by way of contradiction that $l(\lambda_1)=0$. Then by (i) we must have $N_G(J)=N_H(J)$. Since S and M are normal in $N_G(J)$ $(=N_H(J))$, S and M are weakly closed in a 2-Sylow subgroup of G with respect to G by Lemma 5. Let W be a 2-Sylow subgroup of $C_G(\lambda_1)$ containing $C_J(\lambda_1)$. If we denote by $J(W)$ the group generated by all elementary abelian subgroups of W of order 2^{2n}, we have $J(W)=C_J(\lambda_1)$ because $[J: C_J(\lambda_1)]=2$ and $l(\lambda_1)=0$. On the other hand, we have, by (6),

$$H=(W_k\times X_k)\langle\lambda_1\rangle,$$

and the representatives of conjugacy classes of involutions of $W_k\times X_k$ can be taken from S. (This turns out from the knowledge of the conjugacy classes of involutions of W_k which was described in [3; §1] or [4; §1].) Since $N_G(J)=N_H(J)$, a 2-Sylow subgroup of H is that of G by Lemma 3, and then a transfer theorem of Thompson yields that λ_1 must be conjugate in G to some element of S. Therefore S must be conjugate in G a subgroup of $J(W)=C_J(\lambda_1)$. This contradicts the weakly closedness of S, because $C_J(\lambda_1)$ does not contain S. This proves (ii).

LEMMA 7. *Let $k=n-1$. Without loss of generality, we may assume that S and M are normal in $N_G(J)$. (So we shall assume this throughout the present paper.)*

PROOF. If $N_G(J)=N_H(J)$, we have nothing to prove, because S and M are normal in $N_H(J)$ as is seen from the action of $N_H(J)$ on J. So we shall assume $N_G(J)>N_H(J)$. Then we have $N_G(J)/J\cong\mathfrak{S}_n$, because $N_H(J)/J\cong\mathfrak{S}_{n-1}$ by the assumption $k=n-1$ and $N_G(J)/J$ is isomorphic to a subgroup of $\mathfrak{S}_n$ by the fact that $N_G(J)\cap C_G(Z(J))=J$ and Proposition 2. Thus we can find an element σ of $N_G(J)-N_H(J)$ such that $\pi_{n-1}^{\sigma}=\pi_n$ and $\pi_s^{\sigma}=\pi_s$ $(1\leqq s\leqq n-2)$. Then σ must map $\langle\lambda_{n-1},\pi'_{n-1}\rangle$ onto $\langle\lambda_n z,\pi'_n z'\rangle$ where z and z' are some elements of $Z(J)$. Thus we may assume $\lambda_{n-1}^{\sigma}=\lambda_n z$ or $\pi'_n z'$ according as $\pi'^{\sigma}_{n-1}=\pi'_n z'$ or $\lambda_n z$. In particular, this means that S is normal in $N_G(J)$ if and only if M is normal in $N_G(J)$. Next we shall show that S or $\langle\pi'_1,\pi_1,\pi'_2,\pi_2,\cdots,\pi'_{n-1},\pi_{n-1},\lambda_n,\pi_n\rangle$ is normal in $N_G(J)$ according as $\pi'^{\sigma}_{n-1}=\pi'_n z'$ or $\lambda_n z$. In order to this, it will be sufficient to prove that

$$(*)\qquad \pi'^{\sigma}_s\equiv\pi'_s \mod Z(J)\qquad(1\leqq s\leqq n-2).$$

Let $\sigma_1,\sigma_2,\cdots,\sigma_{n-2}$ be elements of $N_H(J)$ which are defined in §1. Then we have

$$\sigma_s^2=1 \text{ and } (\sigma_s\sigma_{s+1})^3=(\sigma_t\sigma_u)^2=1\qquad(1\leqq s,t,u\leqq n-2,\ |t-u|>1),$$
$$(\sigma_s\sigma)^2\equiv1 \mod J\qquad(1\leqq s\leqq n-3),$$
$$(\sigma_{n-2}\sigma)^3\equiv1 \mod J$$

by using $N_G(J)\cap C_G(Z(J))=J$ and the action of the σ_s on $\{\pi_1,\pi_2,\cdots,\pi_n\}$. Now we shall assume $\pi'^{\sigma}_{n-1}=\pi'_n z'$ (i.e. $\pi'^{\sigma}_{n-1}\equiv\pi'_n \mod Z(J)$) and prove $(*)$ by induction on the descending order of $s=1,2,\cdots,n-2$. Note that, also in the case $\pi'^{\sigma}_{n-1}=\lambda_n z$, the following arguments can be applied without any changes. Let $s=n-2$. Since $\pi^{\sigma}_{n-2}=\pi_{n-2}$, we must have $\pi'^{\sigma}_{n-2}\equiv\pi'_{n-2}$ or $\lambda_{n-2} \mod Z(J)$ as we have frequently seen above. Suppose by way of contradiction that $\pi'^{\sigma}_{n-2}\equiv\lambda_{n-2} \mod Z(J)$. Then we easily see $\pi'^{(\sigma_{n-2}\sigma)^3}_{n-2}\equiv\lambda_{n-2} \mod Z(J)$, since we know the action of σ_{n-2} and σ on $\langle\pi'_{n-2},\lambda_{n-2},\pi'_{n-1},\lambda_{n-1},\pi'_n,\lambda_n\rangle \mod Z(J)$. But this contradicts $(\sigma_{n-2}\sigma)^3 \equiv 1 \bmod J$. Thus (*) holds for $s=n-2$. Assume that (*) holds for any s' with $s \leqq s' \leqq n-2$. Since $\pi_s^\sigma = \pi_s$, we have $\pi_s'^\sigma \equiv \pi_s'$ or λ_s mod $Z(J)$. If $\pi_s'^\sigma \equiv \lambda_s \bmod Z(J)$, we get $\pi_s'^{(\sigma_s\sigma)^2} \equiv \lambda_s \bmod Z(J)$ which contradicts $(\sigma_s\sigma)^2 \equiv 1 \bmod J$ $(1 \leqq s \leqq n-2)$. Thus we have proved (*). Firstly we shall assume $m \equiv 3 \bmod 4$. In this case, we see from the structure of H that S is the only elementary abelian subgroup of J of order 2^{2n} with the property that S is not self-centralizing in G. This implies that S is normal in $N_G(J)$ and so M is also normal in $N_G(J)$ by what we have just seen. Secondly let $m \equiv 2 \bmod 4$. Then we have an automorphism ε_0 of H by Lemma 1 such that $S^{\varepsilon_0} = \langle \pi_1', \pi_1, \cdots, \pi_{n-1}', \pi_{n-1}, \lambda_n, \pi_n \rangle$. If we let S^0 be the last group, S^0 can also play the role of S if we change the notations suitably. Since we know that S or S^0 is normal in $N_G(J)$, we may assume without loss of generality that S is normal in $N_G(J)$. This proves Lemma 7.

REMARK. If $k < n-1$, we shall prove later that S and M are always normal in $N_G(J)$.

The configuration in the following lemma will occur in some points of our later discussions.

LEMMA 8. *Suppose that J is a direct product of $\mathfrak{D}^{(s)}$ $(1 \leqq s \leqq n)$ each of which is isomorphic to a dihedral group of order* 8 *and is generated by involutions ξ_s and η_s subject to the relations:*

$$\xi_s^2 = \eta_s^2 = (\xi_s\eta_s)^4 = 1 .$$

Set $\zeta_s = (\xi_s\eta_s)^2$ and $L = \langle \xi_1, \zeta_1, \xi_2, \zeta_2, \cdots, \xi_n, \zeta_n \rangle$. For $i=1$ and 2, let u_i be an element of $N_G(L)$ such that

$$\begin{aligned} &u_i : \zeta_i \to \xi_i \to \xi_i\zeta_i \qquad (i=1, 2), \\ &[u_1, \xi_2\zeta_1] = [u_1, \zeta_2] = [u_2, \xi_1\zeta_2] = [u_2, \zeta_1] = 1, \\ &[u_i, \xi_j] = [u_i, \zeta_j] = 1 \qquad \text{if } j \geqq 3 . \end{aligned}$$

Assume also that L is self-centralizing in G. Then if we set $\delta' = u_1^{-1}u_2u_1\eta_2$ and $\delta = (\eta_1\eta_2)^{\delta'}$ we have $\delta \in N_G(J)$ and $\zeta_1^\delta = \zeta_2$, $[\zeta_j, \delta] = 1$ $(j \geqq 3)$ and $\eta_1^\delta \equiv \eta_2 \bmod L$.

PROOF. From the actions of the elements δ and η_s $(1 \leqq s \leqq n)$ on L, we easily see that η_1^δ and η_2 induce the same action on L, and η_s^δ and η_s $(s \geqq 3)$ also induce the same action on L. This means that $\eta_1^\delta \equiv \eta_2 \bmod L$ and $\eta_s^\delta \equiv \eta_s \bmod L$ $(s \geqq 3)$, because L is self-centralizing in G. Then we see $\delta \in N_G(J)$. Similarly the other parts of the Lemma can be checked.

§ 3. **Fusion of involutions of G.**

3.1. The purpose of this section is to prove the following theorem.

THEOREM 9. *Assume that $G = G_m(k)$ ($m \equiv 2$ or $3 \bmod 4$ and $1 \leqq k < n$) has no subgroup of index* 2. *Then there exist $2n$ elements β_s and γ_s $(1 \leqq s \leqq n)$ with* *the following properties:*

(i) *each* γ_s *is of order* 3,

(ii) $\gamma_s : \pi_s \to \lambda_s \to \lambda_s\pi_s$,

(iii) $[\gamma_s, \pi_t] = [\gamma_s, \lambda_t\pi_s] = [\gamma_s, \pi'_t] = 1$ $(s \neq t)$,

(iv) $\gamma_s^{\pi'_s} = \gamma_s^{-1}$,

(i)′ *each* β_s *is of order* 3,

(ii)′ $\beta_s : \pi_s \to \pi'_s \to \pi'_s\pi_s$,

(iii)′ $[\beta_s, \pi_t] = [\beta_s, \pi'_t] = [\beta_s, \lambda_t] = 1$ $(s \neq t)$,

(iv)′ $\beta_s^{\lambda_s} = \beta_s^{-1}$.

REMARK. If $s > k$, we already have the elements β_s and γ_s with the required properties. In fact, $H_m(k)$ contains $\beta_s = (4s-3, 4s-2, 4s-1)$ $(s > k)$ and $\gamma_s = (4s-3, 4s-1, 4n+1)(4s-2, 4s, 4n+2)$ $(s > k)$. These β_s and γ_s have all the required properties, as is easily checked. So what we shall have to do hereafter is to construct the β_s and γ_s for each s when $1 \leqq s \leqq k$. As remarked at the beginning of § 2, we shall assume $k < n$ in our subsequent discussions.

3.2. The purpose of this paragraph is to prove, under the assumption $l(\lambda_1) > 0$, Lemma 15 which will yield a part of the conclusions of Theorem 9. By Lemma 6 (ii) this assumption is satisfied if G has no subgroup of index 2. We note here that the exact same arguments applied to π'_1 instead of λ_1 yield Lemma 16 under the similar assumption $l(\pi'_1) > 0$. Notice that we do not yet know whether the assumption $l(\pi'_1) > 0$ is satisfied or not even if G has no subgroup of index 2.

For each 2-subgroup D of G, $J(D)$ denotes the group generated by all elementary abelian subgroups of D of order 2^{2n}. Thus if D is a 2-Sylow subgroup of $N_G(J)$, we have $J(D) = J$ by Proposition 2 (ii). From now on we shall assume $l(\lambda_1) > 0$ throughout this paragraph.

Set

$$W = C_J(\lambda_1) = \langle \lambda_1, \pi_1 \rangle \times \langle \lambda_2, \pi'_2 \rangle \times \cdots \times \langle \lambda_n, \pi'_n \rangle .$$

Then we have

$$W' = \langle \pi_2, \pi_3, \cdots, \pi_n \rangle ,$$

$$Z(W) = \langle \lambda_1, \pi_1, \pi_2, \cdots, \pi_n \rangle = \langle \lambda_1 \rangle \times Z(J) .$$

Take a 2-Sylow subgroup of D of $C_G(\lambda_1)$ containing $C_J(\lambda_1)$ and set

$$U = \langle J, J(D) \rangle .$$

We shall find the desired element γ_1 in the group U and obtain the properties which γ_1 should have by examining the action of γ_1 on various subgroups of U. We have

$$J(D) \supseteqq W ,$$

$$J(D) \text{ is conjugate in } U \text{ to } J ,$$

because $l(\lambda_1)>0$ and W is generated by elementary abelian subgroups of order 2^{2n}. Furthermore we have

(11) $$Z(U)=\langle \pi_2, \pi_3, \cdots, \pi_n\rangle,$$

(12) $$U \triangleright W' \text{ and } Z(W),$$

(13) any subgroup of $J\cap J(D)$ containing $Z(W)$ is normal in U.

(13) follows from the fact that $Z(W)$ contains $J'=Z(J)$ and $J(D)'=Z(J(D))=\langle \lambda_1, \pi_2, \cdots, \pi_n\rangle$.

Let x_1 be an element of $Z(J(D))$ of length 1 other than $\pi_2, \pi_3, \cdots, \pi_n$. Such a unique element exists, because $Z(J)$ has exactly n elements of length 1 and $Z(J(D))\ni \pi_2, \pi_3, \cdots, \pi_n$.

LEMMA 10. *$x_1=\lambda_1$, $\lambda_1\pi_2\cdots\pi_k$, $\lambda_1\pi_{k+1}\cdots\pi_n$ or $\lambda_1\pi_2\cdots\pi_n$.*

PROOF. Noting $\lambda_1\in Z(J(D))$, we can write

$$\lambda_1=x_1\pi_{i_1}\pi_{i_2}\cdots\pi_{i_p}\pi_{j_1}\pi_{j_2}\cdots\pi_{j_q},$$

where $\{i_1, i_2, \cdots, i_p\}\subseteqq\{2, 3, \cdots, k\}$ and $\{j_1, j_2, \cdots, j_q\}\subseteqq\{k+1, \cdots, n\}$. Then we have

$$x_1\pi_2\cdots\pi_n=\lambda_1\prod_{I'\ni i}\pi_i\prod_{I''\ni j}\pi_j,$$

where I' and I'' are the complementary sets in $\{2, 3, \cdots, k\}$ and $\{k+1, \cdots, n\}$ of $\{i_1, i_2, \cdots, i_p\}$ and $\{j_1, j_2, \cdots, j_q\}$ respectively. We claim

(14) $$I'=\{2, 3, \cdots, k\} \text{ or } \phi (=\text{the empty set}), \quad \text{and}$$

(15) $$I''=\{k+1, \cdots, n\} \text{ or } \phi,$$

which will yield our lemma. Suppose that (14) is false. Then take an element i_0 of $\{2, \cdots, k\}$ which does not belong to I'. Then we see from the action of $N_H(J)$ on J, which is described in 1.2, that $\lambda_1\prod_{I'\ni i}\pi_i\prod_{I''\ni j}\pi_j$ and $\lambda_1\pi_{i_0}\prod_{I'-\{i_0'\}\ni i}\pi_i\prod_{I''\ni j}\pi_j$ are conjugate in $N_H(J)$ where i_0' is an arbitrary element of I'. Thus these two elements are of length n and are contained in $Z(J(D))$, because $x_1\pi_2\cdots\pi_n$ is of length n. However, this is impossible because $Z(J)$ has exactly one element of length n. This proves (14). Similarly, we can prove that (15) holds, q. e. d.

By (12), we know that U normalizes $Z(W)-W'$. Pick up all elements of length 1 in $Z(W)-W'$. Then we see that

$$\pi_1, x_1, x_1\pi_1$$

are just such elements. In fact, it is obvious that π_1 (resp. x_1) is the only element of length 1 in the coset π_1W' (resp. x_1W'), and since we have $x_1^{\pi_1'}=x_1\pi_1$, $x_1\pi_1$ is also the only element of length 1 in the coset $x_1\pi_1W'$. Thus it turns out that U normalizes a four group $\langle x_1, \pi_1\rangle$. Since $\pi_1\sim x_1\sim x_1\pi_1$ in U, we can find an element γ of U such that

(16) $$\gamma:\ \pi_1 \to x_1 \to x_1\pi_1\,.$$

We may assume without loss of generality that γ is of odd order. Moreover, we have

(17) $$[\gamma, \pi_s]=1 \qquad \text{if} \quad s\geqq 2\,,$$

because the π_s $(s\geqq 2)$ are in $Z(U)$. By (13), we know that γ normalizes $Z(W)$, $Z(W)\times\langle\lambda_t\rangle$ and $Z(W)\times\langle\pi_t'\rangle$ for any t $(2\leqq t\leqq n)$. Then a theorem of Maschke [**1**; p. 66] yields that γ must centralize an element of $Z(W)\times\langle\lambda_t\rangle-Z(W)$ and $Z(W)\times\langle\pi_t'\rangle-Z(W)$. It follows from (17) that γ must centralize one of elements λ_t, $\lambda_t x_1$, $\lambda_t\pi_1$, $\lambda_t x_1\pi_1$ and also one of elements π_t', $\pi_t'x_1$, $\pi_t'\pi_1$, $\pi_t'x_1\pi_1$. More precisely, we have, however,

LEMMA 11. (i) *γ centralizes λ_t or $\lambda_t\pi_1$ for each t $(2\leqq t\leqq n)$.*

(ii) *γ centralizes π_t' or $\pi_t'\pi_1$ if $t>k$.*

PROOF. Firstly we shall prove (i). If (i) is false, we have that γ must centralize $\lambda_t x_1$ or $\lambda_t x_1\pi_1$ by the preceding discussions. Suppose $(\lambda_t x_1)^\gamma=\lambda_t x_1$. Then we have $\lambda_t^\gamma=\lambda_t\pi_1$, because $\gamma:\pi_1\to x_1\to x_1\pi_1$ by (16). If $t>k$, we have $l(\lambda_t^\gamma)=l(\lambda_t)=l(\pi_t)=1$ and $l(\lambda_t\pi_1)=l(\pi_t\pi_1)$ by $\lambda_t=\pi_t^{\gamma_t}$ and $[\gamma_t, \pi_1]=1$. (Note that, by the remark in 3.1, we already have the elements γ_t if $t>k$.) This is a contradiction by Lemma 4. So let $t\leqq k$. Then we have $l(\lambda_t^\gamma)=l(\lambda_1)$ and $l(\lambda_t\pi_1)=l(\lambda_1\pi_t)$, because $\lambda_t\sim\lambda_1$ in H and $\lambda_t\pi_1\sim\lambda_1\pi_t$ in H if $t\leqq k$, as is seen from the action of $N_H(M)$ on M. On the other hand, we have, by (16),

$$\gamma:\ \lambda_1\to\lambda_1\pi_1\to\pi_1(\pi_2\cdots\pi_k)^\delta(\pi_{k+1}\cdots\pi_n)^{\delta'}$$

if we put $x_1=\lambda_1(\pi_2\cdots\pi_k)^\delta(\pi_{k+1}\cdots\pi_n)^{\delta'}$ $(\delta, \delta'=+1$ or $0)$. Then we have $l(\lambda_1)=1+\delta(k-1)+\delta'(n-k)$ and $l(\lambda_1\pi_t)=l((\lambda_1\pi_t)^{\gamma^2})=l(\lambda_1^{\gamma^2}\pi_t)=1+\delta(k-1)+\delta'(n-k)\pm 1$, which contradicts $l(\lambda_t)=l(\lambda_t\pi_1)$. Similarly, if we had $(\lambda_t x_1\pi_1)^\gamma=\lambda_t x_1\pi_1$, we would get a contradiction. This proves (i). Secondly, suppose that (ii) is false. Then γ must centralize $\pi_t'x_1$ or $\pi_t'x_1\pi_1$. If $(\pi_t'x_1)^\gamma=\pi_t'x_1$, we get $\pi_t'^\gamma=\pi_t'\pi_1$ and so $l(\pi_t')=l(\pi_t'\pi_1)$. Since $t>k$, we have $l(\pi_t')=l(\pi_t^{\beta_t})=l(\pi_t)=1$ and $l(\pi_t'\pi_1)=l((\pi_t'\pi_1)^{\beta_t})=l(\pi_t\pi_1)=2$, which contradicts Lemma 4. Similarly we see that $(\pi_t'x_1\pi_1)^\gamma=\pi_t'x_1\pi_1$ does not occur if $t>k$. This proves (ii).

LEMMA 12. $x_1\neq\lambda_1\pi_{k+1}\cdots\pi_n$.

PROOF. Suppose false. We have an element γ of odd order such that

$$\gamma:\ \pi_1\to\lambda_1\pi_{k+1}\cdots\pi_n\to\lambda_1\pi_1\pi_{k+1}\cdots\pi_n\,, \quad \text{and} \quad [\gamma, \pi_s]=1 \qquad (s\geqq 2)\,,$$

by (16) and (17). Furthermore, we know by Lemma 11 (ii) that γ centralize π_t' or $\pi_t'\pi_1$ if $t>k$. Firstly we shall show that γ must centralize $\pi_t'\pi_1$ for each $t>k$. In fact, if $\pi_t'^\gamma=\pi_t'$, we have $(\pi_t'\pi_1)^\gamma=\pi_t'x_1=\lambda_1\pi_t'\pi_{k+1}\cdots\pi_n$. Since $t>k$, we have $l(\pi_t'\pi_1)=l((\pi_t'\pi_1)^{\beta_t^{-1}})=l(\pi_t\pi_1)=2$ and $l(\lambda_1\pi_t'\pi_{k+1}\cdots\pi_n)=l((\lambda_1\pi_t'\pi_{k+1}\cdots\pi_n)^{\beta_t})$

$=l(x_1)=1$, which contradicts Lemma 4. Thus we have $(\pi_t'\pi_1)^\gamma=(\pi_t'\pi_1)$ for each $t>k$. Secondly we shall prove Lemma 12 in the case $k<n-1$. Then, by what we have just proved, we have $(\pi_{k+1}'\pi_{k+2}')^\gamma=\pi_{k+1}'\pi_{k+2}'$. Then we get

$$(\pi_1\pi_{k+1}'\pi_{k+2}')^\gamma=\pi_{k+1}'\pi_{k+2}'x_1=\lambda_1\pi_{k+1}'\pi_{k+2}'\pi_{k+1}\cdots\pi_n .$$

However, we know that

$$l(\pi_1\pi_{k+1}'\pi_{k+2}')=l((\pi_1\pi_{k+1}'\pi_{k+2}')^{\beta_{k+1}^{-1}\beta_{k+2}^{-1}})=l(\pi_1\pi_{k+1}\pi_{k+2})=3 \quad \text{and}$$

$$l((\lambda_1\pi_{k+1}'\pi_{k+2}'\pi_{k+1}\cdots\pi_n)^{\beta_{k+1}\beta_{k+2}})=l(\lambda_1\pi_{k+1}\cdots\pi_n)=l(x_1)=1 .$$

This is a contradiction. Thus we have proved the lemma in this case. Finally let $k=n-1$. Then γ has the following properties:

$$\gamma:\ \pi_1\to\lambda_1\pi_n\to\lambda_1\pi_1\pi_n ,$$
$$[\gamma,\pi_s]=1 \qquad (s\geqq 2),$$
$$[\gamma,\pi_n'\pi_1]=1 .$$

We are now in a position to apply Lemma 8 with

$\xi_1=\lambda_1\pi_n , \qquad \eta_1=\pi_1', \qquad u_1=\gamma ,$

$\xi_2=\pi_n' , \qquad \eta_2=\lambda_n , \qquad u_2=\beta_n ,$

$\xi_s=\lambda_{s-1}$ or $\lambda_{s-1}\pi_1$ $(s\geqq 3)$

according as $[\gamma,\lambda_s]=1$ or $[\gamma,\lambda_s\pi_1]=1$ (cf. Lemma 11) and

$\eta_s=\pi_{s-1}'$ $(s\geqq 3)$.

Then the element δ constructed in Lemma 8 is contained in $N_G(J)$ and $\delta:\lambda_1\pi_n\to\pi_n'$. However, this is impossible because $k=n-1$ and we have $M\lhd N_G(J)$ by Lemma 7.

LEMMA 13. $x_1\neq\lambda_1\pi_2\cdots\pi_n$.

PROOF. Suppose false. Then we have an element γ such that

$$\gamma:\pi_1\to\lambda_1\pi_2\cdots\pi_n\to\lambda_1\pi_1\pi_2\cdots\pi_n ,$$
$$[\gamma,\pi_s]=1 \qquad (2\leqq s\leqq n),$$

by (16) and (17). Furthermore we know by Lemma 11 (ii) that γ must centralize π_t' or $\pi_t'\pi_1$ if $t>k$. As in Lemma 12 we can show that $[\gamma,\pi_t'\pi_1]=1$ for each $t>k$. Also, if $k<n-1$, we can prove Lemma 13 in the same way as in Lemma 12. So let $k=n-1$. Again we shall apply Lemma 8 with

$\xi_1=\lambda_1\pi_2\cdots\pi_n, \quad \eta_1=\pi_1', \quad u_1=\gamma$

$\xi_2=\pi_n', \qquad \eta_2=\lambda_n \quad u_2=\beta_n$

$\xi s=\lambda_{s-1}$ or $\lambda_{s-1}\pi_1$ according as $[\gamma,\lambda_{s-1}]=1$ or $[\gamma,\lambda_{s-1}\pi_1]$,

$\eta_s=\pi_{s-1}'$ $(3\leqq s\leqq n-1)$.

Then we can get a contradiction with Lemma 7.

LEMMA 14. *Without loss of generality, we may assume $x_1 \neq \lambda_1\pi_2 \cdots \pi_k$. In other words, we may assume $x_1=\lambda_1$. (So henceforth we shall assume $x_1=\lambda_1$.)*

PROOF. By Lemma 1 (ii), we have an automorphism ε_2 of H. This means that $\lambda_1\alpha_k$ can also play the role of λ_1 if we interchange the $\lambda_s\alpha_k$ by the λ_s $(1 \leqq s \leqq k)$ when necessary. If $x_1=\lambda_1\pi_2 \cdots \pi_k$, namely, $l(\lambda_1\pi_2 \cdots \pi_k)=1$, we must have $l(\lambda_1\alpha_k)=1$ because $\lambda_1\alpha_k \sim \lambda_1\pi_2 \cdots \pi_k$ in H. Thus if we change the notation as described above, we may assume $l(\lambda_1)=1$, q. e. d.

Now summarizing the preceding results,

LEMMA 15. *Under the assumption $l(\lambda_1)>0$, there exists an element γ_1 such that*

(i) $\gamma_1 : \pi_1 \rightarrow \lambda_1 \rightarrow \lambda_1\pi_1$,

(ii) $[\gamma_1, \pi_t]=1$ $(t \neq 1)$

(iii) *γ_1 centralizes λ_t or $\lambda_t\pi_1$ for each t $(t \neq 1)$ and also centralizes one of elements π_t', $\pi_t'\pi_1$, $\pi_t'\lambda_1$ and $\pi_t'\lambda_1\pi_1$. If $t>k$, γ must centralize π_t' or $\pi_t'\pi_1$.*

PROOF. Set $\gamma_1=\gamma$. Then (i) follows from Lemma 12, 13, 14. (ii) follows from (17). (iii) follows from Lemma 11.

As remarked at the beginning of this paragraph, we can also obtain the following results by the same arguments as the preceding ones.

LEMMA 16. *Under the assumption $l(\pi_1')>0$, there exists an element β_1 such that*

(i) $\beta_1 : \pi_1 \rightarrow \pi_1' \rightarrow \pi_1'\pi_1$

(ii) $[\beta_1, \pi_t]=1$

(iii) *β_1 centralizes π_t' or $\pi_t'\pi_1$ for each t $(t \neq 1)$ and also centralizes one of elements λ_t, $\lambda_t\pi_1$, $\lambda_t\pi_1'$ or $\lambda_t\pi_1'\pi_1$. Moreover, if $t>k$, β must centralize λ_t or $\lambda_t\pi_1$.*

3.3. We recall that $H=H_m(k)$ contains the elements $\sigma_1, \sigma_2, \cdots, \sigma_{k-1}$ of $N_G(J)$ whose action on J were described in § 1. Set

$$\gamma_s=\gamma_1^{\sigma_1\sigma_2\cdots\sigma_{s-1}} \qquad (1 \leqq s \leqq k),$$

where γ_1 is the element in Lemma 15 which exists under the assumption $l(\lambda_1)>0$. Then it is easy to see the following:

$$\gamma_s : \pi_s \rightarrow \lambda_s \rightarrow \lambda_s\pi_s, \tag{18}$$

$$[\gamma_s, \pi_t]=1 \qquad (s \neq t,\ 1 \leqq t \leqq n). \tag{19}$$

Thus we have now the elements γ_s for each s $(1 \leqq s \leqq n)$ with the properties (18) and (19) (under the assumption $l(\lambda_1)>0$).

LEMMA 17. $[\gamma_s, \lambda_t\pi_s]=1$ $(1 \leqq s,\ t \leqq n,\ s \neq t)$.

PROOF. We may assume $1 \leqq s \leqq k$. First of all, we shall prove $[\gamma_1, \lambda_t\pi_1]=1$ $(2 \leqq t \leqq n)$. If this is false, we must have $\lambda_t^{\gamma_1}=\lambda_t$ by Lemma 15 (iii) and so $(\pi_1\lambda_t)^{\gamma_1}=\lambda_1\lambda_t$. But since $(\pi_1\lambda_t)^{\gamma_t^{-1}}=\pi_1\pi_t$, we have $l(\pi_1\lambda_t)=2$, whereas we have $l(\lambda_1\lambda_t)=1$ because

$$l(\lambda_1\lambda_t)=\begin{cases} l(\lambda_1\cdot\lambda_1\pi_1)=l(\pi_1)=1\,, & \text{if } t\leqq k \\ l((\lambda_1\lambda_t)^{\gamma_t^{-1}})=l(\lambda_1)=1\,, & \text{if } t>k\,. \end{cases}$$

This contradicts Lemma 4. Thus we have proved $[\gamma_1, \lambda_t\pi_1]=1$. Then it is easy to see $[\gamma_s, \lambda_t\pi_s]=1$ for $1\leqq s\leqq k$ and $s\neq t$ from the definition of γ_s.

LEMMA 18. *Assume G has no subgroups of index* 2. *Then we have* $l(\lambda_1)>0$ *and* $l(\pi_1')>0$. *Moreover,* $N_G(J)\rhd M$ *and* S.

PROOF. By Lemma 6 (ii), we have $l(\lambda_1)>0$. So we shall prove $l(\pi_1')>0$. Since $l(\lambda_1)>0$, we have the elements γ_s with the properties of (18), (19) and Lemma 17. Now we can apply Lemma 8 with

$$\xi_1=\lambda_k\,,\qquad \eta_1=\pi_k'\,,\qquad u_1=\gamma_k\,,$$

$$\xi_2=\lambda_{k+1}\,,\qquad \eta_2=\pi_{k+2}'\,,\qquad u_2=\gamma_{k+1}\,,$$

$$\xi_s=\begin{cases}\lambda_{s-2} & (3\leqq s\leqq k+1)\,,\\ \lambda_s & (k+2\leqq s\leqq n)\,,\end{cases}\qquad \eta_s=\begin{cases}\pi_{s-2}' & (3\leqq s\leqq k+1)\,,\\ \pi_s' & (k+2\leqq s\leqq n)\,.\end{cases}$$

Then the element δ constructed in Lemma 8 is contained in $N_G(J)$ and $\pi_k^\delta=\pi_{k+1}$ and $[\delta, \pi_s]=1$ $(s\neq k,\ k+1)$. This implies $N_G(J)>N_H(J)$. Then by Lemma 6 (i), we get $l(\pi_1')>0$, as asserted. Furthermore, we have $N_G(J)=\langle N_H(J), \delta\rangle$ as is seen from Proposition 2 and the action of $N_H(J)$ on the set $\{\pi_1, \pi_2, \cdots, \pi_n\}$. Since $N_G(M)\ni\delta$ and $N_H(J)\rhd M$, we must have $N_G(J)\rhd M$ and then $N_G(J)\rhd S$.

From now on *we shall assume that G has no subgroups of index* 2. Then we have n elements γ_s which has been described in the preceding discussions. Furthermore, by Lemma 16 and Lemma 18, we have an element β_1 with the properties in Lemma 16.

Set

$$\beta_s=\beta_1^{\sigma_1\sigma_2\cdots\sigma_{s-1}}\qquad (1\leqq s\leqq k)\,.$$

Then we easily see

$$\beta_s: \pi_s\to\pi_s'\to\pi_s'\pi_s\,, \tag{20}$$

$$[\beta_s, \pi_t]=1\qquad (1\leqq t\leqq n \text{ and } t\neq s)\,. \tag{21}$$

Now we can sharpen the third statement of Lemma 15 and Lemma 16 respectively.

LEMMA 19. *For each* t $(2\leqq t\leqq n)$, *we have*

(i) $[\gamma_1, \lambda_t\pi_1]=1$, (ii) γ_1 *centralizes* π_t' *or* $\pi_t'\pi_1$, (i)$'$ $[\beta_1, \pi_t']=1$, (ii)$'$ β_1 *centralizes* λ_t *or* $\lambda_t\pi_1$.

PROOF. (i) was proved in Lemma 17. We shall prove (ii). If (ii) is false, γ_1 must centralize $\pi_t'\lambda_1$ of $\pi_t'\pi_1\lambda_1$. Suppose $(\pi_t'\lambda_1)^{\gamma_1}=\pi_t'\lambda_1$. Then we have $\pi_t'^{\gamma_1}=\pi_t'\pi_1$ because $\lambda_1^{\gamma_1}=\lambda_1\pi_1$. Since $(\pi_t'\pi_1)^{\beta_t^{-1}}=\pi_t\pi_1$ by (20), we have $l(\pi_t'\pi_1)=2$, which contradicts Lemma 4 because $l(\pi_t')=1$ by (20). Similarly $(\pi_t'\lambda_1\pi_1)^{\gamma_1}=\pi_t'\lambda_1\pi_1$ does not occur. This proves (ii). (i) and (ii)$'$ can be also proved quite similarly by using (18)-(19).

By using Lemma 19 and the definitions of γ_s and β_s, we can easily check the following:

(22) $[\gamma_s, \lambda_t\pi_s]=1$

(23) γ_s centralizes π_t' or $\pi_t'\pi_s$

(24) $[\beta_s, \pi_t']=1$,

(25) β_s centralizes λ_t or $\lambda_t\pi_s$ $(1\leqq s, t\leqq n$ and $s\neq t)$.

LEMMA 20. $[\gamma_s, \pi_t']=[\beta_s, \lambda_t]=1$ $(1\leqq s,\ t\leqq n,\ s\neq t)$.

PROOF. By (23), we know that for each pair s and t we have $[\gamma_s, \pi_t']=1$ or $[\gamma_s, \pi_t'\pi_s]=1$. Suppose by way of contradiction that $[\gamma_s, \pi_t'\pi_s]=1$ for some pair s and t. Then we have $\pi_t'^{\gamma_s}=\pi_t'\pi_s\lambda_s$. If $[\beta_t, \lambda_s]=1$, we would have $1=l(\pi_t'^{\gamma_s})=l(\pi_t'\pi_s\lambda_s)=l((\pi_t'\pi_s\lambda_s)^{\beta_t^{-1}})=l(\pi_t\pi_s\lambda_s)=l((\pi_t\pi_s\lambda_s)^{\gamma_s})=l(\pi_t\pi_s)=2$, a contradiction. So we must have $[\beta_t, \lambda_s\pi_t]=1$. For definiteness, let $s=1$ and $t=2$. Now we can apply Lemma 8 with

$$\xi_1=\lambda_1,\ \eta_1=\pi_1',\ u_1=\gamma_1$$

$$\xi_2=\pi_2',\ \eta_2=\lambda_2,\ u_2=\beta_2$$

$$\xi_i=\pi_i' \text{ or } \pi_i'\pi_1 \text{ according as } [\gamma_1, \pi_i']=1 \text{ or } [\gamma_1, \pi_i'\pi_1]=1,$$

$$\eta_i=\lambda_i \qquad (i\geqq 3).$$

Then we see that the element δ constructed in Lemma 8 does not normalize M which contradicts Lemma 18. Thus we have proved $[\gamma_s, \pi_t']=1$ for each pair s and t. Then if $[\beta_t, \lambda_s\pi_t]=1$ for some pair s and t, we have $\lambda_s^{\beta_t\gamma_s^{-1}}=(\lambda_s\pi_t\pi_t')^{\gamma_s^{-1}}=\pi_s\pi_t\pi_t'$ which is impossible because $l(\lambda_s)=1$ and $l(\pi_s\pi_t\pi_t')=2$. Thus we must have $[\beta_t, \lambda_s]=1$ for each pair s and t. This completes the proof of Lemma 20.

In order to complete the proof of Theorem 9, it now remains to prove (i), (iv), (i)$'$ and (iv)$'$ of Theorem 9. But these can be done easily. So we shall omit the details. We refer the proofs to those of Lemma (3.4) and (3.5) of [**3**].

§ 4. The structure of some 2-local subgroups.

4.1. Let $G=G_m(k)$ as before and we shall assume that G has no subgroup of index 2.

Set

$$N = C_G(\alpha_n) \cap N_G(M),$$

$$E = \langle \lambda_1\lambda_2, \lambda_2\lambda_3, \cdots, \lambda_{n-1}\lambda_n, \pi_1, \pi_2, \cdots, \pi_n \rangle.$$

In this paragraph, we shall determine the precise structures of the group N and $N_G(E)$ by using Theorem 9, which has been established in the preceding section.

LEMMA 21. (α) *$C_G(\alpha_k) \cap N$ contains involutions $\xi_1, \eta_1, \cdots, \eta_{k-1}, \xi_k, \eta_{k+1}, \cdots, \eta_{n-1}, \xi_n$. (Notice that η_k is missing) with the following properties:*

(i) $\lambda_s^{\xi_s} = \lambda_s\pi_s$, $[\xi_s, \lambda_t] = [\xi_s, \lambda_t\pi_t] = 1$ $(s \neq t)$,

(ii) $(\lambda_s\pi_s)^{\eta_s} = \lambda_{s+1}$, $[\eta_s, \lambda_{t+1}] = [\eta_s, \lambda_t\pi_t] = 1$ $(s \neq t)$,

(iii) *ξ_s and η_s are of length* 1.

(β) *The involutions ξ_s and η_s which satisfy the conditions* (i)-(iii) *of* (α) *necessarily have the following properties:*

(iv) *The group $\langle \xi_1, \eta_1, \cdots, \eta_{k-1}, \xi_k \rangle$ (resp. $\langle \xi_{k+1}, \eta_{k+1}, \cdots, \eta_{n-1}, \xi_n \rangle$) is isomorphic to $\mathfrak{S}_{2k}$ (resp. $\mathfrak{S}_{2n-2k}$) and the ordered set $\{\xi_1, \eta_1, \cdots, \eta_{k-1}, \xi_k\}$ (resp. $\{\xi_{k+1}, \eta_{k+1}, \cdots, \eta_{n-1}, \xi_n\}$) is a canonical set of generators of $\mathfrak{S}_{2k}$ (resp. $\mathfrak{S}_{2n-2k}$).*

PROOF. (α) Put $\xi_s = \pi'_s$ $(1 \leqq s \leqq n)$ and $\eta_t = \sigma'_t$ $(1 \leqq t \leqq n-1, t \neq k)$, where the σ'_t are elements of $H_m(k)$ introduced in § 1. Then the ξ_s and η_t satisfy the conditions (i)-(iii) of (α). (β) Let ξ'_s and η'_t be any set of involutions with the properties (i)-(iii). Since M is self-centralizing in G, we must have $\xi'_s = \pi'_s\mu_s$ for some $\mu_s \in M$. Since ξ'_s is an involution, π'_s must commute with μ_s and so we have

$$\mu_s \in \langle \pi_s \rangle \times \prod_{i \neq s}^{n} \langle \lambda_i, \lambda_i\pi_i \rangle.$$

Then in order that ξ'_s is of length 1, it follows from Theorem 9 that we must have $\mu_s = 1$ or π_s. Quite similarly, we must have

$$\eta'_t = \sigma'_t\lambda_{t-1}\pi_{t-1}\lambda_t \quad \text{or} \quad \sigma'_t.$$

Then it can be easily checked that the ξ'_s and the η'_t satisfy the condition (iv), q. e. d.

LEMMA 22. *N is isomorphic to the wreathed product $\boldsymbol{Z}_2 \wr \mathfrak{S}_{2n}$. In particular, N splits over M. More precisely, there exists a complement of N over M which contains $\pi'_1, \pi'_2, \cdots, \pi'_n$.*

PROOF. Set

$$x_s = \pi'_s \qquad (1 \leqq s \leqq n),$$

$$y_s = \gamma_s^{-1}\gamma_{s+1}\gamma_s\pi'_{s+1} \qquad (1 \leqq s \leqq n),$$

where we define $\gamma_{n+1} = \gamma_1$ and $\pi'_{n+1} = \pi'_1$. Let

$$N_1 = \langle x_1, y_1, \cdots, x_{n-1}, y_{n-1}, x_n, M \rangle.$$

Firstly, we shall show $N_1 = N$. We see from Theorem 9 that

$$\lambda_s^{x_s}=\lambda_s\pi_s\,,\qquad [x_s,\lambda_t]=[x_s,\lambda_t\pi_t]=1\qquad (s\neq t)\,,$$
$$(\lambda_s\pi_s)^{y_s}=\lambda_{s+1},\quad [y_s,\lambda_{t+1}]=[y_s,\lambda_t\pi_t]=1\qquad (s\neq t)\,.$$

These relations imply that every element of N_1 induces a permutation on the set $\{\lambda_1, \lambda_1\pi_1, \cdots, \lambda_n, \lambda_n\pi_n\}$ and $N_1/M\cong\mathfrak{S}_{2n}$. On the other hand, we also see from Theorem 9 that every element of N induces a permutation on the set $\{\lambda_1, \lambda_1\pi_1, \cdots, \lambda_n, \lambda_n\pi_n\}$ (cf. [**5**; the proof of Lemma 4.4]). This implies $N=N_1$, because M is self-centralizing. In view of what we have just seen, in order to complete the proof of our lemma, it will be sufficient to see that the group $\langle x_1, y_1, \cdots, y_{n-1}, x_n\rangle$ is isomorphic to $\mathfrak{S}_{2n}$ and the ordered set $\{x_1, y_1, \cdots, y_{n-1}, x_n\}$ is a canonical set of generators of $\mathfrak{S}_{2n}$. If we set $\xi_s=x_s$ and $\eta_s=y_s$ in Lemma 21, then the set of the elements $\{x_s, y_t \mid 1\leqq s\leqq n,\ 1\leqq t\leqq n-1,\ t\neq k\}$ satisfy the conditions (i)-(iii) of Lemma 21. Thus it also satisfies (iv) of Lemma 21. So in order to see that $\{x_1, y_1, \cdots, y_{n-1}, x_n\}$ is a canonical set of generators of $\mathfrak{S}_{2n}$, it will be sufficient to see

$$(x_k y_k)^3=(y_k x_{k+1})^3=1\,,\quad [y_k, x_s]=[y_k, y_t]=1$$
$$(s\neq k,\ k+1,\ 1\leqq t\leqq n-1)\,.$$

Firstly let $n>2$. From the action of N on M we see that N contains an element ε such that

$$\varepsilon:\lambda_s\rightarrow\lambda_{s+1}\quad\text{and}\quad \lambda_s\pi_s\rightarrow\lambda_{s+1}\pi_{s+1}\quad (1\leqq s\leqq n,\ \lambda_{n+1}=\pi_1,\ \pi_{n+1}=\pi_1)\,.$$

By transforming $C_G(\alpha_k)\cap N$ by conjugation by ε, we can apply Lemma 21 to $C_G(\pi_2\pi_3\cdots\pi_{k+1})\cap N$ and a basis $\{\lambda_2, \lambda_2\pi_2, \cdots, \lambda_n, \lambda_n\pi_n, \lambda_1, \lambda_1\pi_1\}$. Then an ordered set of elements

$$\{x_2, y_2, \cdots, y_k, x_{k+1}, x_{k+2}, \cdots, x_n, y_n, x_1\}$$

must have the property (iv) of Lemma 21. This yields

$$(y_k x_k)^3=(y_k x_{k+1})^3=1,\quad [y_k, y_t]=1\qquad (t\neq k+1,\ 1)$$
$$[y_k, x_s]=1\qquad (s\neq k,\ k+1)\,.$$

So it remains to show

$$[y_k, y_{k+1}]=[y_k, y_1]=1\,.$$

Again, we can apply Lemma 21 to $C_G(\pi_3\cdots\pi_{k+2})\cap N$ and a basis of M, $\{\lambda_3, \lambda_3\pi_3, \cdots, \lambda_n, \lambda_n\pi_n, \lambda_1, \lambda_1\pi_1, \lambda_2, \lambda_2\pi_2\}$ and an ordered set of elements

$$\{x_3, y_3, \cdots, y_{k+1}, x_{k+2}, x_{k+3}, \cdots, y_n, x_1, y_1, x_2\}\,.$$

Then the property (iv) of Lemma 21 yields $[y_k, y_{k+1}]=[y_k, y_1]=1$. Finally let $n=2$. Set $\delta=(x_1x_2)^{y_1}$. By Theorem 9, we have

$$\delta:\lambda_1\rightarrow\lambda_2,\ \pi_1\rightarrow\pi_2,\ \pi_1'\rightarrow\pi_2'\quad\text{or}\quad \pi_2'\pi_2\,.$$

The last conjugation follows from the fact that $(\pi_1' y_1)^3 \equiv 1 \bmod M$ and we must have $l(\pi_1'^{\delta}) = 1$. This implies that $\langle \delta, \pi_1', \pi_2' \rangle$ or $\langle \delta, \pi_1', \pi_2'\pi_2 \rangle$ is a complement of $N_G(J) = \langle \delta, J \rangle$ over M. Thus a 2-Sylow subgroup of G, which is equal to $N_G(J)$ in this case, splits over M. Then a theorem of Gaschütz yields that $N_G(M)/M$ splits. Let $Q = \langle t_1, t_2, t_3 \rangle$ be a complement of N over M such that t_1, t_2 and t_3 induce transpositions

$$t_1 : \lambda_1 \leftrightarrow \lambda_1\pi_1$$

$$t_2 : \lambda_1\pi_1 \leftrightarrow \lambda_2$$

$$t_3 : \lambda_2 \leftrightarrow \lambda_2\pi_2$$

on the set $\{\lambda_1, \lambda_1\pi_1, \lambda_2, \lambda_2\pi_2\}$. Such elements exist, because every element of N induces a permutation on the set $\{\lambda_1, \lambda_1\pi_1, \lambda_2, \lambda_2\pi_2\}$. Then an elementary abelian subgroup $\langle t_1, \pi_1, t_3, \pi_2 \rangle$ must be conjugate in N to S. In fact, we easily see that $\langle t_1, \pi_1, t_3, \pi_2 \rangle$ is normal in a 2-Sylow subgroup $\langle M, t_1, t_3, (t_1t_2)^{t_3} \rangle$ of G which must be conjugate in G to $N_G(J)$, and so $\langle t_1, \pi_1, t_3, \pi_2 \rangle$ must be conjugate to S, because S and M are just two such subgroups of $N_G(J)$. Then we may assume $t_1 = \pi_1'$ or $\pi_1'\pi_2$. If we have $t_1 = \pi_1'\pi_2$, interchange $\{t_1, t_2, t_3\}$ by $\{t_1\pi_1\pi_2, t_2\pi_1\pi_2, t_3\pi_1\pi_2\}$. Then we have $t_1^{\lambda_1} = \pi_1'$. Thus we may assume without loss that $t_1 = \pi_1'$ and then $t_3 = \pi_2'$. This completes the proof of Lemma 22.

LEMMA 23. *Let Q be a complement of N over M. Then Q is a complement of $N_G(E)$ over $C_G(E)$ and $C_G(E) = E \times \langle \nu \rangle$ where ν is as in* (2). *In particular, $N_G(E)$ is isomorphic to $C_{\mathfrak{A}_m}(\alpha_n)$.*

PROOF. From the action of Q on M, we see that Q normalizes E. Moreover, from Theorem 9 we also see that any two elements of E are conjugate in G if and only if they are so in $Q \cdot C_G(E)$. This implies

$$[N_G(E) : C_G(\alpha_k) \cap N_G(E)] = [Q \cdot C_G(E) : C_G(\alpha_k) \cap N_G(E)]$$

because $C_G(\alpha_k) \cap N_G(E) = C_G(\alpha_k) \cap Q \cdot C_G(E)$. Therefore we must have $N_G(E) = Q \cdot C_G(E)$, because $N_G(E) \supseteqq Q \cdot C_G(E)$. Since $C_G(E) \subseteqq C_G(\alpha_k) = H_m(k)$, we have $C_G(E) = E \times \langle \nu \rangle$. This means $Q \cap C_G(E) = 1$ and so Q is a complement of $N_G(E)$ over $C_G(E)$. Moreover, since we also know the action of Q on E, it can be easily ckecked that $N_G(E)$ is isomorphic to $C_{\mathfrak{A}_m}(\alpha_n)$, q. e. d.

LEMMA 24. (i) *$N_G(J)$ splits over J.* (ii) *Let D be any 2-Sylow subgroup of G containing S (resp. M). Then S (resp. M) is weakly closed in a 2-Sylow subgroup of G with respect to G and D splits over S.*

PROOF. Let Q be a complement of N over M which contains $\{\pi_1', \pi_2', \cdots, \pi_n'\}$. Such a complement exists by Lemma 22. We know that Q is isomorphic to $\mathfrak{S}_{2n}$ and acts on M as the symmetric group on the set $\{\lambda_1, \lambda_1\pi_1, \cdots, \lambda_n, \lambda_n\pi_n\}$. So we can find $n-1$ involutions $\sigma_1'', \sigma_2'', \cdots, \sigma_{n-1}''$ of Q such that

$$(\lambda_s\pi_s)^{\sigma_s''}=\lambda_{s+1} \qquad (1\leqq s\leqq n-1)$$

$$[\sigma_s'', \lambda_{t+1}]=[\sigma_s', \lambda_t\pi_t]=1 \qquad (t\neq s).$$

Then the ordered set $\{\pi_1', \sigma_1'', \cdots, \sigma_{n-1}'', \pi_n'\}$ is a canonical set of generators of Q. Set $\rho_s=(\pi_s'\pi_{s+1}')^{\sigma_s''}$ $(1\leqq s\leqq n-1)$. Then from the actions of the ρ_s on M, we can easily check that the group $\langle\rho_1, \rho_2, \cdots, \rho_{n-1}\rangle$ normalizes J and is isomorphic to $\mathfrak{S}_n$ and the ordered set $\{\rho_1, \rho_2, \cdots, \rho_{n-1}\}$ is a canonical set of generators of $\mathfrak{S}_n$. Since we know $N_G(J)/J\cong\mathfrak{S}_n$, we must have that $\langle\rho_1, \rho_2, \cdots, \rho_{n-1}\rangle$ is a complement of $N_G(J)$ over J. Furthermore we see that $N_G(J)$ splits over S and M. Indeed, $\langle\rho_1, \rho_2, \cdots, \rho_{n-1}, \lambda_1, \lambda_2, \cdots, \lambda_n\rangle$ (resp. $\langle\rho_1, \rho_2, \cdots, \rho_{n-1}, \pi_1', \pi_2', \cdots, \pi_n'\rangle$) is a complement of $N_G(S)$ over S (resp. $N_G(M)$ over M). We shall prove (ii). We may assume without loss that D is a 2-Sylow subgroup of $N_G(J)$ by Sylow's theorem and Lemma 3. Then it is obvious that D splits over S and M. From Lemma 5 and Lemma 18 it follows that S and M are weakly closed in D with respect to G, q.e.d.

4.2. In order to complete the proof of our main theorem, we will have to determine the centralizer of some involution of $G_m(k)$ other than a given one. We note here that all the information has been obtained which is necessary to apply the arguments of §5 and §6 of [3]. Firstly, let $k=1$ and $G=G_m(1)$. Then we can easily check that we can apply the arguments of §6 of [3] without any changes to get $C_G(\alpha_n)\triangleright E$ where E is an elementary abelian group of order 2^{2n-1} introduced in the preceding paragraph. Thus we obtain $C_G(\alpha_n)=N_G(E)$, because $C_G(\alpha_n)\supseteqq N_G(E)$ by Theorem 9. This yields the desired conclusions $G\cong\mathfrak{A}_m$ by Lemma 23 and [3; Theorem I] (cf. the remark at the end of the proof of Lemma 3). Secondly, let $n>k>1$ and $G=G_m(k)$. Then we can also check that we can apply the arguments of §5 of [3] without any change to determine the structure of $C_G(\alpha_1)$ which will turn out to be isomorphic to $C_{\mathfrak{A}_m}(\alpha_1)\cong H_m(1)$. Thus, in this case, we get the conclusion $G\cong\mathfrak{A}_m$ by reducing the problem to the preceding case $k=1$.

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