J. Math. Soc. Japan
Vol. 23, No. 3, 1971

Axiomatic theory of non-negative fullsuperharmonic functions

By Tosiaki KORI

(Received Aug. 11, 1970)

The axiomatic research of non-negative superharmonic functions on a harmonic space has been treated by M. Brelot, R. M. Herve, H. Bauer, and C. Constantinescu and A. Cornea. In the study of elliptic or parabolic differential equations we can find many applications of their theory ([**4**], [**10**]). Our motivation in the present paper is related to the study of elliptic differential equations with certain lateral conditions. What lateral conditions may be considered on a harmonic space? The problem of this sort was first considered by R. S. Martin in connection with the Dirichlet problem and by Z. Kuramochi in connection with the Neumann problem. In the case of axiomatic theory of harmonic functions K. Gowrisankaran made a study of the Dirichlet problem. The axiomatic formulation of Kuramochi's theory was given by F. Y. Maeda [**15**]. It seems that many known lateral conditions may be considered within the framework of *fullharmonic structure* introduced by Maeda. (We can give the fullharmonic structure that corresponds to the solutions of an elliptic differential equation with a Wentzel's boundary condition lacking the term that indicates, in a probability language, the jumps to the interior.)

Starting from a given fullharmonic structure on a Brelot's harmonic space, we shall make a research of many properties of fullsuperharmonic functions (that is to say, supersolutions of an elliptic differential equation with a lateral condition). Many properties of superharmonic functions that were studied extensively by Brelot and Herve are also valid for fullsuperharmonic functions (Minimum principle, etc.). They are studied by F. Y. Maeda. Maeda has proved a partition theorem, which, in the case of Brelot's theory, was obtained by Herve and is called by the name of Herve's partition theorem. We shall further develop his results, and shall construct potential kernels. We shall give a submarkov resolvent such that the excessive functions relative to this resolvent are exactly the non-negative fullsuperharmonic functions (Sections 3 and 4). P. A. Meyer is the first who constructed a resolvent such that the excessive functions relative to the resolvent are the non-negative superharmonic functions in the axiomatic theory of Brelot. This problem has been studied by N. Boboc, C. Constantinescu and A. Cornea, and W. Hansen in the case of axiomatic theory of Bauer or others.

Every non-negative fullsuperharmonic function is decomposed (uniquely) to the sum of a harmonic fullsuperharmonic function and a fullsuperharmonic function which is étranger to this function relative to the order induced by the cone of non-negative fullsuperharmonic functions. These two functions are characterized as follows: The latter has a Riesz-Martin type integral representation with the aid of a measure on the harmonic space, the first with the aid of a measure on an ideal boundary. The extreme points of a compact base of the cone of non-negative harmonic fullsuperharmonic functions are homeomorphically embedded into this ideal boundary. This boundary of the harmonic space is Kuramochi boundary with respect to the fullharmonic structure. These are studied in sections 5, 6 and 7.

It should be noted that our theory of fullsuperharmonic functions is a slightly different one from that considered by Maeda. We start with the assumption of the existence of a non-zero P-function on the harmonic space (Section 1), while Maeda considered a subdomain of the given harmonic space where the existence of a non-zero P-function was assumed. It can be proved that, if every P-function is zero, then every fullsuperharmonic function on the whole space is fullharmonic there and mutually proportional. Maeda's procedure corresponds to the routine method in function theory of taking off a closed ball from the considering Riemann surface.

The main results of sections 3 and 4 were announced in [**11**] and [**12**]. The result in [**12**] as for the construction of a semigroups of submarkov kernels was false.

§ 1. Preliminaries and basic properties.

Let X be a locally compact, not compact, connected Hausdorff space with a countable base. We adopt a Brelot's harmonic structure. Namely we suppose that we are given a sheaf $\mathcal{H}$ of real vector spaces of real continuous functions such that;

(1) there is a base for the topology of X which is formed by *regular* domains (for $\mathcal{H}$), i.e. relatively compact domains G such that any continuous function f on ∂G has a unique continuous extension $H^G f$ in $\mathcal{H}(G)$ which is non-negative if f is non-negative;

(2) the upper envelope of any upper directed family of functions in $\mathcal{H}(G)$ where G is a domain is either $+\infty$ or an element of $\mathcal{H}(G)$.

Any function in $\mathcal{H}(G)$ is said to be *harmonic on* G. We denote by $\mathcal{S}(G)$ the set of the superharmonic functions on an open set G.

A *fullharmonic structure* subordinate to the harmonic structure $\mathcal{H}$ is introduced as follows. Let $\mathcal{D}$ be the family of domains D in X such that D is not relatively compact and the boundary ∂D of D is compact, and let $\mathcal{G}$ be the family of open subsets with compact boundary. We assume that we are given a class $\tilde{\mathcal{H}}$ of linear subspaces $\tilde{\mathcal{H}}(D)$ of $\mathcal{H}(D)$ where $D \in \mathcal{D}$ such that;

(3) if $D, D' \in \mathcal{D}$, $D' \subset D$, and $u \in \tilde{\mathcal{H}}(D)$, then the restriction $u|_{D'}$ of u on D' belongs to $\tilde{\mathcal{H}}(D')$; if $u \in \mathcal{H}(D)$ and there is a compact set K such that $\mathring{K}$ (the interior of K) $\supset \partial D$ and $u|_{D-K} \in \tilde{\mathcal{H}}(D-K)$, then $u \in \tilde{\mathcal{H}}(D)$.

$D \in \mathcal{D}$ is said to be *regular* (for $\tilde{\mathcal{H}}$) if any continuous function f on ∂D possesses a unique continuous extension to $\bar{D}$ whose restriction $\tilde{H}^D f$ to D belongs to $\tilde{\mathcal{H}}(D)$, and is non-negative if f is non-negative. A set $G \in \mathcal{G}$ is *regular* if every component of G is either regular for $\mathcal{H}$ or regular for $\tilde{\mathcal{H}}$. We suppose that $\tilde{\mathcal{H}}$ satisfies the following axiom;

(4) for any compact set K_0, there is another compact set K such that $\mathring{K} \supset K_0$ and $X-K$ is regular.

For $G \in \mathcal{G}$, we define the set of *fullharmonic functions* on G as follows;

$$\tilde{\mathcal{H}}(G) = \{u \in \mathcal{H}(G)\,;\ u|_D \in \tilde{\mathcal{H}}(D) \text{ for each component } D \text{ of } G \text{ such that } D \in \mathcal{D}\}.$$

A superharmonic function s on $G \in \mathcal{G}$ is said to be *fullsuperharmonic* on G if, for any regular set $D \in \mathcal{D}$, $\bar{D} \subset G$, and for any continuous function f on ∂D, the relation $f \leqq s$ on ∂D implies the relation $\tilde{H}^D f \leqq s$. We denote by $\tilde{\mathcal{S}}(G)$ the set of fullsuperharmonic functions on $G \in \mathcal{G}$. If $G \in \mathcal{G}$ is relatively compact we have $\tilde{\mathcal{H}}(G) = \mathcal{H}(G)$, $\tilde{\mathcal{S}}(G) = \mathcal{S}(G)$.

Throughout this paper we will assume that the constant function 1 is fullsuperharmonic on X;

(5) $1 \in \tilde{\mathcal{S}}(X)$.

From this assumption we see that every fullharmonic function on G is bounded continuous for any $G \in \mathcal{G}_r$.

$C(X)$, $C_b(X)$, $B_b(X)$ respectively are the spaces of continuous functions, bounded continuous functions, bounded Borel measurable functions on X. $C_c(X)$ is the space of continuous functions of compact support. The support of f is denoted as $\mathrm{Supp}[f]$. For any set of functions A we denote by A_+ or A^+ the set of positive elements of A.

We can prove many properties of fullsuperharmonic functions that are similar to those of superharmonic functions. We shall give some of them that will be frequently used. The proofs of these are found in [**15**].

(**a**) The upper envelope of any upper directed family of fullsuperharmonic functions (resp. fullharmonic functions) is either the constant $+\infty$ or a fullsuperharmonic function (resp. fullharmonic function).

(**b**) The lower semi-continuous regularization of the greatest lower bound of a family of fullsuperharmonic functions, that are locally uniformly bounded from below, is a fullsuperharmonic function.

(**c**) (*Minimum Principle*) A fullsuperharmonic function u on $G \in \mathcal{G}$ with the property;

$$\liminf_{G \ni x \to y} u(x) \geqq 0 \qquad \text{for any} \quad y \in \partial G,$$

is non-negative.

(**d**) Let u be a fullsuperharmonic function on X and v be a fullsuperharmonic function on $G \in \mathcal{G}$. If

$$\liminf_{G \ni x \to y} v(x) \geqq u(y) \qquad \text{for any} \quad y \in \partial G,$$

then

$$w = \begin{cases} \inf(u, v) & \text{on} \quad G \\ u & \text{on} \quad X-G \end{cases}$$

is a fullsuperharmonic function on X.

(**e**) (Perron) Let $\mathcal{V}$ be a family of fullsuperharmonic functions on $G \in \mathcal{G}$. Suppose that $\mathcal{V}$ is a Perron's family[1] on G and that, for each compact set K such that $\mathring{K} \supset \partial G$ and $G-K$ is regular, $u \in \mathcal{V}$ implies $u^K \in \mathcal{V}$, where

$$u^K = \begin{cases} \tilde{H}^{G-K}u & \text{on} \quad G-K \\ u & \text{on} \quad G \cap K. \end{cases}$$

Then $\inf\{v;\, v \in \mathcal{V}\}$ is fullharmonic on G.

Let f be a non-negative function on X and F be a subset of X. We define the reduced function of f on F as follows;

$$R^F f = \inf\{v \in \mathcal{S}_+(X),\ v \geqq f \text{ on } F\}.$$

The lower semi-continuous regularization $\hat{R}^F f$ of $R^F f$ is called the balayage of f on F.

(**f**) Let u be a non-negative fullsuperharmonic function on X.

(f-1) $\hat{R}^F u \in \mathcal{S}_+(X)$. $\hat{R}^F u$ is harmonic on $X-\bar{F}$, $\hat{R}^F u$ is fullharmonic on $X-\bar{F}$ if F is relatively compact.

(f-2) $\hat{R}^F u = u$ on $\mathring{F}$, $\hat{R}^{\phi} u = 0$.

(f-3) For any $G \in \mathcal{G}_r$,[2] we have

$$\hat{R}^{X-G} u = \begin{cases} \tilde{H}^G u & \text{on} \quad G \\ u & \text{on} \quad X-G. \end{cases}$$

---

1) As for Perron's family, see [**1**], [**10**], [**15**].

2) $\mathcal{G}_r$, $\mathcal{D}_r$ are the families of regular sets in $\mathcal{G}$ and $\mathcal{D}$ respectively.

Let $\mathcal{P}$ be the set of all non-negative fullsuperharmonic functions p on X with the following property; if a fullsuperharmonic function u on X satisfies $p+u \geqq 0$, then $u \geqq 0$. Any function in $\mathcal{P}$ is called a *P-function*. Any non-negative fullsuperharmonic function u has a unique decomposition $u=p+h$ with h fullharmonic on X and $p \in \mathcal{P}$ (*Riesz decomposition*). In fact let

$$h=\sup\{t;\ -t \in \tilde{\mathcal{S}}(X) \text{ and } u \geqq t\}$$
$$=-\inf\{s;\ s \in \tilde{\mathcal{S}}(X) \text{ and } u+s \geqq 0\}.$$

The family $\mathcal{V}=\{s \in \tilde{\mathcal{S}}(X);\ u+s \geqq 0\}$ satisfies the conditions of (**e**) and therefore h is fullharmonic on X. Let $p=u-h$. $p \in \tilde{\mathcal{S}}_+(X)$. If $v \in \tilde{\mathcal{S}}(X)$ is such that $p+v \geqq 0$, we have $v-h \in \mathcal{V}$, so $v-h \geqq -h$ and $v \geqq 0$. Hence $p \in \mathcal{P}$. We have the decomposition $u=p+h$, $p \in \mathcal{P}$, $h \in \tilde{\mathcal{H}}_+(X)$. The uniqueness is easily proved.

$\mathcal{P}$ is a convex cone. An order defined by the cone $\mathcal{P}$ is called *the specific order in $\mathcal{P}$*, and is denoted by $\prec$.

We adopt the following assumption:

(**6**) For any point x of X there is a $p \in \mathcal{P}$ which is strictly positive at x.

The following properties (**g**) and (**h**) can be shown by a well-known argument (See, for example, Section 5, Chap. II, [**1**]).

(**g**) Let $f \in C_c^+(X)$. We have

$$Rf \equiv R^X f \in \mathcal{P} \cap C(X) \cap \tilde{\mathcal{H}}(X-\mathrm{Supp}[f]).$$

(**h**) For any $x, y \in X$, $x \neq y$, there are $p, q \in \mathcal{P} \cap C_b(X)$ such that $p(x)q(y) \neq q(x)p(y)$.

THEOREM 1.1.

$$\tilde{\mathcal{S}}(X)=\tilde{\mathcal{S}}_+(X)=\mathcal{P}, \qquad \tilde{\mathcal{H}}(X)=\{0\}.$$

PROOF. Let $u \in \tilde{\mathcal{S}}(X)$ and let x_0 be a point of X. From (**6**) there is a P-function p such that $p(x_0)>0$. Since $-u(x_0)<\infty$ we can find a number $\lambda>0$ such that $-u(x_0)<\lambda p(x_0)$. Applying the minimum principle to the function $\lambda p+u \in \tilde{\mathcal{S}}(X-\{x_0\})$ and the domain $X-\{x_0\} \in \mathcal{D}$, we have $\lambda p+u \geqq 0$ on X. λp being a P-function we have $u \geqq 0$. Therefore $\tilde{\mathcal{S}}(X)=\tilde{\mathcal{S}}_+(X)$. From this it follows $\tilde{\mathcal{S}}_+(X)=\mathcal{P}$ and $\tilde{\mathcal{H}}(X)=\{0\}$.

We shall give some definitions as for the decomposition of P-functions. We set:

$$\mathcal{P}_b=\mathcal{P} \cap \mathcal{H}(X),$$

$$\mathcal{P}_i=\{p \in \mathcal{P};\ \text{if there is a } w \in \mathcal{P}_b \text{ such that } w \prec p, \text{ then } w=0\},$$

$$\mathcal{P}_c=\{p \in \mathcal{P};\ p \text{ is fullharmonic out of some compact set}\}.$$

Functions in $\mathcal{P}_b$ and $\mathcal{P}_i$ are called *P_b-functions* and *P_i-functions* respectively. Any P-function p has a unique decomposition $p=q+r$ with $q \in \mathcal{P}_i$ and $r \in \mathcal{P}_b$.

For any $p \in \mathcal{P}$, put

$$Bp = \sup\{w \in \mathcal{P}_b\,;\ w \prec p\}\,.$$

(i) [15] $Bp \in \mathcal{P}_b$, $Bp \prec p$, and Bp is the least upper bound of $\{w \in \mathcal{P}_b\,;\ w \prec p\}$ with respect to the specific order $\prec$. Any $p \in \mathcal{P}$ belongs to $\mathcal{P}_b$ (resp. $\mathcal{P}_i$) if and only if $Bp = p$ (resp. $Bp = 0$). We have; $B(p+q) = Bp + Bq$.

PROPOSITION 1.2. *For any $u \in \mathcal{P}$ there is an increasing sequence of functions $p_n \in \mathcal{P}_c \cap C_b(X)$ such that $p_n \uparrow u$.*

PROOF. Let f_n be a sequence of continuous functions of compact support such that $f_n \uparrow u$. The functions $p_n = Rf_n$ respond to our proposition in view of (g).

LEMMA 1.3. *Let p_n be a sequence of P-functions such that it decreases to 0 with the specific order $\prec$. Let $p_m = p_n + u_{m,n}$ with $u_{m,n} \in \mathcal{P}$ $(n \geqq m)$. If $Bu_{m,n} = 0$ for any m and n, $n \geqq m$, then $Bp_n = 0$ for any n.*

PROOF. From the property (i) we have, for any m, $Bp_m = Bp_n + Bu_{m,n} = Bp_n \leqq p_n$ $(n \geqq m)$. Tending n to infinity we have $Bp_m = 0$.

The next lemma is an immediate consequence of the definition.

LEMMA 1.4. *Let $p, q \in \mathcal{P}$. Then $p \succ q$ implies $Bp \succ Bq$ and $p - Bp \succ q - Bq$.*

LEMMA 1.5. *Let $u \in \mathcal{P}_b$. If $v \prec u$ then $v \in \mathcal{P}_b$.*

PROOF. Let $h \in \mathcal{P}$ be such that $u = v + h$. From the assumption $-h = v - u \in \mathcal{S}(X)$, we have $h \in \mathcal{H}(X)$. Hence $v \in \mathcal{P} \cap \mathcal{H}(X) = \mathcal{P}_b$.

In the above proof we have seen that, for $p, q \in \mathcal{P}_b$, the relation $q \prec p$ is equivalent to the relation $p - q \in \mathcal{P}_b$. Hence the order $\prec$ restricted on $\mathcal{P}_b$ is the order induced by the cone $\mathcal{P}_b$ itself. Also we can easily verify that the order $\prec$ restricted on $\mathcal{P}_i$ is the same as the order induced by the cone $\mathcal{P}_i$.

THEOREM 1.6. *$\mathcal{P}$, $\mathcal{P}_i$, and $\mathcal{P}_b$ are lattices with respect to the specific order $\prec$.*

PROOF. Let $p, q \in \mathcal{P}$. Consider the family

$$\mathcal{U} = \{u \in \mathcal{P}\,;\ u \succ p \text{ and } u \succ q\}\,.$$

Let $r = \widehat{\inf}\, \mathcal{U}$ (the lower semi-continuous regularization of the greatest lower bound of the functions of $\mathcal{U}$). r is a non-negative fullsuperharmonic function, so $r \in \mathcal{P}$. We set

$$u = \widehat{\inf}\, \{v \in \mathcal{P}\,;\ v + p \succ q\}\,.$$

Then we have $u + p = r$ and $u \in \mathcal{P}$, so $r \succ p$. Similarly we have $r \succ q$. Now let $s \in \mathcal{P}$ be such that $s \succ p$ and $s \succ q$. Put

$$t = \begin{cases} s - r & \text{if } r < \infty \\ \infty & \text{if } r = \infty\,. \end{cases}$$

Obviously $t \geqq 0$. We shall prove that t is nearly fullsuperharmonic[3]. It is enough to prove that

$$t(x) \geqq \tilde{H}^G t(x)$$

holds for any $G \in \mathcal{G}_r$ and any $x \in G$ at which t is finite. Take a continuous function φ such that $\varphi \leqq s$ on ∂G, and let

$$f = \begin{cases} \inf(s - \tilde{H}^G\varphi + \tilde{H}^G r,\ r) & \text{on } G \\ r & \text{on } X - G. \end{cases}$$

From (**d**), $f \in \tilde{S}_+(X) = \mathcal{P}$. Similarly

$$g = \begin{cases} \inf(w - \tilde{H}^G\varphi + \tilde{H}^G r,\ u) & \text{on } G \\ u & \text{on } X - G \end{cases}$$

is a P-function, where w is the P-function such that $s = p + w$. Since $g + p = f$ we have $f \succ p$. We have also $f \succ q$ and $f \in \mathcal{V}$. Thus $r \leqq f$, that is,

$$s + \tilde{H}^G r \geqq r + \tilde{H}^G \varphi .$$

Since

$$\tilde{H}^G s = \sup \{\tilde{H}^G \varphi\,;\ \varphi \text{ is continuous and } \varphi \leqq s \text{ on } \partial G\},$$

we have

$$s + \tilde{H}^G r \geqq r + \tilde{H}^G s .$$

If $t(x) < \infty$ at $x \in G$, we have

$$t(x) = s(x) - r(x) \geqq \tilde{H}^G s(x) - \tilde{H}^G r(x) = \tilde{H}^G t(x) .$$

Thus t is nearly fullsuperharmonic, $\geqq 0$. It holds that $\hat{t} \in \mathcal{P}$ and $s = \hat{t} + r$, so $s \succ r$. r is the least upper bound $p \vee q$ of p and q relative to the order $\prec$. We have proved that $(\mathcal{P}, \prec)$ is an upper semi-lattice. Since $\mathcal{P}$ is a convex cone, it is a lattice.

Let p and q be two $\mathcal{P}_b$-functions and $r = p \vee q$. From the property (**i**) it follows $Br \succ p$ and $Br \succ q$, so $Br \succ p \vee q = r$. Hence $r = Br \in \mathcal{P}_b$. $\mathcal{P}_b$ is a lattice.

Finally let p and q be two P_i-functions and $r = p \vee q$. We have $r - Br \succ p - Bp = p$ and $r - Br \succ q$ from Lemma 1.4, so $r - Br \succ p \vee q = r$. Br must be 0 and $r \in \mathcal{P}_i$. This proves that $\mathcal{P}_i$ is a lattice.

We shall proceed to the problem; how many P-functions have we? We shall prove that there are so many functions in $\mathcal{P}_c$ that the linear closure of them relative to the topology of uniform convergence may contain at least the set of all continuous functions of compact support. We denote by $\tilde{\mathcal{P}}$ the

---

3) A function g on $G \in \mathcal{G}$ is said to be *nearly fullsuperharmonic* if it is nearly superharmonic on G, and for any $D \in \mathcal{D}_r$, $\bar{D} \subset G$, it holds $g(x) \geqq \tilde{H}^D g(x)$ for all $x \in D$. If g is nearly fullsuperharmonic then $\hat{g}$ is fullsuperharmonic.

set of bounded continuous P-functions, and by $\widetilde{\mathcal{P}}-\widetilde{\mathcal{P}}$ the functions of the form $p-q$, p and $q\in\widetilde{\mathcal{P}}$.

PROPOSITION 1.7. *Let K be a compact set. Every continuous function on K may be approximated by the functions in $\widetilde{\mathcal{P}}-\widetilde{\mathcal{P}}$ uniformly on K.*

PROOF. Let $\mathcal{E}$ be the set of all functions on K that are restrictions on K of functions from $\widetilde{\mathcal{P}}-\widetilde{\mathcal{P}}$. Obviously $\mathcal{E}$ is a linear subspace of $C(K)$, and from the property (**h**) $\mathcal{E}$ separates the points of K. Let $d\in\mathcal{E}$, $d=p-q$, then $|d|=p+q-2\inf(p,q)$, so $|d|\in\mathcal{E}$, this proves that $\mathcal{E}$ is a lattice. Since $1\in\widetilde{\mathcal{P}}$, applying Stone-Weierstrass' theorem, we have our result, that is, $\mathcal{E}$ is dense in $C(K)$.

For any compact subset K of X such that $X-K$ is regular, we set

$$\boldsymbol{E}^K=C(X)\cap\mathcal{H}(X-K)\,.$$

$\boldsymbol{E}^K$ is a Banach subspace of $C_b(X)$. By virtue of the above proposition, we can find, for any $f\in\boldsymbol{E}^K$ and an $\varepsilon>0$, two functions $p,q\in\widetilde{\mathcal{P}}$ such that $\sup_K|f-(p-q)|<\varepsilon$. Since $f-(\hat{R}^Kp-\hat{R}^Kq)=\widetilde{H}^{X-K}(f-(p-q))$ on $X-K$, we have $|f-(\hat{R}^Kp-\hat{R}^Kq)|<\varepsilon$ uniformly on X. Hence the set of all functions of the form $\hat{R}^Kp-\hat{R}^Kq$, where $p,q\in\widetilde{\mathcal{P}}$, is dense in the Banach space $\boldsymbol{E}^K$. $C(K)$ being separable we have the following proposition.

PROPOSITION 1.8. *Let K be a compact set such that $X-K$ is regular. There is a countable collection (p_n^K) of functions in $\mathcal{P}\cap C(X)\cap\mathcal{H}(X-K)$ such that the differences of these functions are dense in the Banach space $\boldsymbol{E}^K$.*

Let $(K_n)_{n\geqq 1}$ be an exhaustion of X by compact sets K_n with $X-K_n$ being regular, and let $\boldsymbol{E}$ be the set of all continuous functions that are fullharmonic out of some K_n. $\boldsymbol{E}$ is a (LB)-space as the strict inductive limit of the sequence of Banach spaces $\boldsymbol{E}^{K_n}$ [21]. From the above proposition there is a countable collection Q_n of functions in $\mathcal{P}\cap C(X)\cap\mathcal{H}(X-K_n)$ such that Q_n-Q_n is dense in $\boldsymbol{E}^{K_n}$. We put $Q=\bigcup_n Q_n=(p_i)_{i\geqq 1}$, $p_i\in\mathcal{P}_c\cap C(X)$.

THEOREM 1.9. *$Q-Q$ is dense in the (LB)-space $\boldsymbol{E}$. $Q-Q$ is also dense in the uniform closure of the set $\boldsymbol{E}$.*

COROLLARY 1.10. *Any continuous function of compact support can be approximated uniformly on X by the functions in $Q-Q$.*

Let

$$p_0=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{p_k}{\sup p_k}\,,\qquad p_k\in Q\,.$$

This series converges uniformly on X, and from the property (**a**) and Theorem 1.1 it follows $p_0\in\mathcal{P}\cap C_b(X)$.

PROPOSITION 1.11. *p_0 is strictly fullsuperharmonic on X, and therefore, is strictly positive.*

PROOF. If, for some regular set $G\in\mathcal{G}$ and $x\in G$, it holds that $\widetilde{H}^G p_0(x)$

$=p_0(x)$, then $\tilde{H}^G p_n(x)=p_n(x)$ for all n. From Corollary 1.10 it follows that $\tilde{H}^G f(x)=f(x)$ for any $f\in C_c(X)$. Since $\tilde{H}^G(x, \cdot)$ is a Radon measure concentrated on ∂G, this relation is absurd. Hence p_0 is strictly fullsuperharmonic.

REMARK. $p_0\in\mathcal{P}_i$ is proved as follows. Later we will prove the relation $\mathcal{P}_c\subset\mathcal{P}_i$. From this we have $Bq_n=0$, where

$$q_n=\sum_{k=1}^{n}\frac{1}{2^k}\frac{p_k}{\sup p_k}\ .$$

It can be shown that the P-functions p_0-q_n decrease to 0 in the specific order and that $B(q_n-q_m)=0$ $(n\geqq m)$. Hence we have from Lemma 1.3 $B(p_0-q_n)=0$. Thus $Bp_0=0$ and $p_0\in\mathcal{P}_i$.

PROPOSITION 1.12. *Let* $\tilde{\mathcal{P}}=\mathcal{P}\cap C_b(X)$. $(\tilde{\mathcal{P}}-\tilde{\mathcal{P}})\cap C_c(X)$ *is dense in* $C_c(X)$ *with respect to the compact convergence topology on* X.

PROOF. Let

$$A=\left\{\frac{p-q}{p_0}\ ;\ p,\, q\in\tilde{\mathcal{P}},\ p-q\in C_c(X)\right\}.$$

A is a linear subspace of $C_c(X)$. In the same way as in Proposition 1.7 we can verify that A is a lattice. For $d=\frac{1}{p_0}(p-q)\in A$ we have

$$\inf(1,\, d)=\frac{1}{p_0}\{\inf(p_0+q,\, p)-q\}\in A\,.$$

We shall prove that A separates the points of X. Let $x\in X$ and U be a neighborhood of x. Let V be a relatively compact regular neighborhood of x contained in U, and let

$$d=\frac{1}{p_0}(p_0-\hat{R}^{X-V}p_0)\,.$$

From the property (**f**), $d\in A$ and $d=0$ out of V. Since p_0 is strictly fullsuperharmonic $d(x)>0$. Thus A separates the points of X. These being so, applying Stone-Weierstrass' theorem, we conclude that A is dense in $C_c(X)$. Since p_0 is a bounded function we have proved our assertion.

LEMMA 1.13. *Let p and q be two functions in* $\tilde{\mathcal{P}}$ *such that* $p-q\in C_c^+(X)$. *Then for any open neighborhood U of the support of $p-q$ there exist two functions p' and q' of* $\mathcal{P}$ *such that $p'-q'=p-q$ and such that p' and q' are fullharmonic out of* $\bar{U}$.

PROOF. Let $g\in C_c(X)$, $0\leqq g\leqq 1$, equals 1 on the support K of $p-q$, and equals 0 out of U. The functions $p'=R(pg)$, $q'=R(qg)$ belong to $\tilde{\mathcal{P}}\cap\mathcal{H}(X-\bar{U})$ and are equal to p and q on K respectively (Properties (**f**), (**g**)). We shall prove $p'=q'$ on $X-K$. Obviously $p'\geqq q'$. The function

$$u=\begin{cases} p'=p & \text{on } K \\ q' & \text{on } X-K \end{cases}$$

is non-negative fullsuperharmonic on X (Property (**d**)). Since $pg=qg$ on $X-K$, we have $q'\geqq \hat{R}^{X-K}qg=\hat{R}^{X-K}pg\geqq pg$ on $X-K$ and $u\geqq pg$ on X. Thus $q'\geqq p'$ on $X-K$.

COROLLARY 1.14. *$(\tilde{\mathcal{P}}_c-\tilde{\mathcal{P}}_c)\cap C_c(X)$ is dense in $C_c(X)$, where $\tilde{\mathcal{P}}_c=\mathcal{P}_c\cap C_b(X)$.*

§2. Q-compactification of X. Extension of fullsuperharmonic functions.

Let $\boldsymbol{E}$ be the space of continuous functions on X which are fullharmonic out of some compact set, and let $Q=(p_n)_{n\geqq 1}$ be a family of functions in $\mathcal{P}_c\cap C(X)$ such that $Q-Q$ is dense in $\boldsymbol{E}$ with respect to the uniform convergence topology. For any finite subfamily F of functions in $\boldsymbol{E}$ and for any $\varepsilon>0$, let

$$U_{F,\varepsilon}=\{(y, z)\in X\times X\,;\ |f(z)-f(y)|<\varepsilon \text{ for all } f\in F\}\,.$$

The collection $\mathcal{U}_{\boldsymbol{E}}$ of all such sets $U_{F,\varepsilon}$ forms a fundamental system of entourages of some uniformity $\mathcal{U}$ on X. Since $Q-Q$ is dense in $\boldsymbol{E}$, the metric defined by

$$d(x, y)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{|p_k(x)-p_k(y)|}{1+|p_k(x)-p_k(y)|}$$

induces the same uniformity on X as the uniformity $\mathcal{U}$. Also we have, by virtue of Corollary 1.10, the equivalence of the topology induced by the uniformity $\mathcal{U}$ and the initial topology on X.

Let $\hat{X}$ be the completion of X with respect to the uniformity $\mathcal{U}$ (or equivalently with respect to the metric d). The above consideration yields the following theorem.

THEOREM 2.1. *The metric completion $\hat{X}$ of (X, d) is a unique (up to a homeomorphism) compactification of X with the following properties:*

(a) *Each $f\in\boldsymbol{E}$ can be continuously extended over $\hat{X}$.*

(b) *The extensions of the functions of Q separate the points of $\hat{X}$.*

Let

$$p_0=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{p_n}{\sup_x p_n(x)}\,,$$

$$q^N=\sum_{n=1}^{N}\frac{1}{2^n}\frac{p_n}{\sup_x p_n(x)}\,.$$

p_0 is a uniform limit on X of uniformly continuous functions q^N, and so p_0 is uniformly continuous and can be continuously extended over $\hat{X}$.

Since the possibility of the continuous extension over $\hat{X}-X$ of any function depends only on the behavior of the function near the boundary $\hat{X}-X$, any function which is not necessarily continuous on X but is fullharmonic out of some compact set can also be extended uniquely to a function on $\hat{X}$

such that it is continuous near the boundary $\hat{X}-X$. By the same reason if a sequence of functions such that they can be continuously extended over $\hat{X}$ converges uniformly on the complement of some compact set to a function, then the limit function can be extended over $\hat{X}$ to be continuous near the boundary $\hat{X}-X$.

COROLLARY 2.2. *Any function of $\mathcal{P}_c$ can be extended over $\hat{X}$ such that it is continuous near the boundary $\hat{X}-X$.*

REMARK. $\hat{X}$ is the compactification introduced by C. Constantinescu and A. Cornea [7] and they defined it as Kuramochi's compactification. But this is not Kuramochi's original compactification that was introduced in order to get integral representations of P_b-functions. We shall give in section 7 a compactification such that P_b-functions are represented by measures on the ideal boundary.

§ 3. Specific restrictions of fullsuperharmonic functions to subsets of X.

F. Y. Maeda proved a partition theorem of fullsuperharmonic functions and introduced the concept of specific restrictions of fullsuperharmonic functions to the sets of $\mathcal{G}$. In this section we shall develop the result and get some properties of the specific restrictions of fullsuperharmonic functions. Things are different from the situation treated by Herve. Namely the specific restriction of a P-function to a set of $\mathcal{D}$ contains a so-called boundary part (Proposition 3.11).

DEFINITIONS. (1) For any $p\in\mathcal{P}$, $G\in\mathcal{G}$, put

$$\mathcal{B}_G(p)=\{u\in\mathcal{P}\,;\ u=p+s \text{ on } G \text{ for some } s\in\tilde{\mathcal{S}}(G)\}$$

$$p_G=\inf\{u\,;\ u\in\mathcal{B}_G(p)\}\,.$$

THEOREM 3.1 (F. Y. Maeda). *Let $p\in\mathcal{P}$ and $G\in\mathcal{G}$.*

(i) *p_G is a P-function and is harmonic on $X-\bar{G}$.*

(ii) *$p_G\prec p$, more precisely, $p=p_G+w$ with a $w\in\mathcal{P}\cap\tilde{\mathcal{H}}(G)$.*

PROOF. (i) $\hat{p}_G\in\tilde{\mathcal{S}}_+(X)=\mathcal{P}$ follows from the property (**b**) in section 1. Let $\mathcal{U}$ be the family of fullsuperharmonic functions s on G such that $u=p+s$ on G for some $u\in\mathcal{B}_G(p)$. Then we can verify, by virtue of the property (**e**) in section 1, that $s_0=\inf\mathcal{U}$ is fullharmonic on G. We have $\hat{p}_G=p+s_0$ on G and $\hat{p}_G\in\mathcal{B}_G(p)$, hence $\hat{p}_G\geqq p_G$. Therefore $p_G=\hat{p}_G\in\mathcal{P}$. $p_G\in\mathcal{H}(X-\bar{G})$ follows also from the property (**e**) in section 1.

(ii) Let

$$t=\begin{cases} p-p_G & \text{if } p_G<\infty \\ \infty & \text{if } p_G=\infty\,. \end{cases}$$

Take a domain $D\in\mathcal{G}_r$ and a continuous function h on ∂D such that $h\leqq p$ on

∂D. We set

$$g=\begin{cases}\inf(p-\tilde{H}^D h+\tilde{H}^D p_G,\ p_G) & \text{on } D\\ p_G & \text{on } X-D.\end{cases}$$

From the property (**d**) of section 1 it follows $g\in\mathcal{P}$. Similarly

$$f=\begin{cases}\inf(-\tilde{H}^D h+\tilde{H}^D p_G,\ s_0) & \text{on } G\cap D\\ s_0 & \text{on } G-D\end{cases}$$

is a fullsuperharmonic function on G. Since $g=f+p$ on G it follows $g\in\mathcal{B}_G(p)$ and $g\geqq p_G$. Therefore $p-\tilde{H}^D h+\tilde{H}^D p_G\geqq p_G$ on G. Since $p=\sup\{h\in C(\partial D);\ h\leqq p\}$ on ∂D we have $p+\tilde{H}^D p_G\geqq p_G+\tilde{H}^D p$. Thus t is nearly fullsuperharmonic and $\hat{t}\in\mathcal{P}$. From $p_G+\hat{t}=p$ it follows $\hat{t}=-s_0$ on G. The function $w=\hat{t}$ responds to the question.

From this theorem, for any compact subset K, there is a P-function p_K such that $p=p_{X-K}+p_K$ and $p_K\in\mathcal{H}(X-K)$.

PROPOSITION 3.2. (i) *Let $p\in\mathcal{P}$ and $G\in\mathcal{G}$. p_G is the greatest lower bound of $\mathcal{B}_G(p)$ with respect to the specific order $\prec$.*

(ii) *p_K is the greatest specific minorant of p which is fullharmonic out of the compact set K.*

PROOF. Let $u\in\mathcal{B}_G(p)$. $u=p+s$ on G for some $s\in\tilde{\mathcal{S}}(G)$. $p=p_G+w$ for a $w\in\mathcal{P}\cap\tilde{\mathcal{H}}(G)$. We have $u=p_G+s+w$ on G. Let $D\in\mathcal{D}_r$ and let h be a continuous function such that $h\leqq u$ on ∂D. We set

$$g=\begin{cases}\inf(u-\tilde{H}^D h+\tilde{H}^D p_G,\ p_G) & \text{on } D\\ p_G & \text{on } X-D.\end{cases}$$

We can verify from the property (**d**) that $g\in\tilde{\mathcal{S}}_+(X)=\mathcal{P}$ and $g\leqq p_G$. Similarly, the function

$$f=\begin{cases}\inf(s-\tilde{H}^D h+\tilde{H}^D p_G,\ -w) & \text{on } D\cap G\\ -w & \text{on } G-D\end{cases}$$

is fullsuperharmonic on G. (Note that $\liminf_{D\ni x\to y\in\partial D}(s-\tilde{H}^D h+\tilde{H}^D p_G)(x)+w(y)\geqq u(y)-h(y)\geqq 0$.) Since $g\in\mathcal{P}$ and $g=f+p$ on G, we have $g\in\mathcal{B}_G(p)$, so $g\geqq p_G$. Hence $g=p_G$ and

$$u+\tilde{H}^D p_G\geqq p_G+\tilde{H}^D h.$$

h being arbitrary continuous function such that $h\leqq u$ on ∂D, we have

$$u+\tilde{H}^D p_G\geqq p_G+\tilde{H}^D u.$$

Therefore the function

$$v=\begin{cases}u-p_G & \text{if } p_G<\infty\\ \infty & \text{if } p_G=\infty\end{cases}$$

is nearly fullsuperharmonic on X and $\hat{v}+p_G=u$. We have $u\succ p_G$ and p_G is a lower bound of $\mathcal{B}_G(p)$. Obviously it is the greatest lower bound. (ii) follows from (i).

COROLLARY 3.3. *If* $p\in\mathcal{P}\cap\tilde{\mathcal{H}}(G)$, $G\in\mathcal{G}$, *then* $p_G=0$, *in particular, if* $p\in\mathcal{P}\cap\tilde{\mathcal{H}}(X-K)$ *for a compact set* K *then* $p=p_K$.

PROPOSITION 3.4. *Let* $p, q\in\mathcal{P}$ *and* $\alpha\geqq 0$. *We have;*

$$(p+q)_G=p_G+q_G\,;\quad (\alpha p)_G=\alpha p_G \qquad (G\in\mathcal{G})$$

$$(p+q)_K=p_K+q_K\,;\quad (\alpha p)_K=\alpha p_K \qquad (K \text{ is compact}).$$

PROOF. First we shall show that the relation $p\succ q$ implies the relation $p_G\succ q_G$. Let $p=q+u$, $u\in\mathcal{P}$, and $p=p_G+w$, $w\in\mathcal{P}\cap\tilde{\mathcal{H}}(G)$. We have $p_G=q+u-w$ on G and $u-w\in\tilde{\mathcal{S}}(G)$, so $p_G\in\mathcal{B}_G(q)$. Hence $q_G\prec p_G$.

From this remark there is a $u\in\mathcal{P}$ such that $(p+q)_G=p_G+u$. By virtue of Theorem 3.1, $(p+q)=(p+q)_G+s$ and $p=p_G+w$ with $s, w\in\mathcal{P}\cap\tilde{\mathcal{H}}(G)$. On G we have $u=q+w-s$. Hence $u\in\mathcal{B}_G(q)$. Proposition 3.2 yields $q_G\prec u$. Then $p_G+q_G\prec(p+q)_G$. Similarly $p_G+q_G\in\mathcal{B}_G(p+q)$ can be verified, and from Proposition 3.2 we have $p_G+q_G\succ(p+q)_G$. Thus $p_G+q_G=(p+q)_G$. The rest is proved in the same way.

PROPOSITION 3.5. *Let* $p\in\mathcal{P}$, *and* $G, G_i\in\mathcal{G}$ *and let* K, K_i *be compact* $(i=1, 2)$. *We have;*

$$K\subset G \Rightarrow p_K\prec p_G\,;\qquad G\subset K \Rightarrow p_G\prec p_K\,,$$

$$G_1\subset G_2 \Rightarrow p_{G_1}\prec p_{G_2}\,;\qquad K_1\subset K_2 \Rightarrow p_{K_1}\prec p_{K_2}\,.$$

PROOF. Let $K\subset G$, and let

$$t=\begin{cases} p_G-p_K & \text{if } p_K<\infty \\ \infty & \text{if } p_K=\infty \end{cases}$$

For $x\in G$ at which $p_K(x)$ is finite we have $t(x)=p_{X-K}(x)-w(x)$, where $w\in\mathcal{P}\cap\tilde{\mathcal{H}}(G)$ is such that $p=p_G+w$. Hence t is nearly fullsuperharmonic on G and $\hat{t}\in\tilde{\mathcal{S}}(G)$. On the other hand, p_K being fullharmonic on $X-K$, we have $t=p_G-p_K\in\tilde{\mathcal{S}}(X-K)$ and $\hat{t}\in\tilde{\mathcal{S}}(X-K)$. Thus $\hat{t}\in\tilde{\mathcal{S}}(X)$. The relations $p_K\in\mathcal{P}$ and $\hat{t}+p_K=p_G\geqq 0$ imply the relation $\hat{t}\geqq 0$. Hence $\hat{t}\in\tilde{\mathcal{S}}_+(X)=\mathcal{P}$ and $p_K\prec p_G$, which proves the first assertion. The rest is easily proved.

PROPOSITION 3.6. *Let* $p\in\mathcal{P}$, *and* $G, G'\in\mathcal{G}$ *and* K *be compact. We have;*

$$K\subset G \Rightarrow p_K=(p_G)_K$$

$$G\subset G' \Rightarrow p_G=(p_{G'})_G\,.$$

PROOF. Since $p_K\prec p_G\prec p$, we have $(p_K)_K\prec(p_G)_K\prec p_K$. But $(p_K)_K=p_K$ is a consequence of Corollary 3.3, so $(p_G)_K=p_K$. The proof of the second part is as follows. We have $p_G\succ(p_{G'})_G$ from $p\succ p_{G'}$. Choose an element q of

$\mathcal{B}_G(p_{G'})$. Since $q=p_{G'}+s$ on G for some $s\in\tilde{\mathcal{S}}(G)$, and $p_{G'}=p-w$ on G for some $w\in\tilde{\mathcal{H}}(G')$, we have $q=p+(s-w)$ on G, and $q\in\mathcal{B}_G(p)$ follows. Hence $q\succ p_G$ from Proposition 3.2. $q\in\mathcal{B}_G(p_{G'})$ being arbitrary, p_G is a lower bound of $\mathcal{B}_G(p_{G'})$, and the same proposition yields $(p_{G'})_G\succ p_G$. Thus $(p_{G'})_G=p_G$.

It was shown in [15] that

$$Bp=\inf_n p_{X-K_n},$$

where (K_n) is an exhaustion of X by compact sets K_n such that $X-K_n$ is regular. This fact and Corollary 3.3 yield

$$\mathcal{P}_c\subset\mathcal{P}_i.$$

Moreover we have:

PROPOSITION 3.7.

$$\mathcal{P}_i=\left\{p\in\mathcal{P}\,;\ p=\sum_{n=1}^{\infty}p_n\ \textit{with}\ (p_n)\subset\mathcal{P}_c\right\}.$$

PROOF. For every n, there is a $p_n\in\mathcal{P}$ such that $p_{K_n}=p_{K_{n-1}}+p_n$. Both p_{K_n} and $p_{K_{n-1}}$ being fullharmonic on $X-K_n$, p_n is fullharmonic on $X-K_n$, so $p_n\in\mathcal{P}_c$. $p=\sum_{n=1}^{N}p_n+p_{X-K_n}$, where $p_i=p_{K_i}$. If $p\in\mathcal{P}_i$ we have $p_{X-K_n}\downarrow Bp=0$ and $p=\sum_{n=1}^{\infty}p_n$. Conversely, suppose that $p\in\mathcal{P}$ is written in the form $p=\sum_{n=1}^{\infty}p_n$ with $(p_n)\subset\mathcal{P}_c$. Obviously $B\left(\sum_{k=1}^{n}p_k\right)=0$ for any n. Applying Lemma 1.3 to the series

$$\left\{q_n=\sum_{k=n}^{\infty}p_k\,;\ n=1,2,\cdots\right\}\subset\mathcal{P},$$

we have $Bq_n=0$ for all n. Therefore

$$Bp=B\left(\sum_{k=1}^{n-1}p_k\right)+Bq_n=0,$$

and $p\in\mathcal{P}_i$.

Carriers of P_i-functions.

The *carrier* of $p\in\mathcal{P}_i$ is defined as the smallest closed set F such that $X-F\in\mathcal{G}$ and $p\in\tilde{\mathcal{H}}(X-F)$. We denote it by Carr$(p)$ or C(p).

LEMMA 3.8. *If a P_i-function u is harmonic on $G\in\mathcal{G}$, then it is fullharmonic on G.*

PROOF. It is enough to prove our lemma for $G\in\mathcal{D}$. Let K be a compact set such that $X-K$ is regular and is contained in G. Let U be a regular relatively compact set in G. We have $u_{X-K}\geqq H^U u_{X-K}=u-H^U u_K\geqq u-u_K=u_{X-K}$ on U, and $u_{X-K}\in\mathcal{H}(G)$. Since $u_{X-K}\in\mathcal{H}(\mathring{K})$ we have $u_{X-K}\in\mathcal{H}(X)$ and $u_{X-K}\in\mathcal{P}_b$. Hence $u_{X-K}\prec u\in\mathcal{P}_i$ implies $u_{X-K}=0$ and $u=u_K\in\tilde{\mathcal{H}}(X-K)$. $u\in\tilde{\mathcal{H}}(G)$ follows from Axiom (3).

This lemma says that Carr(p), $p\in\mathcal{P}_i$, is the smallest closed set with compact boundary such that p is harmonic out of the set.

PROPOSITION 3.9. *The map* $p \rightarrow \mathrm{Carr}(p)$ *possesses the following properties;*

(i) $p=0 \Leftrightarrow \mathrm{Carr}(p)=\phi$,

(ii) $p \prec q \Rightarrow \mathrm{Carr}(p) \subset \mathrm{Carr}(q)$,

(iii) *for any* $p \in \mathcal{P}_i$ *and for any compact subset* K, *there exist two* P_i-*functions* p_1 *and* p_2 *such that* $p=p_1+p_2$, $\mathrm{Carr}(p_1) \subset K$, $\mathrm{Carr}(p_2) \subset X-\mathring{K}$.

(iv) $C(p_1+p_2)=C(p_1 \curlyvee p_2)=C(p_1) \cup C(p_2)$,

(v) $C(p_1 \curlywedge p_2) \subset C(p_1) \cap C(p_2)$,

where $p_1 \curlyvee p_2$ (*resp.* $p_1 \curlywedge p_2$) *is the maximum* (*resp. minimum*) *of* p_1 *and* p_2 *with respect to the specific order.*

These properties are given in [2] as the theory of abstract carriers and can be proved in the same way. Namely the pair $(\mathcal{P}_i, p \rightarrow C(p))$ is an abstract carrier.

N. Boboc, C. Constantinescu, A. Cornea [2] and W. Hansen [9] developed the theory of abstract carriers and gave a method of constructing potential kernels in connection with H. Bauer's (or some other) axiomatic theory of harmonic functions. Though their method is applicable for our theory [12], we shall here follow R. M. Herve's method for the construction of potential kernels.

LEMMA 3.10. $Bp=B(p_G) \prec p_G$, *for any* $G \in \mathcal{D}$, *in particular*, $p_A \prec p-Bp$, *for any relatively compact open set* A.

PROOF. We have, from Proposition 3.6, $(p_G)_{X-K}=p_{X-K}$ for sufficiently large compact set K. Therefore $Bp=\inf p_{X-K}=\inf (p_G)_{X-K}=B(p_G) \prec p_G$. The second assertion follows from the first and the relation $p_A \prec p_{\bar{A}}$.

PROPOSITION 3.11. *Let* $G \in \mathcal{G}$ *and* G_n *be an increasing sequence of domains in* G *such that* $G=\cup G_n$.

(i) *If either* $G_n \in \mathcal{D}$ *for some* n, *or* G *is relatively compact, then*

$$p_G=\sup p_{G_n}.$$

(ii) *If all* G_n*'s are relatively compact but* $G \in \mathcal{D}$, *then*

$$p_G-Bp=\sup p_{G_n}.$$

Moreover sup *in the above can be taken relative to the specific order.*

PROOF. Let $q=\sup p_{G_n}$. Then $q \in \tilde{S}_+(X)=\mathcal{P}$. Let $p_{G_n}=p_{G_m}+u_{m,n}$ with $u_{m,n} \in \mathcal{P}$ $(n \geqq m)$ (Proposition 3.5). We have $q=p_{G_n}+u_m$, $u_m=\sup u_{m,n} \in \mathcal{P}$. Hence q is a specific majorant of any p_{G_m}. It can be proved similarly that any specific majorant of $\{p_{G_m}; m \geqq 1\}$ is a specific majorant of q. q is the supremum of the p_{G_n}'s relative to the order $\prec$. Let $p=p_{G_n}+w_n$ with a $w_n \in \mathcal{P} \cap \tilde{\mathcal{H}}(G_n)$. We have $p=q+w$, $w=\sup w_n \in P$. w is also harmonic on G.

(i) If $G_n \in \mathcal{D}$ for some n then w is fullharmonic on G, and therefore $q \in \mathcal{B}_G(p)$. Hence $q \geqq p_G$. On the other hand $p_{G_n} \prec p_G$ for every n implies

$q \prec p_G$. Thus $q = p_G$. The same argument holds when G is a relatively compact set.

(ii) We have $p_G - Bp = p_G - B(p_G) \succ (p_G)_{G_n} = p_{G_n}$ from Lemma 3.10 and Proposition 3.6. Hence $p_G - Bp \succ q$. We shall prove the converse. p_G being a specific minorant of p, we have from just the above $p - Bp \succ q$. Let $u \in \mathcal{P}$ be such that $u + q = p - Bp$. $u \prec p - Bp$ implies $u \in \mathcal{P}_i$. Since $u + q = q + w - Bp$, $w \in \mathcal{H}(G)$, we have $u = w - Bp \in \mathcal{H}(G)$. From Lemma 3.8 $u \in \tilde{\mathcal{H}}(G)$. Hence $q + Bp \in \mathcal{B}_G(p)$ and $q + Bp \geqq p_G$. Thus $q + Bp = p_G$.

COROLLARY 3.12. *Let K be the intersection of a decreasing sequence of compact sets K_n. Then $p_K = \inf p_{K_n}$.*

PROPOSITION 3.13.

(i) $p_G = \sup \{p_K\,;\ K \text{ compact } \subset G\} + B(p_G)$,

(ii) $p_K = \inf \{p_G\,;\ G \text{ open } \supset K\}$.

The supremum and the infimum in the above can be taken with regard to the specific order.

PROOF. It can be verified that, for any compact set K and $G \in \mathcal{G}$ with $K \subset G$, $p_{\mathring{K}} \prec p_K = (p_G)_K \prec p_G - B(p_G)$. This and Proposition 3.11 prove our assertion.

PROPOSITION 3.14. *Let $G, G' \in \mathcal{G}$ and K, K' be compact. Then $(p_K)_{K'} = p_{K \cap K'}$, and $(p_G)_{G'} = p_{G \cap G'}$.*

PROOF. We have, from Corollary 3.12, $p_{K \cap K'} = p_K \curlywedge p_{K'}$. $p_{K \cap K'}$ being a specific minorant of p_K which is fullharmonic out of K', we have $p_{K \cap K'} \prec (p_K)_{K'}$. Hence $p_{K \cap K'} \prec (p_K)_{K'} \prec p_K \curlywedge p_{K'} = p_{K \cap K'}$, that is, $(p_K)_{K'} = p_{K \cap K'}$. We shall prove the second equality. Let (K_n) and (K_n') be increasing sequences of compact sets such that $G = \cup K_n$, $G' = \cup K_n'$ respectively. In the following argument p_G, $p_{G'}$, p_{K_n}, $p_{K_n'}$, $p_{G \cap G'}$, $p_{K_n \cap K_n'}$ are written by q, r, q_n, r_n, s, s_n respectively. It can be verified easily that $s \prec q_{G'} \prec q \curlywedge r$. On the other hand we have the equality:

$$s - Bs = \curlyvee s_n = \curlyvee (q_n \curlywedge r_n) = (\curlyvee q_n) \curlywedge (\curlyvee r_n) = (q - Bq) \curlywedge (r - Br),$$

from Proposition 3.13. To prove our assertion we consider the following four cases: (1) $G, G' \in \mathcal{D}$, (2) G, G' are relatively compact sets, (3) $G \in \mathcal{D}$ and G' is relatively compact, (4) $G' \in \mathcal{D}$ and G is relatively compact.

Case (1): We have $Bs = Bq = Br = Bp$ and $s - Bp = (q - Bp) \curlywedge (r - Bp) = q \curlywedge r - Bp$, so $s = q \curlywedge r$. We have $s = q_{G'} = q \curlywedge r$.

Case (2): $Bs = Bq = Br = 0$ and $s = q \curlywedge r = q_{G'}$.

Case (3): We have $Bs = Br = 0$ and $s = (q - Bq) \curlywedge r$. From Lemma 3.10 it follows that $q_{G'} \prec q - Bq$, so $q_{G'} \prec (q - Bq) \curlywedge r = s$. Hence $q_{G'} = s$.

Case (4): In this case $Bs = Bq = 0$, $Br = Bp$, and we have $s = q \curlywedge (r - Bp)$. Lemma 3.10 yields $r - q_{G'} = (p - q)_{G'} \succ B(p - q) = Br$, that is, $q_{G'} \prec r - Br$. Therefore $q_{G'} \prec s$ and $s = q_{G'}$.

Now we shall prove the additivity of the maps $K \to p_K$ and $G \to p_G$.

PROPOSITION 3.15. *Let $G_i \in \mathcal{G}$ and K_i be compact sets $(i=1, 2)$. We have:*

$$p_{G_1 \cup G_2} + p_{G_1 \cap G_2} = p_{G_1} + p_{G_2},$$

$$p_{K_1 \cup K_2} + p_{K_1 \cap K_2} = p_{K_1} + p_{K_2}.$$

PROOF. First we note the following fact, which is easily verified: For $G, G' \in \mathcal{G}$, if $q \in \mathcal{P} \cap \mathcal{H}(G)$ and if $q = q_{G'} + w$ with $w \in \mathcal{H}(G') \cap \mathcal{P}$ then $w \in \mathcal{P} \cap \mathcal{H}(G \cup G')$. Let $p = p_{G_1} + u$ with $u \in \mathcal{P} \cap \mathcal{H}(G_1)$ and $u = u_{G_2} + v$ with $v \in \mathcal{P} \cap \mathcal{H}(G_2)$ (Theorem 3.1). From the above it follows $v \in \mathcal{H}(G_1 \cup G_2)$. Since $u \prec p$, we have $u_{G_2} \prec p_{G_2}$ and $p = p_{G_1} + u_{G_2} + v \prec p_{G_1} + p_{G_2} + v$. Hence $p_{G_1} + p_{G_2} \in \mathcal{B}_{G_1 \cup G_2}(p)$. From Proposition 3.2 we have $p_{G_1} + p_{G_2} \succ p_{G_1 \cup G_2}$. Let $s \in \mathcal{P}$ be such that $s + p_{G_1 \cup G_2} = p_{G_1} + p_{G_2}$. In view of Proposition 3.4 and Proposition 3.14, $s_{G_2} + p_{G_2} = p_{G_1 \cap G_2} + p_{G_2}$, so we have $s \succ s_{G_2} = p_{G_1 \cap G_2}$. Therefore $p_{G_1 \cap G_2} + p_{G_1 \cup G_2} \prec p_{G_1} + p_{G_2}$. Conversely, if $t \in \mathcal{P}$ is such that $p_{G_1 \cup G_2} + p_{G_1 \cap G_2} = p_{G_1} + t$, we have, like the above, $p_{G_2} = t_{G_2} \prec t$ and $p_{G_2} + p_{G_1} \prec p_{G_1 \cup G_2} + p_{G_1 \cap G_2}$. Thus we have $p_{G_1} + p_{G_2} = p_{G_1 \cup G_2} + p_{G_1 \cap G_2}$.

We have seen that, for any $x \in X$, the function $K \to p_K(x)$ defined on the compact sets is positive, increasing, right continuous and additive. Hence it may be extended over all subsets of X as a right continuous Choquet's capacity. If $p(x)$ is finite the restriction of this capacity to the Borel subsets of X is a bounded Radon measure on X. We shall denote this measure by $V^p(x, dy)$.

Let p be a finite P-function. Since $V^p(\cdot, K) = p_K$ is lower semi-continuous for any compact set K, $V^p(\cdot, A)$ is Borel measurable for any Borel set A, that is, $V^p(x, dy)$ is a kernel.

Let A be a Borel set. From the relation

$$V^p(\cdot, A) = \sup\{V^p(\cdot, K);\ K \text{ compact } \subset A\}$$
$$= \sup\{p_K;\ K \text{ compact } \subset A\},$$

it follows that $V^p(\cdot, A)$ is a P-function dominated by p and is fullharmonic out of $\bar{A}$ if A is relatively compact (Property (**a**) in section 1). Similarly, for any non-negative measurable function g, the function

$$V^p g = \int_X g(y) V^p(\cdot, dy)$$

is a P-function if $V^p g \not\equiv \infty$. It is fullharmonic out of the support of g if g has a compact support. From Proposition 3.11 (ii) we have $V^p 1 = p - Bp$. We shall prove that, for any non-negative measurable function g, the function $V^p g$ is a P_i-function if $V^p g \not\equiv \infty$. In case of $g \in B_b^+(X)$, $V^p g$ is a specific minorant of $(\sup_X g) \cdot V^p 1 = (\sup_X g)(p - Bp)$, and so it is a P_i-function. In general $V^p(\min(g, N))$ is a P_i-function. Let $q_N = V^p(g - \min(g, N))$. q_N's are P-functions and, by virtue of Lemma 1.3 applied to the sequence q_N, we have $Bq_N = 0$ for every N. Hence $V^p g = V^p(\min(g, N)) + q_N \in \mathcal{P}_i$.

THEOREM 3.16. (i) *Let p be a finite P-function. There is a kernel V^p which satisfies the following properties:*

(a) *For every non-negative measurable function g, the function $V^p g$ is a P_i-function whenever $V^p g \not\equiv +\infty$. $V^p g$ is fullharmonic out of the support of g if g is of compact support.*

(b) $V^p 1 = p - Bp$.

(ii) *If a kernel V on X has the property;*

(a)′ *$V1 < \infty$ and, for every $g \in C_c^+(X)$, the function Vg is a P-function which is fullharmonic out of* Supp$[g]$, *then there is a finite P_i-function p such that $V = V^p$.*

PROOF. It remains for us to prove (ii). From the above condition (a)′ and the property (**a**) in section 1, Vg is a P-function whenever g is a lower semi-continuous function such that $Vg \not\equiv +\infty$. In particular $V1 = p \in \mathcal{P}$. We shall prove that the relation $V1_K = p_K$ holds for any compact set K. Let K_1 be a compact set such that $K \subset \mathring{K}_1$ and $X - K_1$ is regular. Let $\mathcal{A}$ be the family of continuous functions f such that $0 \leqq f \leqq 1$ and $f = 1$ on K and $f = 0$ out of K_1. Further let $\mathcal{V} = \{Vf;\ f \in \mathcal{A}\}$. Then $V1_K = \inf \mathcal{V}$ is nearly fullsuperharmonic[4] and is equal to the difference $p - V1_{X-K}$, hence $V1_K$ is a P-function and $V1_K \prec p$. Moreover the family $\mathcal{V}$ restricted on $X - K_1$ satisfies the conditions of (**e**) in section 1, and therefore $V1_K = \inf \mathcal{V} \in \mathcal{H}(X - K_1)$. K_1 being arbitrary compact set such that $K \subset \mathring{K}_1$ and $X - K_1$ is regular, we have $V1_K \in \mathcal{H}(X - K)$. Thus $V1_K$ is a specific minorant of p which is fullharmonic out of K, hence Proposition 3.2 yields $V1_K \prec p_K$. In view of this inequality and Proposition 3.13 we have

$$p = V1 = \sup\{V1_H;\ H \text{ compact}\} \leqq \sup\{p_H;\ H \text{ compact}\} = p - Bp,$$

hence $Bp = 0$ and $p \in \mathcal{P}_i$. Now we have

$$\begin{aligned} V1_{X-K} &= \sup\{V1_H;\ H \text{ compact} \subset X - K\} \\ &\leqq \sup\{p_H;\ H \text{ compact} \subset X - K\} \\ &= p_{X-K} - Bp \\ &= p_{X-K}, \end{aligned}$$

4) The greatest lower bound of a family of nearly fullsuperharmonic functions, that are locally uniformly bounded from below, is nearly fullsuperharmonic. If the difference of two finite fullsuperharmonic functions is nearly fullsuperharmonic, then it is fullsuperharmonic.

that is, $V1_K \geqq p - p_{X-K} = p_K$. Therefore $V1_K = p_K$ for any compact set K, and the kernel V and V^p are identical.

DEFINITION. A kernel with the properties (a) and (b) of Theorem 3.16 is called *the potential kernel associated with* p.

Let p be a finite P-function. The potential kernel associated with p is unique. The proof is as follows. Let V be a kernel with the properties (a)$'$ and (b), and let $q = p - Bp$, then $V = V^q$ from Theorem 3.16 (ii) and its proof. Since $(Bp)_K = 0$ (Corollary 3.10), we have $q_K = p_K - (Bp)_K = p_K$ and the kernels V^p and V^q coincide. Thus we have $V = V^p$. We have also shown $V^p = V^{p-Bp}$.

PROPOSITION 3.17. *If p is a continuous P-function, the potential kernel V^p maps $B_b(X)$ into $C(X)$.*

The proof follows from the fact that if q is a specific minorant of p then q is also continuous.

Let $p \in \mathcal{P}_i$. If the carrier of p is the whole space X, then, for each $x \in X$, the support of the measure $V^p(x, dy)$ is the whole space. In fact, suppose that $V^p(x, U) = 0$ for some relatively compact open set U, then $p_U(x) = 0$ and so $p_U = 0$ on X. Therefore $p = p - p_U$ is fullharmonic on U (Theorem 3.1), which contradicts to the fact that the carrier of p is the whole space. Now suppose that the carrier of p is a compact set K. The above argument applied to $p = p_K$ yields that, for each x, the support of the measure $V^p(x, dy)$ contains the compact set K. (Note that $(p_K)_U = p_U$ for every open set contained in K.) On the other hand, let U be a relatively compact open set such that $K \cap \bar{U} = \phi$, then

$$V^p(x, \bar{U}) = p_{\bar{U}}(x) = (p_K)_{\bar{U}}(x) = p_{K \cap \bar{U}}(x) = 0 .$$

Hence the support of $V^p(x, dy)$ is precisely the carrier of p. Applying the above to the function $p - Bp$ whenever p is any P-function, we have the following proposition.

PROPOSITION 3.18. *For every $p \in \mathcal{P}$ and $x \in X$, the support of the measure $V^p(x, dy)$ is exactly the carrier of $p - Bp$.*

PROPOSITION 3.19. *Let $p_n \in P$, $n \geqq 1$, and $p = \sum_{n=1}^{\infty} p_n \in \mathcal{P}$, $p < \infty$, then we have:*

$$Bp = \sum_{n=1}^{\infty} Bp_n ,$$

$$V^p = \sum_{n=1}^{\infty} V^{p_n} .$$

PROOF. We have already known that, for any $p, q \in \mathcal{P}$, $B(p+q) = Bp + Bq$, and $V^{p+q} = V^p + V^q$ (Proposition 3.4). These are also true for finite summand. To prove the first part, put $q = \sum Bp_n$. As an increasing limit of P_b-functions, q is also a P_b-function. On the other hand, applying Lemma 1.3 to the sequence $\left\{q_n=\sum_{k=n}^{n}(p_k-Bp_k)\in\mathscr{P}\,;\ n\geqq 1\right\}$, we have $Bq_n=0$, hence $B(p-q)=0$, and $Bp=Bq=q$. Now let V be the kernel $V=\sum V^{p_n}$. Obviously V satisfies condition (a)′ of Theorem 3.16. From the above it follows that

$$V1=\sum V^{p_n}1=\sum(p_n-Bp_n)=p-\sum Bp_n=p-Bp\,,$$

that is, V satisfies condition (b) of Theorem 3.16. Hence $V=V^p$ from the uniqueness of the potential kernel.

REMARK. Let p be a P-function (not necessarily finite). $p_K(x)$ is finite for $x\notin K$, so we can associate a Radon measure $W^p(x, dy)$ on $X-\{x\}$ such that $W^p1_K(x)=p_K(x)$ for any compact set such that $K\not\ni x$. The support of $W^p(x, dy)$ is $\mathrm{Carr}(p)-\{x\}$.

Let $x_0\in X$ and δ_0 be a regular (relatively compact) neighborhood of x_0. The function $K\rightarrow H^{\delta_0}p_K(x_0)$ defined on the compact sets defines a bounded Radon measure on X whose total mass is equal to $H^{\delta_0}p(x_0)-Bp(x_0)$.

§ 4. Resolvents associated with potential kernels. Excessive functions.

This section is devoted to the investigation of the submarkov resolvent V_λ^p associated with the potential kernel V^p, and also to the investigation of the relation between excessive functions with respect to this resolvent and P-functions. We shall show that, for an appropriately chosen $p\in\mathscr{P}$, the range of V_λ^p and $\boldsymbol{E}$ have the same uniform closure ($\boldsymbol{E}$ is the space introduced in section 2).

DEFINITION. Let W be a kernel. A function $d\in B_+(X)$ is called a *W-dominant function* if, for every pair (f, g) of positive measurable functions, the relation

$$d+Wf\geqq Wg \quad \text{on} \quad \{x\in X\,;\ g(x)>0\}$$

implies the relation

$$d+Wf\geqq Wg \quad \text{on} \quad X.$$

It is well-known that if the function $(I+\lambda W)u$ is a W-dominant function for a $\lambda>0$ then u is non-negative.

PROPOSITION 4.1. *Let $p\in\mathscr{P}\cap C(X)$. Every P-function is a V^p-dominant function.*

PROOF. We may suppose $p\neq 0$. Since V^p is a strictly positive continuous kernel it is enough to prove that; for every pair (f, g) of non-negative continuous functions of compact support, the relation

$$s\in\mathscr{P} \quad \text{and} \quad s+V^pf\geqq V^pg \quad \text{on} \quad \{g>0\}$$

implies the relation

$$s+V^p f \geqq V^p g \quad \text{everywhere}.$$

There is a sequence $(q_n) \subset C(X) \cap \mathcal{P}_c$ such that the carrier of q_n is contained in $\{g>0\}$ and $q_n \uparrow V^p g$ (For instance $q_n = V^p(g \cdot 1_{\{g \geqq 1/n\}})$). For these q_n it holds

$$s+V^p f-q_n \geqq 0 \quad \text{on} \quad \mathrm{Carr}(q_n),$$

and

$$s+V^p f-q_n \in \tilde{\mathcal{S}}(X-\mathrm{Carr}(q_n)).$$

The minimum principle yields $s+V^p f-q_n \geqq 0$, and consequently, $s+V^p f \geqq V^p g$.

In particular V^p satisfies the complete maximum principle from the assumption (5). Hence, if $p \in \mathcal{P} \cap C_b(X)$ there is a submarkov resolvent $(V_\lambda^p)_{\lambda>0}$ such that

$$V^p - V_\lambda^p = \lambda V^p V_\lambda^p = \lambda V_\lambda^p V^p$$

(Th. 10, Chap. X, [17]). In this case every P-function s on X is (V_λ^p)-*supermedian*, that is, $\lambda V_\lambda^p s \leqq s$ for all $\lambda>0$. In fact, take a $q \in \mathcal{P}_c \cap C(X)$ such that $q \leqq s$. The function $V^p q$ is bounded, so the function $h=\lambda(q-\lambda V_\lambda^p q)$ has the finite potential that equals $\lambda V_\lambda^p q$. We have $s \geqq \lambda V_\lambda^p q = V^p h$ on the set $\{s-\lambda V_\lambda^p q>0\}$, in particular, on the set $\{h>0\}$. So $s \geqq \lambda V_\lambda^p q$ everywhere on X, and s being equal to sup $\{q \in \mathcal{P}_c \cap C(X);\ q \leqq s\}$ (Proposition 1.2), it follows $s \geqq \lambda V_\lambda^p s$.

Since $V^p f \in \mathcal{P}_i$ whenever f is non-negative measurable and $V^p f \not\equiv +\infty$, every (V_λ^p)-excessive function is a P-function as an increasing limit of finite potentials. But the converse does not hold in general. (Take a $p \in \mathcal{P}_b$, then $V^p \equiv 0$.) We shall prove that for a suitably chosen $p \in \mathcal{P}$ every P-function is (V_λ^p)-excessive.

LEMMA 4.2. *Let $p \in \mathcal{P} \cap C_b(X)$ and let q be a specific minorant of p. Then*

(i) *$V_\lambda^p s \geqq V_\lambda^q s$ for any $s \in \mathcal{P}$ and $\lambda>0$,*

(ii) *$V_\lambda^p g \leqq V_\lambda^q g + V_\lambda^{p-q} g$ for any $g \in B_b^+(X)$.*

PROOF. From $p \in C_b(X)$ and $q \prec p$ it follows that q and $p-q$ are bounded continuous P-functions, so V_λ^q and V_λ^{p-q} are defined. (i) It is enough to prove the relation for $s \in \mathcal{P}_c \cap C(X)$ (Proposition 1.2). In this case $V^p s$ and $V^q s$ are bounded. Let $t=V_\lambda^p s - V_\lambda^q s$. The resolvent equations for (V_λ^p) and (V_λ^q) imply $t=-\lambda V^p t+(V^p-V^q)h=-\lambda V^p t+V^{p-q}h$ where $h=s-\lambda V_\lambda^q s \geqq 0$. $V^{p-q}h-\lambda V^p t \geqq 0$ on $\{t>0\}$. $V^{p-q}h$ is a (V^p)-dominant function from Proposition 4.1. Therefore $V^{p-q}h-\lambda V^p t \geqq 0$ everywhere. Thus $t \geqq 0$. (ii) Let $u=V_\lambda^q g+V_\lambda^{p-q} g-V_\lambda^p g$. In view of the resolvent equations for these potential kernels and the relation $V^p=V^q+V^{p-q}$, we have the equality;

$$(I+\lambda V^p)u=\lambda V^q V_\lambda^{p-q} g+\lambda V^{p-q} V_\lambda^q g.$$

Since the right hand side belongs to $\mathcal{P}$, $(I+\lambda V^p)u$ is a V^p-dominant function. Therefore $u \geqq 0$.

Let K_m be a compact exhaustion of X such that $X-K_m$ is regular for every m, and let $\boldsymbol{E}^m$ be the space of continuous functions fullharmonic out of K_m. Let $\boldsymbol{E}=\bigcup_m \boldsymbol{E}^m$. From Proposition 1.8 and Theorem 1.9 there is a countable family $Q=(p_k)_{k\geqq 1}$ of functions in $\mathcal{P}_c\cap C(X)$ such that $Q\cap\boldsymbol{E}^n-Q\cap\boldsymbol{E}^n$ is dense in $\boldsymbol{E}^n$, and $Q-Q$ is dense in $\boldsymbol{E}$ (relative to the uniform convergence topology on X). We set

$$p_0=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{p_k}{\sup p_k}, \qquad (p_k)=Q,$$

and

$$q_n=\sum_{r\in Q\cap\boldsymbol{E}^n}\frac{1}{2^{k(r)}}\frac{r}{\sup r},$$

where $k(r)=k$ if $r=p_k\in Q$. $p_0\in\mathcal{P}_i\cap C_b(X)$ and $q_n\in\mathcal{P}\cap\boldsymbol{E}^n$ and $q_n\prec p_0$. We shall denote the potential kernels $V^{p_0}(x, dy)$ and $V^{q_n}(x, dy)$ simply by $V(x, dy)$ and $V^n(x, dy)$ and their associated resolvents by (V_λ) and (V_λ^n)[5].

LEMMA 4.3. *For any $f\in\boldsymbol{E}^n$, the function $\lambda V_\lambda^n f$ converges to f at every point of X as λ tends to infinity.*

PROOF. Let $x\in X$ and let ν_λ be the measure $\lambda V_\lambda^n(x, dy)$, $\lambda>0$. ν_λ is concentrated on the compact set K_n and its total variation is less than 1 because $\lambda V_\lambda^n 1\leqq 1$. For each $\lambda_0>0$, let $M(\lambda_0)$ be the closure (in the weak topology of the space of measures on K_n) of $\{\nu_\lambda;\ \lambda\geqq\lambda_0\}$. $M(\lambda_0)$, $\lambda_0>0$, forms a nested family of non-empty compact sets and hence have non-empty intersection M. Let $\mu\in M$. From Proposition 5.1 it follows $\mu(r)\leqq r(x)$ for every $r\in Q\cap\boldsymbol{E}^n$ and $\mu(q_n)\leqq q_n(x)$. But, since $\mu(q_n)=\lim_{\lambda\to\infty}\lambda V_\lambda^n q_n(x)=\lim_{\lambda\to\infty}\lambda V_\lambda^n V^n 1(x)=q_n(x)$, $\mu(r)=r(x)$ for all $r\in Q\cap\boldsymbol{E}^n$. $Q\cap\boldsymbol{E}^n$ being total in $\boldsymbol{E}^n$ we have $\mu(f)=f(x)$ for any $f\in\boldsymbol{E}^n$. Thus $\lim_{\lambda\to\infty}\lambda V_\lambda^n f(x)=f(x)$ for any $f\in\boldsymbol{E}^n$ and $x\in X$.

THEOREM 4.4. *The excessive functions with respect to the resolvent (V_λ) are exactly the cone $\mathcal{P}$.*

PROOF. It remains for us to prove that any P-function s is (V_λ)-excessive. By virtue of Proposition 1.2 we may suppose $s\in\mathcal{P}_c\cap C(X)$. Let $x\in X$ and let n be a sufficiently large number such that $s\in\boldsymbol{E}^n$. We have $\lim_{\lambda\to\infty}\lambda V_\lambda^n s(x)=s(x)$ from Lemma 4.3. From Lemma 4.2 we have

$$0\leqq s(x)-\lambda V_\lambda s(x)\leqq s(x)-\lambda V_\lambda^n s(x)\to 0 \qquad (\lambda\to\infty).$$

Thus s is (V_λ)-excessive.

THEOREM 4.5. *The space $\boldsymbol{E}$ and the range of the resolvent (V_λ) operating on $B_b(X)$ have the same uniform closure.*

PROOF. Since $q_n\uparrow p_0$ uniformly, $V^n f\to Vf$ uniformly on X for any $f\in B_b(X)$.

5) From Proposition 3.8 the support of the measure $V_\lambda(x, dy)$ is the whole space X, and the support of $V_\lambda^n(x, dy)$ is contained in K_n, for every $x\in X$.

From $V^n f \in \boldsymbol{E}^n \subset \boldsymbol{E}$ it follows $\overline{V(B_b(X))} \subset \bar{\boldsymbol{E}}$. Conversely let $g \in \boldsymbol{E}$. $g \in \boldsymbol{E}^n$ for some n, and we can find, for any $\varepsilon > 0$, two functions $u, v \in Q \cap \boldsymbol{E}^n$ such that $|g-(u-v)| < \varepsilon$ uniformly on X. Hence $\lambda V_\lambda(|g-(u-v)|) < \varepsilon$. On the other hand Theorem 4.4 and Dini's convergence theorem imply that $0 \leqq u - \lambda V_\lambda u < \varepsilon$ uniformly on K_n for sufficiently large λ. Since $\varepsilon + \lambda V_\lambda u - u = \varepsilon + \lambda V(u - \lambda V_\lambda u) - u \in \tilde{\mathcal{S}}(X-K_n)$, we have, from the minimum principle, $\varepsilon + \lambda V_\lambda u - u \geqq 0$ on $X-K_n$, hence $0 \leqq u - \lambda V_\lambda u \leqq \varepsilon$ uniformly on X. Similarly $0 \leqq v - \lambda V_\lambda v \leqq \varepsilon$ uniformly on X for sufficiently large λ. Thus we have $|\lambda V_\lambda g - g| < 4\varepsilon$ uniformly on X for large λ. $g \in \overline{V(B_b(X))}$. Hence $\bar{\boldsymbol{E}} \subset \overline{V(B_b(X))}$.

COROLLARY 4.6. *For every $f \in C_c(X)$, the function $\lambda V_\lambda f$ converges to f uniformly on X as λ tends to infinity.*

§ 5. Integral representation of P_i-functions.

P_i-functions of one-point carrier are introduced and the integral representation of P_i-functions with the aid of these one-point carrier P_i-functions is discussed. The proofs are in parallel with those of Chap. III of [10]. The following lemma is a consequence of axiom (2), [14].

LEMMA 5.1. *Let G be a domain of X.*

(1) *The set $\{h \in \mathcal{H}_+(G) ; h(x_0) = 1\}$, $x_0 \in G$, is equicontinuous at x_0.*

(2) *For any compact subset K of G there is a constant $M \geqq 1$ such that for every $h \in \mathcal{H}_+(G)$ and every pair of points x_1 and x_2 in K the relation*

$$\frac{1}{M} h(x_1) \leqq h(x_2) \leqq M h(x_1)$$

holds.

(3) *The set of non-negative harmonic functions on G that are uniformly bounded is relatively compact with respect to the compact convergence topology on G.*

PROPOSITION 5.2. *Let $x_0 \in X$. For any $y \in X - \{x_0\}$ there is a P_i-function q_y such that* $\mathrm{Carr}(q_y) = \{y\}$ *and* $q_y(x_0) = 1$.

PROOF. Let $p \in \mathcal{P}$, > 0. For any $y \neq x_0$, let U_n be a sequence of relatively compact open neighborhood of y that decreases to y. If we set, for each n,

$$q_n(x) = \frac{\hat{R}^{U_n} p(x)}{\hat{R}^{U_n} p(x_0)},$$

we have $q_n \in \mathcal{P} \cap \tilde{\mathcal{H}}(X - \bar{U}_n)$, $q_n(x_0) = 1$. From Lemma 5.1 we can choose, for any $n \geqq 1$ such that $x_0 \notin U_n$, a subsequence of $\{q_m ; m \geqq n\}$ which converges uniformly on every compact subset of $X - \bar{U}_n$. By the diagonal procedure we can extract a subsequence $q_{n'}$ that converges uniformly on any compact subset of $X - \{y\}$ to a fullharmonic function on $X - \{y\}$. Let

$$q_y = \lim_{n'\to\infty} \widehat{\inf}\, q_{n'}\,.$$

$q_y \in \mathcal{P}$. Hence $q_y \in \mathcal{P} \cap \mathcal{H}(X-\{y\})$. Obviously $q_y(x_0)=1$. From Lemma 5.1 there is a constant $\alpha>0$ such that $q_n(x) \geqq \alpha q_n(x_0)=\alpha$ for sufficiently large n whenever $x \neq y$, so $q_y(x)>0$ whenever $x \neq y$. Thus we get a P_i-function with the carrier $\{y\}$.

PROPOSITION 5.3. *Let μ be a Radon measure, $\geqq 0$, on X and let $g(x, y)$ be a function defined on the set $X\times(X-N)$, where N is a μ-measure null set. Suppose that the function $g(x, y)$ has the following properties:*

(1) *There is an increasing sequence of compact subset K_n such that, for each n, $\mu(X-K_n) \leqq \frac{1}{n}$ and such that the restriction of $g(x, y)$ on $X\times K_n$ is a lower semi-continuous function of (x, y) which is continuous for $x \neq y$.*

(2) *For each $y \in X-N$, $g(\cdot, y)$ is a P_i-function with the carrier y.*

(3) *There is a denumerable dense subset (x_j) of X such that the function $y \to g(x_j, y)$ is μ-integrable. Then the function*

$$u(x)=\int g(x, y)\mu(dy)$$

is a P_i-function and is fullharmonic out of $\mathrm{Supp}[\mu]$ *if $X-\mathrm{Supp}[\mu] \in \mathcal{G}$. Moreover we have, for any compact set K,*

$$u_K(x)=\int_K g(x, y)\mu(dy)\,.$$

PROOF. Let

$$I_n(x)=\int_{K_n} g(x, y)\,\mu(dy)\,.$$

I_n is a lower semi-continuous function (Fatou's theorem). Fubini's theorem yields, for every $G \in \mathcal{G}_r$ and $x \in G$,

$$\begin{aligned}(\tilde{H}^G I_n)(x) &= \int_{K_n} (\tilde{H}^G g(\cdot, y))(x)\mu(dy) \\ &\leqq I_n(x)\,.\end{aligned}$$

From (3) $I_n(x_j) \leqq u(x_j) < \infty$ on the dense points x_j, hence every I_n, and $u=\lim_{n\to\infty} \uparrow I_n$ are non-negative fullsuperharmonic functions. It also follows from (2) that $I_n \in \mathcal{H}(X-K_n)$ and u is harmonic out of $\mathrm{Supp}[\mu]$. u is fullharmonic out of $\mathrm{Supp}[\mu]$ if $X-\mathrm{Supp}[\mu] \in \mathcal{G}$. Let

$$q_n=\int_{X-K_n} g(\cdot, y)\mu(dy)\,.$$

The P-functions q_n satisfy the condition of Lemma 1.3, so $Bq_n=0$. Hence $Bu=Bq_n+BI_n=0$ and $u \in \mathcal{P}_i$. Let K be a compact set. The function

$$I_K(x)=\int_K^{\bullet} g(x, y)\mu(dy)$$

is a P_i-function which is fullharmonic out of K, and it is a specific minorant of u. From Proposition 3.2 we have $I_K \leqq u_K$. We have also

$$\begin{aligned} u_{X-K} &= \sup\{u_H\,;\ H \text{ compact} \subset X-K\} \\ &\geqq \sup\{I_H\,;\ H \text{ compact} \subset X-K\} \\ &= \int_{X-K} g(\cdot, y)\mu(dy)\,. \end{aligned}$$

Here we used Proposition 3.13 (i). Therefore $u_K = I_K$.

COROLLARY 5.4. $\mathrm{Carr}(u)=\mathrm{Supp}[\mu]$.

PROOF. $\mathrm{Carr}(u)\subset \mathrm{Supp}[\mu]$ follows from the above. Suppose that there is a relatively compact regular domain U such that $\mu(U)>0$ and $u\in \mathcal{H}(U)$. Since $g(\cdot, y)\geqq H^U g(\cdot, y)$ on U, we have $g(\cdot, y)=H^U g(\cdot, y)$ on U for μ-almost all y. Take a $y\in U$ where $g(x, y)=(H^U g(\cdot, y))(x)$ holds for any $x\in U$. Then $y=\mathrm{Carr}(g(\cdot, y))\subset X-U$. This is absurd. We have $\mathrm{Carr}(u)=\mathrm{Supp}[\mu]$.

COROLLARY 5.5. *We have, for every relatively compact regular domain* δ,

$$(H^\delta u_K)(x)=\int_K (H^\delta g(\cdot, y))(x)\mu(dy)\,.$$

(*See the remark at the end of section* 3.) *If the function* $y\to g(x, y)$ *is* μ-*integrable for every* $x\in X$, *we have*

$$V^u(x, A)=\int_A g(x, y)\mu(dy)\,.$$

The following lemma is a more perspicuous form of a result of R. M. Herve (Lemma 17.2 and Proposition 18.1 of [10]).

LEMMA 5.6. *Let A be a subset of X with $\mathring{A}\neq\phi$, and let $\{k_y\,;\ y\in A\}$ be a family of superharmonic functions, >0, such that $k_y\in\mathcal{H}(X-\{y\})$. If there is a dense subset B of X such that, for every $x\in B$, the function $y\to k_y(x)$ is continuous on $A-\{x\}$, then the function $(x, y)\to k_y(x)$ is lower semi-continuous on $X\times A$, and is continuous for $x\neq y$.*

PROOF. Let $(a, b)\in X\times A$, $a\neq b$, and let U and V be disjoint neighborhoods of a and b respectively. Let $\varepsilon>0$. From Lemma 5.1 there is a neighborhood U' of a, $U'\subset U$, such that

$$1-\varepsilon\leqq\frac{u(x)}{u(x')}\leqq 1+\varepsilon$$

for any $u\in\mathcal{H}_+(U)$ and $x, x'\in U'$. In particular

$$1-\varepsilon\leqq\frac{k_y(x)}{k_y(x')}\leqq 1+\varepsilon\,,$$

and

$$1-\varepsilon \leqq \frac{k_b(x')}{k_b(a)} \leqq 1+\varepsilon$$

for any $x \in U'$, $x' \in U' \cap B$ and $y \in A \cap V$. From the assumption there is a neighborhood V' of b, $V' \subset V$, such that

$$1-\varepsilon \leqq \frac{k_y(x')}{k_b(x')} \leqq 1+\varepsilon$$

for any $y \in V' \cap A$. Therefore we have, for $x \in U'$ and $y \in V' \cap A$, the relation

$$(1-\varepsilon)^3 \leqq \frac{k_y(x)}{k_b(a)} \leqq (1+\varepsilon)^3 .$$

We see that $(x, y) \rightarrow k_y(x)$ is continuous for $x \neq y$. Now we shall prove the lower semi-continuity. It is enough to prove that, for any $a \in X$ such that $k_a(a) > \lambda > 0$, we can find a neighborhood V of a such that $x \in V$ and $y \in V \cap A$ imply $k_y(x) > \lambda$. Let G be a sufficiently small regular neighborhood of a such that $(H^G k_a)(a) > \lambda$ and let $2\varepsilon = (H^G k_a)(a) - \lambda$. Since $(x, y) \rightarrow k_y(x)$ is continuous for $x \neq y$, there is a neighborhood U of a, $\bar{U} \subset G$, such that

$$|k_y(\xi) - k_a(\xi)| < \varepsilon \cdot (H^G 1(a))^{-1}$$

for all $\xi \in \partial G$ and $y \in U \cap A$. We have

$$(H^G k_y)(a) > \lambda + \varepsilon$$

for all $y \in U \cap A$. On the other hand the family

$$\left\{ \frac{H^G k_y}{H^G k_y(a)} \,;\, y \in U \cap A \right\}$$

being equicontinuous at a, we can find a neighborhood U' of a, $\bar{U}' \subset G$, such that

$$H^G k_y(x) > \frac{\lambda}{\lambda+\varepsilon} H^G k_y(a)$$

for all $x \in U'$. Thus we have

$$k_y(x) \geqq H^G k_y(x) > \lambda$$

for all $(x, y) \in V \times V$, where $V = U \cap U'$.

Before we proceed to the representation theorem of P_i-functions we make a remark. Let δ be a (relatively compact) regular domain and $x \in \delta$. For any $p \in \mathcal{P}$, the set function $K \rightarrow (H^\delta p_K)(x)$ defined on the compact subsets of X is extended to a Radon measure, $\geqq 0$, on X. We denote it by $\mu^{\delta, x}$ (Remark after section 3). Two such measures $\mu^{\delta, x}$ and $\mu^{\delta', x'}$ are mutually absolutely continuous. In fact if $\mu^{\delta, x}(A) = 0$ for a Borel set A then $\inf \{H^\delta p_G(x) \,;\, G$ open and $G \supset A\} = 0$. There is a decreasing sequence of open subsets G_n such that

$\lim H^{\delta}p_{G_n}(x)=0$. We have $H^{\delta}(\lim p_{G_n})(x)=0$. Hence $\widehat{\lim}\, p_{G_n}=0$ on X. Thus $\mu^{\delta',x'}(A)\leqq \lim H^{\delta'}p_{G_n}(x')\leqq \widehat{\lim}\, p_{G_n}(x')=0$.

THEOREM 5.7. *Let $p\in\mathcal{P}_i$, >0, and $x_0\in X$ and δ_0 be a relatively compact regular domain containing x_0. The function p admits an integral representation of the form*

$$p(x)=\int g(x, y)\mu(dy)$$

where μ is a finite measure, $\geqq 0$, on X and $g(x, y)$ is a function with the properties (1)~(3) *of Proposition* 5.3. *$g(x, y)$ satisfies*

$$H^{\delta_0}g(\cdot, y)(x_0)=1\,.$$

μ is uniquely determined by the relation;

$$\mu(A)=\mu^{\delta_0,x_0}(A)\,.$$

PROOF. The uniqueness follows from Corollary 5.5 and the remark after section 3. The existence of $g(x, y)$ and μ is proved as follows. Let $A=(x_j)$ be a countable dense subset of X with $\mu^{\delta_0,x_0}(x_j)=0$ and $p(x_j)<\infty$, and let $\mathcal{B}=(\delta_j)$ be a base formed by relatively compact regular domains δ_j with $\mu^{\delta_0,x_0}(\partial\delta_j)=0$ (Such a base $\mathcal{B}$ does exist from Proposition 7.2 of [**10**]). We set $\mu_0=\mu^{\delta_0,x_0}$ and $\mu_{i,j}=\mu^{\delta_i,x_j}$ whenever $x_j\in\delta_i$. The support of the measures μ_0 and $\mu_{i,j}$ are the carrier of p and $\mu_{i,j}$ is absolutely continuous relative to μ_0. Applying a theorem on the differentiation of measures, we can find a μ_0-negligeable set N and, for each $y\in X-N$, a sequence of compact neighborhoods $B_n(y)$ of y such that; $B_n(y)\downarrow\{y\}$, $\mu_0(B_n(y))>0$ and the limit

$$\text{(1)}\qquad \lim_{n\to\infty}\frac{\mu_{i,j}(B_n(y))}{\mu_0(B_n(y))}$$

exists for every i and j with $x_j\in\delta_i$. This limit function defines a density function of $\mu_{i,j}$ relative to μ_0. (We may suppose $N\supset X-\mathrm{Carr}(p)$.) For every $y\in X-N$ we set

$$g_n(x, y)=\frac{1}{\mu_0(B_n(y))}p_{B_n(y)}(x)\,.$$

$g_n(\cdot, y)$ is fullharmonic on $X-B_n(y)$, and the relation

$$\text{(2)}\qquad (H^{\delta_i}g_n(\cdot, y))(x_j)=\frac{\mu_{i,j}(B_n(y))}{\mu_0(B_n(y))}\,,\qquad x_j\in\delta_i\,,$$

holds for each n. If $\bar{\delta}_i\cap B_n(y)=\phi$ it follows

$$g_n(x_j, y)=\frac{\mu_{i,j}(B_n(y))}{\mu_0(B_n(y))}\,,\qquad x_j\in\delta_i$$

and $g_n(\cdot, y)$ converges at x_j. Moreover $g_n(\cdot, y)$ converges uniformly on every compact subset of $X-\{y\}$ to a fullharmonic function, $\geqq 0$ on $X-\{y\}$ (Lemma 5.1). Let $g(x, y)=\lim_{m\to\infty}\widehat{\inf_{m\geqq n}} g_n(x, y)$, $y\in X-N$, where the operation $\wedge$ is taken with regard to the variable x. The following properties are verified:

(a) $g(\cdot, y)\in \mathcal{P}\cap \bar{\mathcal{H}}(X-\{y\})$.

(b) $g(x, y)>0$. In fact let $x\in X$ and δ be a regular neighborhood of x. Lemma 5.1 applied to the domain $\delta\cup\delta_0$ yields the existence of a constant $\alpha>0$ such that $p_{B_n(y)}(x)\geqq \alpha\mu_0(B_n(y))$ for every n with $B_n(y)\cap\bar{\delta}=\phi$. Thus $g_n(x, y)\geqq\alpha$ for large n if $x\neq y$, so $g(x, y)\geqq\alpha$.

(c) There is a sequence of compact subset K_n such that $\mu_0(X-K_n)\leqq \frac{1}{n}$ and such that, for each n, the restriction of $g(x, y)$ on $X\times K_n$ is lower semi-continuous and is continuous for $x\neq y$. The proof is as follows. Since μ_0 is a bounded Radon measure we can find, for each n, a compact set K_n with $\mu_0(X-K_n)\leqq\frac{1}{n}$ such that the restriction of the function $y\to g(x_j, y)$ on Y_n is continuous for every $x_j\in A$, [**19**]. (We may suppose $K_n\subset X-N$.) By virtue of Lemma 5.6 we have the assertion.

(d) $g(x_j, \cdot)$ is μ_0-integrable and $H^{\delta_i}g(\cdot, y)(x_j)$ is a density function of $\mu_{i,j}$ relative to μ_0. In fact, since the measure $H^{\delta_i}(x_j, dz)$ is concentrated on $\partial\delta_i$ and $g_n(\cdot, y)$ converges uniformly on $\partial\delta_i$ if $y\notin\partial\delta_i$, we have

$$(H^{\delta_i}g(\cdot, y))(x_j)=\lim_n (H^{\delta_i}g_n(\cdot, y))(x_j) \qquad \text{for} \quad y\notin\partial\delta_i\,.$$

But $\mu_0(\partial\delta_i)=0$ implies the validity of this equality for all y up to a μ_0-measure null set. From (1) and (2) $\lim_n(H^{\delta_i}g_n(\cdot, y))(x_j)$ is a density function of $\mu_{i,j}$ relative to μ_0, so we have

$$(3) \qquad H^{\delta_i}p(x_j)=\mu_{i,j}(X)=\int H^{\delta_i}g(\cdot, y)(x_j)\mu_0(dy)$$

and

$$\int_{X-\delta_i} g(x_j, y)\mu_0(dy)\leqq H^{\delta_i}p(x_j)\leqq p(x_j)<\infty\,.$$

This is true for any $\delta_i\in\mathcal{B}$ such that $x_j\in\delta_i$. We can verify the μ_0-integrability of $y\to g(x_j, y)$ if we note $\mu_0(x_j)=0$.

We have seen that the function $g(x, y)$ satisfies the conditions of Proposition 5.3. Hence the function

$$u(x)=\int g(x, y)\mu_0(dy)$$

is a P_i-function. We have

$$H^{\delta_i}u(x_j)=\int H^{\delta_i}g(\cdot, y)(x_j)\mu_0(dy)$$

for any $\delta_i\in\mathcal{B}$ and $x_j\in\delta_i$. From (3) we have $H^{\delta_i}p(x_j)=H^{\delta_i}u(x_j)$. $\mathcal{B}=(\delta_i)$ being a regular base for the topology of X, the superharmonic functions u

and p are identical. It remains for us to prove;

(e) $(H^{\delta_0}g(\cdot, y))(x_0)=1$.

For, $(H^{\delta_0}g_n(\cdot, y))(x_0)=1$ implies $(H^{\delta_0}g(\cdot, y))(x_0)\leqq 1$ and the equality follows from

$$\mu_0(X)=H^{\delta_0}p(x_0)=\int(H^{\delta_0}g(\cdot, y))(x_0)\mu_0(dy).$$

In the rest of this section we treat the case when the following assumption is satisfied.

(7) For every $y\in X$, the P_i-functions with the carrier $\{y\}$ are mutually proportional.

From Proposition 5.2, given a point $y\in X$, there is a P_i-function q_y with its carrier $\{y\}$. Let (x_0, δ_0) be the pair stated in Theorem 5.7. We may suppose $H^{\delta_0}q_y(x_0)=1$ for every y. Then, for any $x\in X$, $q_y(x)$ is a continuous function of $y\in X-(\{x_0\}\cup\{x\})$. In fact let y_0 be a limit point in $X-\{x_0\}$ of a sequence y_n. Suppose that, for a $x\neq y_0$, there is a subsequence n' and an $\alpha>0$ such that

$$|q_{y_0}(x)-q_{y_{n'}}(x)|>\alpha.$$

For large n', the functions $q_{y_{n'}}$ is fullharmonic out of some neighborhood of y_0, and is equal to 1 at x_0. Hence, applying Lemma 5.1 and the diagonal procedure, we can find a subsequence n'' such that $q_{y_{n''}}$ converges locally uniformly on $X-\{y_0\}$ to a fullharmonic function on $X-\{y_0\}$. $\widehat{\liminf_{n''\to\infty}}\, q_{y_{n''}}$ is a P-function with its carrier y_0, and is equal to 1 at x_0. Our assumption (7) yields $q_{y_0}=\widehat{\liminf_{n''\to\infty}}\, q_{y_{n''}}$ and $q_{y_0}(x)=\lim_{n''\to\infty} q_{y_{n''}}(x)$. This is a contradiction, and $y\to q_y(x)$ is continuous on $X-(\{x_0\}\cup\{x\})$ for every $x\in X$.

PROPOSITION 5.8. *We can take, for every $y\in X$, a P_i-function p_y with the carrier $\{y\}$ such that; $(x, y)\to p_y(x)$ is lower semi-continuous on $X\times X$, and is continuous for $x\neq y$.*

PROOF. Let (δ_0, x_0) and (δ_1, x_1) be two pairs of relatively compact regular domains δ_0 and δ_1 such that $\delta_0\ni x_0$, $\delta_1\ni x_1$ and $\bar{\delta}_0\cap\bar{\delta}_1=\phi$. Let q_y and r_y be P_i-functions with their carriers $\{y\}$ such that $H^{\delta_0}q_y(x_0)=1$ and $H^{\delta_1}r_y(x_1)=1$. We have $r_y(x)=q_y(x)H^{\delta_0}r_y(x_0)$. Take an open set U such that $\bar{\delta}_0\subset U\subset\bar{U}\subset X-\bar{\delta}_1$ and put

$$p_y(x)=\begin{cases} r_y(x) & \text{if } y\in\bar{U}\\ q_y(x)\cdot c(y) & \text{if } y\in X-\bar{U},\end{cases}$$

where $c(y)$ is a continuous function >0 on $X-\bar{\delta}_0$ which coincides with $H^{\delta_0}r_y(x_0)$ on $\bar{U}-\bar{\delta}_0$. p_y is a P_i-function with the carrier y and the function $y\to p_y(x)$ is continuous on $X-\{x\}$. Applying Lemma 5.6 we have our result.

THEOREM 5.9. *Let q_y and p_y be the P_i-functions in the above.*

(a) *Every P_i-function p has a unique integral representation of the form:*

$$p(x)=\int q_y(x)\mu(dy)$$

by a measure $\mu\geqq 0$ *on* X $(\mu=\mu^{\delta_0,x_0})$.

(b) *p has a unique integral representation of the form:*

$$p(x)=\int p_y(x)\lambda(dy)$$

by a measure $\lambda\geqq 0$ *on* X, λ *is given by*

$$\lambda(dy)=\frac{\mu(dy)}{(H^{\delta_0}p_y)(x_0)}\,.$$

PROOF. Theorem 5.7 and Corollary 5.4 yield our theorem. (Note $g(x, y)=q_y(x)$ where they are defined.)

§ 6. Locally convex topology on the cone $\mathcal{P}$. Integral representation of P-functions by measures on the extreme elements of a base of $\mathcal{P}$.

In order to get a representation theorem of Martin type for the cone $\mathcal{P}$, we shall proceed in the following way: 1) $\mathcal{P}$ is embedded into a locally convex separated topological vector space. It is proved that $\mathcal{P}$ has a compact base. 2) Applying Choquet's representation theorem [20], we prove that; for a given compact base $\mathcal{K}$ of $\mathcal{P}$, every P-function p admits a unique integral representation;

$$p(x)=\int u(x)d\nu(u)\,,$$

by a Radon measure $\nu\geqq 0$ on $\mathcal{K}$ which is supported by the extreme points of K.

First we shall introduce a vector lattice of which the positive elements are exactly $\mathcal{P}$. The relation

$$(p, p')\sim(q, q')$$

defined by $p+q'=q+p'$ is an equivalence relation of $\mathcal{P}\times\mathcal{P}$. We denote by $[\mathcal{P}]$ the quotient set of $\mathcal{P}\times\mathcal{P}$ with respect to this relation and by $[p, p']$ the equivalence class of the element (p, p'). Defining the sum of two elements of $[\mathcal{P}]$ by

$$[p, p']+[q, q']=[p+q, p'+q']$$

and the multiplication with a real number by

$$\alpha[p, p']=[\alpha p, \alpha p']$$

$$-\alpha[p, p']=[\alpha p', \alpha p]$$

for $\alpha \geqq 0$, $[\mathcal{P}]$ becomes a real vector space. We shall identify $\mathcal{P}$ with the set $\{[p, 0]; p \in \mathcal{P}\}$ and consider as $\mathcal{P} \subset [\mathcal{P}]$. The *specific order in* $[\mathcal{P}]$ is defined by the cone $\mathcal{P}$: $[p, p'] \succ [q, q']$ if $[p, p'] = [q, q'] + u$ for some $u \in \mathcal{P}$.

THEOREM. $[\mathcal{P}]$ *with the specific order is a conditionally complete vector lattice* (*espace de Riesz complètement réticulé*), [**2, 10**].

Before we introduce a locally convex topology on $[\mathcal{P}]$ we shall extend the notion of potential kernels. Let $X_0 = X \cup \{\partial\}$ be a one point compactification of X. Let $p \in \mathcal{P}$, $x \in X$ and $f \in C_+(X_0)$. We set

$$U^p f(x) = \int g(x, y) f(y) d\mu(y) + f(\partial) Bp(x),$$

where μ is a Radon measure $\geqq 0$ on X representing the P_i-function $p - Bp$;

$$p - Bp = \int g(\cdot, y) d\mu(y) \qquad \text{(Theorem 5.7)}.$$

From the remark at the end of section 3 we have

$$U^p f(x) = W^p f(x) + f(\partial) Bp(x)$$

whenever $x \notin \mathrm{Supp}[f]$[6], and, for a finite valued $p \in \mathcal{P}$, we have

$$U^p f = V^p f + f(\partial) Bp.$$

The map $L = U^p$ from $C_+(X_0)$ into $\mathcal{P}$ satisfies the following properties: (a) $f \to Lf$ *is a linear map from* $C_+(X_0)$ *into* $\mathcal{P}$, (b) $Lf \in \mathcal{P}_i$ *if* $\mathrm{Supp}[f] \subset X$, (c) Lf *is harmonic on* $X - \mathrm{Supp}[f]$, (d) $L1 = p$, *and* (e) *for any* $f, g \in C_+(X_0)$ *and* $q = U^p f$, *we have* $U^q g = U^p(f \cdot g)$.

PROPOSITION 6.1. *Let* L *be a linear map from* $C_+(X_0)$ *into* $\mathcal{P}$. L *is given by the form* $L = U^p$ *with a* $p \in \mathcal{P}$ *if and only if* L *satisfies the above properties* (a)$\sim$(c).

PROOF. We shall prove 'if' part only. For each $x \in X$, the linear form $f \to Lf(x)$, $f \in C_+(X_0)$, is extended to a measure L_x on X_0. Let $p = L1 \in \mathcal{P}$ and let μ be the Radon measure on X which represents the P_i-function $p - Bp$;

$$p - Bp = \int_X g(\cdot, y) \mu(dy).$$

For any $f \in C_c^+(X)$ we have $Lf \in \mathcal{P}_i \cap \mathcal{H}(X - \mathrm{Supp}[f])$, so from Lemma 3.8, $Lf \in \mathcal{P}_i \cap \tilde{\mathcal{H}}(X - \mathrm{Supp}[f])$. We have, like the proof of Theorem 3.16,

$$L1_K = p_K = \int_K g(\cdot, y) d\mu(y)$$

for any compact subset K of X. Hence we have;

$$L(f|_X) = \int_X g(\cdot, y) f(y) d\mu(y),$$

6) Here $\mathrm{Supp}[f]$ is considered in X_0, so $\mathrm{Supp}[f] \subset X$ means $f \in C_c(X)$.

$$L(1_X)=p-Bp,$$

and

$$L(1_{\{\partial\}})=L1-L(1_X)=Bp.$$

Therefore

$$Lf=\int_X g(\cdot, y)f(y)\mu(dy)+f(\partial)Bp$$
$$=U^p f.$$

R. M. Herve has shown that; if a sequence of non-negative superharmonic functions s_n is such that, for each $x\in X$, their associated measures $\mu_{s_n}^x$ converge vaguely in the space of measures on $X_0-\{x\}$, then the map $f\to \widehat{\liminf_{n\to\infty}}\, \mu_{s_n}^x(f)$ is linear and positively homogeneous on $C_+(X_0)$. (Proposition 21.1 of [**10**].) An adaptation of her proof shows the following:

LEMMA 6.2. *Let $l_{n,x}$, $n\geqq 1$, $x\in X$, be a family of measures on X_0 such that*

$$l_{n,\cdot}(f)\in \mathcal{S}_+(X)\cap \mathcal{H}(X-\mathrm{Supp}[f]) \qquad \textit{for any} \quad f\in C_+(X_0).$$

Suppose that; for each $x\in X$, the measures $\{l_{n,x}\,;\, n\geqq 1\}$ converges vaguely in the space of measures on $X_0-\{x\}$. Then the map

$$f\to Uf=\widehat{\liminf_{n\to\infty}}\, l_{n,\cdot}(f)$$

from $C_+(X_0)$ into $\mathcal{S}_+(X)$ satisfies

$$U(af+bg)=aUf+bUg,$$

for any $a, b\geqq 0$ and $f, g\in C_+(X_0)$.

The next lemma is a general version of Lemma 21.2 of [**10**]. Herve's proof is applicable without any change.

LEMMA 6.3. *Let $l_{n,x}$, $n\geqq 1$, $x\in X$, be the same as above. Suppose that, for a number $\alpha>0$ and a relatively compact regular domain δ_0 with a point $x_0\in\delta_0$, it holds*

$$(H^{\delta_0} l_{n,\cdot}(1))(x_0)\leqq \alpha.$$

Then there is a subsequence n' such that, for each $x\in X$, the measures $l_{n',x}$ converges vaguely in the space of measures on $X_0-\{x\}$ to a measure ν_x.

Now we shall introduce a locally convex topology on $[\mathcal{P}]$. The pair (f, x) with $f\in C_c^+(X_0-\{x\})$ is called a couple (f, x). For any couple (f, x) the map

$$[p, p']\longrightarrow U^p f(x)-U^{p'}f(x)$$

is a linear form on $[\mathcal{P}]$. We denote by $\tilde{\mathrm{T}}$ the least fine topology on $[\mathcal{P}]$ for which all these linear forms are continuous. $[\mathcal{P}]$ provided with this topology is a locally convex separated topological vector space. From the property (e) of U^p listed previous to Proposition 6.1 we see that the map $p\to U^p f$ is continuous on $(\mathcal{P}, \tilde{\mathrm{T}})$ for any $f \in C_+(X_0)$.

Let $\mathcal{X} = (x_j)$ be a countably dense subset of X and F be a countable family of continuous functions on X_0 with the following properties: For any $x \in \mathcal{X}$ and $f \in C_c^+(X_0 - \{x\})$ and any compact neighborhood K of x such that $K \cap \mathrm{Supp}[f] = \phi$, and for an $\varepsilon > 0$, one can find two functions f_i and f_j of F such that they vanish on K and such that

$$f_j \leqq f \leqq f_i \quad \text{and} \quad |f - f_k| < \varepsilon \qquad \text{on} \quad X_0 \ (k = i, j).$$

We can verify that the family of all sets

$$\left\{ p \in \mathcal{P} ;\ |U^p f_n(x_j)| \leqq \frac{1}{r} \right\}, \qquad f_n \in F, \quad x_j \in \mathcal{X} \quad \text{and} \quad r > 0,$$

forms a subbase of a fundamental system of neighborhoods of 0 of the topology $\tilde{\mathrm{T}}$ induced on the positive cone $\mathcal{P}$. Hence $(\mathcal{P}, \tilde{\mathrm{T}})$ is metrisable.

PROPOSITION 6.4. *Let $p_n \in \mathcal{P}$, $n \geqq 1$. If, for each $x \in X$, the measure $U^{p_n}(x, dy)$ considered on $X_0 - \{x\}$ converges vaguely to a measure on $X_0 - \{x\}$, then the functions p_n are $\tilde{\mathrm{T}}$-convergent to $\widehat{\liminf_{n\to\infty}} p_n$. In particular if p_n's are $\tilde{\mathrm{T}}$-convergent in $\mathcal{P}$ the limit is equal to $\widehat{\liminf} p_n$.*

PROOF. From Lemma 6.2 the map $f \to Uf = \widehat{\liminf}\, U^{p_n} f$, $f \in C_+(X_0)$, is linear. Uf is a P-function and $U^{p_n} f$ converges locally uniformly on $X - \mathrm{Supp}[f]$ to a harmonic function on $X - \mathrm{Supp}[f]$. If $\mathrm{Supp}[f]$ is contained in X every $U^{p_n} f$ is fullharmonic on $X - \mathrm{Supp}[f]$, and hence, Uf is so. Thus U satisfies the conditions of Proposition 6.1 and we have from this proposition $U = U^p$ with $p = \widehat{\liminf} p_n$. Let (f, x) be a couple. $U^{p_n} f$ being convergent on $X - \mathrm{Supp}[f]$ to $Uf = U^p f$, we have $\lim U^{p_n} f(x) = U^p f(x)$. Hence p_n's are $\tilde{\mathrm{T}}$-convergent to p.

Let $x_0 \in X$ and let δ_0 be a relatively compact regular domain containing x_0. Let $D(u)$ denote $H^{\delta_0} u(x_0)$ for $u \in \mathcal{P}$. Let

$$\mathcal{P}^\alpha = \{ u \in \mathcal{P} ;\ D(u) \leqq \alpha \}, \qquad \alpha > 0 .$$

If p_n is a sequence of functions of $\mathcal{P}^\alpha$, then $\widehat{\liminf_{n\to\infty}} p_n \in \mathcal{P}^\alpha$. For,

$$\begin{aligned} H^{\delta_0}(\widehat{\liminf} p_n)(x_0) &= H^{\delta_0}(\liminf p_n)(x_0) \\ &\leqq \liminf p_n(x_0) \\ &\leqq \alpha . \end{aligned}$$

THEOREM 6.5. (1) *$\mathcal{P}^\alpha$ is compact in $(\mathcal{P}, \tilde{\mathrm{T}})$.* (2) *The cone $\mathcal{P}$ has a compact base. The set*

$$\mathcal{K} = \{ p \in \mathcal{P} :\ U^p f(x_1) + U^p(1-f)(x_2) = 1 \},$$

where $f\in C_+(X_0)$, $0\leqq f\leqq 1$, $f=0$ *on a neighborhood of* x_1 *and* $f=1$ *on a neighborhood of* x_2, *is a compact base of* $\mathcal{P}$.

PROOF. (1) follows from Lemma 6.3, Proposition 6.4 and the above remark. Let $p\in\mathcal{K}$ and let δ_1 and δ_2 be regular neighborhoods ot x_1 and x_2 respectively such that $f=0$ on δ_1 and $f=1$ on δ_2. From Lemma 5.1 there is a constant $\alpha>0$ such that

$$D(U^pf)\leqq \alpha H^{\delta_1}U^pf(x_1)=\alpha U^pf(x_1)\,,$$

and

$$D(U^p(1-f))\leqq \alpha U^p(1-f)(x_2)\,,$$

hence it follows

$$D(p)\leqq \alpha(U^pf(x_1)+U^p(1-f)(x_2))\,.$$

From this we see $\mathcal{K}\subset P^\alpha$. $\mathcal{K}$ is $\tilde{\mathrm{T}}$-closed, and compact in $\mathcal{P}$ from (1). We shall show that $\mathcal{K}$ is a base. $U^pf(x_1)+U^p(1-f)(x_2)=0$ implies $p=U^pf+U^p(1-f)=0$. Therefore we have $\alpha p\in K$ with $\alpha=(U^pf(x_1)+U^p(1-f)(x_2))^{-1}$ for every $p\in\mathcal{P}$, $p>0$.

THEOREM 6.6. (1) $\mathcal{P}$ *is complete.*

(2) $\mathcal{P}_b$ *is closed in* $\mathcal{P}$.

PROOF. (1) Let p_n be a Cauchy sequence in $\mathcal{P}$. Let (x, f) be a couple. Then there is a number $\alpha>0$ such that $D(p_n)\leqq\alpha U^{p_n}f(x)$ for all n (Lemma 5.1). Since the set (p_n) is bounded in $\mathcal{P}$, $\sup_n U^{p_n}f(x)<\beta$ for some $\beta>0$, hence $\sup D(p_n)<\alpha\cdot\beta$, and $(p_n)\subset P^\gamma$, where $\gamma=\alpha\cdot\beta$. $\mathcal{P}^\gamma$ being compact there is a subsequence $p_{n'}$ which converges in $\mathcal{P}^\gamma$. Since p_n is a Cauchy sequence p_n converges to the same limit.

(2) Let $p_n\in\mathcal{P}_b$ and let p_n be $\tilde{\mathrm{T}}$-convergent to a limit $p\in\mathcal{P}$. We have $p_n=U^{p_n}f$ for any $f\in C_+(X_0)$ which is equal to 1 at $\{\partial\}$. Hence (p_n) is convergent pointwisely on X. From Lemma 5.1 this convergence is uniform on every compact subset, and the limit is a P_b-function q. Therefore we have $U^pf(x)=f(\partial)q(x)$ for any couple (f, x). We shall show $p=q$. Let x be a point such that $p(x)<\infty$ and $\mu^{\delta_0,x_0}(x)=0$, and let $f_n\in C_c^+(X-\{x\})$, $n\geqq 1$, be such that $f_n\uparrow 1_{X_0-\{x\}}$. The measure $U^p(x, dy)$ being absolutely continuous relative to $\mu^{\delta_0,x_0}(dy)$, we have $p(x)=U^p1_{X_0-\{x\}}=\lim U^pf_n(x)=q(x)$. Since the points where $\mu^{\delta_0,x_0}(\{x\})>0$ are at most countable and the set of points of infinity of p is a polar set, we have $p=q$.

REMARK. $\tilde{\mathrm{T}}$-topology induced on $\mathcal{P}_b$ coincides with the compact convergence topology on X.

We have seen that $\mathcal{P}$ is a metrisable convex cone with a compact base in the locally convex separated topological vector space $[\mathcal{P}]$ and $\mathcal{P}$ is a lattice relative to the specific order defined in $[\mathcal{P}]$. Thus we can apply Choquet's representation theorem.

THEOREM 6.7. *Let $\mathcal{K}$ be a compact base of $\mathcal{P}$. Every $p \in \mathcal{P}$ has a unique representation*

$$p=\int u\nu(du) \tag{6.1}$$

with the aid of a Radon measure $\nu \geqq 0$ on $\mathcal{K}$ which is supported by the set of extreme points of $\mathcal{K}$[7].

The vectorial integral (6.1) means that, if $[p, p'] \to L(p, p')$ is a continuous linear form on $[\mathcal{P}]$, we have

$$L(p)=\int L(u)\nu(du) .$$

REMARK. From a general property as for semi-continuous affine functionals over a compact base of a convex cone in a locally convex space (Lemma 9.7 of [**20**]) we have the following formula:

$$\Phi(p)=\int \Phi(u)\nu(du) \tag{6.1$'$}$$

for any affine lower (or upper) semi-continuous functional Φ on $\mathcal{K}$ whenever $p \in \mathcal{P}$ is represented as (6.1).

Here we shall study the extreme rays of $\mathcal{P}$.

A *ray* ρ of a convex cone C is a set of the form $R^+x=\{\lambda x;\ \lambda \geqq 0\}$ where $x \in C$, $x \neq 0$. A ray ρ of C is called an *extreme ray* of C if whenever $x \in \rho$ and $x=\lambda y+(1-\lambda)z$, $(y, z \in C)$, then $y, z \in \rho$. We denote by $(C)_{ex}$ the union of extreme rays of C; this set has the following useful descriptions: Suppose $x \in C$, then x is in an extreme ray if and only if $x=y+z (y, z \in C)$ implies $y, z \in R^+x$. Let $\prec$ denote the partial ordering induced by the cone C on the linear space $C-C$. $x \in C$ is an extreme ray of C if and only if $y=\lambda x$ (for some $\lambda \geqq 0$) whenever $0 \prec y \prec x$. If C has a base B then ρ is an extreme ray of C if and only if ρ intersects B in an extreme point of B. It holds that the intersection of extreme rays with B is exactly the extreme points of B; $\mathrm{ex}\, B=B \cap (C)_{ex}$.

In the sequel we denote the set $(\mathcal{P})_{ex}$, $(\mathcal{P}_i)_{ex}$ and $(\mathcal{P}_b)_{ex}$ by $\mathcal{E}$, $\mathcal{E}_i$ and $\mathcal{E}_b$ respectively. Let $p \in \mathcal{E}$. Then from the above description of extreme rays it follows that $p-Bp$, $Bp \in R^+p$. Hence p must be either in $\mathcal{P}_b$ or in $\mathcal{P}_i$.

PROPOSITION 6.8.

$$\mathcal{E}=\mathcal{E}_i \cup \mathcal{E}_b\,, \qquad \mathcal{E}_i=\mathcal{E} \cap \mathcal{P}_i\,, \qquad \mathcal{E}_b=\mathcal{E} \cap \mathcal{P}_b\,.$$

PROOF. Let $p \in \mathcal{E}_i$ and $p=\alpha r+(1-\alpha)s$ for some $r, s \in \mathcal{P}$ and $0<\alpha<1$. It follows $\alpha r \prec p$ and $(1-\alpha)s \prec p$. These relations are also true for the order induced by the cone $\mathcal{P}_i$ (Theorem 1.6), so we have $r, s \in R^+p$. Thus $\mathcal{E}_i \subset \mathcal{E} \cap \mathcal{P}_i$.

7) The extreme points of $\mathcal{K}$ is a G_δ-set.

The converse is obvious and we have $\mathcal{E}_i=\mathcal{E}\cap\mathcal{P}_i$. Now let $p\in\mathcal{E}_b$ and $p=\alpha r+(1-\alpha)s$ for $r, s\in\mathcal{P}$ and $0<\alpha<1$. From Lemma 1.5 we have $\alpha r, (1-\alpha)s\in\mathcal{P}_b$. Hence $r, s\in R^+p$. $\mathcal{E}_b\subset\mathcal{E}\cap\mathcal{P}_b$. The converse is obvious and we have $\mathcal{E}_b=\mathcal{E}\cap\mathcal{P}_b$.

Every element $p\in\mathcal{E}_i$, >0, is of one-point carrier. In fact, suppose that $\mathrm{Carr}(p)$ contains two distinct points y_1 and y_2. We have, for a compact set K such that $y_1\in\mathring{K}$ and $y_2\notin K$, $p=p_K+p_{X-K}$. Since $p\in\mathcal{E}$, p_{X-K} must be proportional to p. It follows $p\in\mathcal{H}(\mathring{K})$ and $y_1\in\mathring{K}\cap\mathrm{Carr}(p)=\phi$, this is a contradiction. Conversely every point $y\in X$ is the carrier of an element of $\mathcal{E}_i$. Because the set of all P_i-functions with carrier $\{y\}$, which are equal to 1 at a point $x_0\neq y$, forms a closed convex subset of $\mathcal{P}_i\cap\mathcal{P}^1=\{p\in\mathcal{P}_i\,;\,D(p)\leqq 1\}$, hence this is a compact convex set and has an extreme point.

We shall study the topology $\tilde{\mathrm{T}}$ induced on $\mathcal{E}$ or on $\mathcal{E}\cap\mathcal{K}$ for some compact base $\mathcal{K}$ of $\mathcal{P}$. In the sequel we define, for the convenience, the carrier of a P_b-function as the point $\{\partial\}$; $\mathrm{C}(p)\in X$ if $p\in\mathcal{E}_i$ and $\mathrm{C}(p)=\{\partial\}$ if $p\in\mathcal{P}_b$. Let $p\in\mathcal{E}$. We have

$$U^pf(x)=W^pf(x)+f(\partial)Bp(x)$$
$$=\begin{cases}0 & \text{if } x=\mathrm{C}(p)\in X\\ f(\mathrm{C}(p))p(x) & \text{if } x\neq\mathrm{C}(p),\end{cases}$$

for any couple (f, x).

Proposition 6.9. (a) *The map* $\mathrm{C}: p\to\mathrm{C}(p)$ *from* $\mathcal{E}_i\cup\mathcal{P}_b$ *to* X_0 *is continuous.*

(b) $p\to p(x)$ *is continuous on* $\mathcal{E}_i\cup\mathcal{P}_b$ *for* $x\neq\mathrm{C}(p)$.

Proof. (a) Let $p_0\in\mathcal{E}_i$ and K be a compact neighborhood of $\mathrm{C}(p_0)$ in X. We take a point $x\in X-K$ and a continuous function f which vanishes on $X-K$ and >0 at $\mathrm{C}(p_0)$. $U^{p_0}f(x)=p_0(x)\cdot f(\mathrm{C}(p_0))>0$. Hence $\mathcal{V}=\{p\in\mathcal{E}_i\,;\,U^pf(x)=p(x)f(\mathrm{C}(p))>0\}$ is a neighborhood of p_0 and $p\in\mathcal{V}$ implies $\mathrm{C}(p)\in K$. Thus $p\to\mathrm{C}(p)$ is continuous at p_0. For $p_0\in\mathcal{P}_b$ we can prove by the same way if we take a $G\in\mathcal{D}$, $x\in X-G$, and a $f\in C_c^+(X_0)$ with $f(\partial)>0$ and $f=0$ on $X-\bar{G}$. (b) Let $x\neq\mathrm{C}(p_0)$ and G be a neighborhood of $\mathrm{C}(p_0)$ in X_0 such that $x\notin\bar{G}$. Let $f\in C_c^+(X_0-\{x\})$ be such that $f=1$ on G. $\mathcal{V}=\{p\in\mathcal{E}_i\cup\mathcal{P}_b\,;\,|U^pf(x)-U^{p_0}f(x)|<\varepsilon\}\cap\{p\in\mathcal{E}\,;\,\mathrm{C}(p)\in G\}$ is a neighborhood of p_0. We have, for $p\in\mathcal{V}$, $U^pf(x)=p(x)f(\mathrm{C}(p))=p(x)$, so we have $|p(x)-p_0(x)|<\varepsilon$.

From the above (a) and (b) we see that;

(c) *the map* $(p, x)\to p(x)$ *is lower semi-continuous on* $(\mathcal{E}_i\cup\mathcal{P}_b)\times X$ *and is continuous for* $x\neq\mathrm{C}(p)$. (The proof is the same as Lemma 5.7).

Proposition 6.10. *Let* p_y *be a* P_i*-function of carrier* $\{y\}$ *such that, for any* $x\in X$, *the map* $y\to p_y(x)$ *is continuous on* $X-\{x\}$. *If* $p_y\in\mathcal{E}_i$ *for any* $y\in X$, *the map* $y\to p_y$ *from* X *into* $\mathcal{E}_i$ *is continuous.*

Proof. Let y_n be a sequence that converges to y_0 in X. For any $x\neq y_0$

and any $f \in C_c^+(X_0 - \{x\})$, we have $U^{p_n}f(x) = f(y_n)p_n(x) \to f(y_0)p_0(x) = U^{p_0}f(x)$, where $p_n = p_{y_n}$ and $p_0 = p_{y_0}$. For $x = y_0$ and $f \in C_c^+(X_0 - \{y_0\})$ we have $U^{p_n}f(x) = U^{p_0}f(x) = 0$ for large n. Therefore $y \to p_y$ from X to $\mathcal{E}_i$ is continuous.

In the following we shall prove that the formula (6.1) holds as a function on X (Theorem 6.13).

PROPOSITION 6.11. *Let $\mathcal{K}$ be a compact base of $\mathcal{P}$ and let ν be a Radon measure, $\geqq 0$, on $\mathcal{K} \cap \mathcal{E}$. Then we have;*

$$\int u\nu(du) \in \mathcal{P}, \int_{\mathcal{E}_i \cap \mathcal{K}} u\nu(du) \in \mathcal{P}_i,$$

$$\int_{\mathcal{E}_b \cap \mathcal{K}} u\nu(du) \in \mathcal{P}_b.$$

PROOF. From Proposition 6.9 (c) the function

$$q(x) = \int u(x)\nu(du)$$

is lower semi-continuous. If

$$\int H^\delta u(x)\nu(du) < \infty$$

for any x and any regular neighborhood δ of x, we have, by using Fubini's theorem, $q \in \mathcal{S}_+(X)$. We have also $q \in \tilde{\mathcal{S}}_+(X) = \mathcal{P}$. The finiteness of

$$\int H^\delta u(x)\nu(du)$$

is verified as follows. Let f, x_1, x_2 be the same as in Theorem 6.5. From Lemma 5.1 there is a constant $\alpha > 0$ such that $H^\delta(U^p f)(x) \leqq \alpha U^p f(x_1)$ and $H^\delta(U^p(1-f))(x) \leqq \alpha U^p(1-f)(x_2)$ for any $p \in \mathcal{P}$. Since $p \to U^p f(x_1)$ and $p \to U^p(1-f)(x_2)$ are continuous, they are bounded on $\mathcal{K}$. Hence $H^\delta p(x)$ is also bounded on $\mathcal{K}$, and it follows the finiteness of the above integral. Fubini's theorem also yields

$$\int_{\mathcal{E}_b \cap \mathcal{K}} u\nu(du) \in \mathcal{P}_b$$

and

$$\int_{\mathrm{C}^{-1}(A)} u\nu(du) \in \mathcal{P} \cap \tilde{\mathcal{H}}(X-A),$$

where A is a compact subset of X and $\mathrm{C}^{-1}(A) = \{u \in \mathcal{E} \cap \mathcal{K} ; \mathrm{C}(u) \in A\}$. We have

$$(1) \qquad Bq \succ \int_{\mathcal{E}_b \cap \mathcal{K}} u\nu(du),$$

and

$$q_A \succ \int_{\mathrm{C}^{-1}(A)} u\nu(du)$$

(The property (i) in section 1 and Proposition 3.2). It follows from Lemma 3.10 that

$$q-Bq \succ \int_{C^{-1}(A)} u\nu(du).$$

Therefore

$$(2) \qquad q-Bq \geqq \int_{C^{-1}(X)} u\nu(du) = \int_{\mathcal{E}_i \cap \mathcal{K}} u\nu(du).$$

The inequalities (1) and (2) imply

$$Bq = \int_{\mathcal{E}_b \cap \mathcal{K}} u\nu(du),$$

and

$$q-Bq = \int_{\mathcal{E}_i \cap \mathcal{K}} u\nu(du) \in \mathcal{P}_i.$$

COROLLARY 6.12.

$$q_K(x) = \int_{C^{-1}(K)} u(x)\nu(du)$$

for any compact set K *of* X *and* $x \notin K$.

$$W^q f(x) = \int_{\mathcal{E}_i \cap \mathcal{K}} f(C(u))u(x)\nu(du)$$

for any couple (f, x).

The proof is the same as Proposition 5.4.

THEOREM 6.13. *Let* $\mathcal{K}$ *be a compact base of* $\mathcal{P}$. *Every* $p \in \mathcal{P}$ *has a unique representation;*

$$(6.2) \qquad p(x) = \int u(x)\nu(du), \qquad x \in X,$$

by a Radon measure $\nu \geqq 0$ *on* $\mathcal{K}$, *which is supported by* $\mathcal{E} \cap \mathcal{K}$.

PROOF. Let ν be the Radon measure on $\mathcal{K}$ which appeared in the formula (6.1). We have, for any couple (f, x),

$$U^p f(x) = \int U^q f(x)\nu(dq) = \int q(x) f(C(q))\nu(dq).$$

For every $x \in X$ let $f_n \in C_c^+(X_0 - \{x\})$ be such that $f_n \uparrow 1_{X_0 - \{x\}}$. If x is a point of X such that $p(x) < \infty$ and such that $H^{\delta_0} U^p 1_{\{x\}}(x_0) = 0$ and $\nu(\mathcal{E} \cap \mathcal{K} \cap C^{-1}(x)) = 0$, we have $p(x) = U^p 1_{X_0 - \{x\}}(x) = \lim \uparrow U^p f_n(x) = \lim \uparrow \int q(x) f_n(C(q))\nu(dq) = \int q(x)\nu(dq)$. Here we have used the fact that $U^p(x, dy)$ is absolutely continuous relative to the measure $H^{\delta_0}U^p(\cdot, dy)(x_0)$. The points of X where $\nu(\mathcal{E} \cap \mathcal{K} \cap \mathrm{C}^{-1}(x)) > 0$ or $H^{\delta_0}U^p 1_{\{x\}}(x_0) > 0$ holds are at most countable, and the set of points of infinity of p is a polar set. Therefore the superharmonic functions p and $\int q\nu(dq)$ must be identical. We shall prove the uniqueness part of the theorem. Suppose $p \in \mathcal{P}$ is represented as

$$p(x) = \int_{\mathcal{K}} q(x)\mu(dq)\,.$$

For any couple (f, x) we have, from Proposition 6.11 and Corollary 6.12,

$$U^p f(x) = W^p f(x) + f(\partial)Bp(x) = \int f(C(q))q(x)\mu(dq)$$
$$= \int U^q f(x)\mu(dq)\,.$$

Hence we have, for any continuous linear form L on $[\mathcal{P}]$,

$$L(p) = \int L(q)\mu(dq)\,.$$

Choquet's theorem (Theorem 6.7) yields the uniqueness of μ.

COROLLARY 6.14. *Every $p \in \mathcal{P}_b$ admits a unique representation;*

$$p(x) = \int_{\mathcal{E}_b \cap \mathcal{K}} q(x)\nu(dq)\,, \tag{6.3}$$

and every $p \in \mathcal{P}_i$ admits a unique representation;

$$p(x) = \int_{\mathcal{E}_i \cap \mathcal{K}} q(x)\nu(dq)\,. \tag{6.4}$$

Further properties of the topology $\tilde{\mathrm{T}}$.

Let p_y be a P_i-function with carrier at y such that $y \to p_y(x)$ is continuous on $X - \{x\}$ for every x. Let a P_i-function p be represented by

$$p = \int p_y \lambda_p(dy)\,, \qquad \lambda_p = \frac{\mu^{\delta_0, x_0}}{D(p_y)}$$

(Theorem 5.9). We know $\mathrm{Carr}(p) = \mathrm{Supp}[\lambda_p]$. Consider the map $p \to \lambda_p$ from $\mathcal{P}_i$ into the set of Radon measures $\geqq 0$ on X. This map is bijective, (Theorem 5.9).

(6.5) *Let K be a compact set of X. The topology on the set $\mathrm{C}^{-1}(K) = \{p \in \mathcal{P}_i\,;\ \mathrm{Carr}(p) \subset K\}$ induced by $\tilde{\mathrm{T}}$ is the inverse image by the map $p \to \lambda_p$ of the vague topology on the space of Radon measures $\geqq 0$ on K.*

First we note that $U^p f(x) = \int p_y(x) f(y) \lambda_p(dy)$ for any couple (f, x) and $p \in \mathcal{P}_i$. Since $y \to p_y(x)$ is continuous on $X - \{x\}$ the map $\lambda(dy) \to p_y(x)\lambda(dy)$

from $\mathcal{M}_+(K)$ into $\mathcal{M}_+(X_0-\{x\})$[8] is continuous whenever these spaces are given the vague topologies. Hence the map $p\rightarrow p_y(x)\lambda_p(dy)$ from $C^{-1}(K)$ into $\mathcal{M}_+(X_0-\{x\})$ is continuous whenever $C^{-1}(K)$ is given the inverse image of the vague topology on $\mathcal{M}_+(K)$. This topology is finer than $\tilde{T}$-topology on $C^{-1}(K)$. Conversely, choose a point $x\in X-K$, and let $\varphi_x(y)=(p_y(x))^{-1}$, $y\in K$. Since the topology $\tilde{T}$ lets the map $p\rightarrow U^p(\varphi_x\cdot f)(x)=\int f(y)\lambda_p(dy)$ continuous for any $f\in C_+(K)$, $\tilde{T}$ is finer than the inverse image of the vague topology on $\mathcal{M}_+(K)$.

(6.6) *Let $p_n\in\mathcal{P}$ be a decreasing sequence (resp. increasing sequence with $\sup_n p_n\in\mathcal{P}$). Then p_n is $\tilde{T}$-convergent to $\widehat{\inf_n} p_n$ (resp. $\sup p_n$).*

Every p_n belongs to $\mathcal{P}^\alpha$ for some $\alpha>0$. Hence there is a subsequence $p_{n'}$ which is $\tilde{T}$-convergent to $\widehat{\liminf} p_{n'}$. This proves (6.6).

(6.7) *Let $\mathcal{K}$ be a compact base of the cone $\mathcal{P}$. For every $p\in\mathcal{E}_b\cap\mathcal{K}$, there is a sequence $p_n\in\mathcal{E}_i\cap\mathcal{K}$ such that it is $\tilde{T}$-convergent to p.*

Since $p\in\mathcal{E}_b\cap\mathcal{K}$ is an extreme point of the compact convex set $\mathcal{K}$, the traces on $\mathcal{E}\cap\mathcal{K}$ of the open half-spaces in $[\mathcal{P}]$ that contain p form a fundamental system $\mathcal{V}(p)$ of neighborhoods in $\mathcal{E}\cap\mathcal{K}$ of p ([**6**], p. 108). Let $\mathcal{V}\in\mathcal{V}(p)$ be such that; $\mathcal{V}=\{q\in\mathcal{K}\cap\mathcal{E}\,;\,\varphi(q)>\alpha\}$, where φ is a continuous linear form, $\neq 0$, on $[\mathcal{P}]$ and $\alpha>0$. By virtue of Proposition 1.2 and (6.6) we can find a $q\in\mathcal{P}_i\cap\mathcal{V}$. q is represented as

$$q=\int u\nu(du)$$

by a probability measure on $\mathcal{K}$ which is supported by $\mathcal{E}_i\cap\mathcal{K}$ (Corollary 6.15 and Proposition 6.7). If $\mathcal{E}_i\cap\mathcal{K}\cap\mathcal{V}=\phi$ then

$$\alpha<\varphi(q)=\int\varphi(u)\nu(du)\leqq\alpha$$

which is contradictory. $\mathcal{E}_i\cap\mathcal{K}\cap\mathcal{V}\neq\phi$ for any $\mathcal{V}\in\mathcal{V}(p)$. Hence there is a sequence $p_n\in\mathcal{E}_i\cap\mathcal{K}$ that converges to p.

§7. Representation of P_b-functions. Ideal boundary of X.

In this section we shall adopt the hypothesis of proportionality, that is, for every point y of X all P_i-functions >0 with carrier at y are proportional. In this case we can identify X homeomorphically with the extreme P_i-functions of some compact base of P, and the closure of this set contains the

---

8) $\mathcal{M}_+(A)$ is the set of finite measures $\geqq 0$ on the locally compact space A.

extreme P_b-functions of the compact base. This is Kuramochi compactification of X with respect to the fullharmonic structure $\tilde{\mathcal{H}}$.

Let x_1 and x_2 be two distinct points of X and let δ_1 and δ_2 be disjoint regular neighborhoods of x_1 and x_2 respectively. Let $f_0 \in C_+(X_0)$ be such that $f_0 = 0$ on $\bar{\delta}_1$, and $f_0 = 1$ on $\bar{\delta}_2$, and $0 < f_0 < 1$ on $X_0 - \bar{\delta}_1 \cup \bar{\delta}_2$. The set

$$\mathcal{K}_0 = \{p \in \mathcal{P} \,;\, U^p f_0(x_1) + U^p(1-f_0)(x_2) = 1\}$$

forms a compact base of $\mathcal{P}$ as we have seen in Theorem 6.5. We define a function $\alpha_0(y)$ on X as follows;

$$\alpha_0(y) = \begin{cases} \dfrac{1}{f_0(y)p_y(x_1) + (1-f_0)(y)p_y(x_2)} & \text{for } y \neq x_1,\ y \neq x_2 \\ \dfrac{1}{p_{x_1}(x_2)} & \text{for } y = x_1 \\ \dfrac{1}{p_{x_2}(x_1)} & \text{for } y = x_2 \,. \end{cases}$$

$\alpha_0(y)$ is a continuous function and $\alpha_0(y)p_y \in \mathcal{K}_0 \cap \mathcal{E}_i$ for any $y \in X$. We shall denote it by $k_y(\cdot)$.

PROPOSITION 7.1. *The correspondence $y \to k_y$ gives a homeomorphism from X onto $\mathcal{E}_i \cap \mathcal{K}_0$.*

PROOF. Let $y \in X$. The set of all functions $p \in \mathcal{K}_0 \cap \mathcal{P}_i$ with $C(p) = \{y\}$ is not empty and forms a closed convex subset of $\mathcal{K}_0$, hence this set is compact convex and has an extreme point $q \in \mathcal{E}_i \cap \mathcal{K}_0$. From our assumption (7) q is written as $q = \beta(y)p_y$. It is easy to see $\beta(y) = \alpha(y)$. Thus $y \to \alpha(y)p_y = k_y$ is bijective. From Propositions 6.9 and 6.10 this map is a homeomorphism from X onto $\mathcal{E}_i \cap \mathcal{K}_0$.

We recall that, for any subset $\mathcal{A}$ of $\mathcal{P}$, the collection of all sets in $\mathcal{A} \times \mathcal{A}$ of the form

$$\{(p, q) \in \mathcal{A} \times \mathcal{A} \,;\, |U^p f_j(x_j) - U^q f_j(x_j)| < \varepsilon,\ 1 \leqq j \leqq n\},$$

where each (f_j, x_j) is a couple and $\varepsilon > 0$, forms a fundamental system of entourages of the uniformity induced on $\mathcal{A}$ by that of $[\mathcal{P}]$. This uniformity defines $\tilde{T}$-topology on $\mathcal{A}$ and the $\tilde{T}$-closure $\bar{\mathcal{A}}$ is complete (Theorem 6.6).

Let $\mathcal{U}^*$ be the uniformity on X which is the inverse image of the uniformity on $\mathcal{E}_i \cap \mathcal{K}_0$ under the map $y \to k_y$, that is, $\mathcal{U}^*$ is the coarest uniformity on X for which the map $y \to k_y$ is uniformly continuous. The fundamental system of entourages of $\mathcal{U}^*$ is given by the sets of the form

$$\{(y_1, y_2) \in X \times X \,;\, |f_j(y_1)k_{y_1}(x_j) - f_j(y_2)k_{y_2}(x_j)| < \varepsilon,\ 1 \leqq j \leqq n\},$$

where (f_j, x_j)'s are couples and $\varepsilon > 0$. The topology on X induced by $\mathcal{U}^*$ is the inverse image under the map $y \to k_y$ of $\tilde{T}$-topology, hence this topology is the original topology on X. Since $y \to k_y$ is bijective from X onto $\mathcal{E}_i \cap \mathcal{K}_0$, the entourages on $\mathcal{E}_i \cap \mathcal{K}_0$ are the direct images under this map of the entourages in $\mathcal{V}^*$, hence X and $\mathcal{E}_i \cap \mathcal{K}_0$ are isomorphic as uniform spaces. It follows that the inverse of the map $y \to k_y$ is also uniformly continuous [**5**, Chap. II, 2.4].

Let X^* be the completion of X with respect to $\mathcal{V}^*$. The isomorphism of X onto $E_i \cap K_0$ extends to an isomorphism of X^* onto $\overline{\mathcal{E}_i \cap \mathcal{K}_0}$ [**5**, Chap. II, 3.6]. In particular X^* and $\overline{\mathcal{E}_i \cap \mathcal{K}_0}$ are homeomorphic. The map $y \to k_y$ is continuously extended over X^*, which we shall denote by $\xi \to k_\xi$, $\xi \in X^*$.

THEOREM 7.2. (1) *X^* is a compactification of X.*

(2) *The function $\xi \to k_\xi(x)$ is continuous on $X^* - \{x\}$, for every $x \in X$.*

(3) $\mathcal{E}_b \cap \mathcal{K}_0 \subset \{k_\xi\,;\, \xi \in X^* - X\} \subset \mathcal{P}_b \cap \mathcal{K}_0$.

(4) *There is a constant $\gamma > 0$ such that*

$$\frac{1}{\gamma} \leqq D(k_\xi) \leqq \gamma \qquad \text{for every } \xi \in X^*.$$

PROOF. We have already proved (1). We shall prove (3). The first inclusion relation follows from (6.7). For any $\xi \in X^* - X$, k_ξ is a $\tilde{\mathrm{T}}$-limit of a Cauchy sequence k_{y_n}, $y_n \in X$, such that y_n has no accumulation point in X. We see from the remark after Theorem 6.6 that k_{y_n} converges locally uniformly on X. Hence k_ξ is harmonic and $k_\xi \in \mathcal{P}_b \cap \mathcal{K}_0$. (2) From Proposition 6.9 $p \to p(x)$ is continuous on $\mathcal{E}_i \cup \mathcal{P}_b$ whenever $x \neq \mathrm{C}(p)$. On the other hand the map $\xi \to k_\xi$ from X^* onto $\overline{\mathcal{E}_i \cap \mathcal{K}_0}$ is continuous, and $\overline{\mathcal{E}_i \cap \mathcal{K}_0} \subset \mathcal{E}_i \cup \mathcal{P}_b$ from the above. Hence $\xi \to k_\xi(x)$ is continuous on $X^* - \{x\}$ for any $x \in X$. (4) From Lemma 5.1 there is a constant $\gamma > 0$ such that

$$\frac{1}{\gamma} U^p f_0(x_1) \leqq H^{\delta_0} U^p f_0(x_0) \leqq \gamma U^p f_0(x_1)$$

and

$$\frac{1}{\gamma} U^p(1-f_0)(x_2) \leqq H^{\delta_0} U^p(1-f_0)(x_0)$$
$$\leqq \gamma U^p(1-f_0)(x_2).$$

Therefore we have $\frac{1}{\gamma} \leqq D(p) \leqq \gamma$ for any $p \in \mathcal{K}_0$.

DEFINITION.

$$\varDelta = X^* - X,$$
$$\varDelta_1 = \{\xi \in \varDelta\,;\, k_\xi \in \mathcal{E}_b \cap \mathcal{K}_0\}.$$

The function $(x, \xi) \to k_\xi(x)$ on $X \times \varDelta$ is continuous. In fact, let $(a, b) \in X \times \varDelta$ and $\varepsilon > 0$. Since k_ξ, $\xi \in \varDelta$, is harmonic, we have, from Lemma 5.1,

$$1 - \varepsilon \leqq \frac{k_\xi(x)}{k_\xi(a)} \leqq 1 + \varepsilon$$

on a relatively compact neighborhood U of a for any $\xi \in \Delta$. Take a neighborhood V of b in X^* such that

$$1-\varepsilon \leqq \frac{k_\xi(a)}{k_b(a)} \leqq 1+\varepsilon$$

for every $\xi \in V$. Then we have

$$(1-\varepsilon)^2 \leqq \frac{k_\xi(x)}{k_b(a)} \leqq (1+\varepsilon)^2 .$$

Thus $(x, \xi) \to k_\xi(x)$ is continuous at (a, b).

Like the proof of Proposition 5.8, it is verified that $(x, \xi) \to k_\xi(x)$ is lower semi-continuous on $X \times X^*$.

PROPOSITION 7.3. *Let μ be a Radon measure $\geqq 0$ on X^*. Then we have*

$$p = \int k_\xi \mu(d\xi) \in \mathcal{P} .$$

If μ is supported by the set Δ we have $p \in \mathcal{P}_b$.

The proof is the same as Proposition 6.11.

THEOREM 7.4. *Every $p \in \mathcal{P}$ (resp. $\mathcal{P}_b$) has a unique representation*

$$p = \int k_\xi \nu(d\xi)$$

by a Radon measure $\nu \geqq 0$ on X^* which is supported by $\Delta_1 \cup X$ (resp. Δ_1). The total variation of ν is given by

$$\nu(X^*) = U^p f_0(x_1) + U^p(1-f_0)(x_2) .$$

This theorem follows from Theorem 6.13.

The measure ν is called the canonical measure of p.

LEMMA 7.5. *Let μ_n, $n \geqq 1$, be a sequence of Radon measures $\geqq 0$ on X^*. Let*

$$p_n = \int k_\xi \mu_n(d\xi) .$$

If μ_n converges vaguely to a Radon measure μ on X^* and if p_n converges to a $p \in \mathcal{P}$ in $\tilde{\mathrm{T}}$-topology, then

$$p = \int k_\xi \mu(d\xi) .$$

PROOF. Let L be a continuous linear form on $[\mathcal{P}]$ of the form $L(p, p') = U^p f(x) - U^{p'} f(x)$, for a couple (f, x). Since $\xi \to k_\xi$ is a continuous map from X^* into K, and p_n is $\tilde{\mathrm{T}}$-convergent to p, we have

$$L(p) = \lim L(p_n) = \lim \int L(k_\xi) \mu_n(d\xi)$$

$$= \int L(k_\xi) \mu(d\xi) = L\Big(\int k_\xi \mu(d\xi)\Big) .$$

Therefore

$$p=\int k_\xi \mu(d\xi)$$

in $[\mathcal{P}]$. Like the proof of Theorem 6.13 we get the above equality as functions on X.

THEOREM 7.6. *Let $p\in\mathcal{P}$ and F be a subset of X. There is a Radon measure $\mu\geqq 0$ on X^* which is supported by the closure of F in X^* such that*

$$\hat{R}^F p=\int k_\xi \mu(d\xi).$$

PROOF. Let (K_n) be an exhaustion of X by compact sets. We have $\hat{R}^F p=\lim_{n\to\infty}\uparrow \hat{R}^{F\cap K_n}p$. From (6.6) $\hat{R}^{F\cap K_n}p$ are $\tilde{\mathrm{T}}$-convergent to $\hat{R}^F p$. The fact that $\hat{R}^{F\cap K_n}p\in\mathcal{P}\cap\tilde{\mathcal{H}}(X-K_n)$ and Theorem 5.9 imply the existence of a Radon measure μ_n on X such that

$$\hat{R}^{F\cap K_n}p=\int k_y \mu_n(dy).$$

Let γ be the constant of Theorem 7.2 (4), then we have; the total mass of $\mu_n\leqq\gamma\cdot D(p)$. Hence we can extract from the sequence μ_n, considered as measures on X^*, a subsequence $\mu_{n'}$ that converges vaguely to a Radon measure μ on X^*. In view of Lemma 7.5 we have

$$\hat{R}^F p=\int k_\xi \mu(d\xi).$$

μ_n is supported by $\overline{F\cap K_n}$ (Corollary 5.4), so μ is supported by the closure of F in X^*.

LEMMA 7.7. *Let G be an open set of X. Then, for every $x\in X$, the map $p\to R^G p(x)$ on $\mathcal{P}$ is lower semi-continuous.*

PROOF. Let a sequence $p_n\in\mathcal{P}$ be convergent to a $p\in\mathcal{P}$. We have $p=\widehat{\liminf}\, p_n$. Let $u_n=R^G p_n$ and $u=\liminf u_n$. Then $u_n\in\mathcal{P}$, $u_n=p_n$ on G $\hat{u}\in\mathcal{P}$. Since $u(y)=\liminf p_n(y)\geqq p(y)$ for all $y\in G$, it follows $\hat{u}\geqq p$ on G. Hence $u\geqq\hat{u}\geqq R^G p$ on X. This proves the lower semi-continuity of $p\to R^G p(x)$.

THEOREM 7.8. *Let G be an open set of X. Let $p\in\mathcal{P}$ and let ν be the canonical measure of p. Then we have*

$$R^G p=\int R^G k_\xi \nu(d\xi).$$

PROOF. The function $q\to R^G q(x)$ being affine lower semi-continuous on $\mathcal{K}_0$, we have,

$$R^G p(x)=\int_{\mathcal{E}\cap\mathcal{K}_0} R^G u(x)\nu'(du)$$

from (6.1)′, where ν' is the unique measure on $\mathcal{E}\cap\mathcal{K}_0$ that represents p. Therefore

$$R^G p(x)=\int R^G k_\xi(x)\nu(d\xi).$$

Shizuoka University Present address: Waseda University

References

[1] H. Bauer, Harmonische Räume und ihre Potentialtheorie, Lecture Notes in Mathematics 22, Springer, Berlin-Heidelberg-New York, 1966.

[2] N. Boboc, C. Constantinescu and A. Cornea, Axiomatic theory of harmonic functions—Nonnegative superharmonic functions, Ann. Inst. Fourier, **15**-1 (1965), 283–312.

[3] N. Boboc, C. Constantinescu and A. Cornea, Semigroups of transitions on harmonic spaces, Rev. Roumaine Math. Pures Appl., **12**-6 (1967), 763–805.

[4] N. Boboc et P. Mustată, Espaces harmoniques associés aux opérateurs différentiels linéaires du second ordre de type elliptique, Lecture Notes in Mathematics 68, Springer, Berlin-Heidelberg-New York, 1968.

[5] N. Bourbaki, Topologie générale, Hermann, Paris, 1964.

[6] N. Bourbaki, Espaces vectoriels topologiques, Hermann, Paris, 1964.

[7] C. Constantinescu und A. Cornea, Ideale Ränder Riemannscher Flächen, Ergebnisse der Mathematik und ihre Grenzgebiete 32, Springer, Berlin-Göttingen-Heidelberg, 1963.

[8] K. Gowrisankaran, Extreme harmonic functions and boundary value problems, Ann. Inst. Fourier, **13**-2 (1963), 307–356.

[9] W. Hansen, Charakterisierung von Familien exzessiver Funktionen, Invent. Math., **5** (1968), 335–348.

[10] R. M. Hervé, Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiels, Ann. Inst. Fourier, **12** (1962), 415–571.

[11] T. Kori, Axiomatic treatment of fullsuperharmonic functions and submarkov resolvents, Proc. Japan Acad., **44** (1968), 981–986.

[12] T. Kori, Fullsuperharmonic functions and excessive functions, USSR-Japan symposium on probability held at Xabarovsk (1969).

[13] Z. Kuramochi, Mass distributions on the ideal boundaries of abstract Riemann surfaces II, Osaka Math. J., **8** (1956), 145–186.

[14] P. A. Loeb and B. Walsh, The equivalence of Harnack's principle and Harnack's inequality in the axiomatic system of Brelot, Ann. Inst. Fourier, **15** (1965), 597–600.

[15] F. Y. Maeda, Axiomatic treatment of fullsuperharmonic functions, J. Sci. Hiroshima Univ., Ser. A-1, **30** (1966), 197–215.

[16] F. Y. Maeda, Kuramochi Boundaries of Riemann surfaces, Lecture Notes in Mathematics 58, Springer, Berlin-Heidelberg-New York, 1968.

[17] P. A. Meyer, Probabilités et potentiel, Hermann, Paris, 1966.

[18] P. A. Meyer, Brelot's axiomatic theory of the Dirichlet problem and Hunt's theory, Ann. Inst. Fourier, 13-2 (1963), 357–372.
[19] K. R. Parthasarathy, Probability measures on metric spaces, Academic press, New York-London, 1967.
[20] R. R. Phelps, Lectures on Choquet's theorem, Van Nostrand, Princeton, 1965.
[21] H. H. Schaefer, Topological vector spaces, Macmillan, New York, 1966.