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On radii of convexity and starlikeness of some classes of functions

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§ 1. Introduction.

Let S_β^* be the class of functions

$$F(z)=z+a_2z^2+\cdots, \tag{1.1}$$

which are regular, univalent and starlike of order β $(0\leqq\beta<1)$ in the unit circle E $(|z|<1)$. Let H_β^* be the class of functions

$$f(z)=\frac{1}{z}+b_0+b_1z+\cdots, \tag{1.2}$$

which are regular, univalent and starlike of order β $(0\leqq\beta<1)$ in $0<|z|<1$. Let V_β be the class of functions

$$g(z)=\frac{1}{2}[F(z)+zF'(z)], \tag{1.3}$$

where $F(z)\in S_\beta^*$. Let T be the class of functions

$$F(z)=z+a_2z^2+\cdots,$$

which are regular, univalent and starlike in $|z|<1$ and satisfy the condition

$$\left|\frac{zF'(z)}{F(z)}-\alpha\right|<\alpha, \qquad \left(\alpha>\frac{1}{2}\right) \qquad \text{for } |z|<1. \tag{1.4}$$

If P_β is the class of regular, analytic functions in E $(|z|<1)$ whose real part is not less than β and which take the value 1 at the origin, then it is easily seen that for every $u(z)\in P_\beta$ there exists a unique $p(z)\in P_0$ such that

$$u(z)=\beta+(1-\beta)p(z). \tag{1.5}$$

Hence, according as $h(z)\in S_\beta^*$ or H_β^* we have

$$1+\frac{zh''(z)}{h'(z)}=\pm[\beta+(1-\beta)p(z)]+\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}, \tag{1.6}$$

negative sign pertaining to the case where $h(z)\in H_\beta^*$. Thus, in order to find the radius of convexity for the class S_β^* or H_β^* one needs to find the extreme values of

$$\text{(1.7)} \qquad \operatorname{Re}\left[\pm(\beta+(1-\beta)p(z))+\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}\right]$$

for $p(z)\in P_0$ and $|z|=r$.

The following theorem due to Robertson [1] has been found extremely useful in obtaining explicit results for the extreme values of expressions of the form (1.7).

THEOREM. *If* $F(u, v)$ *is analytic in the* v*-plane and the halfplane* $\operatorname{Re} u>0$ *and if* $p(z)\in P_0$, *then on* $|z|=r$

$$\min_{p(z)\in P_0} \operatorname{Re}\,[F(p(z),\, zp'(z))]$$

is attained only for a function $p(z)=p_0(z)$ *of the form*

$$\text{(1.8)} \qquad p_0(z)=\frac{1+\alpha}{2}\cdot\frac{1+ze^{i\theta}}{1-ze^{i\theta}}+\frac{1-\alpha}{2}\cdot\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}},$$

where $-1\leqq\alpha\leqq 1,\ 0\leqq\theta\leqq 2\pi$.

The proof of the above theorem depends upon a variation formula obtained by Robertson [2] for the functions of the class P_0.

V. A. Zomorovic [3] obtained the radii of convexity for the classes S^*_β and H^*_β by applying the above theorem. A. Schild [4] and K. S. Padmanabhan [5] tried to find the radii of convexity for the classes S^*_β and H^*_β by a different method but were able to obtain only partial solutions.

Using the fact that every function $p(z)\in P_0$ can be expressed in the form

$$\text{(1.9)} \qquad p(z)=\frac{1-w(z)}{1+w(z)},$$

where $w(z)$ is a regular analytic function in E, and $w(0)=0$, $|w(z)|<1$, we develop an alternative technique for finding the extreme values of expressions of the form (1.7) which is independent of the variational techniques of Robertson [2] and Sakaguchi [6]. This technique seems to be more powerful than the variational techniques, for it can as well be used to find the extreme values of functionals of the form (1.7) where $p(z)$ belongs to a sub-class of P_0 or where $p(z)$ has a fixed second coefficient because in the latter case the representation of the functions giving the extreme values will not be as simple as (1.8).

§ 2.

We give below some elementary results which are useful in what follows.

Let B denote the class of regular, analytic functions $w(z)$ in $|z|<1$ which satisfy the conditions (i) $w(0)=0$ and (ii) $|w(z)|<1$ for $|z|<1$.

LEMMA 1. *If* $u(z)\in P_\beta$, *then for* $0\leqq\beta<1$,

$$u(z)=\frac{1+(2\beta-1)w(z)}{1+w(z)}, \tag{2.1}$$

where $w(z)\in B$.

The proof follows from the equations (1.5) and (1.9).

LEMMA 2. *If* $w(z)\in B$, *then for* $|z|<1$,

$$|zw'(z)-w(z)|\leqq\frac{|z|^2-|w(z)|^2}{1-|z|^2}. \tag{2.2}$$

PROOF. Let $\varphi(z)=w(z)/z$, $w(z)\in B$. Then $|\varphi(z)|<1$ and $\varphi(z)$ is regular in $|z|<1$. It is well known [8, p. 18] that $|\varphi'(z)|\leqq\frac{1-|\varphi(z)|^2}{1-|z|^2}$. Substituting for $\varphi(z)$ in terms of $w(z)$, we obtain (2.2).

It is easily seen that the equality in (2.2) holds for any function $w(z)$ for which $w(z)/z$ is a mapping of the unit circle on to itself.

LEMMA 3. *For* $w(z)\in B$, *we have*

$$\operatorname{Re}\left[\frac{zw'(z)}{(1+w(z))(1+nw(z))}\right]\geqq-\frac{1}{(1-n)^2}\operatorname{Re}\left[p(z)+\frac{n}{p(z)}-1-n\right]-\frac{1}{(1-n)^2}\cdot\frac{r^2|p(z)-n|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}, \tag{2.3}$$

$$\operatorname{Re}\left[\frac{zw'(z)}{(1+w(z))(1+nw(z))}\right]\leqq-\frac{1}{(1-n)^2}\operatorname{Re}\left[p(z)+\frac{n}{p(z)}-1-n\right]+\frac{1}{(1-n)^2}\cdot\frac{r^2|p(z)-n|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}, \tag{2.4}$$

where $p(z)=(1+nw(z))/(1+w(z))$, $r=|z|$ *and* $-1\leqq n<1$.

PROOF. (2.3) and (2.4) follow immediately from (2.2).

REMARK 1. The transformation $p(z)=(1+nw(z))/(1+w(z))$ maps the circle $|w(z)|\leqq r$ on to the circle $\left|p(z)-\frac{1-nr^2}{1-r^2}\right|\leqq\frac{(1-n)r}{1-r^2}$.

LEMMA 4. *Let* $p(z)=\frac{1+nw(z)}{1+w(z)}$, $a=\frac{1-nr^2}{1-r^2}$, $d=\frac{(1-n)r}{1-r^2}$, *then for* $|z|=r$, $0\leqq r<1$, *we have*

$$\operatorname{Re}\left[kp(z)+\frac{n}{p(z)}\right]+\frac{r^2|p(z)-n|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\leqq\frac{k+n-2n(1+k)r+n(1+nk)r^2}{(1-r)(1-nr)} \tag{2.5}$$

for $k\geqq1$, $1>n\geqq0$, *or for* $k=2-n$ *and* $-1\leqq n\leqq0$;

$$-\operatorname{Re}\left[np(z)+\frac{n}{p(z)}\right]+\frac{r^2|p(z)-n|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}\leqq-2(1+n)\sqrt{a}+2a \quad \textit{for } -1\leqq n<1; \tag{2.6}$$

and

$$\text{(2.7)}\qquad \operatorname{Re}\left[kp(z)+\frac{n}{p(z)}\right]-\frac{r^2|p(z)-n|^2-|1-p(z)|^2}{(1-r^2)|p(z)|}$$
$$\geqq\begin{cases} 2\sqrt{(1+k)(1+n)a}-2a & \text{for } R_0\geqq R_1\\ \dfrac{k+n+2n(1+k)r+n(1+nk)r^2}{(1+r)(1+nr)} & \text{for } R_0\leqq R_1,\end{cases}$$

where $R_0^2=\dfrac{(1+n)a}{1+k}$, $R_1=a-d$, $k\geqq 1$, $-1\leqq n<1$.

PROOF. Put $p(z)=\mathrm{Re}^{i\theta}$ and denote the left hand side of (2.5) by $S_1(R,\theta)$. Then

$$\text{(2.8)}\qquad S_1(R,\theta)=\left(kR+\frac{n}{R}+2a\right)\cos\theta-R-\frac{a^2-d^2}{R}.$$

$$\text{(2.9)}\qquad \frac{\partial S_1}{\partial\theta}=-\sin\theta\cdot T(R)$$

where $T(R)=kR+\dfrac{n}{R}+2a$.

For $n\geqq 0$, $T(R)$ is evidently positive and therefore the maximum of $S_1(R,\theta)$ inside the circle $|p(z)-a|\leqq d$ is attained on the diameter $\theta=0$. But for $k=2-n$, $-1\leqq n<0$, $T(R)$ may change sign. That it remains positive for $a-d\leqq R\leqq a+d$ can, in view of the fact that $T(R)$ is monotone increasing function of R, be determined from its sign at $R=a-d$. We have

$$\text{(2.10)}\qquad T(a-d)=\frac{(2-n)(1+nr)}{1+r}+\frac{n(1+r)}{1+nr}+\frac{2(1-nr^2)}{1-r^2}$$
$$=\frac{4(1-r)^3+(1+n)r[10(1-r)^2+(1+n)(6-n)r(1-r)-2n+2r]}{(1+nr)(1-r^2)}$$
$$>0\quad\text{for } -1\leqq n<0.$$

Hence the maximum of $S_1(R,\theta)$ for all values of n $(-1\leqq n<1)$ inside the circle $|p(z)-a|\leqq d$ is attained at $\theta=0$. By putting $\theta=0$ in (2.8) we obtain

$$\text{(2.11)}\qquad S_1(R,0)=2a+(k-1)R-\frac{a^2-d^2-n}{R},$$

where $a-d\leqq R\leqq a+d$.

Since $k\geqq 1$ and

$$\text{(2.12)}\qquad a^2-d^2-n=\frac{(1-n)(1+nr^2)}{1-r^2}>0,\qquad -1\leqq n<1,$$

it is evident that $S_1(R,0)$ is a monotone increasing function of R for $-1\leqq n<1$. Hence its maximum is attained at $R=a+d$ and equals

$$\text{(2.13)}\qquad \frac{k+n-2n(1+k)r+n(1+nk)r^2}{(1-r)(1-nr)}.$$

In order to prove (2.6) and (2.7) let us put $p(z)=a+u+iv$, $R^2=(a+u)^2+v^2$ and denote their respective left hand sides by $S_2(u,v)$ and $S_3(u,v)$. Then we have

(2.14) $$S_2(u, v) = -n(a+u) - n(a+u)R^{-2} + (d^2-u^2-v^2)R^{-1},$$

(2.15) $$S_3(u, v) = k(a+u) + n(a+u)R^{-2} - (d^2-u^2-v^2)R^{-1},$$

and

$$\frac{\partial S_2}{\partial v} = -\frac{\partial S_3}{\partial v} = -vR^{-4}T_1(u, v),$$

$$\begin{aligned} T_1(u, v) &= -2n(a+u) + (d^2-u^2-v^2)R + 2R^3 \\ &\geqq -2n(a+u) + 2R^3 \geqq 2(a+u)\{-n+(a-d)^2\} \\ &= 2(a+u)(1-n)(1-nr^2)/(1+r)^2 \\ &> 0. \end{aligned}$$

Hence the maximum of $S_2(u, v)$ and the minimum of $S_3(u, v)$ inside the circle $|p(z)-a| \leqq d$ are attained on the diameter $v=0$. On putting $v=0$ in (2.14) we obtain

(2.16) $$L_2(R) \equiv S_2(u, 0) = -(n+1)R - (n+1)aR^{-1} + 2a,$$

where $R = a+u$ and $a-d \leqq R \leqq a+d$.

The absolute maximum of $L_2(R)$ in $(0, \infty)$ is attained at $R = \sqrt{a}$ and equals

$$-2(1+n)\sqrt{a} + 2a$$

which proves (2.6).

Again on putting $v=0$ in (2.15) we obtain

(2.17) $$L_3(R) \equiv S_3(u, 0) = (k+1)R + (n+1)aR^{-1} - 2a,$$

where $R = a+u$ and $a-d \leqq R \leqq a+d$.

The absolute minimum of $L_3(R)$ in $(0, \infty)$ is attained at

(2.18) $$R_0 = \sqrt{\frac{(1+n)a}{1+k}}$$

and equals

(2.19) $$L_3(R_0) = 2\sqrt{(1+k)(1+n)a} - 2a.$$

It is easy to see that $R_0 < a+d$, but R_0 is not always greater than $a-d$. In such a case when $R_0 \notin [a-d, a+d]$ the minimum of $L_3(R)$ on the segment $[a-d, a+d]$ is attained at $R_1 = a-d$ and equals

(2.20) $$L_3(R_1) = L_3(a-d) = \frac{k+n+2n(1+k)r+n(1+kn)r^2}{(1+r)(1+nr)}.$$

$L_3(R_0) = L_3(R_1)$ for such values of k, n for which $R_0 = R_1$. The inequality (2.7) follows from (2.19) and (2.20).

§ 3. Inequalities for the class V_β.

THEOREM 3.1. *If $g(z)\in V_\beta$, then for $|z|=r$, $0\leqq r<1$,*

$$\text{(3.1)}\qquad \operatorname{Re}\left[\frac{zg'(z)}{g(z)}\right]\leqq\frac{1+(2-4\beta)r+\beta(2\beta-1)r^2}{(1-r)(1-\beta r)},$$

$$\text{(3.2)}\qquad \operatorname{Re}\left[\frac{zg'(z)}{g(z)}\right]\geqq\begin{cases}\dfrac{2}{1-\beta}\cdot\{\sqrt{(1+\beta)(4-2\beta)a}-1-a\} & \text{for } R_0\geqq R_1,\\[2ex] \dfrac{1-(2-4\beta)r+\beta(2\beta-1)r^2}{(1+r)(1+\beta r)} & \text{for } R_0\leqq R_1,\end{cases}$$

where $R_0^2=\dfrac{(1+\beta)a}{4-2\beta}$, $R_1=\dfrac{1+\beta r}{1+r}$, $a=\dfrac{1-\beta r^2}{1-r^2}$. *The above bounds are sharp.*

PROOF. Differentiating (1.3) and using Lemma 1, we obtain

$$\text{(3.3)}\qquad \frac{zg'(z)}{g(z)}=\frac{1+(2\beta-1)w(z)}{1+w(z)}-\frac{(1-\beta)zw'(z)}{(1+w(z))(1+\beta w(z))}.$$

If we put $n=\beta$ in (2.3) and (2.4), then (3.1) and (3.2) follow from (3.3) using respectively, (2.5) and (2.7) and taking $k=3-2\beta$ and $n=\beta$.

The equality sign in (3.1) is attained for the function $F(z)=z(1-z)^{2\beta-2}$.

The signs of equality in (3.2) are attained respectively for the second and first inequality for the functions given by the following equations

$$\text{(3.4)}\qquad F(z)=z(1-z)^{2\beta-2},$$

$$\text{(3.5)}\qquad \frac{zF'(z)}{F(z)}=\frac{1-2\beta\cos\theta\cdot z+(2\beta-1)z^2}{1-2\cos\theta\, z+z^2},$$

where $\cos\theta$ is determined from

$$\text{(3.6)}\qquad \frac{1-(1+\beta)\cos\theta\cdot r+\beta r^2}{1-2\cos\theta\cdot r+r^2}=R_0.$$

THEOREM 3.2. *Let $g(z)\in V_\beta$. Let β_0 denote the smallest positive root of the equation $4x^3-4x^2-10x+1=0$. Then*

(i) *for $0\leqq\beta\leqq\beta_0$, $g(z)$ is starlike in*

$$|z|<[\sqrt{(1-\beta)(1-2\beta)}+1-2\beta]^{-1},$$

(ii) *for $\beta_0\leqq\beta<1$, $g(z)$ is starlike in*

$$|z|<\left[\frac{2\beta}{\sqrt{\beta(2-\beta)(1-\beta^2)}+\beta(1+\beta)}\right]^{\frac{1}{2}}.$$

These bounds are sharp.

PROOF. By (3.2) the radii of starlikeness for the class V_β are determined from the following equations

$$\text{(3.7)}\qquad \beta(2\beta-1)r^2-(2-4\beta)r+1=0,$$

(3.8) $$\sqrt{(1+\beta)(4-2\beta)a}-1-a=0\,.$$

Equation (3.8) reduces to

(3.9) $$(1-2\beta)(1+\beta)r^4+2\beta(1+\beta)r^2-2\beta=0\,.$$

Also the two minima given by (3.2) become equal to each other for such a β $(0\leqq\beta<1)$ for which

(3.10) $$R_0=R_1\,.$$

As we are interested in those real roots of (3.7) and (3.9) for which $0<r<1$, it is easily seen that the radii of starlikeness for the class V_β are given by

(3.11) $$r=[\sqrt{(1-\beta)(1-2\beta)}+1-2\beta]^{-1}\,,$$

(3.12) $$r=\Big[\frac{2\beta}{\sqrt{\beta(2-\beta)(1-\beta^2)}+\beta(1+\beta)}\Big]^{\frac{1}{2}}\,.$$

For some value of β, the two values of r given by (3.11) and (3.12) may become equal. Such values of β will be obtained by eliminating r from (3.7) and (3.10) and are roots of the equation

(3.13) $$4\beta^3-4\beta^2-10\beta+1=0\,.$$

The roots of (3.13) are situated in the intervals $(-2,-1)$, $(1/12,1/10)$, $(1,3)$. It is evident that β_0 in the theorem is the smallest positive root of (3.13).

Functions given by equations (3.4) and (3.5) show that the bounds are sharp.

REMARK 2. For $\beta=0$, (3.11) gives $r=1/2$, which is a result obtained earlier by Livingston [7].

§ 4. Inequalities for the class S_β^*.

THEOREM 4.1. *If* $F(z)\in S_\beta^*$, *then for* $|z|=r$, $0\leqq r<1$,

(4.1) $$\mathrm{Re}\Big[1+\frac{zF''(z)}{F'(z)}\Big]\leqq\frac{1+(4-6\beta)r+(1-2\beta)^2r^2}{(1-r)(1+(1-2\beta)r)}\,,$$

(4.2) $$\mathrm{Re}\Big[1+\frac{zF''(z)}{F'(z)}\Big]\geqq\begin{cases}\dfrac{1}{1-\beta}(2\sqrt{\beta(2-\beta)a}-\beta-a) & \textit{for } R_0\geqq R_1\,,\\[2ex] \dfrac{1-(4-6\beta)r+(2\beta-1)^2r^2}{(1+r)(1+(2\beta-1)r)} & \textit{for } R_0\leqq R_1\,,\end{cases}$$

where $R_0^2=\dfrac{\beta a}{2-\beta}$, $R_1=\dfrac{1+(2\beta-1)r}{1+r}$, $a=\dfrac{1-(2\beta-1)r^2}{1-r^2}$.

PROOF. Since $F(z)\in S_\beta^*$, Lemma 1 gives

(4.3) $$\frac{zF'(z)}{F(z)}=\frac{1+(2\beta-1)w(z)}{1+w(z)}\,,\qquad w(z)\in B\,.$$

Taking logarithmic derivative of (4.3) we have

$$(4.4)\qquad 1+\frac{zF''(z)}{F'(z)}=\frac{1+(2\beta-1)w(z)}{1+w(z)}-\frac{2(1-\beta)zw'(z)}{(1+w(z))(1+(2\beta-1)w(z))}\,.$$

If we put $n=2\beta-1$ in (2.3) and (2.4), then (4.1) and (4.2) follow from (4.4), respectively, using (2.5) and (2.7) by taking $K=3-2\beta$ and $n=2\beta-1$. Equality in (4.1) is attained for the function $F(z)=z(1-z)^{2\beta-2}$. Sharp results in (4.2) are obtained respectively for the second and first inequality for the functions given by the following equations.

$$(4.5)\qquad F(z)=z(1-z)^{2\beta-2}\,,$$

$$(4.6)\qquad F(z)=z(1-2\cos\theta\cdot z+z^2)^{-1+\beta}\,,$$

where $\cos\theta$ is determined from

$$(4.7)\qquad \frac{1-2r\beta\cos\theta+(2\beta-1)r^2}{1-2r\cos\theta+r^2}=R_0\,.$$

THEOREM 4.2. *Let $F(z)\in S_\beta^*$ and let β_0 denote the smallest positive root of the equation $20x^4-52x^3+15x^2+12x-4=0$. Then*

(i) *for $0\leqq\beta\leqq\beta_0$, $F(z)$ is convex in*

$$|z|<[2-3\beta+\sqrt{(1-\beta)(3-5\beta)}]^{-1}\,,$$

(ii) *for $\beta_0\leqq\beta<1$, $F(z)$ is convex in*

$$|z|<\left[\frac{5\beta-1}{4\beta^2-\beta+1+4\beta\sqrt{\beta^2-3\beta+2}}\right]^{\frac{1}{2}}.$$

All these bounds are sharp.

PROOF. By (4.2) the radii of convexity for the class S_β^* are determined from the following equations

$$(4.7)\qquad (2\beta-1)^2r^2-(4-6\beta)r+1=0\,,$$

$$(4.8)\qquad 2\sqrt{\beta(2-\beta)a}-\beta-a=0\,.$$

(4.8) reduces to

$$(4.9)\qquad (8\beta^2-3\beta-1)r^4-(8\beta^2-2\beta+2)r^2+5\beta-1=0\,.$$

By an argument similar to that given in the proof of Theorem (3.2) we find that the radii of convexity for the class S_β^* are given by

$$(4.10)\qquad r=[2-3\beta+\sqrt{(1-\beta)(3-5\beta)}]^{-1}\qquad \text{for } 0\leqq\beta\leqq\beta_0\,,$$

$$(4.11)\qquad r=\left[\frac{5\beta-1}{4\beta^2-\beta+1+4\beta\sqrt{\beta^2-3\beta+2}}\right]^{\frac{1}{2}}\qquad \text{for } \beta_0\leqq\beta<1\,,$$

where β_0 is the smallest positive root of the equation $20\beta^4-52\beta^3+15\beta^2+12\beta-4=0$.

Functions given by (4.5) and (4.6) show that the results obtained are sharp.

REMARK 3. Since the extremal functions given by (4.5) and (4.6) are typically real starlike functions of order β, they also solve a corresponding problem for typically real starlike functions of order β.

§ 5. Inequalities for the class H_β^*.

THEOREM 5.1. *If* $f(z)\in H_\beta^*$, *then for* $|z|=r$, $0\leqq r<1$,

$$-\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]\leqq\frac{a-2\beta\sqrt{a}+\beta}{1-\beta}, \tag{5.1}$$

$$-\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]\geqq\begin{cases}2\sqrt{\dfrac{1-2\beta+\beta a}{1-\beta}}+\dfrac{\beta-a}{1-\beta} & \text{for } R_0\leqq R_2,\\[2ex] \dfrac{1-2\beta r+(2\beta-1)^2r^2}{(1-r)(1-(2\beta-1)r)} & \text{for } R_0\geqq R_2,\end{cases} \tag{5.2}$$

where $R_0^2=\dfrac{1-2\beta+\beta a}{1-\beta}$, $R_2=\dfrac{1-(2\beta-1)r}{1-r}$, $a=\dfrac{1-(2\beta-1)r^2}{1-r^2}$.

PROOF. Since $f(z)\in H_\beta^*$, Lemma 1 gives

$$-\frac{zf'(z)}{f(z)}=\frac{1+(2\beta-1)w(z)}{1+w(z)},\qquad w(z)\in B. \tag{5.3}$$

Taking logarithmic derivative of (5.3) we get

$$-\left[1+\frac{zf''(z)}{f'(z)}\right]=\frac{1+(2\beta-1)w(z)}{1+w(z)}+\frac{2(1-\beta)zw'(z)}{(1+w(z))(1+(2\beta-1)w(z))}. \tag{5.4}$$

If we put $n=2\beta-1$ in (2.4), then (5.1) follows from (5.4) in conjunction with (2.6) by taking $n=2\beta-1$. To see that the result is sharp, consider the function $f(z)$ defined by

$$-\frac{zf'(z)}{f(z)}=\frac{1-2\beta\cos\theta\cdot z+(2\beta-1)z^2}{1-2\cos\theta\cdot z+z^2}, \tag{5.5}$$

where $\cos\theta$ is determined from

$$\frac{1-2r\beta\cos\theta+(2\beta-1)r^2}{1-2r\cos\theta+r^2}=\left[\frac{1-(2\beta-1)r^2}{1-r^2}\right]^{\frac{1}{2}}. \tag{5.6}$$

We shall now prove inequality (5.2).

(5.4) yields in conjunction with (2.3) for $n=2\beta-1$

$$-\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]\geqq\frac{\beta}{1-\beta}+\frac{1-2\beta}{2(1-\beta)}\cdot\mathrm{Re}\left[p(z)+\frac{1}{p(z)}\right]-\frac{r^2|p(z)-(2\beta-1)|^2-|1-p(z)|^2}{2(1-\beta)(1-r^2)|p(z)|}. \tag{5.7}$$

Putting in (5.7), $p(z)=\mathrm{R}e^{i\theta}$, we get

$$(5.8)\qquad -\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]$$

$$\geqq \frac{\beta}{1-\beta}+\frac{1-2\beta}{2(1-\beta)}\cdot\left(R+\frac{1}{R}\right)\cos\theta-\frac{d^2-a^2-R^2+2Ra\cos\theta}{2(1-\beta)R}\,.$$

where $a=\dfrac{1-(2\beta-1)r^2}{1-r^2}$, $d=\dfrac{2(1-\beta)r}{1-r^2}$. Let

$$(5.9)\qquad M(R,\theta)=\frac{\beta}{1-\beta}+\frac{1-2\beta}{2(1-\beta)}\cdot\left(R+\frac{1}{R}\right)\cos\theta-\frac{d^2-a^2-R^2+2Ra\cos\theta}{2(1-\beta)R}\,.$$

Then

$$(5.10)\qquad \frac{\partial M}{\partial\theta}=\frac{1}{2(1-\beta)}\cdot N(R)\sin\theta\,,$$

where $N(R)=(2\beta-1)(R+1/R)+2a$. For $\beta\geqq 0$ we have $N(R)\geqq -(R+1/R)+2a$. The minimum of $-(R+1/R)+2a$, $a-d\leqq R\leqq a+d$, is attained at either $R=a-d$ or $R=a+d$. As

$$(5.11)\qquad a^2-d^2-1=\frac{r^2(1-(2\beta-1)^2)}{1-r^2}\geqq 0\,,$$

we see that the values of $-(R+1/R)+2a$ at $R=a-d$ and $R=a+d$ are both non-negative. Therefore $N\geqq 0$ for any β such that $0\leqq\beta<1$ and the minimum of $M(R,\theta)$ is attained at $\theta=0$. We get from (5.9)

$$(5.12)\qquad M(R,0)=R+\frac{1-2\beta+\beta a}{(1-\beta)R}+\frac{\beta-a}{1-\beta}\,,$$

where $a-d\leqq R\leqq a+d$. The absolute minimum of $M(R,0)$ in $(0,\infty)$ is attained at $R_0=\left(\dfrac{1-2\beta+\beta a}{1-\beta}\right)^{\frac{1}{2}}$ and equals

$$(5.13)\qquad M(R_0,0)=2\left(\frac{1-2\beta+\beta a}{1-\beta}\right)^{\frac{1}{2}}+\frac{\beta-a}{1-\beta}\,.$$

It is easy to see that $R_0>a-d$ but R_0 is not always less than $a+d$. In such a case the minimum of $M(R,0)$ on the segment $[a-d, a+d]$ is attained at $R_2=a+d$ and equals

$$(5.14)\qquad M(R_2,0)=\frac{1-2\beta r+(2\beta-1)^2r^2}{(1-r)(1-(2\beta-1)r)}\,.$$

These two minima coincide for such a β $(0\leqq\beta<1)$ for which

$$(5.15)\qquad R_0=R_2\,.$$

The inequality (5.2) follows from (5.13) and (5.14).

The equality signs in (5.2) are attained respectively for the second and first inequality for the functions defined by the following equations

$$(5.16)\qquad f(z)=z^{-1}\cdot(1-z)^{2-2\beta}\,,$$

$$(5.17)\qquad f(z)=\frac{1}{z}\cdot[(1-z)^{1+m}\cdot(1+z)^{1-m}]^{1-\beta}\,,$$

where m is determined from

$$R_0 = a + md. \tag{5.18}$$

THEOREM 5.2. *Let $f(z) \in H_\beta^*$. Let x_0 denote the unique positive root of the equation $x^4 - 4x^3 + 2x^2 - 8 = 0$ and $\beta_0 = (x_0^2 - 4)/(4x_0 - 4)$. Then*

(i) *for $0 \leqq \beta \leqq \beta_0$, $f(z)$ is convex in*

$$0 < |z| < \left[\frac{4\beta - 5 + 4\sqrt{\beta^2 - \beta + 1}}{8\beta - 3}\right]^{\frac{1}{2}},$$

(ii) *for $\beta_0 \leqq \beta < 1$, $f(z)$ is convex in*

$$0 < |z| < [\beta + \sqrt{(1-\beta)(3\beta - 1)}\,]^{-1}.$$

All these bounds are sharp.

PROOF. By (5.2) the radii of convexity for the class H_β^* can be determined from the following equations

$$(2\beta - 1)^2 r^2 - 2\beta r + 1 = 0, \tag{5.19}$$

$$2\sqrt{\frac{1 - 2\beta + \beta a}{1 - \beta}} + \frac{\beta - a}{1 - \beta} = 0. \tag{5.20}$$

Equation (5.20) reduces to

$$(8\beta - 3) r^4 - 2(4\beta - 5) r^2 - 3 = 0. \tag{5.21}$$

As we are interested in those real roots of (5.19) and (5.21) for which $0 < r < 1$, hence the radii of convexity for the class H_β^* are given by

$$r = [\beta + \sqrt{(1-\beta)(3\beta - 1)}\,]^{-1}, \tag{5.22}$$

$$r = \left[\frac{4\beta - 5 + 4\sqrt{\beta^2 - \beta + 1}}{8\beta - 3}\right]^{\frac{1}{2}}. \tag{5.23}$$

The formula (5.22) cannot be applied for $0 \leqq \beta \leqq \frac{1}{2}$ because for $0 \leqq \beta < \frac{1}{3}$ it gives imaginary values and for $\frac{1}{3} \leqq \beta \leqq \frac{1}{2}$, it gives $r > 1$. The formula (5.23) is, however, true for $0 \leqq \beta \leqq \frac{1}{2}$. The value of β for which (5.22) and (5.23) give equal values of r must be in $\left(\frac{1}{2}, 1\right)$. Such values of β are obtained by eliminating r from (5.22) and (5.23). Finally we get the following result, if $\beta_0 = (x_0^2 - 4)/(4x_0 - 4)$, where x_0 is the unique positive root of $x^4 - 4x^3 + 2x^2 - 8 = 0$, then for $0 \leqq \beta \leqq \beta_0$, we use the formula (5.23) and for $\beta_0 \leqq \beta < 1$, we use (5.22).

Functions given by (5.16) and (5.17) show that the bounds are sharp.

REMARK 4. Since the extremal functions given by (5.16) and (5.17) are typically real starlike functions of order β, they also solve a corresponding problem for typically real starlike functions of order β.

§ 6. Inequalities for the class T.

As an illustration of the use of this method we prove

THEOREM 6. *If* $F(z)\in T$, *then for* $|z|=r$, $0\leqq r<1$,

$$\text{(6.1)}\qquad \operatorname{Re}\left[1+\frac{zF''(z)}{F'(z)}\right]\geqq\frac{1-(3-A)r+r^2}{(1-r)(1-Ar)},$$

$$\text{(6.2)}\qquad \operatorname{Re}\left[1+\frac{zF''(z)}{F'(z)}\right]\leqq\begin{cases}\dfrac{1+(3-A)r+r^2}{(1+r)(1+Ar)} & \text{for } R_0\leqq R_1,\\[2ex] \dfrac{1}{(1-A)}\cdot[1+A+2a-2\sqrt{2(a^2-d^2+2A-1)}] & \text{for } R_0\geqq R_1,\end{cases}$$

where

$$A=\frac{1}{\alpha}-1,\ a=\frac{1-Ar^2}{1-r^2},\ d=\frac{(1-A)r}{1-r^2},\ R_1=\frac{1+Ar}{1+r}\ \text{and}\ R_0^2=\frac{a^2-d^2+2A-1}{2}.$$

All these bounds are sharp.

PROOF. In the present case it is easily seen that

$$\text{(6.3)}\qquad \frac{zF'(z)}{F(z)}=\frac{1+w(z)}{1+Aw(z)},\qquad w(z)\in B.$$

Differentiating logarithmically (6.3) we get

$$\text{(6.4)}\qquad 1+\frac{zF''(z)}{F'(z)}=\frac{1+w(z)}{1+Aw(z)}+\frac{(1-A)zw'(z)}{(1+w(z))(1+Aw(z))}.$$

We shall prove (6.2) because the proof of (6.1) is similar to it and moreover it has been done by Mr. Ram Singh (to be published).

In view of (2.4) by taking $n=A$, (6.4) yields

$$\text{(6.5)}\qquad \operatorname{Re}\left[1+\frac{zF''(z)}{F'(z)}\right]\leqq\frac{1}{1-A}\left[1+A-\operatorname{Re}\left(p(z)+\frac{2A-1}{p(z)}\right)\right]+\frac{1}{1-A}\cdot\frac{r^2|p(z)-A|^2-|1-p(z)|^2}{(1-r^2)|p(z)|},$$

where $p(z)=(1+Aw(z))/(1+w(z))$ and $|p(z)-a|\leqq d$.

Hereafter the argument used here is similar to the one given in the proof of (2.7). Putting $p(z)=a+u+iv$, $R^2=(a+u)^2+v^2$, and denoting the right hand side of (6.5) by $S_4(u, v)$ we get

$$\text{(6.6)}\qquad S_4(u,v)=\frac{1}{1-A}\cdot\left[1+A-(a+u)+\frac{(1-2A)(a+u)}{R^2}+\frac{d^2-u^2-v^2}{R}\right].$$

By analysis similar to the previous we find that the maximum of $S_4(u, v)$ on each chord $u=$ constant inside the circle $|p(z)-a|\leqq d$ is attained on the diameter $v=0$. We get from (6.6)

(6.7) $$L_4(R) \equiv S_4(u, 0) = \frac{1}{1-A} \cdot \left[1+A+2a-2R+\frac{(1-2A+d^2-a^2)}{R}\right]$$

where $R=a+u$ and $(a-d) \leqq R \leqq (a+d)$. The absolute maximum of $L_4(R)$ in $(0, \infty)$ is attained at $R_0^2=(a^2-d^2+2A-1)/2$ and equals

(6.8) $$L_4(R) = \frac{1}{1-A}\left[1+A+2a-2\sqrt{2(a^2-d^2+2A-1)}\right].$$

It is easy to see that $R_0 < a+d$ but R_0 is not always greater than $a-d$. In such a case the minimum of $L_4(R)$ on the segment $[a-d, a+d]$ is attained at $R=R_1=a-d$ and equals

(6.9) $$L_4(R_1) = \frac{1+(3-A)r+r^2}{(1+r)(1+Ar)}.$$

The two maxima coincide where $R_0=R_1$. The inequality (6.2) follows from (6.8) and (6.9).

The equality signs in (6.2) are attained respectively for the first and second inequality for the functions defined by the following equations

(6.10) $$F(z) = z(1+Az)^{(1-A)/A} \qquad \text{for } R_0 \leqq R_1,$$

(6.11) $$\frac{zF'(z)}{F(z)} = \frac{1-2z\cos\theta+z^2}{1-(1+A)\cos\theta\cdot z+Az^2} \qquad \text{for } R_0 \geqq R_1,$$

where $\cos\theta$ is given by

(6.12) $$R_0 = \frac{1-(1+A)r\cos\theta+Ar^2}{1-2r\cos\theta+r^2}.$$

COROLLARY. *Each function $F(z)$ in T maps $|z| < \frac{3-A-\sqrt{5-6A+A^2}}{2}$ onto a convex domain.*

§ 7. Radius of starlikeness for the class P.

As another illustration of the use of our method we determine the radius of starlikeness for the class P of functions $p(z)=1-2bz+\cdots$, belonging to P_0 with the additional condition that b is fixed.

Since $p(ze^{i\theta})$ belongs to P whenever $p(z) \in P$, we may without loss of generality assume $b>0$.

THEOREM 7. *If $p(z) \in P$ and $|p'(0)|=2b>0$, then for $0 \leqq |z| < b$,*

$$\operatorname{Re}\left[\frac{zp'(z)}{p(z)-1}\right] \geqq \frac{b-2r+br^2}{(b-r)(1-r^2)}, \qquad |z|=r.$$

The result is sharp.

PROOF. We note that no due restriction is involved by taking $0 \leqq |z| < b$. Indeed, if $p(z)$ is expressed in the form

(7.1) $$p(z) = \frac{1-w(z)}{1+w(z)},$$

where $w(z)\in B$ and $|w'(0)|=b$, then it is well known [9, p. 171] that $w(z)$ is univalent in $|z|\leqq b/(1+\sqrt{1-b^2})$ which is always less than b.

Using (7.1) we obtain

$$\frac{zp'(z)}{p(z)-1}=\frac{zw'(z)}{w(z)(1+w(z))}\,. \tag{7.2}$$

Such functions $w(z)$ are known [9, p. 167] to satisfy the inequalities

$$\frac{r(b-r)}{1-br}\leqq|w(z)|\leqq\frac{r(b+r)}{1+br}\,. \tag{7.3}$$

On using Lemma 2, we obtain from (7.2)

$$\mathrm{Re}\left[\frac{zp'(z)}{p(z)-1}\right]\geqq\mathrm{Re}\left[\frac{1}{1+w(z)}\right]-\frac{r^2-|w(z)|^2}{(1-r^2)|w(z)|\,|1+w(z)|}\,, \tag{7.4}$$

which on substituting

$$w_1(z)=\frac{w(z)}{1+w(z)} \tag{7.5}$$

reduces to

$$\mathrm{Re}\left[\frac{zp'(z)}{p(z)-1}\right]\geqq 1-\mathrm{Re}\,w_1(z)-\frac{r^2|1-w_1(z)|^2-|w_1(z)|^2}{(1-r^2)|w_1(z)|}\,. \tag{7.6}$$

In view of the second inequality in (7.3) it is easily seen that

$$\left|w_1(z)+\frac{r^2}{1-r^2}\right|\leqq\frac{r}{1-r^2}\,. \tag{7.7}$$

However, because of the first inequality in (7.3) the values in the interior of the circle given by (7.7) are not all taken.

Putting $w_1(z)=u+iv$, $R^2=u^2+v^2$ and then denoting the right hand side of (7.6) by $S_5(u, v)$ we get

$$S_5(u, v)=1-u+R+\frac{(2u-1)r^2}{(1-r^2)R}\,. \tag{7.8}$$

Since (7.8) is symmetric with respect to v it is enough to confine ourselves to the case $v\geqq 0$. We note first that $S_5(u, v)$ is, for a fixed u, a monotone increasing function of v for $v\geqq 0$,

$$\frac{\partial S_5}{\partial v}=vR^{-3}T_5(u, v)\,,$$

where

$$T_5(u, v)=R^2-\frac{(2u-1)r^2}{1-r^2}\,.$$

Since $u=\mathrm{Re}\,w_1(z)\leqq\frac{r}{1+r}$, it follows that $2u-1\leqq\frac{r-1}{r+1}<0$ and hence $T_5(u, v)>0$ $(r\neq 0)$. Thus the minimum of $S_5(u, v)$ on each chord $u=$ constant is either attained on the real axis within the circle (7.7) or on the image of $|w(z)|=r(b-r)/(1-br)$ under the transformation (7.5). In the latter case putting $w(z)=R'e^{i\theta}$, $R'=r(b-r)/(1-br)$ in the right hand side of (7.4) we have

$$\text{(7.9)}\quad \operatorname{Re}\Big[\frac{zp'(z)}{p(z)-1}\Big] \geqq \frac{1+R'\cos\theta}{1+R'^2+2R'\cos\theta} - \frac{r^2-R'^2}{R'(1-r^2)} \cdot \frac{1}{\sqrt{1+R'^2+2R'\cos\theta}}\,.$$

If $G(\theta)$ denotes the right hand side of (7.9) we get

$$\text{(7.10)}\quad \frac{dG}{d\theta} = \frac{R'\sin\theta}{(1+R'^2+2R'\cos\theta)^{3/2}} \cdot \Big[\frac{1-R'^2}{\sqrt{1+R'^2+2R'\cos\theta}} - \frac{r^2-R'^2}{R'(1-r^2)}\Big].$$

Thus the minimum of $G(\theta)$ is attained either at $\sin\theta=0$, that is, the points of intersection of the circle with the real axis, or at the points where

$$\text{(7.11)}\quad \cos\theta = \frac{1}{2R'} \cdot \Big[\frac{R'^2(1-r^2)^2 \cdot (1-R'^2)^2}{(r^2-R'^2)^2} - 1 - R'^2\Big].$$

Since R' vanishes at $r=0$ and $r=b$, the right hand side of (7.11) does not lie between -1 and $+1$. The value of $\cos\theta$ lies in $0 \leqq \cos\theta < 1$, provided

$$\text{(7.12)}\quad K(r) \equiv (b-r)^2(1-r^2)^2(1-2br+r^2)^2 - r^2(1-b^2)^2[(1-br)^2 + r^2(b-r)^2] \geqq 0$$

and

$$\text{(7.13)}\quad K(r) - 2r^3(1-br)(b-r)(1-b^2)^2 < 0\,.$$

On simplification condition (7.13) reduces to

$$\text{(7.14)}\quad [(b-r)(1-2br+r^2) + r(1-b^2)][(b-r)(1-2br+r^2) - r(1-b^2)] < 0\,.$$

Thus we find that $0 \leqq \cos\theta < 1$ for $r_1 < r \leqq r_3$ where r_3 and r_1 denote respectively the smallest positive roots of $K(r)=0$ and

$$\text{(7.15)}\quad f_1(r) \equiv (b-r)(1-2br+r^2) - r(1-b^2) = 0\,.$$

Similarly we can show that $-1 < \cos\theta \leqq 0$ for $r_3 \leqq r < r_2$ where r_3 and r_2 denote respectively the smallest positive roots of $K(r)=0$ and

$$\text{(7.16)}\quad f_2(r) \equiv (b-r)(1-r^2) - r(1-b^2) = 0\,.$$

It is easy to see that $f_1(r)$ and $f_2(r)$ are monotone decreasing functions of r in $0 < r < b$ and further $f_1(r) < 0$ for $r > r_1$ and $f_2(r) > 0$ for $r < r_2$. Thus, we notice that the value of $\cos\theta$ lies in $-1 < \cos\theta < 1$ if $r_1 < r < r_2$.

On substituting the value of $\cos\theta$ from (7.11) in (7.9) we get

$$\text{(7.17)}\quad \operatorname{Re}\Big[\frac{zp'(z)}{p(z)-1}\Big] \geqq \frac{1}{2} \cdot \Big[1 - \frac{r^2(1-b^2)^2}{(1-r^2)(b-r)^2(1-2br+r^2)}\Big] \quad \text{for } r_1 < r < r_2\,.$$

We shall now consider the case when the minimum of $S_5(u, v)$ is attained on the diameter $v=0$. Putting $v=0$ in (7.8) we obtain

$$\text{(7.18)}\quad S_5(u, 0) = \begin{cases} \dfrac{1+r^2}{1-r^2} - \dfrac{r^2}{1-r^2} \cdot \dfrac{1}{u} & \text{for } u > 0\,, \\[2ex] \dfrac{1-3r^2}{1-r^2} - 2u + \dfrac{r^2}{1-r^2} \cdot \dfrac{1}{u} & \text{for } u < 0\,. \end{cases}$$

We notice that $S_5(u, 0)$ is a monotone increasing function of u for $u>0$ and $S_5(u, 0)$ is a monotone decreasing function of u for $u<0$. For $u>0$ and $v=0$ we have

$$u=\frac{|w(z)|}{1+|w(z)|}=1-\frac{1}{1+|w(z)|}\geqq\frac{r(b-r)}{1-r^2},$$

where we have used the first inequality in (7.3). Hence the minimum of the first expression of the right hand side of (7.18) is equal to

$$\frac{b-2r+br^2}{(b-r)(1-r^2)}. \tag{7.19}$$

Likewise for $u<0$ the minimum of the second expression of the right hand side of (7.18) is attained at $u=-r(b-r)/(1-2br+r^2)$ and equals

$$\frac{b-2r+br^2}{(b-r)(1-2br+r^2)}. \tag{7.20}$$

It is easy to see that

$$\frac{b-2r+br^2}{(b-r)(1-r^2)}\leqq\frac{b-2r+br^2}{(b-r)(1-2br+r^2)} \qquad \text{for } r<b.$$

Hence finally we obtain

$$\operatorname{Re}\left[\frac{zp'(z)}{p(z)-1}\right]\geqq\frac{b-2r+br^2}{(b-r)(1-r^2)} \qquad \text{for } 0\leqq r<b. \tag{7.21}$$

In order to establish the theorem it is enough to show that for $r_1<r<r_2$,

$$\frac{1}{2}\left[1-\frac{r^2(1-b^2)^2}{(1-r^2)(b-r)^2(1-2br+r^2)}\right]\geqq\frac{br^2-2r+b}{(b-r)(1-r^2)}.$$

This follows easily from (7.14).

The equality sign in (7.21) is attained for the function $p(z)=\frac{1-2bz+z^2}{1-z^2}$ which obviously belongs to the class P.

This completes the proof of the theorem.

COROLLARY 1. *Each function* $p(z)\in P$ *maps* $|z|<\frac{b}{(1+\sqrt{1-b^2})}$ *onto a starlike domain (with centre at the point '1').*

COROLLARY 2. *If* $F(z)=z+a_2z^2+\cdots$ *belongs to* S_β^* *where* a_2 *is fixed, then for* $0\leqq|z|<\frac{|a_2|}{2(1-\beta)}$,

$$\operatorname{Re}\left[\frac{z(zF'(z)/F(z))'}{\frac{zF'(z)}{F(z)}-1}\right]\geqq\frac{|a_2|r^2-4(1-\beta)r+|a_2|}{(|a_2|-2(1-\beta)r)(1-r^2)}, \qquad |z|=r.$$

The result follows immediately from (7.21) on noting that in this case

$$\frac{zF'(z)}{F(z)}=(1-\beta)p(z)+\beta, \qquad b=-\frac{a_2}{2(1-\beta)}.$$

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