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Semigroups in an ordered Banach space

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This note gives the basic theory of linear and nonlinear semigroups in the class of Banach spaces X with a cone (which includes Banach lattices and Banach spaces). A Hille-Yosida theorem and a perturbation theorem are given for normal cones. Spaces with two cones are considered, obtaining convergence of some integral curves $x(t)$, where $\frac{dx}{dt}(t)=Ax(t)$ and A is a nonlinear dispersive operator. We refrain from generalizing all the Banach lattice results; also some of the theory extends readily to study $\frac{dx}{dt}(t)\in A_t x(t)$, where A_t depends on t, is multivalued, and X is an ordered l. c. s.

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Operators in ordered spaces.

Suppose X a Banach space over the real numbers R or the complex numbers C. Suppose K a closed convex subset invariant under multiplication by positive numbers. K is called a cone, and defines an ordering, $x\leqq y$ if $y-x$ is in K. Let $K^*=\{f \text{ in } X^*: \operatorname{Re}(f, x)\geqq 0 \text{ for all } x \text{ in } K\}$, $B^*=\{f \text{ in } X^*: \|f\|\leqq 1\}$ and for x in X define $|x|_K=\sup\{\operatorname{Re}(f, x): f \text{ in } B^*\cap K^*\}$ the support functional of $B^*\cap K^*$. Rockafeller [13] page 39 points out that such a functional has subgradient $J_K(x)$ consisting of elements f of $B^*\cap K^*$ where $\operatorname{Re}(f, x)=|x|_K$. We recall that f is in the subgradient of $|x|_K$ means that for all y in X

$$|y|_K\geqq|x|_K+\operatorname{Re}(f, y-x).$$

LEMMA 1. *Suppose $x(t)$, $y(t)$ are strongly continuous, once weakly differentiable on the left,*

$$\frac{dx}{dt}(t)=-Ax(t), \qquad \frac{dy}{dt}(t)=-Ay(t),$$

and for each $x(t)$, $y(t)$, there exists $w(t)$ in $J_K(x(t)-y(t))$ with $\operatorname{Re}(w(t), Ax(t)-Ay(t))\geqq 0$. *Then $|x(t)-y(t)|_K$ is nonincreasing.*

PROOF.

$$|x(t-h)-y(t-h)|_K-|x(t)-y(t)|_K$$
$$\geqq \operatorname{Re}(w(t), (x(t-h)-x(t))-(y(t-h)-y(t)))\,,$$

with $h>0$, $w(t)$ as above, since $w(t)$ is in the subgradient of $|x(t)-y(t)|_K$. Consequently

$$\varliminf_{h\to 0} h^{-1}(|x(t-h)-y(t-h)|_K-|x(t)-y(t)|_K)$$
$$\geqq \operatorname{Re}(w(t), Ax(t)-Ay(t))$$
$$\geqq 0\,,$$

giving the result, since $|x(t)-y(t)|_K$ is continuous. Q. E. D.

LEMMA 2. *Suppose $x(t)$, $y(t)$ are once weakly differentiable on the right,*

$$\frac{dx}{dt}(t)=-Ax(t)\,,\qquad \frac{dy}{dt}(t)=-Ay(t)\,,$$

and $|x(t)-y(t)|_K$ is nonincreasing. Then for any $w(t)$ in $J_K(x(t)-y(t))$, $\operatorname{Re}(w(t), Ax(t)-Ay(t))\geqq 0$.

PROOF. Take any $w(t)$ as above, $h>0$, then

$$|x(t+h)-y(t+h)|_K\geqq|x(t)-y(t)|_K$$
$$+\operatorname{Re}(w(t), (x(t+h)-x(t))-(y(t+h)-y(t)))\,.$$

Dividing by h and letting $h\to 0$ give the result. Q. E. D.

REMARK. If X is a Banach lattice, then $|x|_K=\|x^+\|$, and $J_1(x^+)\subset J_K x \subset J_{(0)}(x^+)$ where $J_{(0)}$ is the subgradient of the support functional of B^* and $J_1(x)$ is any positive duality map. We recall a positive duality map is a function J_1 from X to X^* with $\|J_1x\|=1$, $(J_1x, x)=\|x\|$, $(J_1x, y)=0$ if $x\perp y$, and $(J_1x, y)\geqq 0$ if $x\geqq 0$ and $y\geqq 0$. The first assertion follows from [2] Proposition 1.1, and the second from [2] Proposition 1.2, noting that if $w(x)$ is in $J_K(x)$ then $w(x)\geqq 0$.

We recall (Krasnoselskii [9]) that K is normal if and only if there is an equivalent monotonic norm. The norm is monotonic means that $0\leqq x\leqq y$ implies $\|x\|\leqq\|y\|$.

THEOREM 3. *K is normal if and only if $|u|_K+|u|_{-K}$ is an equivalent norm.*

PROOF. In any case, $|u|_K\leqq\|u\|$, giving $|u|_K+|u|_{-K}\leqq 2\|u\|$. Suppose K is normal. As a consequence of Theorem 3.3, page 219 of Schaefer [18], there exists n in Z^+ such that for f in B^* there exist f_1, f_2 in K^*, $f=f_1-f_2$, $\|f_1\|\leqq n$, $\|f_2\|\leqq n$. For u in X, f in B^*, $|(f, u)|=|(f_1, u)-(f_2, u)|\leqq n|u|_K+n|u|_{-K}$. Consequently, $\|u\|=\sup\{|(f, u)| : f \text{ in } B^*\}\leqq n(|u|_K+|u|_{-K})$.

Conversely, suppose K not normal. Then, by Theorem 1 of Krasnoselskii [9], there exist sequences x_n, y_n in K, $\|x_n\|=\|y_n\|=1$, with $\|x_n+y_n\|\to 0$. Therefore $|x_n+y_n|_K\to 0$, giving $|x_n|_K\to 0$. Since $|x_n|_K+|x_n|_{-K}\to 0$, this is not an equivalent norm. Q. E. D.

We say $U: D(U) \to X$, $D(U) \subset X$, is K nonexpansive if for x, y in $D(U)$, $|U(x)-U(y)|_K \leqq |x-y|_K$. We say A: $D(A) \to X$, $D(A) \subset X$, is K accretive if for x, y in $D(A)$, there exists w in $J_K(x-y)$ with $\mathrm{Re}(w, Ax-Ay) \geqq 0$. Clearly U is K nonexpansive if and only if $-K$ nonexpansive. We say A is K dispersive if $-A$ is K accretive. We say U is nonexpansive if $K=\{0\}$, and A is g accretive or $-A$ is g dissipative if $K=\{0\}$. See Lumer and Phillips [10].

LEMMA 4. *A is K accretive if and only if for all $d>0$ $(I+dA)^{-1}$ is K nonexpansive.*

PROOF. Similar to Theorem 9.1 of Browder [1], using the fact that $x \to |x|_K J_K x$ is upper semicontinuous from the strong topology of X to subsets of X^* with the weak* topology (c. f. Cudia [4], Theorem 4.3.).

The functionals

$$\varphi_0(f, g) = \lim_{d \to 0^+} d^{-1}(|f+dg|_K - |f|_K),$$

$$\varphi_0'(f, g) = -\varphi_0(f, -g)$$

were introduced by Sato [15] as the maximum and minimum (0) gauge functionals on a Banach lattice.

THEOREM 5.

$$\varphi_0(f, g) = \sup\{\mathrm{Re}(h, g) : h \text{ in } J_K(f)\},$$

$$\varphi_0'(f, g) = \inf\{\mathrm{Re}(h, g) : h \text{ in } J_K(f)\}.$$

PROOF. We prove the first assertion; the second is similar. By definition of subgradient for $d>0$ and h in $J_K(f)$,

$$|f+dg|_K \geqq |f|_K + \mathrm{Re}(h, dg),$$

giving $\varphi_0(f, g) \geqq \mathrm{Re}(h, g)$. Fixing f, g $\varphi(h) = \varphi_0(f, h)$ defined on the space spanned by f, g has $\varphi(\alpha h) = \alpha\varphi(h)$, $\alpha \geqq 0$, and $\varphi(h) \leqq |h|_K$.

By Theorem 5.4 of Schaefer [18], we can extend φ to φ in $B^* \cap K^*$, i.e. a positive linear functional with norm $\leqq 1$. Since $\varphi(f) = |f|_K$, φ is in $J_K(f)$. Q. E. D.

For a comparison of various functionals as in Sato [14], Phillips [12], Hasegawa [7], Dorroh [5] we refer to Sato [15].

COROLLARY 6. *A is K accretive if and only if $\varphi_0(f-g, Af-Ag) \geqq 0$ for f, g in $D(A)$. If $-A$ generates a K nonexpansive semigroup then $\varphi_0'(f-g, Af-Ag) \geqq 0$ for f, g in $D(A)$.*

The first assertion follows from Theorem 5, the second from Lemma 2.

THEOREM 7. *Suppose X a Banach space with cone K.*

(a) *Suppose $U(t)$ a continuous bounded semigroup of positive linear operators on X. Then the generator A is K dispersive in an equivalent norm, and $R(I-A)=X$.*

(b) *Suppose K is normal, A is a densely defined K accretive linear operator with $R(I+A)=X$.*

Then $-A$ generates a continuous bounded semigroup of positive linear operators on X.

PROOF. (a) Renorm X by $\|x\|=\sup\{\|U(t)x\|: t\geqq 0\}$, so $\|U(t)\|\leqq 1$. Given f in $B^*\cap K^*$, and $t\geqq 0$, then $U(t)^*f$ is in $B^*\cap K^*$. For x in X, $t\geqq 0$,

$$\begin{aligned}|U(t)x|_K&=\sup\{\operatorname{Re}(f, U(t)x): f \text{ in } B^*\cap K^*\}\\&=\sup\{\operatorname{Re}(U(t)^*f, x): f \text{ in } B^*\cap K^*\}\\&\leqq |x|_K.\end{aligned}$$

By Lemma 2, A is K dispersive.

(b) It is enough to show A is g accretive in an equivalent norm. For $d>0$, $(I+dA)^{-1}$ is K nonexpansive, and consequently nonexpansive in $|x|_K+|x|_{-K}$, giving A g accretive in this norm, by Theorem 3. Q. E. D.

COROLLARY. *That a densely defined linear operator A is the generator of a continuous semigroup of positive linear operators on X implies that there exists m in R with $A-mI$ K dispersive in an equivalent norm, $R(A-nI)=X$ for $n>m$, and conversely if K is normal.*

PROOF. Supposing A generates a positive semigroup T_t of class C_0, then $\|T_t\|\leqq Me^{mt}$ with constants m, M, by Hille's theorem (Page 232 of Yosida [19]). Then $U_t=e^{-mt}T_t$ is positive, and bounded, so the generator $A-mI$ is K dispersive in an equivalent norm by (a) of Theorem 7.

Conversely, by (b), U_t generated by $A-mI$ is positive, continuous and bounded, so $e^{mt}U_t=T_t$ is positive and continuous. Its generator is A. Q. E. D.

This answers a question of Sato [15]. Next we look at pseudo-resolvents and see that some results hold without their being a resolvent in the usual sense.

THEOREM 8. *Suppose X a Banach space and $\{J_\lambda: \lambda\geqq\lambda_0\}$ a pseudo-resolvent, i.e. a family of bounded operators in X with*

$$J_\lambda-J_\mu=(\mu-\lambda)J_\lambda J_\mu.$$

Suppose $\|\lambda J_\lambda\|\leqq M$ for all λ.

(a) *Then the multivalued operator $A=\lambda I-J_\lambda^{-1}$ is well defined.*

(b) *If X is reflexive, there is a bounded semigroup $U(t)$ on the closure of $D(A)$ with infinitesimal generator A_1 having $D(A_1)=D(A)$, and for x in $D(A)$, $A_1(x)$ is in $A^0(x)=\{y \text{ in } A(x): \|y\|=d(0, Ax)\}$.*

(c) *If X is ordered and J_λ are all positive then so is $U(t)$.*

PROOF. (a) By Lemma 1′, page 217 of Yosida [19], $N(J)\cap cl(R(J))=\{0\}$, where $R(J)$ is the common range and $N(J)$ the common nullspace of the family $\{J_\lambda\}$. As in Yosida we have for any λ, μ

$$
\begin{aligned}
J_\lambda J_\mu(\lambda I - J_\lambda^{-1} - \mu I + J_\mu^{-1}) &= (\lambda-\mu) J_\lambda J_\mu - J_\lambda J_\mu (J_\lambda^{-1} - J_\mu^{-1}) \\
&= (\lambda-\mu) J_\lambda J_\mu - (J_\mu - J_\lambda) \\
&= 0 .
\end{aligned}
$$

It follows that for x in X, $J_\mu(\lambda I - J_\lambda^{-1} - \mu I + J_\mu^{-1})x$ is in $N(J)$, and hence in $N(J) \cap cl(R(J)) = \{0\}$. Consequently $(\lambda I - J_\lambda^{-1})x = (\mu I - J_\mu^{-1})x$.

(b) This may be proved as on page 246 of Yosida [**19**] using a little of the technique of Theorem 9.23 of Browder [**1**]. Reflexivity is needed because $D(A)$ will not be dense unless A is single valued.

(c) As in (a) of Theorem 7 there is an equivalent norm in which $\{\lambda J_\lambda\}$ are nonexpansive, hence K nonexpansive, giving A K dispersive, by Lemma 4, and $U(t)$ K nonexpansive by Lemma 1. Q. E. D.

REMARK. There should be a nonlinear extension of this. We note the condition for the pseudo-resolvent to admit a potential operator V can be stated in terms of the multivalued A above, giving $V = -A^{-1}$, (see Sato [**16**]).

Phillips [**12**], Gustafson and Sato [**17**] ask if their (linear) dispersive operators A are always dissipative. It is understood that A should be a densely defined operator; counterexamples defined on a one dimensional subspace are known. If Bohnenblusts' property P holds, which says, in the setting of cones, $|x|_K = |y|_K$ and $|x|_{-K} = |y|_{-K}$ imply $\|x\| = \|y\|$, then as in Calvert [**3**], K nonexpansive implies nonexpansive, so that by Lemma 4, K accretive implies g accretive. This result is contained in Theorem 5.1 of Sato [**15**] for linear operators in a lattice.

To study perturbation theory, as in Sato and Gustafson [**17**], a related question is whether a K accretive operator is g accretive in some equivalent norm. By Theorem 3, the answer is yes if K is normal. The multiplicative perturbation results of [**17**] then follow from the results of Gustafson in the g accretive case.

THEOREM 9. *Suppose A is K accretive, $(I+A)$ surjective, $D(B) \supset D(A)$, and there exists $a < 1$, b in R^+, with $\|Bx\| \leqq a\|Ax\| + b\|x\|$ for x in $D(A)$. Suppose K normal. Suppose*

(a) *B is K accretive, $(1+B)$ surjective, J_K uniformly continuous on bounded sets, and X reflexive, or*

(b) *A and B are linear, and $A+B$ is K accretive.*

Then $-(A+B)$ generates a K nonexpansive semigroup on $cl(D(A))$.

PROOF. (a) Similar to Theorem 9.22 of Browder [**1**]. (Cases where J_K is uniformly continuous on bounded sets include X^* uniformly convex and either $K = \{0\}$ or X is a Banach lattice.)

(b) The result follows from the accretivity in an equivalent norm $\|\ \|_e$, for $a < a_0$, some small a_0, for then $\|Bx\|_e \leqq a\|Ax\|_e + b\|x\|_e$ with $a < 1$. The result follows by Lemma 5.1 of Gustafson [6]; there exist $c_j > 0$ $(j=1, \cdots, n)$, $a^1 < a_0$, $b^1 < \infty$, with $\sum c_j = 1$ and for $k = 1, 2, \cdots, n$, x in $D(A)$,

$$\|c_k B(x)\| \leqq a' \left\| \left(A + \sum_{j=1}^{k-1} c_j B \right) x \right\| + b' \|x\| . \qquad \text{Q. E. D.}$$

LEMMA 10. *Suppose $x(t)$ strongly continuous, once weakly differentiable on the left, $\frac{dx}{dt}(t) = -Ax(t)$, and for $t \geqq s$ there is w in $J_{-K}(x(t)-x(s))$ with* $\mathrm{Re}\,(w, Ax(t)-Ax(s)) \geqq 0$. *Then $|Ax(t)|_K$ is nonincreasing.*

PROOF. Let $y(t) = x(t-h)$, $h > 0$, then $\frac{dy}{dt}(t) = -Ay(t)$, and by Lemma 1 $|x(t)-y(t)|_{-K}$ is nonincreasing. Dividing by h and letting $h \to 0$ give the result. Q. E. D.

Lemmas 1, 2, 10 compare with Propositions 1.10, 1.11, 1.12 of Calvert [2].

THEOREM 11. *Suppose X a Banach space with cones K and H, $H \subset K$. Suppose $x(t)$ is once weakly differentiable, strongly continuous, $\frac{dx}{dt}(t) = -Ax(t)$, and $x(t)$ is increasing with respect to K. Suppose the norm is monotonic with respect to K, and A is K accretive or g accretive. Then $|Ax(t)|_H$ is nonincreasing.*

PROOF. By Lemma 10, it suffices to show that for $t \geqq s$ there is w in $J_{-H}(x(t)-x(s))$ with $\mathrm{Re}\,(w, Ax(t)-Ax(s)) \geqq 0$. By Lemma 2, if A is K accretive, any w in $J_{-K}(x(t)-x(s))$ gives $\mathrm{Re}\,(w, Ax(t)-Ax(s)) \geqq 0$, and if A is g accretive, any w in $J_{(0)}(x(t)-x(s))$ gives $\mathrm{Re}\,(w, Ax(t)-Ax(s)) \geqq 0$. Putting $x = x(t)-x(s)$, it suffices to show that x in K implies $J_K(x) \cap J_H(x)$ is nonempty and $J_{(0)}(x) \cap J_H(x)$ is nonempty. Since the norm is monotonic with respect to K, Proposition 1.1 of Calvert [2] tells us there is f in $K^* \cap B^*$ with $\mathrm{Re}\,(f, x) = \|x\|$. Since $K^* \subset H^*$, this f is in $J_K(x) \cap J_H(x) \cap J_{(0)}(x)$. Q. E. D.

COROLLARY. *If in addition $Ax(0)$ is in $-H$ then $Ax(t)$ is in $-H$ as long as $x(t)$ is defined, and $x(t)$ is increasing with respect to H.*

REMARK. A broad sufficient condition for $x(t)$ to be increasing if $\frac{dx}{dt}(t) = -Ax(t)$ and $Ax(0)$ is in the cone $-K$ is: for z in $D(A)$ there exists k in R^+ and N a neighborhood of z in X, such that $A+kI$ restricted to $N \cap D(A)$ is K accretive. The proof is similar to Proposition 1.12 of Calvert [2]. Applications to the problems of [8] can be made: if the resistance function is differentiable and monotonic, then it satisfies the above by Proposition 1.3 of Calvert [3]. Integral $x(t)$ curves increasing with t were considered by Olubummo [11].

THEOREM 12. *Suppose X a Banach space with monotonic norm with respect to a cone K. Suppose A is K accretive, demiclosed, and $x(t)$, t in $[0, \infty)$, is a strongly continuous once weakly differentiable curve with $\frac{dx}{dt}(t) = -Ax(t)$, and*

$Ax(0) \leqq 0$. *Suppose* A^{-1} *bounded, then* $x(t)$ *converges to a zero of* A.

PROOF. Suppose e in K, and for x in K put $\sup x = \inf\{p \text{ in } R : x \leqq pe\}$ (possibly ∞) and $\inf x = \sup\{p \text{ in } R : pe \leqq x\}$. For b in $(0,1]$ let $K(b) = \{x \text{ in } K : \inf x \geqq b \sup x\}$ c. f. Krasnoselskii [**9**] page 27. $K(b)$ is a cone which allows plastering (i. e. there exists f in X^*, k in R, with $\operatorname{Re}(f, x) \geqq k\|x\|$ for x in $K(b)$), and consequently is fully regular (i. e. any bounded set directed under $\leqq$ is convergent).

By the Corollary, $x(t)$ is increasing in K. Let $H = K(b)$, with any b in $(0, 1]$ and $e = -Ax(0)$. By the Corollary, since $Ax(0)$ is in $-H$, $x(t)$ is increasing with respect to H. By Theorem 11, $|Ax(t)|_{-K}$ is decreasing as t increases. By Theorem 3, $\{Ax(t) : t \geqq 0\}$ is bounded. Since A^{-1} is bounded, $x(t)$ is bounded. Since H is fully regular, $x(t)$ converges as $t \to \infty$ to a point x. For f in $-K^*$, $(f, Ax(t))$ must converge to 0. Since K^* is reproducing, i. e. $X^* = K^* - K^*$, $Ax(t)$ converges weakly to zero. Since A is demiclosed, x is in $D(A)$ and $A(x) = 0$. Q. E. D.

EXAMPLE 13. Suppose X a Banach space with monotonic norm and $U(t)$ a bounded positive continuous linear semigroup, with generator $-A$, A^{-1} bounded. Then for x_0 in $D(A)$, g in X, $g \geqq A(x)$, there is a solution $x(t)$ to

$$\frac{dx}{dt}(t) = g - Ax(t), \quad \text{with} \quad x(0) = x_0,$$

and $x(t)$ converges to x with $A(x) = g$.

PROOF. By Theorem 7, A is K accretive. The operator A_g taking x to $A(x) - g$ is K accretive. For x_0 in $D(A) = D(A_g)$ there is a strong solution to $\frac{dx}{dt}(t) = -A(x(t)) + g$, $x(0) = x_0$. A_g is demiclosed and A_g^{-1} is bounded since A has these properties. $A_g(x_0) \leqq 0$, so that the conclusion follows from Theorem 12. Q. E. D.

DEFINITION. Suppose X a Banach space with fully regular normal cone K. Suppose A is K accretive, $R(I+A) = X$. We say an element x of X is harmonic if $Ax = 0$, subharmonic if $Ax \leqq 0$, (as in Yosida [**19**] page 411).

THEOREM 14. *Suppose there exists* z *with* $\{(I+dA)^{-1}z : d > 0\}$ *bounded. Then any subharmonic element* x *has a least harmonic majorant* $x_h = \lim_{d\to\infty} (1+dA)^{-1}x$.

PROOF. Given $d > 0$, $(I+dA)x \leqq x$. Since $(I+dA)^{-1}$ is K nonexpansive, $x_d = (I+dA)^{-1}x \geqq x$. Moreover if $d > e > 0$, $(I+eA)x_d = ed^{-1}x + (1-ed^{-1})x_d$, which is $\geqq x$. Consequently $x_d \geqq (I+eA)^{-1}x = x_e$. Now $\{x_d : d > 0\}$ is bounded, since K normal implies $\{(I+dA)^{-1} : d > 0\}$ is equi-Lipschitz. Since K is fully regular, and x_d increases as d increases, there exists $x_h = \lim_{d\to\infty} x_d$. Now $(I+A)x_d = d^{-1}x + (1-d^{-1})x_d = y_d$ converges to x_h. $(1+A)^{-1}y_d = x_d$ gives $(1+A)^{-1}x_h = x_h$ by continuity, so that x_h is harmonic.

Suppose $x_H \geqq x$, $Ax_H = 0$, then $x_H = (1+dA)^{-1}x_H \geqq (I+dA)^{-1}x$, and taking limits gives $x_H \geqq x_h$.

REMARK. There exists z with $\{(1+dA)^{-1}z : d>0\}$ bounded if A is linear ($z=0$) or A^{-1} is locally bounded, by Theorem 3.2 of Calvert [3].

Generalizations.

Given φ a proper convex function of X, we say A is $d\varphi$ accretive if for x, y in $D(A)$, there is f in $d\varphi(x-y)$ with $\mathrm{Re}(f, Ax-Ay) \geqq 0$.

The basic Lemmas 1 and 2 hold in this context. If $d\varphi$ satisfies conditions of the type of Chapter 3 of Browder [1], then the basic existence theorem holds, in the following form for nonlinear operators.

THEOREM 15. *Suppose X a reflexive Banach space, N an open neighborhood of* 0, *$T: N \to X^*$ uniformly continuous, and there exists $c, k>0$ with* $\mathrm{Re}(Tx, x) \geqq c\|x\|^2$ *and $\|Tx\| \leqq k\|x\|$ for x in N. Suppose T is cyclically monotone, i.e. (Rockafeller [13]) for any n-tuple $x_1 \cdots x_n$,* $\mathrm{Re}\left(\sum_{i=1}^{n}(Tx_i, x_i-x_{i+1})+(Tx_n, x_n-x_1)\right) \geqq 0$. *Suppose $B: D(B) \subset X \to X$ is T accretive, i.e.* $\mathrm{Re}(T(x-y), Bx-By) \geqq 0$ *for x, y in $D(B)$, $x-y$ in N. Suppose $R(I+B)=X$.*

Then for x_0 in $D(B)$ there exists a unique strongly continuous weakly C^1 function $x: [0, \infty) \to X$ with $x(0)=x_0$, $\frac{dx}{dt}(t)=-Bx(t)$. The strong derivative exists almost everywhere and equals $-Bx(t)$.

PROOF. By Rockafeller [13], we have a C^1 convex function $\varphi: N \to R$, with derivative T, there are constants $a, b>0$ with

$$a\|x\|^2 \geqq \varphi(x) \geqq c\|x\|^2$$

for x in N. Using $\varphi(x)$ instead of $\|x\|^2$ we may easily generalize Theorem 9.15 of Browder [1]. Q. E. D.

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