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Some closed subalgebras of measure algebras and a generalization of P. J. Cohen's theorem

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§ 0. Introduction.

Let G_1 and G_2 be locally compact abelian groups, and let $L^1(G_1)$ and $M(G_2)$ be the group algebra of G_1 and the measure algebra of G_2, respectively. Homomorphisms of $L^1(G_1)$ into $M(G_2)$ have been studied by H. Helson, W. Rudin, J. P. Kahane, Z. L. Leibenson, P. J. Cohen and others; and P. J. Cohen [1], [2] determined all the homomorphisms of $L^1(G_1)$ into $M(G_2)$ by the notion of the coset ring and piecewise affine maps. He also proved that every homomorphism of $L^1(G_1)$ into $M(G_2)$ has a natural norm-preserving extension to a homomorphism of $M(G_1)$ into $M(G_2)$, but in general an extension to a homomorphism of $M(G_1)$ into $M(G_2)$ is not unique.

The purpose of this paper is to introduce some closed subalgebra $L^*(G_1)$ of $M(G_1)$, which contains $L^1(G_1)$ properly if G_1 is not discrete, to determine the maximal ideal space of $L^*(G_1)$, and to determine all the homomorphisms of $L^*(G_1)$ into $M(G_2)$ as a generalization of P. J. Cohen's theorem.

We give in § 1 some preliminaries, and in § 2 we introduce a closed subalgebra $L^*(G_1)$ of $M(G_1)$. In § 3 we investigate the maximal ideal space of $L^*(G_1)$, and obtain it as a semi-group. Finally we determine in § 4 all the homomorphisms of $L^*(G_1)$ into $M(G_2)$ as a generalization of P. J. Cohen's theorem.

§ 1. Preliminaries.

Throughout this paper G_1 and G_2 denote locally compact abelian groups ($=LCA$ groups), and Γ_1 and Γ_2 denote their dual groups, respectively. The notations G^τ and Γ_τ are also used to express an LCA group with underlying group G and topology τ, and its dual group, respectively. Thus by G^τ and $G^{\tau'}$, we mean that they have the same underlying group G.

$L^1(G_1)$ is the group algebra of G_1, i. e. the Banach algebra of all the Haar integrable functions on G_1 under convolution multiplication, and $M(G_2)$ is the measure algebra of G_2, the Banach algebra of all the regular bounded complex

Borel measures on G_2 under convolution multiplication.

If f is an element of $L^1(G_1)$, and if we define $\mu_f(E)=\int_E f(x)dx$ for each Borel set E in G_1, μ_f is a regular bounded complex Borel measure on G_1 and

$$L^1(G_1)\ni f\longmapsto \mu_f\in M(G_1)$$

is a norm-preserving isomorphism of $L^1(G_1)$ into $M(G_1)$. Through this isomorphism we identify $L^1(G_1)$ with a subset of $M(G_1)$, and then $L^1(G_1)$ is a closed ideal of $M(G_1)$. The set $L^1(G_1)$ is characterized as the set of all absolutely continuous measures in $M(G_1)$ with respect to the Haar measure of G_1 (cf. [4] Chap. 1).

$B(\Gamma_1)$ denotes the set of all the Fourier Stieltjes transforms of elements in $M(G_1)$.

DEFINITION 1.1. We mean by an open coset of Γ_2 a coset of some open subgroup of Γ_2. The coset ring of Γ_2 is the smallest collection Σ of subsets of Γ_2 which satisfies the following conditions:

1) Σ contains all the open cosets of Γ_2.

2) If $\Sigma\ni A, B$ then $A\cup B$, $A^c\in\Sigma$.

DEFINITION 1.2. If E is an open coset of Γ_2 and α is a continuous mapping from E into Γ_1, then α is called affine if

$$\alpha(r+r'-r'')=\alpha(r)+\alpha(r')-\alpha(r'') \qquad (r, r', r''\in E)$$

holds. Suppose that

(a) $S_1, S_2, \cdots, S_n$ are pairwise disjoint sets belonging to the coset ring of Γ_2.

(b) Each set S_i is contained in an open coset K_i of Γ_2.

(c) For each i, α_i is an affine map of K_i into Γ_1.

(d) α is the map of $Y=S_1\cup S_2\cup\cdots\cup S_n$ into Γ_1, which coincides on S_i with α_i $(i=1,2,\cdots,n)$.

Then α is said to be a piecewise affine map of Y into Γ_1.

THEOREM 1 (Cohen). *Suppose Y belongs to the coset ring of Γ_2, and α is a piecewise affine map from Y into Γ_1.*

(i) *For each $f\in L^1(G_1)$, put*

$$(\hat{f}\circ\alpha)(r)=\begin{cases}\hat{f}(\alpha(r)); & r\in Y\\ 0 \quad ; & r\notin Y,\end{cases}$$

where $\hat{f}$ is the Fourier transform of f. Then $\hat{f}\circ\alpha$ belongs to $B(\Gamma_2)$, and there exists a unique element $h(f)$ of $M(G_2)$ such that $\hat{f}\circ\alpha$ is the Fourier-Stieltjes transform of $h(f)$. The mapping h of $L^1(G_1)$ into $M(G_2)$ is a homomorphism, and conversely every homomorphism of $L^1(G_1)$ into $M(G_2)$ is obtained in this way.

(ii) *For each $\mu\in M(G_1)$, put*

$$(\hat{\mu}\circ\alpha)(r)=\begin{cases}\hat{\mu}(\alpha(r)); & r\in Y\\ 0 \quad ; & r\notin Y,\end{cases}$$

where $\hat{\mu}$ *is the Fourier-Stieltjes transform of* μ. *Then we have* $\hat{\mu}\circ\alpha\in B(\Gamma_2)$, *and we can choose a unique element* $h_1(\mu)$ *of* $M(G_2)$ *such that* $\hat{\mu}\circ\alpha$ *is the Fourier-Stieltjes transform of* $h_1(\mu)$. h_1 *is a norm-preserving extension of* h *to a homomorphism of* $M(G_1)$ *into* $M(G_2)$ (cf. [1], [2] and [4] Chap. 4).

§ 2. A closed subalgebra $L^*(G_1)$ of $M(G_1)$.

We denote by C the complex number field, and by T the set of all the complex numbers of absolute value 1. T is an LCA group with respect to multiplication and usual topology.

PROPOSITION 2.1. *Let* G_1 *and* G_2 *be two LCA groups, and let* η *be a continuous isomorphism of* G_1 *onto* G_2. *Then*

(i) *There exists a natural norm-preserving isomorphism* π *of* $M(G_1)$ *into* $M(G_2)$, *given by*

$$\pi(\mu)(E)=\mu(\eta^{-1}(E)) \qquad (E\text{: Borel set of } G_2\,;\ \mu\in M(G_1))\,.$$

(ii) *If* $\nu\in M(G_2)$, ν *belongs to* $\pi(M(G_1))$ *if and only if there exists a* σ-*compact subset* K *of* G_1 *such that* ν *is concentrated in* $\eta(K)$.

PROOF. (i) Suppose $\mu\in M(G_1)$. Choose a σ-compact open subset K of G_1, in which μ is concentrated. Since η is continuous, $\eta(K)$ is also σ-compact in G_2, and hence $\eta(K)$ is a Borel set in G_2. Choose compact sets Q_i in G_1 such that $\bigcup_{i=1}^{\infty} Q_i=K$. Let U be an open set in G_1 which is contained in K. Then $\eta(Q_i-U)$ is compact, and $\eta(Q_i\cap U)$ $(i=1,2,\cdots)$ is a Borel set in G_2, and hence $\eta(U)=\bigcup_{i=1}^{\infty}\eta(Q_i\cap U)$ is a Borel set in G_2. Thus if we put

$$\Omega=\{E\colon E \text{ is a Borel set in } G_1 \text{ and } \eta(E\cap K) \text{ is a Borel set in } G_2\}\,,$$

then Ω contains all the Borel sets in G_1. Therefore we see that a subset E of K is a Borel set in G_1 if and only if $\eta(E)$ is a Borel set in G_2.

Define $\pi(\mu)$ by

$$\pi(\mu)(E)=\mu(\eta^{-1}(E)) \qquad (E\,;\ \text{Borel set of } G_2)$$

then $\pi(\mu)$ is an element of $M(G_2)$, and from the above discussion we see that $\pi(\mu)$ has the same norm as μ, and hence

$$\pi:\qquad M(G_1)\ni\mu\longmapsto\pi(\mu)\in M(G_2)$$

is a norm-preserving isomorphism, and this completes the proof of (i).

(ii) Necessity is clear from the definition of the mapping π. Suppose that K is a σ-compact set in G_1 such that $\nu\in M(G_2)$ is concentrated in $\eta(K)$. We can assume without loss of generality that K is open in G_1. By the paragraph in (i), $\eta(E\cap K)$ is a Borel set in G_2 for each Borel set E of G_1.

We put

$$\nu_1(E)=\nu(\eta(E\cap K)) \qquad (E\,;\ \text{Borel set in } G_1)\,.$$

Then ν_1 is a bounded complex Borel measure on G_1.

To show the regularity of ν_1, we remark here that the total variation of ν_1 is associated to the total variation of ν, that is $|\nu_1|(E)=|\nu|(\eta(E\cap K))$ holds for each Borel set E in G_1, and thus we can assume without loss of generality that ν is a positive measure.

Let Q_i $(i=1,2,\cdots)$ be a sequence of compact subsets of K such that $Q_1\subset Q_2\subset Q_3\subset\cdots$, and $\bigcup_{i=1}^{\infty}Q_i=K$. Given $\varepsilon>0$ and a Borel set E in G_1, which is contained in K, choose a compact subset F of $\eta(E)$ such that $\nu(\eta(E)-F)\leqq\varepsilon/2$, and choose a positive integer n such that $\nu_1(\eta^{-1}(F))-\varepsilon/2\leqq\nu_1(\eta^{-1}(F)\cap Q_n)$, and then we have

$$\begin{aligned}\nu_1(\eta^{-1}(F)\cap Q_n)&\geqq\nu_1(\eta^{-1}(F))-\varepsilon/2\\&=\nu(F)-\varepsilon/2\\&=\nu(\eta(E))-\nu(\eta(E)-F)-\varepsilon/2\\&\geqq\nu(\eta(E))-\varepsilon\\&=\nu_1(E)-\varepsilon\,.\end{aligned}$$

Since the restriction of η to Q_i is a homeomorphism for each i $(i=1,2,3,\cdots)$, $\eta^{-1}(F)\cap Q_n$ is a compact subset of E, and hence ν_1 is inner regular. Since ν_1 is bounded, ν_1 is also outer regular and this shows that ν_1 is an element of $M(G_1)$ and $\nu=\pi(\nu_1)\in\pi(M(G_1))$.

DEFINITION 2.1. Let G^{τ} and $G^{\tau'}$ be two LCA groups with the same underlying group G and $\tau\leqq\tau'$. By Proposition 2.1 we can define the norm-preserving isomorphism π of $M(G^{\tau'})$ into $M(G^{\tau})$. We identify $L^1(G^{\tau'})$ and $M(G^{\tau'})$ with subalgebras of $M(G^{\tau})$ through π, respectively.

DEFINITION 2.2. If λ and μ are elements of $M(G^{\tau})$, we say that λ and μ are orthogonal each other (notation $\lambda\perp\mu$) if there exist two disjoint Borel sets A and B in G^{τ} such that λ is concentrated in A and μ is concentrated in B. If Λ and Λ' are subsets of $M(G^{\tau})$, we say that Λ and Λ' are orthogonal each other if $\lambda\perp\mu$ for each pair (λ,μ), where $\lambda\in\Lambda$, $\mu\in\Lambda'$.

PROPOSITION 2.2. *Let G^{τ} and $G^{\tau'}$ be two LCA groups with the same underlying group G with $\tau\leqq\tau'$, and let η be the natural continuous isomorphism of $G^{\tau'}$ onto G^{τ}. If μ is an element of $M(G^{\tau})$, following* a), b) *and* c) *are equivalent each other.*

a) $\mu\perp M(G^{\tau'})$,

b) $\mu=\mu_1+\mu_2$, $\mu_1\in M(G^{\tau'})$ *and* $\mu_1\perp\mu_2$ *implies* $\mu_1=0$,

c) $|\mu|(\eta(K))=0$ *for every compact set K in $G^{\tau'}$, where $|\mu|$ is the total variation of μ.*

PROOF. a) implies b); Suppose a), and if $\mu=\mu_1+\mu_2$, $\mu_1 \perp \mu_2$ and $0 \neq \mu_1 \in M(G^{\tau'})$, then μ and μ_1 are not orthogonal each other and this contradicts a).

b) implies c); Suppose b), and if there exists a compact set K in $G^{\tau'}$ with $|\mu|(\eta(K)) \neq 0$, we set μ_1 the restriction of μ to $\eta(K)$, that is

$$\mu_1(E)=\mu(\eta(K)\cap E) \qquad (E\,;\ \text{Borel set of } G^{\tau})$$

then we have $\mu_1 \in M(G^{\tau'})$ by Proposition 2.1 (ii), and that $\mu=(\mu-\mu_1)+\mu_1$, $\mu_1 \neq 0$ and $\mu_1 \perp (\mu-\mu_1)$, contradicting b).

c) implies a); Soppose c), and let λ be an element of $M(G^{\tau'})$. There exists a σ-compact subset E of $G^{\tau'}$ such that λ is concentrated in $\eta(E)$. Then by c), $|\mu|(\eta(E))=0$ and this implies $\mu \perp \lambda$. Since λ was an arbitrary element of $M(G^{\tau'})$, we have $\mu \perp M(G^{\tau'})$.

DEFINITION 2.3. Let G^{τ} be an *LCA* group. We denote by $\mathfrak{T}(G^{\tau})$ the class of all locally compact group topologies of G, which are equal or stronger than τ.

LEMMA 2.3. *Let G^{τ} be an LCA group and let $\mathfrak{T}(G^{\tau}) \ni \tau_1, \tau_2$ with $\tau_2 \subseteqq \tau_1$. If $\eta_{\tau_1}^{\tau_2}$ is the natural continuous isomorphism of G^{τ_1} onto G^{τ_2}, then $r\circ\eta_{\tau_1}^{\tau_2}$ $(r \in \Gamma_{\tau_2})$ is an element of Γ_{τ_1}, which we denote by $\varphi_{\tau_1}^{\tau_2}(r)$. $\varphi_{\tau_1}^{\tau_2}$ is a continuous isomorphism of Γ_{τ_2} onto a dense subgroup of Γ_{τ_1}.*

PROOF. It is clear that $\varphi_{\tau_1}^{\tau_2}$ is an isomorphism of Γ_{τ_2} into Γ_{τ_1}. Let W be a neighbourhood of 0 in Γ_{τ_1}. There exists a compact subset K of G^{τ_1} and $\varepsilon>0$ such that $N(K, \varepsilon)=\{r \in \Gamma_{\tau_1}\,;\ |(x, r)-1|<\varepsilon,\ x \in K\} \subseteqq W$. Since $\eta_{\tau_1}^{\tau_2}(K)$ is also compact in G^{τ_2}, $V=N(\eta_{\tau_1}^{\tau_2}(K), \varepsilon)$ is a neighbourhood of 0 in Γ_{τ_2} and that $\varphi_{\tau_1}^{\tau_2}(V) \subseteqq W$. This shows that $\varphi_{\tau_1}^{\tau_2}$ is continuous.

Suppose that $\overline{\varphi_{\tau_1}^{\tau_2}(\Gamma_{\tau_2})}=H \subsetneqq \Gamma_{\tau_1}$. Γ_{τ_1}/H is a non-trivial *LCA* group and there exists a continuous homomorphism $\bar{\beta} \neq 0$ of Γ_{τ_1}/H into T. $\bar{\beta}$ induces a non-trivial continuous homomorphism β of Γ_{τ_1} into T such that

$$\beta(r)=\bar{\beta}(\bar{r}) \qquad (r \in \Gamma_{\tau_1}),$$

where $\bar{r}$ is a coset of H which contains r. There exists $0 \neq x \in G^{\tau_1}$ such that

$$\beta(r)=(x, \gamma) \qquad (r \in \Gamma_{\tau_1}),$$

and hence we have

$$1=\beta(\varphi_{\tau_1}^{\tau_2}(r))=(x, \varphi_{\tau_1}^{\tau_2}(r))=(\eta_{\tau_1}^{\tau_2}(x), r) \qquad (r \in \Gamma_{\tau_2}). \tag{2.1}$$

From (2.1) we have $\eta_{\tau_1}^{\tau_2}(x)=0$ and this is a contradiction. This proves that $\overline{\varphi_{\tau_1}^{\tau_2}(\Gamma_{\tau_2})}=H=\Gamma_{\tau_1}$ and thus $\varphi_{\tau_1}^{\tau_2}(\Gamma_{\tau_2})$ is a dense subgroup of Γ_{τ_1}.

DEFINITION 2.4. Let G^{τ} be an *LCA* group and let $\mathfrak{T}(G^{\tau}) \ni \tau_1, \tau_2$ with $\tau_1 \supseteqq \tau_2$, and let $\eta_{\tau_1}^{\tau_2}$ be the natural continuous isomorphism of G^{τ_1} onto G^{τ_2}. By the Lemma 2.3 we define the natural continuous isomorphism $\varphi_{\tau_1}^{\tau_2}$ of Γ_{τ_2} onto a dense subgroup of Γ_{τ_1} such that

$$(\eta_{\tau_1}^{\tau_2}(x), r)=(x, \varphi_{\tau_1}^{\tau_2}(r)) \qquad (x \in G^{\tau_1},\ r \in \Gamma_{\tau_2}).$$

THEOREM 2.4. *Suppose G^{τ} is an LCA group and $\mathfrak{T}(G^{\tau})\ni\tau_1, \tau_2$. If $L^1(G^{\tau_1})\cap L^1(G^{\tau_2})\neq\{0\}$, then we have $L^1(G^{\tau_1})=L^1(G^{\tau_2})$.*

PROOF. Put $L^1(G^{\tau_1})\cap L^1(G^{\tau_2})=I\neq 0$. Since $L^1(G^{\tau_1})$ and $L^1(G^{\tau_2})$ are translation invariant closed subspaces of $M(G^{\tau})$, I is also a translation invariant closed subspace of $M(G^{\tau})$, and hence of $L^1(G^{\tau_i})$ $(i=1,2)$. Therefore I is a closed ideal of $L^1(G^{\tau_i})$ $(i=1,2)$. Set $Z(I)=\{r\in\Gamma_{\tau_1}: \hat{f}(r)=0,\ f\in I\}$, where $\hat{f}$ denotes the Fourier transform of f. If $r\in\Gamma_{\tau}$, we have $L^1(G^{\tau_i})\varphi^{\tau}_{\tau_i}(r)\subseteqq L^1(G^{\tau_i})$ $(i=1,2)$, and hence $I\varphi^{\tau}_{\tau_i}(r)=I$. This implies that

$$Z(I)+\varphi^{\tau}_{\tau_1}(r)=Z(I)\qquad (r\in\Gamma_{\tau}).$$

Since $\varphi^{\tau}_{\tau_1}(\Gamma_{\tau})$ is dense in Γ_{τ_1}, $Z(I)$ is either ϕ or Γ_{τ_1}, and since $I\neq 0$ we conclude that $Z(I)=\phi$. By the general Tauberian theorem, we get $I=L^1(G^{\tau_1})$. In the same way we have $I=L^1(G^{\tau_2})$ and this completes the proof.

THEOREM 2.5. *Let G^{τ} be an LCA group and $\mathfrak{T}(G^{\tau})\ni\tau_1, \tau_2$. If $M(G^{\tau_1})\supseteqq L^1(G^{\tau_2})$, then we have $\tau_1\subseteqq\tau_2$.*

PROOF. Let η be the natural isomorphism from G^{τ_2} onto G^{τ_1}. We shall prove that η is continuous, and this will complete the proof.

Let $r\in\Gamma_{\tau_1}$, and there exists a unique $\varphi(r)\in\Gamma_{\tau_2}$ such that

$$\int_{G^{\tau_2}}\varphi(r)(-x)d\mu(x)=\int_{G^{\tau_1}}r(-x)d\mu(x)\qquad (\mu\in L^1(G^{\tau_2})).$$

We shall show that φ is continuous, and that r and $\varphi(r)$ induce the same function on the underlying group G. If these are proved, we can easily show that η is continuous. Thus for each neighbourhood $N(K,\varepsilon)=\{x\in G^{\tau_1}: |(x,r)-1|<\varepsilon,\ r\in K\}$ of 0 in G^{τ_1}, where K is a compact subset of Γ_{τ_1} and $\varepsilon>0$, $\varphi(K)$ is a compact set in Γ_{τ_2}, and $\eta(N(\varphi(K),\varepsilon))=N(K,\varepsilon)$, and hence η is continuous.

Let $\mu\in L^1(G^{\tau_2})$ and let $\hat{\mu}_{(1)}$ and $\hat{\mu}_{(2)}$ be the Fourier-Stieltjes transform of μ into Γ_{τ_1}, and the Fourier transform of μ into Γ_{τ_2}, respectively. Thus we have the relation

$$\hat{\mu}_{(2)}(\varphi(r))=\hat{\mu}_{(1)}(r)\qquad (r\in\Gamma_{\tau_1}).$$

If U is an open set in C, then $\hat{\mu}^{-1}_{(1)}(U)=\varphi^{-1}(\hat{\mu}^{-1}_{(2)}(U))$ is an open set in Γ_{τ_1}. Since $\hat{\mu}^{-1}_{(2)}(U)$ is open and the topology of Γ_{τ_2} is the weakest one such that each $\hat{\mu}_{(2)}$ is continuous, we conclude that φ is continuous.

If $r\in\varphi^{\tau}_{\tau_1}(\Gamma_{\tau})$, it is clear that r and $\varphi(r)$ induce the same function on G. For $r_0\in\Gamma_{\tau_1}$ and $x\in G^{\tau_2}$, let $N(K,\varepsilon)+\varphi(r_0)$ be a neighbourhood of $\varphi(r_0)$, where $\varepsilon>0$ and K is a compact set in G^{τ_2}, which contains x. Since φ is continuous, there exist a compact set K' in G^{τ_1} and $\varepsilon'>0$ such that $\varphi(N(K'\cup\eta(x),\varepsilon')+r_0)\subseteqq N(K,\varepsilon)+\varphi(r_0)$. Since $\varphi^{\tau}_{\tau_1}(\Gamma_{\tau})$ is dense in Γ_{τ_1}, we can choose an element r_1 in $(N(K'\cup\eta(x),\varepsilon')+r_0)\cap\varphi^{\tau}_{\tau_1}(\Gamma_{\tau})$, and we have

$$(2.2)\qquad \begin{aligned} &|(\eta(x), r_1)-(\eta(x), r_0)|<\varepsilon' \\ &|(x, \varphi(r_1))-(x, \varphi(r_0))|<\varepsilon . \end{aligned}$$

The fact that $r_1 \in \varphi_{\tau_1}^{\tau}(\Gamma_\tau)$ gives

$$(2.3)\qquad (\eta(x), r_1)=(x, \varphi(r_1)) .$$

From (2.2) and (2.3), we get

$$\begin{aligned} &|(\eta(x), r_0)-(x, \varphi(r_0))| \\ &\quad \leqq |(\eta(x), r_0)-(\eta(x), r_1)|+|(x, \varphi(r_1))-(x, \varphi(r_0))| \\ &\quad \leqq \varepsilon+\varepsilon' . \end{aligned}$$

Since we can take ε and ε' arbitrary, we have

$$(\eta(x), r_0)=(x, \varphi(r_0)) \qquad (x \in G^{\tau_2}),$$

and hence r_0 and $\varphi(r_0)$ induce the same function on G. This completes the proof of the theorem.

COROLLARY 2.6. *If $\mathfrak{T}(G^\tau) \ni \tau_1, \tau_2$ and $L^1(G^{\tau_1})=L^1(G^{\tau_2})$, then we have $\tau_1=\tau_2$.*

COROLLARY 2.7. *If $\tau_1, \tau_2 \in \mathfrak{T}(G^\tau)$ and $\tau_1 \neq \tau_2$, then we have $L^1(G^{\tau_1}) \perp L^1(G^{\tau_2})$.*

PROOF. Suppose that $L^1(G^{\tau_1})$ and $L^1(G^{\tau_2})$ are not orthogonal each other, and choose $\mu \in L^1(G^{\tau_1})$ and $\nu \in L^1(G^{\tau_2})$ such that μ is not orthogonal to ν. By Proposition 2.1 there exists a σ-compact set K in G^{τ_1} such that μ is concentrated in $\eta_{\tau_1}^{\tau}(K)$. If ν_1 is the restriction of ν to $\eta_{\tau_1}^{\tau}(K)$, then we have $0 \neq \nu_1 \in M(G^{\tau_1})$. Let $\nu_1=\nu_1'+\nu_1''$ be the Lebesgue decomposition of ν_1 such that $\nu_1' \ll \mu$, $\nu_1'' \perp \mu$. Then $\nu_1' \neq 0$ and $\nu_1' \in L^1(G^{\tau_1}) \cap L^1(G^{\tau_2})$, that is $L^1(G^{\tau_1}) \cap L^1(G^{\tau_2}) \neq 0$. From Theorem 2.4 we have $L^1(G^{\tau_1})=L^1(G^{\tau_2})$, and from Corollary 2.6 we have $\tau_1=\tau_2$, and this is a contradiction.

THEOREM 2.8. *If $\tau_1, \tau_2 \in \mathfrak{T}(G^\tau)$, then there exists a unique $\tau_3 \in \mathfrak{T}(G^\tau)$ such that $L^1(G^{\tau_1}) * L^1(G^{\tau_2}) \subseteqq L^1(G^{\tau_3})$. Moreover τ_3 enjoys the additional property such that $\tau_3 \subseteqq \tau_1, \tau_2$, and if $\tau_0 \in \mathfrak{T}(G^\tau)$ with $\tau_0 \subseteqq \tau_1, \tau_2$, then $\tau_0 \subseteqq \tau_3$.*

To prove the theorem we provide the following lemma. R^n denotes the n-dimensional Euclidean space, and Z denotes the discrete group of all rational integers.

LEMMA 2.9. *Let $H_1=R^p \times K_1$, $H_2=R^q \times K_2$ and $H=H_1 \times H_2/K$ be LCA groups, where p and q are non-negative integers, K_1 and K_2 are compact groups, and K is a closed subgroup of $H_1 \times H_2$. B_0 denotes the ring of all the bounded Borel sets of H, and f denotes the natural homomorphism of $H_1 \times H_2$ onto H.*

(i) *If φ denotes the projection of $H_1 \times H_2$ onto $R^p \times R^q$, then $\varphi(K)$ is a closed subgroup of $R^p \times R^q$, and hence there exists a basis $\{u_1, \cdots, u_{n_1}, \cdots, u_{n_2}, \cdots, u_{p+q}\}$ of the vector space $R^p \times R^q$ over R such that $\varphi(K)=\sum_{i=1}^{n_1} R u_i+\sum_{j=n_1+1}^{n_2} Z u_j$.*

(ii) *Put $V^{(r)}=\{x \in H_1 \times H_2:\ \varphi(x)=\sum_{i=1}^{p+q} \alpha_i u_i,\ 0 \leqq \alpha_i<1\ (i=1,2, \cdots, n_2),\ |\alpha_i|<r$*

$(i=n_2+1, \cdots, p+q)\}$, *for each positive number* r. *If* E *is an element of* B_0, *and if* r *and* r' *are positive numbers such that* $f(V^{(r)}) \supseteqq E$, $f(V^{(r')}) \supseteqq E$, *then*

$$f^{-1}(E) \cap V^{(r)} = f^{-1}(E) \cap V^{(r')} .$$

(iii) *For each* $E \in B_0$, *choose a positive number* r *such that* $f(V^{(r)}) \supseteqq E$, *and put*

$$m^*(E) = m(f^{-1}(E) \cap V^{(r)}) .$$

Then m^* *is well defined by* (ii), *and* m^* *is a non-negative finite translation invariant measure on* B_0.

(iv) *We can extend* m^* *to a Borel measure* $\bar{m}^*$ *of* H *in a unique way, and* $\bar{m}^*$ *is the Haar measure of* H.

PROOF. (i) Since the latter of (i) is well known, we only prove that $\varphi(K)$ is closed. Suppose x is an element of $\overline{\varphi(K)}-\varphi(K)$. We can choose a sequence $\{x_i\}_{i=1}^{\infty}$ of elements in K such that $\lim_{i\to\infty} \varphi(x_i)=x$. Let ψ be the projection of $H_1 \times H_2$ onto $K_1 \times K_2$. Then we have either $\{\psi(x_i): i=1,2,\cdots\}$ is a finite set, or $\{\psi(x_i): i=1,2,\cdots\}$ has accumulating points in $K_1 \times K_2$. In either cases $\{x_i\}=\{\varphi(x_i)+\psi(x_i)\}$ has an accumulating point z in $H_1 \times H_2$, and since K is closed, z belongs to K. Thus we have $x=\varphi(z) \in \varphi(K)$. This is a contradiction and hence we have $\overline{\varphi(K)}=\varphi(K)$.

(ii) Suppose $r' \geqq r$ and x is an element of $f^{-1}(E) \cap V^{(r')}$. Then $f(x)$ belongs to E, and since $f(V^{(r)}) \supseteqq E$ there exists an element y of $V^{(r)}$ such that $f(x) = f(y)$. We have $x-y \in K$ and so $\varphi(x)$ and $\varphi(y)$ differ only on $u_1, \cdots, u_{n_2}$ components, therefore $x \in V^{(r)}$. This shows that $f^{-1}(E) \cap V^{(r')} = f^{-1}(E) \cap V^{(r)}$.

(iii) That m^* is a non-negative finite measure is clear, and we only prove that m^* is translation invariant. Let $E \in B_0$, and let r be a positive number such that $f(V^{(r)}) \supseteqq E$, $E+\bar{x}$, where $\bar{x} \in H$. If we choose an element x in $f^{-1}(\bar{x})$, we have $(f^{-1}(E)+x) \cap V^{(r)} = f^{-1}(E+\bar{x}) \cap V^{(r)}$, and hence

$$\begin{aligned} m^*(E) &= m(f^{-1}(E) \cap V^{(r)}) = m((f^{-1}(E)+x) \cap V^{(r)}) \\ &= m(f^{-1}(E+\bar{x}) \cap V^{(r)}) = m^*(E+\bar{x}) . \end{aligned}$$

(iv) Since m^* is a finite non-negative translation invariant measure on B_0, we can extend m^* uniquely to a σ-finite translation invariant measure $\bar{m}^*$ on $S(B_0)$, the σ-ring generated by B_0. Since H is σ-compact, $S(B_0)$ is the class of all the Borel sets in H, and hence $\bar{m}^*$ is a Borel measure on H.

To prove that $\bar{m}^*$ is the Haar measure of H, we have only to prove that $\bar{m}^*$ is regular in the sense:

(a) For every open set U in H, we have

$$\bar{m}^*(U) = \sup \{\bar{m}^*(F): F \text{ is compact and } F \subseteqq U\} ,$$

(b) For each Borel set A in H, we have

$$\bar{m}^*(A)=\inf\{\bar{m}^*(U): U \text{ is open and } U \supseteqq A\}.$$

Suppose first that E is a bounded Borel set in H, r is a positive number such that $f(V^{(r)}) \supseteqq E$, and $\varepsilon>0$. There exists a compact subset F of $f^{-1}(E) \cap V^{(r)}$ such that

$$m(f^{-1}(E) \cap V^{(r)}) \leqq m(F)+\varepsilon.$$

Then $f(F)$ is a compact subset of H and $\bar{m}^*(f(F))+\varepsilon \geqq \bar{m}^*(E)$. Since H is σ-compact, this proves (a) for every open set in H. Next choose a bounded open set W which contains E, and by what we have proved in (a) there exists a compact set $F_1 \subseteqq W-E$ such that $\bar{m}^*(F_1)+\varepsilon \geqq \bar{m}^*(W-E)=\bar{m}^*(W)-\bar{m}^*(E)$, and so we have $\bar{m}^*(E)+\varepsilon \geqq \bar{m}^*(W-F_1)$, and again this proves (b) for every Borel set E in H.

PROOF OF THEOREM 2.8. Let H_i be an open subgroup of G^{τ_i} $(i=1, 2)$ such that

$$H_1 \cong R^p \times K_1, \qquad H_2 \cong R^q \times K_2,$$

where K_1 and K_2 are compact groups. We identify H_1 and H_2 with $R^p \times K_1$ and $R^q \times K_2$, respectively. Let f be a continuous homomorphism of $H_1 \times H_2$ into G^τ,

$$f;\quad H_1 \times H_2 \ni (x, y) \longmapsto x+y \in G^\tau.$$

We can introduce in $H=H_1+H_2=f(H_1 \times H_2)$ a locally compact group topology τ_3' in H such that f becomes an open continuous map of $H_1 \times H_2$ onto $H^{\tau_3'}$. This topology τ_3' in H can be extended uniquely to a locally compact group topology τ_3 in G such that H is open in G^{τ_3} and $\tau_3|_H=\tau_3'$. We shall show that if $\lambda \in L^1(G^{\tau_1})$, $\mu \in L^1(G^{\tau_2})$, then $\lambda * \mu \in L^1(G^{\tau_3})$ and this will complete the proof.

First suppose that λ is concentrated in H_1 and μ is concentrated in H_2. Then $\lambda * \mu$ is concentrated in H. Since $\tau_3 \subseteqq \tau_1, \tau_2$, and by Proposition 2.1 we have $L^1(G^{\tau_1}) * L^1(G^{\tau_2}) \subseteqq M(G^{\tau_3})$. Thus we have only to show that $\lambda * \mu$ is absolutely continuous with respect to the Haar measure of G^{τ_3}. We remark here that the Haar measure of $H^{\tau_3'}$ is obtained by restricting the Haar measure of G^{τ_3} to H. The same relation also holds between G^{τ_i} and H_i $(i=1, 2)$. We apply the preceding lemma for the present H_1, H_2 and the closed subgroup $K=\{(x, y) \in H_1 \times H_2 : x+y=0\}$ of $H_1 \times H_2$ and introduce the Haar measure $\bar{m}^*$ on $H_1 \times H_2/K \cong H^{\tau_3'}$. We extend $\bar{m}^*$ to the Haar measure of G^{τ_3} and we also represent it by $\bar{m}^*$.

To prove that $\lambda * \mu$ is absolutely continuous with respect to $\bar{m}^*$, suppose first that E is a bounded Borel set in $H^{\tau_3'}$ with $\bar{m}^*(E)=0$. We can suppose without loss of generality that $\lambda \geqq 0$ and $\mu \geqq 0$. For each $\varepsilon>0$, there exist a compact set C_i in H_i $(i=1, 2)$, $\lambda' \in L^1(G^{\tau_1})$, $\mu' \in L^1(G^{\tau_2})$, and $d>0$, such that

$$\begin{cases} dm_1|_{C_1} \geqq \lambda' \geqq 0\,, \\ dm_2|_{C_2} \geqq \mu' \geqq 0\,, \\ \|\lambda * \mu - \lambda' * \mu'\| < \varepsilon\,, \end{cases}$$

where m_i denotes the Haar measure of H_i $(i=1,2)$, and $dm_i|_{C_i}$ $(\mathrm{i}=1,2)$ denotes the restriction of dm_i to C_i. Choose a positive number r such that $f(V^{(r)}) \supseteqq E$, and a finite number of elements $x_1, x_2, \cdots, x_t \in H_1 \times H_2$ such that $\bigcup_{i=1}^{t} (V^{(r)}+x_i) \supseteqq C_1 \times C_2$. Then

$$\begin{aligned} \lambda * \mu(E) &\leqq \lambda' * \mu'(E) + \varepsilon \\ &\leqq (dm_1|_{C_1}) * (dm_2|_{C_2})(E) + \varepsilon \\ &= d^2 (m_1|_{C_1}) \times (m_2|_{C_2})(E_{(2)}) + \varepsilon \\ &= d^2 (m_1 \times m_2)(f^{-1}(E) \cap C_1 \times C_2) + \varepsilon \\ &\leqq d^2 \sum_{i=1}^{t} (m_1 \times m_2)(f^{-1}(E) \cap (V^{(r)}+x_i)) + \varepsilon \\ &= d^2 \sum_{i=1}^{t} (m_1 \times m_2)(f^{-1}(E-f(x_i)) \cap V^{(r)}) + \varepsilon \\ &\leqq d^2 \sum_{i=1}^{t} \bar{m}^*(E-f(x_i)) + \varepsilon \\ &= \varepsilon\,, \end{aligned}$$

where we put $E_{(2)} = \{(x, y) \in G^{\tau_3} \times G^{\tau_3} : x+y \in E\}$. Since $\varepsilon > 0$ was arbitrary, we have $\lambda * \mu(E) = 0$. If $\bar{m}^*(E) = 0$ for a Borel set in G^{τ_3}, then E is a union of a subset of $G^{\tau_3} - H$ and a countably many bounded Borel sets in $H^{\tau'_3}$, and so $\lambda * \mu(E) = 0$.

Next let us consider the general case. Since λ and μ are regular, they are concentrated in at most countably many cosets of H_1 and H_2, respectively. Thus we may assume without loss of generality that λ is concentrated in $H_1 + x$, and μ is concentrated in $H_2 + y$, where $x \in G^{\tau_1}$, and $y \in G^{\tau_2}$. Let $\lambda - x$ and $\mu - y$ be the translations of λ and μ by x and y respectively, that is $(\lambda - x)(E - x) = \lambda(E)$, etc. Then we have

$$\lambda * \mu(E) = ((\lambda - x) * (\mu - y))(E - x - y) \tag{2.4}$$

and if $\bar{m}^*(E) = 0$, the right side of (2.4) is 0 by the above result, and hence $\lambda * \mu \in L^1(G^{\tau_3})$. The uniqueness of τ_3 follows from Corollary 2.7.

Now let us prove the remainder of the assertions of the theorem and complete the proof.

Suppose that $\tau_0 \in \mathfrak{T}(G^{\tau})$ and $\tau_0 \leqq \tau_1, \tau_2$. Then we have $M(G^{\tau_0}) \supset L^1(G^{\tau_1})$, $L^1(G^{\tau_2})$, and hence $M(G^{\tau_0}) \supset L^1(G^{\tau_1}) * L^1(G^{\tau_2})$. Let $\mathfrak{A}$ be the closed subspace generated by $\{\lambda * \mu : \lambda \in L^1(G^{\tau_1}),\ \mu \in L^1(G^{\tau_2})\}$. $\mathfrak{A}$ is a translation invariant subspace and hence an ideal of $L^1(G^{\tau_3})$. It is easy to see that $Z(\mathfrak{A})=\{r\in\Gamma_{\tau_3}: \hat{\nu}(r)=0,\ \nu\in\mathfrak{A}\}=\phi$, and from the general Tauberian theorem we have $\mathfrak{A}=L^1(G^{\tau_3})$, and so $L^1(G^{\tau_3})\subset M(G^{\tau_0})$. From Theorem 2.5 we get $\tau_3\supset\tau_0$ and this completes the proof of Theorem 2.8.

DEFINITION 2.5. Let G^τ be an LCA group. By Theorem 2.8 $\sum_{\tau'\in\mathfrak{T}(G^\tau)} L^1(G^{\tau'})$ is a subalgebra and hence $\overline{\sum_{\tau'\in\mathfrak{T}(G^\tau)} L^1(G^{\tau'})}$ is a closed subalgebra of $M(G^\tau)$, which we denote by $L^*(G^\tau)$. $L^*(G^\tau)$ contains the identity of $M(G^\tau)$, and hence $L^*(G^\tau)$ properly contains $L^1(G^\tau)$ if G^τ is not discrete.

§ 3. The maximal ideal space of $L^*(G^\tau)$.

If μ is an element of $L^*(G^\tau)$, we denote by $\hat{\mu}$ the Gelfand transform of μ.

DEFINITION 3.1. Let G^τ be an LCA group. We introduce a partial order $\geqq$ in $\mathfrak{T}(G^\tau)$ such that, if $\tau_1,\tau_2\in\mathfrak{T}(G^\tau)$ then $\tau_1\geqq\tau_2$ if and only if $\tau_1\subset\tau_2$. $\mathfrak{T}(G^\tau)$ is a directed set under this binary relation $\geqq$, that is for each pair $\tau_1,\tau_2\in\mathfrak{T}(G^\tau)$, there exists $\tau_3\in\mathfrak{T}(G^\tau)$ such that $\tau_3\geqq\tau_1,\tau_2$ (cf. Theorem 2.8). A directed subset S of $\mathfrak{T}(G^\tau)$ is a non-empty subset of $\mathfrak{T}(G^\tau)$ such that; 1) S is itself a directed set under $\geqq$; 2) If $S\ni\tau_1$, $\mathfrak{T}(G^\tau)\ni\tau_2$ and $\tau_1\geqq\tau_2$, then we have $\tau_2\in S$.

PROPOSITION 3.1. *Let G^τ be an LCA group and let h be a non-zero complex homomorphism of $L^*(G^\tau)$. Then*

1) $S=\{\tau'\in\mathfrak{T}(G^\tau): h|_{L^1(G^{\tau'})}\neq 0\}$ *is a directed subset of* $\mathfrak{T}(G^\tau)$.

2) *If $\tau_1,\tau_2\in S$ and $\tau_1\geqq\tau_2$, with*

$$h(\lambda)=\int_{G^{\tau_1}} r_{\tau_1}(-x)d\lambda(x) \qquad (\lambda\in L^1(G^{\tau_1}))\,,$$

$$h(\mu)=\int_{G^{\tau_2}} r_{\tau_2}(-x)d\mu(x) \qquad (\mu\in L^1(G^{\tau_2}))\,,$$

where $r_{\tau_1}\in\Gamma_{\tau_1}$, $r_{\tau_2}\in\Gamma_{\tau_2}$, then $\varphi^{\tau_1}_{\tau_2}(r_{\tau_1})=r_{\tau_2}$.

3) *Conversely if S is a directed subset of $\mathfrak{T}(G^\tau)$, and if $(r_{\tau'})_{\tau'\in S}$ is an element of $\prod_{\tau'\in S}\Gamma_{\tau'}$ such that*

$$\varphi^{\tau_1}_{\tau_2}(r_{\tau_1})=r_{\tau_2} \qquad (\tau_1,\ \tau_2\in S \text{ and } \tau_1\geqq\tau_2)\,,$$

then $(r_{\tau'})_{\tau'\in S}$ induces a non-zero complex homomorphism h' of $L^*(G^\tau)$ such that

$$h'(\lambda)=\begin{cases}\int_{G^{\tau'}} r_{\tau'}(-x)d\lambda(x): & \lambda\in L^1(G^{\tau'}),\ \tau'\in S\,,\\ 0 \qquad\qquad : & \lambda\in L^1(G^{\tau'}),\ \tau'\notin S\,.\end{cases} \tag{3.1}$$

PROOF. 1) Since $h\neq 0$, it is clear that S is not empty. If $S\ni\tau_1,\tau_2$ then there exist $\lambda\in L^1(G^{\tau_1})$ and $\mu\in L^1(G^{\tau_2})$ such that $h(\lambda)\neq 0$, $h(\mu)\neq 0$, and hence $h(\lambda*\mu)\neq 0$. By Theorem 2.8 there exists $\tau_3\in\mathfrak{T}(G^\tau)$ such that $\tau_3\geqq\tau_1,\tau_2$ and $\lambda*\mu\in L^1(G^{\tau_3})$, and so $\tau_3\in S$.

If $\tau_1 \in S$, $\tau_2 \in \mathfrak{T}(G^\tau)$ and $\tau_1 \geqq \tau_2$, then there exist $r_{\tau_1} \in \Gamma_{\tau_1}$ and $\lambda_1 \in L^1(G^{\tau_1})$ such that

$$\begin{cases} h(\lambda) = \int_{G^{\tau_1}} r_{\tau_1}(-x) d\lambda(x) \qquad (\lambda \in L^1(G^{\tau_1})), \\ h(\lambda_1) = \int_{G^{\tau_1}} r_{\tau_1}(-x) d\lambda_1(x) \neq 0 . \end{cases}$$

Choose $\mu_1 \in L^1(G^{\tau_2})$ such that

$$\int_{G^{\tau_2}} \varphi_{\tau_2}^{\tau_1}(r_{\tau_1})(-x) d\mu_1(x) \neq 0 .$$

Then we have

$$\begin{aligned} h(\lambda_1)h(\mu_1) &= h(\lambda_1 * \mu_1) \\ &= \int_{G^{\tau_1}} r_{\tau_1}(-x) d\lambda_1 * \mu_1(x) \\ &= \int_{G^{\tau_1}} r_{\tau_1}(-x) d\lambda_1(x) \int_{G^{\tau_1}} r_{\tau_1}(-x) d\mu_1(x) \\ &= \int_{G^{\tau_1}} r_{\tau_1}(-x) d\lambda_1(x) \int_{G^{\tau_2}} \varphi_{\tau_2}^{\tau_1}(r_{\tau_1})(-x) d\mu_1(x) \\ &\neq 0 . \end{aligned} \tag{3.2}$$

Therefore we have $h(\mu_1) \neq 0$, and hence τ_2 belongs to S.

2) If $\tau_1, \tau_2 \in S$ and $\tau_1 \geqq \tau_2$, then we have from (3.2)

$$h(\mu_1) = \int_{G^{\tau_2}} \varphi_{\tau_2}^{\tau_1}(r_{\tau_1})(-x) d\mu_1(x) \qquad (\mu_1 \in L^1(G^{\tau_2}))$$

and hence we have $\varphi_{\tau_2}^{\tau_1}(r_{\tau_1}) = r_{\tau_2}$.

3) Since $L^*(G^\tau) = \overline{\sum_{\tau' \in \mathfrak{T}(G^\tau)} L^1(G^{\tau'})}$, it is obvious from Corollary 2.7 that there exists a linear functional h' such that (3.1) holds. We shall show that h' is a complex homomorphism of $L^*(G^\tau)$.

Let $\tau_1, \tau_2 \in \mathfrak{T}(G^\tau)$, and let $\lambda \in L^1(G^{\tau_1})$ and $\mu \in L^1(G^{\tau_2})$. We have only to prove that $h'(\lambda * \mu) = h'(\lambda)h'(\mu)$. By Theorem 2.8 there exists $\tau_3 \in \mathfrak{T}(G^\tau)$ such that $\lambda * \mu \in L^1(G^{\tau_3})$ and $\tau_3 \geqq \tau_1, \tau_2$. If $\tau_1 \notin S$, then τ_3 does not belong to S, and we have

$$h'(\lambda * \mu) = h'(\lambda)h'(\mu) = 0 . \tag{3.3}$$

If $\tau_2 \notin S$, we can prove the same relation as (3.3). If $\tau_1 \in S$ and $\tau_2 \in S$, then by Theorem 2.8 τ_3 belongs to S, and

$$\begin{aligned} h'(\lambda * \mu) &= \int_{G^{\tau_3}} r_{\tau_3}(-x) d\lambda * \mu(x) \\ &= \int_{G^{\tau_3}} r_{\tau_3}(-x) d\lambda(x) \int_{G^{\tau_3}} r_{\tau_3}(-x) d\mu(x) \end{aligned}$$

$$=\int_{G^{\tau_1}} r_{\tau_1}(-x)\,d\lambda(x)\int_{G^{\tau_2}} r_{\tau_2}(-x)\,d\mu(x)$$
$$=h'(\lambda)h'(\mu),$$

and this completes the proof.

DEFINITION 3.2. If S is a directed subset of $\mathfrak{T}(G^\tau)$, then

$$\Gamma_S=\{(r_{\tau'})_{\tau'\in S}\in\prod_{\tau'\in S}\Gamma_{\tau'}:\ \varphi_{\tau_2}^{\tau_1}(r_{\tau_1})=r_{\tau_2},\ \text{if}\ \tau_1\geqq\tau_2\,;\ \tau_1,\tau_2\in S\}$$

forms a group with respect to the pointwise addition. By Proposition 3.1, $\Gamma^*=\bigcup_{S\subset\mathfrak{T}(G^\tau)}\Gamma_S$ constitutes the maximal ideal space of $L^*(G^\tau)$.

If S is a directed subset of $\mathfrak{T}(G^\tau)$ and $\tau_0\in S$, we denote by $\varphi_{\tau_0}^S$ the natural homomorphism of Γ_S into Γ_{τ_0}, given by

$$\varphi_{\tau_0}^S((r_{\tau'})_{\tau'\in S})=r_{\tau_0}\qquad((r_{\tau'})_{\tau'\in S}\in\Gamma_S)\,. \tag{3.4}$$

PROPOSITION 3.2. *For each* $\Gamma_{S_1}\times\Gamma_{S_2}\ni((r_{\tau'})_{\tau'\in S_1},(r'_{\tau'})_{\tau'\in S_2})$, *we define*

$$(r_{\tau'})_{\tau'\in S_1}+(r'_{\tau'})_{\tau'\in S_2}=(r_{\tau'}+r'_{\tau'})_{\tau'\in S_1\cap S_2}\,. \tag{3.5}$$

Then Γ^* *becomes a semi-group with unit.*

PROOF. Since intersection of two directed subsets of $\mathfrak{T}(G^\tau)$ is again a directed subset of $\mathfrak{T}(G^\tau)$, it is obvious that Γ^* forms a semi-group with unit $(0_{\tau'})_{\tau'\in\mathfrak{T}(G^\tau)}$, where $0_{\tau'}$ is the unit of $\Gamma_{\tau'}$.

PROPOSITION 3.3. *Suppose that* $\Gamma^*\supseteqq\Gamma_S\ni r_0$. *For each* $\tau_0\in S$, *a neighbourhood* U *of* $\varphi_{\tau_0}^S(r_0)$ *in* Γ_{τ_0} *and a finite subset* $\{\tau_1,\tau_2,\cdots,\tau_m\}$ *of* $\mathfrak{T}(G^\tau)-S$, *and a compact subset* K_i *of* Γ_{τ_i} $(i=1,2,\cdots,m)$, *put*

$$U_{\tau_0}^{(K_1,\tau_1),(K_2,\tau_2),\cdots,(K_m,\tau_m)}=\bigcup_{S'\ni\tau_0}\{r\in\Gamma_{S'}:\ \varphi_{\tau_0}^{S'}(r)\in U,\ \text{and if}\ S'\ni\tau_i\ \text{then}\ \varphi_{\tau_i}^{S'}(r)\notin K_i\ (i=1,\cdots,m)\}. \tag{3.6}$$

Then the class of all the sets of the form (3.6) *constitutes a basis of neighbourhoods of* r_0 *with respect to the Gelfand topology of* Γ^*.

PROOF. The Gelfand topology of Γ^* is the weakest one such that every Gelfand transform $\hat{\mu}$ $(\mu\in L^*(G^\tau))$ is continuous on Γ^*. Since each element $\hat{\mu}$ $(\mu\in L^*(G^\tau))$ is a uniform limit of some sequence of elements in $\{\hat{\hat{\lambda}}:\ \lambda\in\sum_{\tau'\in\mathfrak{T}(G^\tau)}L^1(G^{\tau'})\}$, it can be said that the Gelfand topology of Γ^* is the weakest one such that each $\hat{\mu}$ $(\mu\in L^1(G^{\tau'}):\tau'\in\mathfrak{T}(G^\tau))$ is continuous on Γ^*.

Suppose $\tau_*\in\mathfrak{T}(G^\tau)$, $\mu\in L^1(G^{\tau_*})$, and W is a neighbourhood of $\hat{\mu}(r_0)$ in C, where $W\not\ni 0$ if $\hat{\mu}(r_0)\neq 0$. If $\tau_*\notin S'$, $\hat{\mu}(r)=0$ for every $r\in\Gamma_{S'}$. If $\tau_*\in S'$, then $\hat{\mu}(r)=\hat{\mu}(\varphi_{\tau_*}^{S'}(r))$, where $\hat{\mu}$ is the Fourier transform of μ into Γ_{τ_*}. Thus we have

$$(3.7)\qquad \hat{\mu}^{-1}(W)=\begin{cases}(\bigcup_{S'\not\ni\tau_*}\Gamma_{S'})\cup[\bigcup_{S'\ni\tau_*}\{r\in\Gamma_{S'}:\ \varphi_{\tau_*}^{S'}(r)\in\hat{\mu}^{-1}(W)\}]: & \text{if } \hat{\mu}(r_0)=0\\ \bigcup_{S'\ni\tau_*}\{r\in\Gamma_{S'}:\ \varphi_{\tau_*}^{S'}(r)\in\hat{\mu}^{-1}(W)\}: & \text{if } \hat{\mu}(r_0)\neq 0\,.\end{cases}$$

Suppose $\tau_1,\tau_2,\cdots,\tau_m\in\mathfrak{T}(G^\tau)-S$, $\tau_{m+1},\tau_{m+2},\cdots,\tau_n\in S$ $(m<n)$ and $\mu_1\in L^1(G^{\tau_1}),\cdots,\mu_n\in L^1(G^{\tau_n})$, and let W_i be an open neighbourhood of $\hat{\mu}_i(r_0)$ $(i=1,2,\cdots,n)$. Let $\tau_0\in\mathfrak{T}(G^\tau)$ be the least upper bound of $\{\tau_{m+1},\cdots,\tau_n\}$ (cf. Theorem 2.8). Since $\varphi_{\tau_i}^{\tau_0}$ is continuous and $\varphi_{\tau_i}^{\tau_0}\circ\varphi_{\tau_0}^{S}=\varphi_{\tau_i}^{S}$, $U=\bigcap_{i=m+1}^{n}\varphi_{\tau_i}^{\tau_0\,-1}(\hat{\mu}_i^{-1}(W_i))$ is a neighbourhood of $\varphi_{\tau_0}^{S}(r_0)$, and we have from (3.7)

$$(3.8)\qquad U_{\tau_0}=\bigcup_{S'\ni\tau_0}\{r\in\Gamma_{S'}:\ \varphi_{\tau_0}^{S'}(r)\in U\}\subseteqq\hat{\mu}_{m+1}^{-1}(W_{m+1})\cap\cdots\cap\hat{\mu}_n^{-1}(W_n)\,.$$

Put $(\hat{\mu}_j^{-1}(W_j))^c=K_j$ $(j=1,2,\cdots,m)$, and since W_j is an open neighbourhood of 0 $(j=1,2,\cdots,m)$, K_j is a compact subset of Γ_{τ_j}. By (3.7) we have

$$(3.9)\qquad \hat{\mu}_j^{-1}(W_j)=(\bigcup_{S'\not\ni\tau_j}\Gamma_{S'})\cup[\bigcup_{S'\ni\tau_j}\{r\in\Gamma_{S'}:\ \varphi_{\tau_j}^{S'}(r)\notin K_j\}]\qquad (j=1,2,\cdots,m)\,.$$

If we put $U_{\tau_0}^{(K_1,\tau_1),\cdots,(K_m,\tau_m)}$ as (3.6), we get from (3.8) and (3.9)

$$U_{\tau_0}^{(K_1,\tau_1),\cdots,(K_m,\tau_m)}\subseteqq\bigcap_{j=1}^{n}\hat{\mu}_j^{-1}(W_j)\,.$$

Conversely, let $\tau_0\in S$, $\tau_1,\cdots,\tau_m\in\mathfrak{T}(G^\tau)-S$, and let U be a neighbourhood of $\varphi_{\tau_0}^{S}(r_0)$, and suppose K_j is a compact subset of Γ_{τ_j} $(j=1,2,\cdots,m)$. Then we can choose $\mu_i\in L^1(G^{\tau_i})$ $(i=0,1,\cdots,m)$ and a neighbourhood V of $\hat{\mu}_0(r_0)\in C$ such that

$$\begin{cases}\hat{\mu}_0(\varphi_{\tau_0}^{S}(r_0))\neq 0\,,\\ U\supseteqq\hat{\mu}_0^{-1}(V)\,,\qquad V\not\ni 0\,,\\ \hat{\mu}_j(r)\geqq 1\ (r\in K_j)\,,\qquad (j=1,2,\cdots,m)\,.\end{cases}$$

Then we get

$$U_{\tau_0}^{(K_1,\tau_1),\cdots,(K_m,\tau_m)}\supseteqq[\bigcap_{j=1}^{m}\hat{\mu}_j^{-1}(\Delta)]\cap\hat{\mu}_0^{-1}(V)\,,$$

where $\Delta=\{\alpha\in C:\ |\alpha|<1\}$, and hence the set of the form (3.6) is a neighbourhood of r_0.

What we have proved above and the fact that

$$\{\hat{\mu}^{-1}(W):\ \mu\in L^1(G^{\tau'}),\ \tau'\in\mathfrak{T}(G^\tau),\ W\ni\hat{\mu}(r_0)\}$$

forms a sub-basis of neighbourhoods of r_0 show that the class of the set of the form (3.6) constitutes a basis of neighbourhoods of r_0 in Γ^*.

REMARK. If τ_0 is an element of $\mathfrak{T}(G^\tau)$, then $S_{\tau_0}=\{\tau'\in\mathfrak{T}(G^\tau):\ \tau'\leqq\tau_0\}$ is a directed subset of $\mathfrak{T}(G^\tau)$. It is easy to see from Proposition 3.3 that $\varphi_{\tau_0}^{S_{\tau_0}}$ is a homeomorphic isomorphism from $\Gamma_{S_{\tau_0}}$ (as a subspace of Γ^*) onto Γ_{τ_0}.

PROPOSITION 3.4. *Suppose S is a directed subset of $\mathfrak{T}(G^{\tau})$ and μ is an element of $M(G^{\tau})$. Then there exists a unique decomposition $\mu=\mu_1+\mu_2$, where $\mu_1\in\overline{\sum_{\tau'\in S}M(G^{\tau'})}$ and $\mu_2\perp\overline{\sum_{\tau'\in S}M(G^{\tau'})}$.*

PROOF. We can assume without loss of generality that $\mu\geqq 0$. Put $\Sigma=\{\mu'\in\overline{\sum_{\tau'\in S}M(G^{\tau'})}: \mu'\perp(\mu-\mu')\}$. It is clear that Σ is an inductive set with respect to the usual partial order in $M(G^{\tau})$, and so there exists a maximal element in Σ. Let μ_1 be a maximal element in Σ, and put $\mu_2=\mu-\mu_1$.

If there exists $\tau_0\in S$ such that μ_2 is not orthogonal to $M(G^{\tau_0})$, then by Proposition 2.2, there is a decomposition

$$\mu_2=\mu_2'+\mu_2'', \qquad 0\neq\mu_2'\in M(G^{\tau_0}), \qquad \mu_2'\perp\mu_2''.$$

Then $\mu_1+\mu_2'\in\Sigma$, and $\mu_1+\mu_2'\gneqq\mu_1$, and this contradicts the maximality of μ_1 and thus $\mu=\mu_1+\mu_2$ is the desired decomposition.

THEOREM 3.5. *Each complex homomorphism of $L^*(G^{\tau})$ can be extended to a complex homomorphism of $M(G^{\tau})$, and so Γ^* is contained in the maximal ideal space of $M(G^{\tau})$.*

PROOF. Let S be a directed subset of $\mathfrak{T}(G^{\tau})$, and suppose $\mu\in M(G^{\tau})$. Then by Proposition 3.4, we have a decomposition

$$\mu=\mu_1+\mu_2, \qquad \mu_1\in\overline{\sum_{S\ni\tau'}M(G^{\tau'})}, \qquad \mu_2\perp\overline{\sum_{S\ni\tau'}M(G^{\tau'})}.$$

μ_1 has an expression $\mu_1=\lim_{i\to\infty}\mu_{1i}$, where $\mu_{1i}\in M(G^{\tau_i})$, $S\ni\tau_i$ $(i=1, 2, \cdots)$. Define a function $\hat{\hat{\mu}}$ by

$$\hat{\hat{\mu}}(r)=\lim_{i\to\infty}\int_{G^{\tau_i}}\varphi_{\tau_i}^{S}(r)(-x)d\mu_{1i}(x) \qquad (r\in\Gamma_S,\ \mu\in M(G^{\tau})). \tag{3.10}$$

It is clear that the above definition is well posed and $\hat{\hat{\mu}}$ is equal to the Gelfand transform of μ if μ is an element of $L^*(G^{\tau})$. For each fixed $r\in\Gamma^*$, the mapping

$$M(G^{\tau})\ni\mu\longmapsto\hat{\hat{\mu}}(r)\in C$$

is a complex homomorphism, and hence Γ^* is contained in the maximal ideal space of $M(G^{\tau})$.

§ 4. Homomorphisms of $L^*(G^{\tau})$ into $M(G_2)$.

Let h be a homomorphism of $L^*(G^{\tau})$ into $M(G_2)$. For each $r\in\Gamma_2$, we have either $\widehat{h(\mu)}(r)=0$ for every $\mu\in L^*(G^{\tau})$, or there exists a unique $\alpha(r)\in\Gamma^*$ such that

$$\widehat{h(\mu)}(r)=\hat{\mu}(\alpha(r)) \qquad (\mu\in L^*(G^{\tau})). \tag{4.1}$$

We put

$$Y=\{r\in\Gamma_2 : {}^{\exists}\mu\in L^*(G^\tau),\ \widehat{h(\mu)}(r)\neq 0\}\,. \tag{4.2}$$

For each $\tau'\in\mathfrak{T}(G^\tau)$, we define

$$Y_{\tau'}=\bigcup_{S\ni\tau'}\{r\in Y:\ \alpha(r)\in\Gamma_S\} \tag{4.3}$$

$$\alpha_{\tau'}(r)=\begin{cases}\varphi^S_{\tau'}(\alpha(r)): & r\in Y_{\tau'}\\ 0 \quad : & r\notin Y_{\tau'}\,.\end{cases}$$

THEOREM 4.1. (i) *Let h be a homomorphism of $L^*(G^\tau)$ into $M(G_2)$, and let $\{(Y,\alpha),(Y_{\tau'},\alpha_{\tau'});\ \tau'\in\mathfrak{T}(G^\tau)\}$ be defined by* (4.1), (4.2) *and* (4.3). *Then*

1) *$Y_{\tau'}$ is an element of the coset ring of Γ_2, and $\alpha_{\tau'}$ is a piecewise affine map of $Y_{\tau'}$ into $\Gamma_{\tau'}$.*

2) *If we express by $h_{\tau'}$ a homomorphism of $L^1(G^{\tau'})$ into $M(G_2)$ determined by $(Y_{\tau'},\alpha_{\tau'})$, then $\{\|h_{\tau'}\|:\ \tau'\in\mathfrak{T}(G^\tau)\}$ is bounded, where $\|h_{\tau'}\|$ denotes $\sup_{\mu\in L^1(G^{\tau'})}\|h_{\tau'}(\mu)\|/\|\mu\|$.*

(ii) *Conversely, let Y be a subset of Γ_2 and let α be a map of Y into Γ^*. We define $Y_{\tau'},\alpha_{\tau'}$ $(\tau'\in\mathfrak{T}(G^\tau))$ by* (4.3). *Suppose that $\{(Y_{\tau'},\alpha_{\tau'}):\ \tau'\in\mathfrak{T}(G^\tau)\}$ satisfies* 1), 2) *of* (i). *Then for each $\mu\in L^*(G^\tau)$, there exists an element $h'(\mu)$ of $M(G_2)$ such that*

$$\widehat{h'(\mu)}(r)=\begin{cases}\hat\mu(\alpha(r)): & r\in Y\\ 0 \quad : & r\notin Y\end{cases}\qquad (r\in\Gamma_2)$$

and h' is a homomorphism of $L^*(G^\tau)$ into $M(G_2)$.

PROOF. (i) For each $\tau'\in\mathfrak{T}(G^\tau)$, let $h_{\tau'}$ be the restriction of h to $L^1(G^{\tau'})$. By Theorem 1, there exists an element $Y'_{\tau'}$ of the coset ring of Γ_2 and a piecewise affine map $\alpha'_{\tau'}$ of $Y'_{\tau'}$ into $\Gamma_{\tau'}$ such that

$$\widehat{h(\mu)}(r)=\widehat{h_{\tau'}(\mu)}(r)=\begin{cases}\hat\mu(\alpha'_{\tau'}(r)): & r\in Y'_{\tau'}\\ 0 \quad : & r\notin Y'_{\tau'}\end{cases}\qquad (\mu\in L^1(G^{\tau'}))\,. \tag{4.4}$$

On the other hand, we have from the definition of $Y_{\tau'}$ and $\alpha_{\tau'}$,

$$\widehat{h(\mu)}(r)=\begin{cases}\hat\mu(\alpha(r))=\hat\mu(\varphi^S_{\tau'}(\alpha(r))): & r\in Y_{\tau'}\\ 0 \quad : & r\notin Y_{\tau'}\end{cases}\qquad (\mu\in L^1(G^{\tau'}))\,. \tag{4.5}$$

From (4.4) and (4.5), we have $Y'_{\tau'}=Y_{\tau'}$ and $\alpha'_{\tau'}=\alpha_{\tau'}$, and 1) follows from this, and since 2) is trivial, this completes the proof of (i).

(ii) For each $\mu\in L^*(G^\tau)$, put

$$\alpha_\mu(r)=\begin{cases}\hat\mu(\alpha(r)): & r\in Y\\ 0 \quad : & r\notin Y\end{cases}\qquad (r\in\Gamma_2)\,.$$

Suppose τ_0 is an element of $\mathfrak{T}(G^\tau)$, and $\mu \in L^1(G^{\tau_0})$. Then by the definition of $(Y_{\tau_0}, \alpha_{\tau_0})$, we have

$$\alpha_\mu(r) = \begin{cases} \hat{\mu}(\alpha_{\tau_0}(r)) : & r \in Y_{\tau_0} \\ 0 \qquad : & r \notin Y_{\tau_0}, \end{cases}$$

and by the condition 1) of (i), $\alpha_\mu \in B(\Gamma_2)$. Therefore we have $\alpha_\mu \in B(\Gamma_2)$ for each $\mu \in \sum_{\tau' \in \mathfrak{T}(G^\tau)} L^1(G^{\tau'})$.

If $\mu \in L^*(G^\tau)$, choose a sequence of elements $\mu_i \in \sum_{\tau' \in \mathfrak{T}(G^\tau)} L^1(G^{\tau'})$ $(i=1, 2, \cdots)$ such that $\lim_{i \to \infty} \mu_i = \mu$, and since α_μ is the uniform limit of $\{\alpha_{\mu_i} : i=1, 2, \cdots\}$, we have $\alpha_\mu \in B(\Gamma_2)$.

Thus for each $\mu \in L^*(G^\tau)$, there exists a unique $h'(\mu) \in M(G_2)$ such that $\alpha_\mu = \widehat{h'(\mu)}$, and it is easy to see that

$$h' : \quad L^*(G^\tau) \ni \mu \longmapsto h'(\mu) \in M(G_2)$$

is the desired homomorphism of $L^*(G^\tau)$ into $M(G_2)$ and this completes the proof of the theorem.

REMARKS. If G^τ is not discrete, it is easy to see that $L^*(G^\tau)$ is symmetric, and hence $L^*(G^\tau)$ is contained properly in $M(G^\tau)$. Thus $L^*(G^\tau)$ contains $L^1(G^\tau)$ properly, and is contained in $M(G^\tau)$ properly, if G^τ is not discrete.

It is natural to think about how large the set $\mathfrak{T}(G^\tau)$ is. For this we can refer to [5].

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