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Pseudogroups associated with the one dimensional foliation group (II)

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CHAPTER III: THE EQUIVALENCE PROBLEM AND THE PSEUDOGROUPS

The purpose of this chapter is to complete the proof of the theorem below:

THEOREM 14. *The following is a complete list of all pseudogroups associated with the one-dimensional foliation subgroup of* $GL(N, R)$. (Note: We omit, throughout this list, the conditions $\frac{\partial f^1}{\partial x_1}>0$ and $\det\left(\frac{\partial f^i}{\partial x_j}\right)>0$, which guarantee that $\left(\frac{\partial f^i}{\partial x_j}\right)$ belongs to the *connected* group G^1.)

1. $$\begin{cases} y_1=\sum_{j=1}^{N} a_{1j}x_j+b_1 \\ y_i=\sum_{j=2}^{N} a_{ij}x_j+b_i \end{cases}$$

2. $$\begin{cases} y_1=cx_1+g(x_2, \cdots, x_n) \\ y_i=\sum_{j=2}^{N} a_{ij}x_j+b_i \end{cases}$$

2n. $$\begin{cases} y_1=cx_1+P_n(x_2, \cdots, x_n) \\ y_i=\sum_{j=2}^{N} a_{ij}x_j+b_i \end{cases}$$ (P_n *any polynomial of degree* $\leqq n$, $n=2, 3, \cdots$)

3. $$\begin{cases} y_1=cx_1+g(x_2, \cdots, x_n) \\ y_i=\dfrac{\sum_{j=2}^{N} a_{ij}x_j+b_i}{\sum_{j=2}^{N} c_jx_j+d} \end{cases}$$

4. $$\begin{cases} y_1=cx_1+g(x_2, \cdots, x_n) \\ y_i=f_i(x_2, \cdots, x_n) \end{cases}$$ $\left(\det\left(\frac{\partial f^i}{\partial x_j}\right)= constant;\ N\geqq 3\right)$

5. $$\begin{cases} y_1=cx_1+g(x_2, \cdots, x_n) \\ y_i=f_i(x_2, \cdots, x_n) \end{cases}$$

6. $$\begin{cases} y_1 = x_1 g(x_2, \cdots, x_n) + h(x_2, \cdots, x_n) \\ y_i = \sum\limits_{j=2}^{N} a_{ij} x_j + b_i \end{cases}$$

6n. $$\begin{cases} y_1 = x_1 e^{P_n(x_2, \cdots, x_n)} + h(x_2, \cdots, x_n) \\ y_i = \sum\limits_{j=2}^{N} a_{ij} x_j + b_i \end{cases}$$ (P_n *any polynomial of degree* $\leqq n,\ n = 1, 2, \cdots$)

7. $$\begin{cases} y_1 = x_1 g(x_2, \cdots, x_n) + h(x_2, \cdots, x_n) \\ y_i = \dfrac{\sum\limits_{j=2}^{N} a_{ij} x_j + b_i}{\sum\limits_{j=2}^{N} c_i x_i + d} \end{cases}$$

8. $$\begin{cases} y_1 = x_1 g(x_2, \cdots, x_n) + h(x_2, \cdots, x_n) \\ y_i = f_i(x_2, \cdots, x_n) \end{cases}$$ $\left(\det\left(\dfrac{\partial f^i}{\partial x_j}\right) = constant;\ N \geqq \text{[illegible]}\right)$

9. $$\begin{cases} y_1 = x_1 g(x_2, \cdots, x_n) + h(x_2, \cdots, x_n) \\ y_i = f_i(x_2, \cdots, x_n) \end{cases}$$

10λ. $$\begin{cases} y_1 = \dfrac{A x_1 + h(x_2, \cdots, x_n)}{\left(\sum\limits_{j=2}^{N} c_j x_j + d\right)^{\lambda}} \\ y_i = \dfrac{\sum\limits_{j=2}^{N} a_{ij} x_j + b_i}{\sum\limits_{j=2}^{N} c_j x_j + d} \end{cases}$$ (λ *fixed*, $\lambda \neq 0$)

10n. $$\begin{cases} y_1 = \dfrac{A x_1 + P_n(x_2, \cdots, x_n)}{\left(\sum\limits_{j=2}^{N} c_j x_j + d\right)^{n}} \\ y_i = \dfrac{\sum\limits_{j=2}^{N} a_{ij} x_j + b_i}{\sum\limits_{j=2}^{N} c_j x_j + d} \end{cases}$$ (P_n *any polynomial of degree* $\leqq n,\ n = 1, 2, \cdots$)

11λ. $$\begin{cases} y_1 = C x_1 \left[\det\left(\dfrac{\partial f^i}{\partial x_j}\right)\right]^{\lambda} + h(x_2, \cdots, x_n) \\ y_i = f_i(x_2, \cdots, x_n) \end{cases}$$ (λ *fixed*, $\lambda \neq 0$)

12. $$\begin{cases} y_1 = \dfrac{a(x_2, \cdots, x_n) x_1 + b(x_2, \cdots, x_n)}{c(x_2, \cdots, x_n) x_1 + d(x_2, \cdots, x_n)} \\ y_i = \sum\limits_{j=2}^{N} a_{ij} x_j + b_i \end{cases}$$

13. $$\begin{cases} y_1=\dfrac{a(x_2,\cdots,x_n)x_1+b(x_2,\cdots,x_n)}{c(x_2,\cdots,x_n)x_1+d(x_2,\cdots,x_n)} \\ y_i=\dfrac{\sum_{j=2}^{N} a_{ij}x_j+b_i}{\sum_{j=2}^{N} c_j x_j+d} \end{cases}$$

14. $$\begin{cases} y_1=\dfrac{a(x_2,\cdots,x_n)x_1+b(x_2,\cdots,x_n)}{c(x_2,\cdots,x_n)x_1+d(x_2,\cdots,x_n)} \\ y_i=f_i(x_2,\cdots,x_n) \end{cases} \qquad \left(\det\left(\frac{\partial f^i}{\partial x_j}\right)=constant;\ N\geqq 3\right)$$

15. $$\begin{cases} y_1=\dfrac{a(x_2,\cdots,x_n)x_1+b(x_2,\cdots,x_n)}{c(x_2,\cdots,x_n)x_1+d(x_2,\cdots,x_n)} \\ y_i=f_i(x_2,\cdots,x_n) \end{cases}$$

16. $$\begin{cases} y_1=f_1(x_1,\cdots,x_n) \\ y_i=\sum_{j=2}^{N} a_{ij}x_j+b_i \end{cases}$$

17. $$\begin{cases} y_1=f_1(x_1,\cdots,x_n) \\ y_i=\dfrac{\sum_{j=2}^{N} a_{ij}x_j+b_i}{\sum_{j=2}^{N} c_j x_j+d} \end{cases}$$

18. $$\begin{cases} y_1=f_1(x_1,\cdots,x_n) \\ y_i=f_i(x_2,\cdots,x_n) \end{cases} \qquad \left(\det\left(\frac{\partial f^i}{\partial x_j}\right)=constant;\ N\geqq 3\right)$$

19. $$\begin{cases} y_1=f_1(x_1,\cdots,x_n) \\ y_i=f_i(x_2,\cdots,x_n) \end{cases}$$

1. G^2

We will constantly refer to the previous list and to the list of kernels (the numbering on the two lists is different). In this section we will solve the equivalence problem for G^2 and determine the corresponding pseudogroups (which belong to k^2, $\Lambda^3 k^2, \cdots$). In each case we must find f such that $\theta^2(f)-\varphi\in k^2$; then letting $\varphi=0$, we must find the most general solution to get the pseudogroup. As $\left(\frac{\partial f^i}{\partial x_j}\right)\in G^1$, $y_1=f_1(x_1,\cdots,x_n)$, $y_i=f_i(x_2,\cdots,x_n)$ $(i\geqq 2)$. The equation to be solved can be written

$$f^i{}_{j_1j_2}=\sum(\varphi^i{}_{k_1k_2}+k^i{}_{k_1k_2})f^{k_1}{}_{j_1}f^{k_2}{}_{j_2}$$

where φ determines g^2 and so is fixed, but $k^i{}_{k_1k_2}$ may vary with $x_1,\cdots,x_n$, remaining in k^2.

Recall that $\varphi \in \Lambda^2 k^1/k^2$.

LEMMA 47. *Let* $\xi^i{}_{j_1 j_2} = \begin{pmatrix} e & f_j & g_{j_1 j_2} \\ 0 & 0 & h^i{}_{j_1 j_2} \end{pmatrix}$, $\xi^i{}_j = \begin{pmatrix} a & b_i \\ 0 & d^i{}_j \end{pmatrix}$. *Then*

$$\sum \xi^i{}_{k_1 k_2} \xi^{k_1}{}_{j_1} \xi^{k_2}{}_{j_2}$$

$$= \begin{pmatrix} ea^2 & eab_j + \sum f_k d^k{}_j a & (eb_{j_1} b_{j_2} + \sum f_k d^k{}_{j_1} b_{j_2} + \sum f_k d^k{}_{j_2} b_{j_1} + \sum g_{k_1 k_2} d^{k_1}{}_{j_1} d^{k_2}{}_{j_2}) \\ 0 & 0 & h^i{}_{k_1 k_2} d^{k_1}{}_{j_1} d^{k_2}{}_{j_2} \end{pmatrix}.$$

PROOF. Calculate.

LEMMA 48. *If* $\xi^i{}_{j_1 j_2} = \begin{pmatrix} e & f_i & g_{j_1 j_2} \\ 0 & 0 & h^i{}_{j_1 j_2} \end{pmatrix}$, *the equations*

$$\frac{\partial^2 \phi^i}{\partial x_{j_1} \partial x_{j_2}} = \sum \xi^i{}_{k_1 k_2} \frac{\partial \phi^{k_1}}{\partial x_{j_1}} \frac{\partial \phi^{k_2}}{\partial x_{j_2}}$$

can be written explicitly as:

(A) $$\frac{\partial^2 \phi^1}{\partial x_1{}^2} = e\left(\frac{\partial \phi^1}{\partial x_1}\right)^2$$

(B) $$\frac{\partial^2 \phi^1}{\partial x_1 \partial x_j} = e\left(\frac{\partial \phi^1}{\partial x_1}\right)\left(\frac{\partial \phi^1}{\partial x_j}\right) + \sum f_k \left(\frac{\partial \phi^k}{\partial x_j}\right)\left(\frac{\partial \phi^1}{\partial x_1}\right)$$

(C) $$\frac{\partial^2 \phi^1}{\partial x_{j_1} \partial x_{j_2}} = e\left(\frac{\partial \phi^1}{\partial x_{j_1}}\right)\left(\frac{\partial \phi^1}{\partial x_{j_2}}\right) + \sum f_k \left(\frac{\partial \phi^k}{\partial x_{j_1}}\right)\left(\frac{\partial \phi^1}{\partial x_{j_2}}\right) + \sum f_k \left(\frac{\partial \phi^k}{\partial x_{j_2}}\right)\left(\frac{\partial \phi^1}{\partial x_{j_1}}\right) + \sum g_{k_1 k_2} \left(\frac{\partial \phi^{k_1}}{\partial x_{j_1}}\right)\left(\frac{\partial \phi^{k_2}}{\partial x_{j_2}}\right)$$

(D) $$\frac{\partial^2 \phi^i}{\partial x_{j_1} \partial x_{j_2}} = \sum h^i{}_{k_1 k_2} \left(\frac{\partial \phi^{k_1}}{\partial x_{j_1}}\right)\left(\frac{\partial \phi^{k_2}}{\partial x_{j_2}}\right).$$

PROOF. A trivial consequence of Lemma 47.

LEMMA 49. *If* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T^2 \end{pmatrix} \in k^2$, (D) *is irrelevant. If* $\begin{pmatrix} 0 & 0 & g_{ij} \\ 0 & 0 & 0 \end{pmatrix} \in k^2$, (C) *is irrelevant. If* $\begin{pmatrix} 0 & f_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \in k^2$, (B) *is irrelevant.*

PROOF. For example, if the second condition holds, we can choose $g_{ij}(x_1, \cdots, x_n)$ so that (C) is satisfied, since $\frac{\partial \phi^i}{\partial x_j}$ is non-singular. The other cases follow in the same way.

LEMMA 50. *The most general solution of* (A), *if* $e=0$, *is* $x_1 g(x_2, \cdots, x_n) + h(x_2, \cdots, x_n)$. *The most general solution of* (A), *if e is a non-zero constant, is* $-\frac{1}{e} \ln (x_1 + g(x_2, \cdots, x_n)) + h(x_2, \cdots, x_n)$.

PROOF. Obvious.

We now discuss the list of kernels case by case, proceeding from the easier to the more difficult examples. Notice that (13) and (16) on the list of kernels are done, since $\Lambda^2 k^1/k^2 = 0$. These obviously give rise to pseudogroups

(19) on the list of pseudogroups.

Now turn to (8) and (11) on the kernel list. We can pick $\varphi=\begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $k^2=\begin{pmatrix} 0 & f_j & g_{jk} \\ 0 & 0 & h^i{}_{jk} \end{pmatrix}$. Consider the equivalence problem; lemmas 49 and 50 show that $\psi^1=-\frac{1}{e}\ln(x_1+1)$, $\psi^i=x_i$ is a solution. Notice that ψ is defined in a neighborhood of the origin and is a local diffeomorphism there; $\psi(0)=0$.

To find the pseudogroups, suppose $e=0$. Then $\psi^1=x_1g(x_2,\cdots,x_n)+h(x_2,\cdots,x_n)$ by lemma 50. This is example (9) on our list of pseudogroups.

Next turn to (3), (6), (17), and (18) and (18b) in the list of kernels. Each of these contains $\begin{pmatrix} 0 & 0 & g_{jk} \\ 0 & 0 & 0 \end{pmatrix}\in k^2$, so equation (C) is irrelevant. In each case we can pick $\varphi=\begin{pmatrix} e & f_i & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For the purpose of the equivalence problem consider $k^i{}_{j_1j_2}=\begin{pmatrix} 0 & 0 & e_{j_1j_2} \\ 0 & 0 & 0 \end{pmatrix}$ and let $\psi^i=x_i$. Then only

$$\frac{\partial^2\psi^1}{\partial x_1{}^2}=e\Big(\frac{\partial\psi^1}{\partial x_1}\Big)^2$$

$$\frac{\partial^2\psi^1}{\partial x_j\partial x_1}=e\Big(\frac{\partial\psi^1}{\partial x_j}\Big)\Big(\frac{\partial\psi^1}{\partial x_1}\Big)+f_j\frac{\partial\psi^1}{\partial x_1}$$

need be solved.

But if $e=0$, $\psi^1=x_1l_1(x_2,\cdots,x_n)+l_2(x_2,\cdots,x_n)$ and we have only to solve $\frac{\partial l_1}{\partial x_j}=f_jl_1$. If $l_1=e^{s(x_2,\cdots,x_n)}$, this becomes $\frac{\partial s}{\partial x_j}=f_j$ and as the f_j are constants, this can be solved. If $e\neq 0$, we can pick $\psi^1=-\frac{1}{e}\ln(x_1+1)+l(x_2,\cdots,x_n)$. Then we need only solve

$$0=e\Big(-\frac{1}{e}\frac{1}{x+1}\Big)\Big(\frac{\partial l}{\partial x_j}\Big)+f_j\Big(-\frac{1}{e}\frac{1}{x+1}\Big)$$

or $\frac{\partial l}{\partial x_j}=-\frac{1}{e}f_j$ which again is easily done.

Next we determine the pseudogroups. Examine (3) and (6) first. By lemma 49, equations (C) and (D) are irrelevant. Hence

$$\frac{\partial^2\psi^1}{\partial x_1{}^2}=0 \qquad \frac{\partial^2\psi^1}{\partial x_1\partial x_j}=0\,.$$

Therefore $\psi^1=cx_1+g(x_2,\cdots,x_n)$, $\psi^i=f_i(x_2,\cdots,x_n)$, which is (5) on the list of pseudogroups. In (17) and (18) we must solve

$$\frac{\partial^2\psi^1}{\partial x_1{}^2}=0$$

$$\frac{\partial^2\psi^1}{\partial x_1\partial x_j}=\sum\lambda\, d^{(2)\,k}{}_{kk}\frac{\partial\psi^1}{\partial x_1}\frac{\partial\psi^k}{\partial x_j}$$

$$\frac{\partial^2\psi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum d^i{}_{k_1k_2}\frac{\partial\psi^{k_1}}{\partial x_{j_1}}\frac{\partial\psi^{k_2}}{\partial x_{j_2}}$$

with $d^i{}_{j_1j_2}$ functions. Hence $\psi^1=x_1l_1(x_2,\cdots,x_n)+l_2(x_2,\cdots,x_n)$, $\psi^i=f_i(x_2,\cdots,x_n)$ and

$$\frac{\partial l_1}{\partial x_j}=\sum\lambda d^{(2)k}{}_{kk}\frac{\partial f^k}{\partial x_j}l_1 .$$

Now $l_1\neq 0$ since $\frac{\partial\psi^1}{\partial x_1}\neq 0$, so $l_1=e^{s(x_2,\cdots,x_n)}$, at least locally, and

$$\frac{\partial s}{\partial x_j}=\sum\lambda d^{(2)k}{}_{kk}\frac{\partial f^k}{\partial x_j} .$$

LEMMA 51. *Suppose* $\{f^i(x_2,\cdots,x_N)\}$ *satisfy*

$$\frac{\partial^2 f^i}{\partial x_{j_1}\partial x_{j_2}}=\sum d^i{}_{k_1k_2}\frac{\partial f^{k_1}}{\partial x_{j_1}}\frac{\partial f^{k_2}}{\partial x_{j_2}} .$$

Then

$$\sum d^{(2)k}{}_{kk}\frac{\partial f^k}{\partial x_j}=\frac{2}{N}\frac{\partial}{\partial x_j}\ln\det\left(\frac{\partial f^i}{\partial x_j}\right).$$

PROOF. The identity

$$\frac{\partial}{\partial x_j}\det\left(\frac{\partial f^i}{\partial x_j}\right)=\det\left(\frac{\partial f^i}{\partial x_j}\right)\sum\frac{\partial^2 f^k}{\partial x_r\partial x_j}\left[\left(\frac{\partial f^i}{\partial x_j}\right)^{-1}\right]^r_k$$

is well known. Applying the formula for $\frac{\partial^2 f^k}{\partial x_r\partial x_j}$, we have

$$\begin{aligned}\frac{\partial}{\partial x_j}\det\left(\frac{\partial f^i}{\partial x_j}\right)&=\det\left(\frac{\partial f^i}{\partial x_j}\right)\sum h^k{}_{st}\frac{\partial f^s}{\partial x_r}\frac{\partial f^t}{\partial x_j}\left[\left(\frac{\partial f}{\partial x}\right)^{-1}\right]^r_k\\&=\det\left(\frac{\partial f^i}{\partial x_j}\right)\sum h^k{}_{kr}\frac{\partial f^r}{\partial x_j} .\end{aligned}$$

Thus

$$\begin{aligned}\frac{\frac{\partial}{\partial x_j}\det\left(\frac{\partial f^i}{\partial x_k}\right)}{\det\left(\frac{\partial f^i}{\partial x_k}\right)}&=\frac{\partial}{\partial x_j}\ln\det\left(\frac{\partial f^i}{\partial x_j}\right)\\&=\sum\left[\sum h^{(1)k}{}_{kr}+\sum h^{(2)k}{}_{kr}\right]\frac{\partial f^r}{\partial x_j} ,\end{aligned}$$

where $h=h^{(1)}+h^{(2)}$, $h^{(1)}\in(1-\Omega)(T^2)$, $h^{(2)}\in\Omega(T^2)$. But then $\sum h^{(1)k}{}_{kr}=0$, $h^{(2)k}{}_{kr}=\frac{1}{2}[\delta^k{}_kh^{(2)r}{}_{rr}+\delta^k{}_rh^{(2)k}{}_{kk}]$, so we have

$$\begin{aligned}\frac{1}{2}((N-1)+1)\sum h^{(2)r}{}_{rr}\frac{\partial f^r}{\partial x_j}&=\frac{N}{2}\sum h^{(2)r}{}_{rr}\frac{\partial f^r}{\partial x_j}\\&=\frac{\partial}{\partial x_j}\ln\det\left(\frac{\partial f^i}{\partial x_j}\right).\end{aligned}$$

Hence $\frac{\partial s}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\frac{2\lambda}{N}\ ln\ \det\frac{\partial f^i}{\partial x_j}\right]$, so $\phi^1=Cx_1\det\left(\frac{\partial f^i}{\partial x_j}\right)^{\frac{2\lambda}{N}}+h(x_2,\cdots,x_n)$. Of course we can let $\tilde{\lambda}=\frac{2\lambda}{N}$; then we obtain (11) on the list of pseudogroups.

Consider next (7), (9), (10), (12), (14), (15) on the kernel list; in each of these $\begin{pmatrix}0 & f_i & g_{ij}\\ 0 & 0 & 0\end{pmatrix}\in k^2$, so by lemma 49, equations (B) and (C) are irrelevant. Moreover, $\varphi=\begin{pmatrix}e & 0 & 0\\ 0 & 0 & h^i{}_{jk}\end{pmatrix}$. In each case, k^2 can be taken in the form $\begin{pmatrix}0 & f_i & g_{j_1j_2}\\ 0 & 0 & k^i{}_{j_1j_2}\end{pmatrix}$, with $k^i{}_{j_1j_2}\in\Omega(T^2)$, $(1-\Omega)(T^2)$, or T^2, depending on the particular example considered. We must solve

$$\frac{\partial^2\phi^1}{\partial x_1{}^2}=e\left(\frac{\partial\phi^1}{\partial x_1}\right)^2$$

$$\frac{\partial^2\phi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum(h^i{}_{k_1k_2}+k^i{}_{k_1k_2})\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}.$$

But the first can always be solved, and the second can be solved by theorem 11. (To apply theorem 11, we need to know the required integrability conditions. Thus there must exist $\varphi^i{}_{jkr}$ such that $\langle 1,\varphi^i{}_{jk},\varphi^i{}_{jkr}\rangle$ is in Λ^3g^2. But it is easy to check that if this holds in the present N dimensional case, it also holds in the lower $N-1$ dimensional block.)

Next we compute the pseudogroups. Consider $k^2=\begin{pmatrix}e & e_i & e_{ij}\\ 0 & 0 & k^i{}_{j_1j_2}\end{pmatrix}$; since (B) and (C) are still irrelevant we must solve

$$\frac{\partial^2\phi^1}{\partial x_1{}^2}=e\left(\frac{\partial\phi^1}{\partial x_1}\right)^2$$

$$\frac{\partial^2\phi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum k^i{}_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}.$$

As k^2 varies over the several cases, either $e=0$ or e is arbitrary. If $e=0$, $\phi^1=x_1g(x_2,\cdots,x_n)+h(x_2,\cdots,x_n)$; if e is arbitrary, $\phi^1=f_1(x_1,\cdots,x_n)$. Moreover, $k^i{}_{jk}$ varies over $\Omega(T^2)$, $(1-\Omega)(T^2)$ and T^2, and thus generates one of the pseudogroups of theorem 12. We get, then, (6), (7), (8), (16), (17), and (18) on the pseudogroup list.

Next consider (2), (4), and (5) in the list of kernels. Each k^2 contains $\begin{pmatrix}0 & 0 & g_{ij}\\ 0 & 0 & 0\end{pmatrix}$, so by lemma 49 equation (C) is irrelevant.

Letting $\varphi=\begin{pmatrix}e & f_i & 0\\ 0 & 0 & h^i{}_{jk}\end{pmatrix}$ and $k^2=\begin{pmatrix}0 & 0 & g_{jk}\\ 0 & 0 & k^i{}_{j_1j_2}\end{pmatrix}$ we must solve

$$\frac{\partial^2\phi^1}{\partial x_1{}^2}=e\left(\frac{\partial\phi^1}{\partial x_1}\right)^2$$

$$\frac{\partial^2\phi^1}{\partial x_1\partial x_{j_1}}=e\frac{\partial\phi^1}{\partial x_1}\frac{\partial\phi^1}{\partial x_{j_1}}+\sum f_k\frac{\partial\phi^1}{\partial x_1}\frac{\partial\phi^k}{\partial x_{j_1}}$$

$$\frac{\partial^2\psi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum(h^i{}_{k_1k_2}+k^i{}_{k_1k_2})\frac{\partial\psi^{k_1}}{\partial x_{j_1}}\frac{\partial\psi^{k_2}}{\partial x_{j_2}}.$$

As before, $\Lambda^3g^2\to g^2$ onto implies the required integrability condition for $h^i{}_{k_1k_2}$, so the last equation can be solved.

If $e=0$, $\psi^1=x_1l_1(x_2,\cdots,x_n)+l_2(x_2,\cdots,x_n)$ and we must only solve $\frac{\partial l_1}{\partial x_{j_1}}=\sum f_k\frac{\partial\psi^k}{\partial x_{j_1}}l_1$. But $l_1\neq 0$, so locally $l_1=e^{s(x_2,\cdots,x_n)}$; thus $\frac{\partial s}{\partial x_j}=\sum f_k\frac{\partial\psi^k}{\partial x_j}$. The integrability conditions for this equation are just

$$\frac{\partial}{\partial x_r}\Big[\sum f_k\frac{\partial\psi^k}{\partial x_j}\Big]=\frac{\partial}{\partial x_j}\Big[\sum f_k\frac{\partial\psi^k}{\partial x_r}\Big],$$

which hold because the f_k are constant.

Now say $e\neq 0$. Then $\psi^1=-\frac{1}{e}ln(x_1+1)l(x_2,\cdots,x_n)$ and we need only solve

$$0=e\Big(-\frac{1}{e(x_1+1)}\Big)\frac{\partial l}{\partial x_j}+\sum f_k\frac{\partial\psi^k}{\partial x_j}\Big(-\frac{1}{e(x_1+1)}\Big)$$

or $\frac{\partial l}{\partial x_j}=-\frac{1}{e}\sum f_k\frac{\partial\psi^k}{\partial x_j}$ which again can be done.

To find the pseudogroups, we solve

$$\frac{\partial^2\psi^1}{\partial x_1^2}=0$$

$$\frac{\partial^2\psi^1}{\partial x_1\partial x_j}=0$$

$$\frac{\partial^2\psi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum k^i{}_{k_1k_2}\frac{\partial\psi^{k_1}}{\partial x_{j_1}}\frac{\partial\psi^{k_2}}{\partial x_{j_2}}.$$

Here $k^i{}_{k_1k_2}\in\Omega(T^2)$, $(1-\Omega)(T^2)$, or T^2. But then $\psi^1=Cx_1+h(x_2,\cdots,x_n)$ and ψ^i belongs to one of the pseudogroups of theorem 12, so w; have (2), (3), and (4) of the list of pseudogroups.

Consider (19) and (19b) in the list of kernels. $\begin{pmatrix}0 & 0 & g_{jk}\\ 0 & 0 & 0\end{pmatrix}\in k^2$, so equation (C) is irrelevant. We have

$$\varphi=\begin{pmatrix}e & f_i & 0\\ 0 & 0 & k^i{}_{rs}\end{pmatrix},\qquad k^i{}_{jk}=\begin{pmatrix}0 & \lambda d^j{}_{jj} & g_{jk}\\ 0 & 0 & d^i{}_{jk}\end{pmatrix}$$

and so must solve

$$\frac{\partial^2\psi^1}{\partial x_1^2}=e\Big(\frac{\partial\psi^1}{\partial x_1}\Big)^2$$

$$\frac{\partial^2\psi^1}{\partial x_1\partial x_{j_1}}=e\frac{\partial\psi^1}{\partial x_1}\frac{\partial\psi^1}{\partial x_{j_1}}+\sum(f_k+\lambda d^k{}_{kk})\frac{\partial\psi^1}{\partial x_1}\frac{\partial\psi^k}{\partial x_{j_1}}$$

$$\frac{\partial^2\psi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum(h^i{}_{k_1k_2}+d^i{}_{k_1k_2})\frac{\partial\psi^{k_1}}{\partial x_{j_1}}\frac{\partial\psi^{k_2}}{\partial x_{j_2}}.$$

But these are precisely the equations we considered above with the addition of $\lambda d^k{}_{kk}$, so we are done if

$$\frac{\partial}{\partial x_j}\Big[\sum(f_k+\lambda d^k{}_{kk})\frac{\partial\psi^k}{\partial x_r}\Big]=\frac{\partial}{\partial x_r}\Big[\sum(f_k+\lambda d^k{}_{kk})\frac{\partial\psi^k}{\partial x_j}\Big]$$

or, as f_k is constant,

$$\frac{\partial}{\partial x_j}\Big[\sum d^k{}_{kk}\frac{\partial\psi^k}{\partial x_r}\Big]=\frac{\partial}{\partial x_r}\Big[\sum d^k{}_{kk}\frac{\partial\psi^k}{\partial x_j}\Big].$$

By lemma 51,

$$\sum(\varphi^{(2)k}{}_{kk}+d^{(2)k}{}_{kk})\frac{\partial\psi^k}{\partial x_j}=\frac{2}{N}\frac{\partial}{\partial x_j}\ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big).$$

In this case $d\in\Omega(T^2)$, $\varphi\in(1-\Omega)(T^2)$, so $d^{(2)}=d$, $\varphi^{(2)}=0$, and we must have

$$\frac{\partial}{\partial x_j}\Big[\frac{2}{N}\frac{\partial}{\partial x_r}\ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big)\Big]=\frac{\partial}{\partial x_r}\Big[\frac{2}{N}\frac{\partial}{\partial x_j}\ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big)\Big]$$

which, of course, holds.

To find the pseudogroup, we put e, f_i, $\varphi^i{}_{rs}=0$. Thus

$$\frac{\partial^2\psi^1}{\partial x_1{}^2}=0$$

$$\frac{\partial^2\psi^1}{\partial x_1\partial x_j}=\lambda\frac{2}{N}\frac{\partial}{\partial x_j}\Big[\ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big)\Big]\frac{\partial\psi^1}{\partial x_1}$$

$$\frac{\partial^2\psi^1}{\partial x_{j_1}\partial x_{j_2}}=\sum d^i{}_{k_1k_2}\frac{\partial\psi^{k_1}}{\partial x_{j_1}}\frac{\partial\psi^{k_2}}{\partial x_{j_2}}.$$

Then $\psi^1=x_1e^{s(x_2,\cdots,x_n)}+l_2(x_2,\cdots,x_n)$

$$\frac{\partial s}{\partial x_j}=\frac{2\lambda}{N}\frac{\partial}{\partial x_j}\Big[\ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big)\Big],$$

and ψ^i belongs to the projective pseudogroup $\psi^i=\dfrac{\sum a_{ij}x_j+b_i}{\sum c_jx_j+d}$. We have

$$\psi^1=Cx_1\Big(\det\frac{\partial\psi^i}{\partial x_j}\Big)^{\frac{2\lambda}{N}}+h(x_2,\cdots,x_n)$$

$$\psi^i=\frac{\sum a_{ij}x_j+b_i}{\sum c_jx_j+d};$$

since $\det\dfrac{\partial\psi^i}{\partial x_j}\neq 0$, this is (10) on the list of pseudogroups, provided:

LEMMA 52.

$$\det\frac{\partial\Big(\dfrac{\sum a_{ij}x_j+b_i}{\sum c_jx_j+d}\Big)}{\partial x_j}=\frac{C}{(\sum c_jx_j+d)^N}\qquad (C\ \textit{a constant}).$$

PROOF.

$$\frac{\partial\psi^i}{\partial x_j}=\frac{\partial}{\partial x_j}\Big(\frac{\sum a_{ij}x_j+b_i}{\sum c_jx_j+d}\Big)$$

$$=\frac{(\sum c_kx_k+d)a_{ij}-(\sum a_{ik}x_k+b_i)c_j}{(\sum c_kx_k+d)^2}=\frac{a_{ij}}{\sum c_kx_k+d}-\frac{c_j}{\sum c_kx_k+d}\psi^i$$

$$\frac{\partial^2\psi^i}{\partial x_{j_1}\partial x_{j_2}}=\frac{-a_{ij_1}c_{j_2}}{(\sum c_kx_k+d)^2}+\frac{c_{j_1}c_{j_2}}{(\sum c_kx_k+d)^2}\psi^i-\frac{c_{j_1}}{\sum c_kx_k+d}\frac{\partial\psi^i}{\partial x_{j_2}}$$

$$=\frac{\partial\psi^i}{\partial x_{j_1}}\Big(\frac{-c_{j_2}}{\sum c_kx_k+d}\Big)+\frac{\partial\psi^i}{\partial x_{j_2}}\Big(\frac{-c_{j_1}}{\sum c_kx_k+d}\Big).$$

But also,

$$\frac{\partial^2\psi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum h^i{}_{k_1k_2}\frac{\partial\psi^{k_1}}{\partial x_{j_1}}\frac{\partial\psi^{k_2}}{\partial x_{j_2}}$$

$$=\frac{1}{2}[\delta^i{}_{k_1}h^{k_2}{}_{k_2k_2}+\delta^i{}_{k_2}h^{k_1}{}_{k_1k_1}]\frac{\partial\psi^{k_1}}{\partial x_{j_1}}\frac{\partial\psi^{k_2}}{\partial x_{j_2}}$$

$$=\frac{\partial\psi^i}{\partial x_{j_1}}\frac{1}{N}\frac{\partial}{\partial x_{j_2}}\,ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big)+\frac{\partial\psi^i}{\partial x_{j_2}}\frac{1}{N}\frac{\partial}{\partial x_{j_1}}\,ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big).$$

Hence

$$0=\frac{\partial\psi^i}{\partial x_{j_1}}\Big[\frac{1}{N}\frac{\partial}{\partial x_{j_2}}\,ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big)+\frac{c_{j_2}}{\sum c_kx_k+d}\Big]$$

$$+\frac{\partial\psi^i}{\partial x_{j_2}}\Big[\frac{1}{N}\frac{\partial}{\partial x_{j_1}}\,ln\det\Big(\frac{\partial\psi^i}{\partial x_j}\Big)+\frac{c_{j_1}}{\sum c_kx_k+d}\Big].$$

Since there is some $\langle i,j\rangle$ with $\frac{\partial\psi^i}{\partial x_j}\neq 0$, letting $j_1=j_2$ we get

$$\frac{1}{N}\frac{\partial}{\partial x_j}\,ln\det\frac{\partial\psi^i}{\partial x_j}=-\frac{c_j}{\sum c_kx_k+d}=-\frac{\partial}{\partial x_j}ln(\sum c_kx_k+d).$$

Hence

$$\frac{\partial}{\partial x_j}\Big[\frac{1}{N}\,ln\det\frac{\partial\psi^i}{\partial x_j}+ln(\sum c_kx_k+d)\Big]=0,$$

so

$$e^{\frac{1}{N}ln\det\frac{\partial\psi^i}{\partial x_j}}e^{ln(\sum c_kx_k+d)}=D$$

or

$$\Big(\det\frac{\partial\psi^i}{\partial x_j}\Big)^{\frac{1}{N}}=\frac{D}{\sum c_kx_k+d}$$

and we are done.

The final two examples force us to pay more attention to the requirement that $\Lambda^3g^2\to g^2$ be onto. Look at (1) in the list of kernels. Then $k^2=0$, and we are to solve the equations in lemma 48 with $\varphi=\begin{pmatrix}e & f_i & g_{ij}\\ 0 & 0 & h^i{}_{jk}\end{pmatrix}$. But since $\Lambda^3g^2\to g^2$ is onto, we can find $\varphi^i{}_{jkr}$ so that $\langle 1,\varphi^i{}_{jk},\varphi^i{}_{jkr}\rangle\in\Lambda^3g^2$. Then

$$\varphi^i{}_{kj_1j_2}=\sum[\varphi^i{}_{kr}\varphi^r{}_{j_1j_2}-\varphi^i{}_{j_1r}\varphi^r{}_{kj_2}-\varphi^i{}_{j_2r}\varphi^r{}_{kj_1}].$$

This must be symmetric in k and j_1, so

$$\sum\varphi^i{}_{j_1r}\varphi^r{}_{j_2j_3}=\sum\varphi^i{}_{j_2r}\varphi^r{}_{j_1j_3}.$$

In particular if $j_1=1$, $j_2, j_3\geqq 2$, $i=1$, we have

$$\varphi^1{}_{11}\varphi^1{}_{j_2j_3}+\sum_{k\geqq 2}\varphi^1{}_{1k}\varphi^k{}_{j_2j_3}=\varphi^1{}_{j_21}\varphi^1{}_{j_31}$$

or

$$\boxed{e\,g_{j_2j_3}+\sum f_kh^k{}_{j_2j_3}=f_{j_2}f_{j_3}}$$

Similarly, $j_1, j_2, j_3\geqq 2$, $i=1$ implies

$$\boxed{f_{j_1}g_{j_2j_3}+\sum g_{kj_1}h^k{}_{j_2j_3}=f_{j_2}g_{j_1j_3}+\sum g_{kj_2}h^k{}_{j_1j_3}}$$

Let us now solve the equations of lemma 48. Clearly $\Lambda^3g^2\to g^2$ onto implies the usual integrability conditions on $h^i{}_{j_1j_2}$, so (D) can be solved.

Let $e\neq 0$. Then $\phi^1=-\frac{1}{e}ln(x_1+1)+l(x_2,\cdots,x_n)$ solves (A); (B) becomes

$$0=\frac{-\frac{\partial l}{\partial x_j}}{x_1+1}+\sum f_k\frac{\partial\phi^k}{\partial x_j}\left(-\frac{1}{e(x_1+1)}\right)$$

or $\frac{\partial l}{\partial x_j}=-\frac{1}{e}\sum f_k\frac{\partial\phi^k}{\partial x_j}$. Hence we can let $l=-\frac{1}{e}\sum f_k\phi^k$. Consider (C); it now reads

$$\begin{aligned}&-\frac{1}{e}\sum f_k\frac{\partial^2\phi^k}{\partial x_{j_1}\partial x_{j_2}}\\&=\frac{1}{e}\sum f_{k_1}f_{k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}+\sum f_{k_1}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\left[\frac{-1}{e}\sum f_{k_2}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\right]\\&\quad+\sum f_k\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\left[\frac{-1}{e}\sum f_{k_1}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\right]+\sum g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\end{aligned}$$

or

$$\begin{aligned}&\frac{-1}{e}\sum f_kh^k{}_{r_1r_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}\\&=\frac{-1}{e}\sum f_{r_1}f_{r_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}+\frac{1}{e^2}\sum g_{k_1k_2}f^{k_1}{}_{r_1}f^{k_2}{}_{r_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}.\end{aligned}$$

Since $\frac{\partial^2\phi^k}{\partial x_{j_1}\partial x_{j_2}}=\sum h^k{}_{r_1r_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}$, this equation holds because $eg_{j_2j_3}+\sum f_kh^k{}_{j_2j_3}=f_{j_2}f_{j_3}$.

Suppose $e=0$. Then $\phi^1=e^{s(x_2,\cdots,x_n)}(x_1+l(x_2,\cdots,x_n))$ solves (A), and (B) reads $\dfrac{\partial s}{\partial x_j}=\sum f_k\dfrac{\partial\phi^k}{\partial x_j}$, so $s=\sum f_k\phi^k$. But

$$\frac{\partial^2 s}{\partial x_{j_1}\partial x_{j_2}}=\sum f_k h^k{}_{r_1r_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}=\sum f_{r_1}f_{r_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}=\frac{\partial s}{\partial x_{j_1}}\frac{\partial s}{\partial x_{j_2}}$$

since $\sum f_k h^k{}_{j_1j_2}=f_{j_1}f_{j_2}$ by the first boxed-in formula above.

Finally, (C) reads

$$\begin{aligned}
&\frac{\partial}{\partial x_{j_1}}\Big[e^s\frac{\partial s}{\partial x_{j_2}}(x_1+l)+e^s\frac{\partial l}{\partial x_{j_2}}\Big]\\
&=e^s\frac{\partial s}{\partial x_{j_1}}\frac{\partial s}{\partial x_{j_2}}(x_1+l)+e^s\frac{\partial^2 s}{\partial x_{j_1}\partial x_{j_2}}(x_1+l)+e^s\frac{\partial s}{\partial x_{j_1}}\frac{\partial l}{\partial x_{j_2}}\\
&\quad+e^s\frac{\partial s}{\partial x_{j_2}}\frac{\partial l}{\partial x_{j_1}}+e^s\frac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}}\\
&=\frac{\partial s}{\partial x_{j_1}}\Big[e^s\frac{\partial s}{\partial x_{j_2}}(x_1+l)+e^s\frac{\partial l}{\partial x_{j_2}}\Big]+\frac{\partial s}{\partial x_{j_2}}\Big[e^s\frac{\partial s}{\partial x_{j_1}}(x_1+l)+e^s\frac{\partial l}{\partial x_{j_1}}\Big]\\
&\quad+\sum g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}
\end{aligned}$$

or

$$\frac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}}=e^{-s}\sum g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}.$$

Now in general $\dfrac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}}=R_{j_1j_2}$ can be solved if and only if $\dfrac{\partial R_{j_1j_2}}{\partial x_{j_3}}$ is a symmetric tensor. But

$$\begin{aligned}
&\frac{\partial}{\partial x_{j_3}}\Big[e^{-s}\sum g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\Big]\\
&=e^{-s}\sum g_{k_1k_2}h^{k_1}{}_{r_1r_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}\frac{\partial\phi^{k_2}}{\partial x_{j_3}}+e^{-s}\sum g_{k_1k_2}h^{k_2}{}_{r_1r_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_1}}{\partial x_{j_2}}\frac{\partial\phi^{r_2}}{\partial x_{j_3}}\\
&\quad-e^{-s}\sum f_{k_1}g_{k_2k_3}\frac{\partial\phi^{k_2}}{\partial x_{j_1}}\frac{\partial\phi^{k_3}}{\partial x_{j_2}}\frac{\partial\phi^{k_1}}{\partial x_{j_3}}.
\end{aligned}$$

Hence it suffices if $\sum[g_{j_1k}h^k{}_{j_2j_3}+g_{kj_2}h^k{}_{j_1j_3}]-f_{j_3}g_{j_1j_2}$ is symmetric, or subtracting $\mathcal{S}\sum g_{j_1k}h^k{}_{j_2j_3}$, if $f_{j_3}g_{j_1j_2}+\sum g_{kj_3}h^k{}_{j_1j_2}$ is symmetric, and this is the second boxed-in condition. The resulting pseudogroup obviously consists of all linear maps, (1) in our list.

Consider (19a) on the list of kernels. $\varphi=\begin{pmatrix} e & f_i & g_{ij}\\ 0 & 0 & h^i{}_{jk}\end{pmatrix}$, $h^i{}_{jk}\in(1-\Omega)(T^2)$ and

$$k^2=\begin{pmatrix} 0 & \frac{1}{2}d^j{}_{jj} & 0\\ 0 & 0 & \Omega(T^2)\end{pmatrix}.$$

Again, the equation for ϕ^i can be solved by theorem 11.

We now discuss the requirement that $\Lambda^3 g^2 \to g^2$ be onto. There must exist a $\varphi^i{}_{jkr}$ so $\langle 1, \varphi^i{}_{jk}, \varphi^i{}_{jkr}\rangle \in \Lambda^3 g^2$; this means that for each fixed j_0, ther is ${}^{j_0}k^i{}_{j_1j_2}$ such that

$$\varphi^i{}_{j_0j_1j_2} = -\sum[\varphi^i{}_{j_0r}\varphi^r{}_{j_1j_2} - \varphi^i{}_{j_1r}\varphi^r{}_{j_0j_2} - \varphi^i{}_{j_2r}\varphi^r{}_{j_0j_1}] + {}^{j_0}k^i{}_{j_1j_2}.$$

But if $j_0 \geqq 2$, $j_1 \geqq 2$, $j_2 \geqq 2$, $i=1$, ${}^{j_0}k^i{}_{j_1j_2}=0$ since

$$k^2 = \begin{pmatrix} 0 & \frac{1}{2}d^j{}_{jj} & 0 \\ 0 & 0 & d^i{}_{jk} \end{pmatrix},$$

so

$$\sum[\varphi^1{}_{j_0r}\varphi^r{}_{j_1j_2} - \varphi^1{}_{j_1r}\varphi^r{}_{j_0j_2} - \varphi^1{}_{j_2r}\varphi^r{}_{j_0j_1}]$$

must be symmetric in j_0, j_1; therefore $\sum\varphi^1{}_{j_0r}\varphi^r{}_{j_1j_2} = \sum\varphi^1{}_{j_1r}\varphi^r{}_{j_0j_2}$, that is

$$\boxed{f_{j_0}g_{j_1j_2} + \sum g_{j_0k}h^k{}_{j_1j_2} \text{ is symmetric.}}$$

But let $i=j_2=1$, $j_0 \geqq 2$, $j_1 \geqq 2$. Th n

$$\varphi^1{}_{j_0j_11} = -[f_{j_0}f_{j_1} - f_{j_1}f_{j_0} - e\, g_{j_0j_1} - \sum f_k h^k{}_{j_0j_1}] + {}^{j_0}k^i{}_{j_11}$$

$$\varphi^1{}_{1j_0j_1} = -[e\, g_{j_0j_1} + \sum f_k h^k{}_{j_0j_1} - f_{j_0}f_{j_1} - f_{j_1}f_{j_0}]$$

or

$$e\, g_{j_0j_1} + \sum f_k h^k{}_{j_0j_1} - f_{j_0j_1} = -\frac{{}^{j_0}k^{j_1}{}_{j_1j_1}}{4}.$$

Hence ${}^{j_0}k^{j_1}{}_{j_1j_1} = {}^{j_1}k^{j_0}{}_{j_0j_0}$.

Looking at $i \geqq 2$, $j_0 \geqq 2$, $j_1 \geqq 2$, $j_2 \geqq 2$, we see that

$$\sum[h^i{}_{j_0r}h^r{}_{j_1j_2} - h^i{}_{j_1r}h^r{}_{j_0j_2} - h^i{}_{j_2r}h^r{}_{j_0j_1}] + \frac{1}{2}[\delta^i{}_{j_1}{}^{j_0}k^{j_2}{}_{j_2j_2} + \delta^i{}_{j_2}{}^{j_0}k^{j_1}{}_{j_1j_1}]$$

must be symmetric in j_0, j_1, and j_2. Thus interchanging j_0 and j_1 and subtracting,

$$\sum 2(h^i{}_{j_0r}h^r{}_{j_1j_2} - h^i{}_{j_1r}h^r{}_{j_0j_2}) + \frac{1}{2}[\delta^i{}_{j_1}{}^{j_0}k^{j_2}{}_{j_2j_2} - \delta^i{}_{j_0}{}^{j_1}k^{j_2}{}_{j_2j_2}] = 0.$$

If $j_0 \neq j_1$, and $i=j_0$, we have

$$\sum(h^{j_0}{}_{j_0r}h^r{}_{j_1j_2} - h^{j_0}{}_{j_1r}h^r{}_{j_0j_2}) = -\frac{1}{4}{}^{j_1}k^{j_2}{}_{j_2j_2}.$$

If $j_0 = j_1$, the left-hand side is zero. Summing over all j_0, we conclude, since $\varphi \in (1-\Omega)(T^2)$,

$$-\frac{(N-2)}{4}{}^{j_1}k^{j_2}{}_{j_2j_2} = -\sum h^{j_0}{}_{j_0r}h^r{}_{j_0j_2}.$$

Hence, since in (19a) $N>2$, we have

$$\boxed{e\, g_{j_0 j_1} + \sum f_k h^k{}_{j_0 j_1} - f_{j_0 j_1} = -\frac{1}{N-2} \sum h^r{}_{j_0 k} h^k{}_{r j_1}}$$

These are sufficient conditions to solve our equations.

Now let $e \neq 0$. Let $\phi^1 = -\frac{1}{e} \ln (x_1+1) + s(x_2, \cdots, x_n)$. Equation (B) becomes

$$0 = \frac{-\frac{\partial s}{\partial x_j}}{x_1+1} + \sum\Big(f_k + \frac{1}{2} d^k{}_{kk}\Big)\Big(-\frac{1}{e(x_1+1)}\Big)\frac{\partial \phi^k}{\partial x_j}$$

or

$$\frac{\partial s}{\partial x_j} = -\frac{1}{e}\sum\Big(f_k + \frac{1}{2} d^k{}_{kk}\Big)\frac{\partial \phi^k}{\partial x_j}$$

$$= -\frac{1}{e}\frac{\partial}{\partial x_j}\Big[\sum f_k \phi^k + \frac{1}{N} \ln \det \frac{\partial \phi^i}{\partial x_j}\Big].$$

Hence, let $s = -\frac{1}{e}\Big(\sum f_k \phi^k + \frac{1}{N} \ln \det \frac{\partial \phi^i}{\partial x_j}\Big)$. Equation (C) becomes $\Big($writing let $\frac{\partial \phi^i}{\partial x_j} = \Delta\Big)$:

$$\frac{\partial^2}{\partial x_{j_1}\partial x_{j_2}}\Big[-\frac{1}{e}\Big(\sum f_k \phi^k + \frac{1}{N} \ln \Delta\Big)\Big]$$

$$= \frac{1}{e}\frac{\partial}{\partial x_{j_1}}\Big[\sum f_k \phi^k + \frac{1}{N} \ln \Delta\Big]\frac{\partial}{\partial x_{j_2}}\Big[\sum f_k \phi^k + \frac{1}{N} \ln \Delta\Big]$$

$$+ \sum\Big(f_k + \frac{1}{2} d^k{}_{kk}\Big)\frac{\partial \phi^k}{\partial x_{j_1}}\frac{\partial}{\partial x_{j_2}}\Big[-\frac{1}{e}\Big(\sum f_k \phi^k + \frac{1}{N} \ln \Delta\Big)\Big]$$

$$+ \sum\Big(f_k + \frac{1}{2} d^k{}_{kk}\Big)\frac{\partial \phi^k}{\partial x_{j_2}}\frac{\partial}{\partial x_{j_1}}\Big[-\frac{1}{e}\Big(\sum f_k \phi^k + \frac{1}{N} \ln \Delta\Big)\Big]$$

$$+ g_{k_1 k_2}\frac{\partial \phi^{k_1}}{\partial x_{j_1}}\frac{\partial \phi^{k_2}}{\partial x_{j_2}}.$$

Hence it suffices if $l = \sum f_k \phi^k + \frac{1}{N} \ln \Delta$ and

$$\frac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}} = \frac{\partial l}{\partial x_{j_1}}\frac{\partial l}{\partial x_{j_2}} - \sum e\, g_{k_1 k_2}\frac{\partial \phi^{k_1}}{\partial x_{j_1}}\frac{\partial \phi^{k_2}}{\partial x_{j_2}}.$$

But

$$\frac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}} = \sum f_k (h^k{}_{r_1 r_2} + d^k{}_{r_1 r_2})\frac{\partial \phi^{r_1}}{\partial x_{j_1}}\frac{\partial \phi^{r_2}}{\partial x_{j_2}} + \frac{1}{N}\frac{\partial^2 \ln \Delta}{\partial x_{j_1}\partial x_{j_2}}$$

$$= \sum f_k h^k{}_{r_1 r_2}\frac{\partial \phi^{r_1}}{\partial x_{j_1}}\frac{\partial \phi^{r_2}}{\partial x_{j_2}} + \sum f_s \frac{\partial \phi^s}{\partial x_{j_1}}\frac{1}{N}\frac{\partial}{\partial x_{j_2}} \ln \Delta$$

$$+ \sum f_s \frac{\partial \phi^s}{\partial x_{j_2}}\frac{1}{N}\frac{\partial}{\partial x_{j_1}} \ln \Delta + \frac{1}{N}\frac{\partial^2}{\partial x_{j_1}\partial x_{j_2}} \ln \Delta.$$

$$\frac{\partial l}{\partial x_{j_1}}\frac{\partial l}{\partial x_{j_2}}=\Big[\sum f_k\frac{\partial\phi^k}{\partial x_{j_1}}+\frac{1}{N}\frac{\partial\ln\Delta}{\partial x_{j_1}}\Big]\Big[\sum f_k\frac{\partial\phi^k}{\partial x_{j_2}}+\frac{1}{N}\frac{\partial\ln\Delta}{\partial x_{j_2}}\Big]$$

so

$$\frac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}}-\frac{\partial l}{\partial x_{j_1}}\frac{\partial l}{\partial x_{j_2}}=\sum[\sum f_k h^k{}_{r_1r_2}-f_{r_1}f_{r_2}]\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}$$
$$+\frac{1}{N}\frac{\partial^2\ln\Delta}{\partial x_{j_1}\partial x_{j_2}}-\frac{1}{N^2}\frac{\partial\ln\Delta}{\partial x_{j_1}}\frac{\partial\ln\Delta}{\partial x_{j_2}}.$$

However

$$\frac{\partial^2\phi^i}{\partial x_{j_1}\partial x_{j_2}}=\sum h^i{}_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}+\frac{1}{N}\frac{\partial\phi^i}{\partial x_{j_1}}\frac{\partial\ln\Delta}{\partial x_{j_2}}+\frac{1}{N}\frac{\partial\phi^i}{\partial x_{j_2}}\frac{\partial\ln\Delta}{\partial x_{j_1}}.$$

A short calculation then shows that

$$\frac{\partial^3\phi}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}=\sum[\mathcal{S}\sum h^i{}_{kr_1}h^k{}_{r_2r_3}]\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}\frac{\partial\phi^{r_3}}{\partial x_{j_3}}$$
$$+\frac{2}{N}\mathcal{S}\sum\varphi^i{}_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\frac{\partial\ln\Delta}{\partial x_{j_3}}$$
$$+\frac{1}{N^2}\mathcal{S}\frac{\partial\phi^i}{\partial x_{j_1}}\frac{\partial\ln\Delta}{\partial x_{j_2}}\frac{\partial\ln\Delta}{\partial x_{j_3}}+\mathcal{S}\frac{1}{N}\frac{\partial\phi^i}{\partial x_{j_1}}\frac{\partial^2\ln\Delta}{\partial x_{j_2}\partial x_{j_3}}$$
$$+\frac{\partial\phi^i}{\partial x_{j_1}}\Big[\frac{1}{N^2}\frac{\partial\ln\Delta}{\partial x_{j_2}}\frac{\partial\ln\Delta}{\partial x_{j_3}}-\frac{1}{N}\frac{\partial^2\ln\Delta}{\partial x_{j_2}\partial x_{j_3}}\Big]$$
$$-\sum h^i{}_{kr_1}h^k{}_{r_2r_3}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}\frac{\partial\phi^{r_3}}{\partial x_{j_3}}.$$

But this is symmetric, so

$$\frac{\partial\phi^i}{\partial x_{j_1}}\Big[\frac{1}{N^2}\frac{\partial\ln\Delta}{\partial x_{j_2}}\frac{\partial\ln\Delta}{\partial x_{j_3}}-\frac{1}{N}\frac{\partial^2\ln\Delta}{\partial x_{j_2}\partial x_{j_3}}\Big]\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_1}_{s_1}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_2}_{s_2}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_3}_{s_3}$$
$$-\Big[\sum h^i{}_{kr_1}h^k{}_{r_2r_3}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}\frac{\partial\phi^{r_3}}{\partial x_{j_3}}\Big]\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_1}_{s_1}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_2}_{s_2}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_3}_{s_3}$$
$$=\delta^i{}_{s_1}\Big[\frac{1}{N^2}\frac{\partial\ln\Delta}{\partial x_{j_2}}\frac{\partial\ln\Delta}{\partial x_{j_3}}-\frac{1}{N}\frac{\partial^2\ln\Delta}{\partial x_{j_2}\partial x_{j_3}}\Big]\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_2}_{s_2}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_3}_{s_3}$$
$$-\sum h^i{}_{ks_1}h^k{}_{s_2s_3}=\vartheta^i{}_{s_1s_2s_3}$$

is symmetric. But

$$\sum\vartheta^i{}_{is_1s_2}=(N-1)\Big[\frac{1}{N^2}\frac{\partial\ln\Delta}{\partial x_{j_1}}\frac{\partial\ln\Delta}{\partial x_{j_2}}-\frac{1}{N}\frac{\partial^2\ln\Delta}{\partial x_{j_1}\partial x_{j_2}}\Big]\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_1}_{s_1}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_2}_{s_2}$$

(since $\sum h^i{}_{ik}=0$)

$$=\sum\vartheta^i{}_{s_1is_2}=\Big[\frac{1}{N^2}\frac{\partial\ln\Delta}{\partial x_{j_1}}\frac{\partial\ln\Delta}{\partial x_{j_2}}-\frac{1}{N^2}\frac{\partial^2\ln\Delta}{\partial x_{j_1}\partial x_{j_2}}\Big]\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_1}_{s_1}\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{j_2}_{s_2}$$
$$-\sum h^i{}_{ks_1}h^k{}_{is_2}$$

so

$$(N-2)\Big[\frac{1}{N^2}\frac{\partial\, ln\, \Delta}{\partial x_{j_1}}\frac{\partial\, ln\, \Delta}{\partial x_{j_2}}-\frac{1}{N}\frac{\partial^2 ln\, \Delta}{\partial x_{j_1}\partial x_{j_2}}\Big]=-\sum h^i{}_{ks_1}h^k{}_{is_2}\frac{\partial\phi^{s_1}}{\partial x_{j_1}}\frac{\partial\phi^{s_2}}{\partial x_{j_2}}\,.$$

Thus

$$\frac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}}-\frac{\partial l}{\partial x_{j_1}}\frac{\partial l}{\partial x_{j_2}}=\sum(f_k h^k{}_{r_1r_2}-f_{r_1}f_{r_2})\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}$$
$$+\frac{1}{N-2}\sum h^i{}_{kr_1}h^k{}_{ir_2}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}-\sum e\, g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}$$

by the second boxed-in formula in the discussion of 19a.

Now let $e=0$. Write $\phi^1=e^l(x_1+r(x_2,\cdots,x_n))$. This solves the first and second equations. But $e=0$, so $(N>2)\dfrac{\partial^2 l}{\partial x_{j_1}\partial x_{j_2}}=\dfrac{\partial l}{\partial x_{j_1}}\dfrac{\partial l}{\partial x_{j_2}}$. Looking at the previous calculations, we conclude that it is enough to solve

$$\frac{\partial^2 r}{\partial x_{j_1}\partial x_{j_2}}=e^{-l}\sum g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\,.$$

Thus we must show that

$$\frac{\partial}{\partial x_{j_3}}\Big[e^{-l}\sum g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\Big]=\vartheta_{j_1j_2j_3}$$

is symmetric. But this is

$$e^{-l}\sum g_{k_1k_2}(h^{k_1}{}_{r_1r_2}+d^{k_1}{}_{r_1r_2})\frac{\partial\phi^{r_1}}{\partial x_{j_3}}\frac{\partial\phi^{r_2}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}$$
$$+e^{-l}\sum g_{k_1k_2}(h^{k_1}{}_{r_1r_2}+d^{k_1}{}_{r_1r_2})\frac{\partial\phi^{r_1}}{\partial x_{j_3}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}\frac{\partial\phi^{k_2}}{\partial x_{j_1}}$$
$$-e^{-l}\sum f_k g_{k_1k_2}\frac{\partial\phi^{k}}{\partial x_{j_3}}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}-e^{-l}\frac{\partial}{\partial x_{j_3}}\Big(\frac{1}{N}\,ln\,\Delta\Big)g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\,.$$

It suffices if

$$\sum f_k g_{k_1k_2}\frac{\partial\phi^{k}}{\partial x_{j_3}}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}+\sum\frac{\partial}{\partial x_{j_3}}\Big(\frac{1}{N}\,ln\,\Delta\Big)g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}$$
$$+\sum g_{k_1k_2}(h^{k_1}{}_{r_1r_2}+d^{k_1}{}_{r_1r_2})\frac{\partial\phi^{k_2}}{\partial x_{j_3}}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}$$

is symmetric.

Applying the first boxed-in expression to $f_k g_{k_1k_2}+\sum g_{kr}h^r{}_{k_1k_2}$, we conclude that it is enough if

$$\sum\frac{\partial}{\partial x_{j_3}}\Big(\frac{1}{N}\,ln\,\Delta\Big)g_{k_1k_2}\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\frac{\partial\phi^{k_2}}{\partial x_{j_2}}$$
$$+\sum g_{k_1k_2}\Big[\frac{1}{2}\Big\{\delta^{k_1}{}_{r_1}d^{r_2}{}_{r_2r_2}+\delta^{k_1}{}_{r_2}d^{r_1}{}_{r_1r_1}\Big\}\Big]\frac{\partial\phi^{k_1}}{\partial x_{j_3}}\frac{\partial\phi^{r_1}}{\partial x_{j_1}}\frac{\partial\phi^{r_2}}{\partial x_{j_2}}$$

is symmetric, and by a now-familiar calculation it is.

Finally we determine the pseudogroup. If $e=f_i=g_{ij}=h^i{}_{rs}=0$,

$$\psi^i=\frac{\sum a_{ij}x_j+b_i}{\sum c_jx_j+d} \quad \text{and} \quad l=\frac{1}{N}\ ln\, \varDelta\,,$$

$$\begin{aligned}\psi^1&=Ce^{\frac{1}{N}ln\,\varDelta}(x_1+r(x_2,\cdots,x_n))\\&=C(ln\,\varDelta)^{\frac{1}{N}}(x_1+r(x_2,\cdots,x_n))\\&=(\text{by lemma 51})\ \frac{Ax_1+r(x_2,\cdots,x_n)}{\sum c_jx_j+d}\,.\end{aligned}$$

Since $\dfrac{\partial^2 r}{\partial x_{j_1}\partial x_{j_2}}=0$, we get 10_1 on the list of pseudogroups.

It remains to look at (18a) on the kernel list. In this case, $N=2$, and $\varphi=\begin{pmatrix} e & f & g\\ 0 & 0 & 0\end{pmatrix}$, $k=\begin{pmatrix} 0 & k & 0\\ 0 & 0 & 2k\end{pmatrix}$; the equations become

$$\begin{aligned}&\frac{\partial^2\psi^1}{\partial x_1{}^2}=e\Big(\frac{\partial\psi^1}{\partial x_1}\Big)^2\\&\frac{\partial^2\psi^1}{\partial x_1\partial x_2}=e\frac{\partial\psi^1}{\partial x_1}\frac{\partial\psi^1}{\partial x_2}+(f+k)\frac{\partial\psi^1}{\partial x_1}\frac{\partial\psi^2}{\partial x_2}\\&\frac{\partial^2\psi^1}{\partial x_2{}^2}=e\Big(\frac{\partial\psi^1}{\partial x_2}\Big)^2+2(f+k)\frac{\partial\psi^1}{\partial x_2}\frac{\partial\psi^2}{\partial x_2}+g\Big(\frac{\partial\psi^2}{\partial x_2}\Big)^2\\&\frac{\partial^2\psi^2}{\partial x_2{}^2}=2k\Big(\frac{\partial\psi^2}{\partial x_2}\Big)^2.\end{aligned}$$

If $e=0$, let $k=-f$. Then $\psi^1=x_1+l(x_2)$, $\psi^2=s(x_2)$. If $f=0$, let $s(x_2)=x_2$; $\dfrac{d^2l}{dx_2{}^2}=g$. If $f\neq 0$, $s(x_2)=-\dfrac{1}{2k}\,ln\,(x_2+1)$; $\dfrac{d^2l}{dx_2{}^2}=\dfrac{g}{4k^2}\dfrac{1}{(x_2+1)^2}$, which can be solved.

If $e\neq 0$, $\psi^1=-\dfrac{1}{e}\,ln\,(x_1+1)+m(x_2)$; to solve the second equation, we want

$$\frac{dm}{dx_2}=-\frac{f+k}{e}\frac{d\psi^2}{dx_2}$$

which can, of course, be solved.

The third equation becomes

$$\Big[-\frac{(f+k)^2}{e}+g\Big]\Big(\frac{d\psi^2}{dx_2}\Big)^2=-\frac{\frac{dk}{dx_2}}{e}\frac{d\psi^2}{dx_2}-\frac{f+k}{e}\frac{d^2\psi^2}{dx_2{}^2}$$

and the last equation

$$\frac{d^2\psi^2}{dx_2{}^2}=2k\Big(\frac{d\psi^2}{dx_2}\Big)^2.$$

This is equivalent to

$$(k^2-f^2+eg)\left(\frac{d\phi^2}{dx_2}\right)^2=-\frac{dk}{dx_2}\frac{d\phi^2}{dx_2}.$$

Let $k(x_2)=x_2$. Then

$$\frac{d\phi^2}{dx_2}=-\frac{1}{x_2{}^2-f^2+eg}$$

can be solved, and

$$\frac{d^2\phi^2}{dx_2{}^2}=\frac{2x_2}{(x_2{}^2-f^2+eg)^2}=2x_2\left(\frac{-1}{x_2{}^2-f^2+eg}\right)^2.$$

We are done.

Finally, we find the pseudogroup. Then $e=f=g=0$, and

$$\phi^1=e^{l(x_2)}[x_1+h(x_2)]$$

$$\frac{dl}{dx_2}=k\frac{d\phi^2}{dx_2}=\frac{1}{2}\frac{\frac{d^2\phi^2}{dx_2{}^2}}{\frac{d\phi^2}{dx_2}}=\frac{1}{2}\frac{d}{dx_2}\ ln\frac{d\phi^2}{dx_2}.$$

Hence $l=\frac{1}{2}\ ln\frac{d\phi^2}{dx_2}+C$, so

$$\phi^1=\left(\frac{d\phi^2}{dx_2}\right)^{\frac{1}{2}}[ax_1+h(x_2)]$$

and

$$\frac{d}{dx_2}\left[\frac{1}{2}\left(\frac{d\phi^2}{dx_2}\right)^{-\frac{1}{2}}\frac{d^2\phi^2}{dx_2{}^2}\{ax_1+h(x_2)\}+\left(\frac{d\phi^2}{dx_2}\right)^{\frac{1}{2}}\frac{dh}{dx_2}\right]$$

$$=\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{d\phi^2}{dx_2}\right)^{-\frac{3}{2}}\left(\frac{d^2\phi^2}{dx_2{}^2}\right)^2\{ax_1+h(x_2)\}+\frac{1}{2}\left(\frac{d\phi^2}{dx_2}\right)^{-\frac{1}{2}}\frac{d^3\phi^2}{dx_2{}^3}\{ax_1+h(x_2)\}$$

$$+\frac{1}{2}\left(\frac{d\phi^2}{dx_2}\right)^{-\frac{1}{2}}\frac{d^2\phi^2}{dx_2{}^2}\frac{dh}{dx_2}+\frac{1}{2}\left(\frac{d\phi^2}{dx_2}\right)^{-\frac{1}{2}}\frac{d^2\phi^2}{dx_2{}^2}\frac{dh}{dx_2}+\frac{1}{2}\left(\frac{d\phi^2}{dx_2}\right)^{\frac{1}{2}}\frac{d^2h}{dx_2{}^2}$$

$$=\frac{\frac{d^2\phi}{dx_2{}^2}}{\frac{d\phi}{dx_2}}\left[\frac{1}{2}\left(\frac{d\phi^2}{dx_2}\right)^{-\frac{1}{2}}\frac{d^2\phi}{dx_2{}^2}\{ax_1+h(x_2)\}\left(\frac{d\phi^2}{dx_2}\right)^{\frac{1}{2}}\frac{dh}{dx_2}\right].$$

Equating the coefficients of x_1, we have

$$\frac{d^3\phi^2}{dx_2{}^3}\frac{d\phi^2}{dx_2{}^2}-\frac{3}{2}\left(\frac{d^2\phi^2}{dx_2{}^2}\right)^2=0.$$

This is the Schwarzian differential equation, whose solutions are linear fractional transformations $\frac{ax_2+b}{cx_2+d}$. The rest of the above equation becomes $\frac{d^2h}{dx_2{}^2}=0$. Therefore we obtain 10_1 on the list of pseudogroups.

There is an interesting feature in the above calculation. The Schwarzian differential equation, normally a defining equation for G^3, appears here in the examination of G^2.

2. G^3

Due to the above equivalence result for G^2, we can assume $g^2=\langle k^1, k^2\rangle$ and use the lemmas in chapter I to construct G^3. Thus we are given $\varphi\in\Lambda^3 k^2/k^2$ and must solve $\theta_0{}^3(\psi)-\varphi\in k^3$; the complete pseudogroup consists of those elements f in the pseudogroup defined by G^2 which satisfy $\theta_0{}^3(f)\in k^3$.

$\theta^3(f)-\varphi\in k^3$ can be rewritten

$$\xi^i{}_{j_1j_2j_3}=\frac{1}{2}[\mathcal{S}\sum\theta^2(\xi)^i{}_{r_1k}\theta^2(\xi)^k{}_{r_2r_3}]\xi^{r_1}{}_{j_1}\xi^{r_2}{}_{j_2}\xi^{r_3}{}_{j_3}$$
$$+\frac{1}{2}\sum(\varphi^i{}_{k_1k_2k_3}+k^i{}_{k_1k_2k_3})\xi^{k_1}{}_{j_1}\xi^{k_2}{}_{j_2}\xi^{k_3}{}_{j_3}.$$

LEMMA 53. *Let* $\varphi^i{}_{j_1j_2j_3}=\begin{pmatrix} e & e_i & e_{j_1j_2} & e_{j_1j_2j_3} \\ 0 & 0 & 0 & h^i{}_{j_1j_2j_3}\end{pmatrix}$, $\xi^i{}_k=\begin{pmatrix} a & b_i \\ 0 & d^i{}_j\end{pmatrix}$. *Then*

$$\sum\varphi^i{}_{k_1k_2k_3}\xi^{k_1}{}_{j_1}\xi^{k_2}{}_{j_2}\xi^{k_3}{}_{j_3}$$
$$=\begin{pmatrix} ea^2 & ea^2b_j+\sum a^2e_rd^r{}_j & [eab_{j_1}b_{j_2}+\sum ae_rd^r{}_{j_1}b_{j_2}+\sum ae_rd^r{}_{j_2}b_{j_1}+\sum ae_{r_1r_2}d^{r_1}{}_{j_1}d^{r_2}{}_{j_2}] \\ 0 & 0 & 0\end{pmatrix.}$$
$$\begin{matrix} [eb_{j_1}b_{j_2}b_{j_3}+\sum e_r(\mathcal{S}b_{j_1}b_{j_2}d^r{}_{j_3})+\sum e_{r_1r_2}(\mathcal{S}b_{j_1}d^{r_1}{}_{j_2}d^{r_2}{}_{j_3})+\sum e_{r_1r_2r_3}d^{r_1}{}_{j_1}d^{r_2}{}_{j_2}d^{r_3}{}_{j_3}] \\ \sum h^i{}_{r_1r_2r_3}d^{r_1}{}_{j_1}d^{r_2}{}_{j_2}d^{r_3}{}_{j_3}\end{matrix}\Bigg).$$

PROOF. A calculation.

LEMMA 54. *Let* $\theta^2(\psi)^i{}_{j_1j_2}=\begin{pmatrix} f & f_j & f_{j_1j_2} \\ 0 & 0 & r^i{}_{j_1j_2}\end{pmatrix}$. *Then*

$$\mathcal{S}\sum\theta^2(\psi)^i{}_{j_1k}\theta^2(\psi)^k{}_{j_2j_3}=\begin{pmatrix} 3f^2 & 3ff_j & [ff_{j_1j_2}+\sum f_rr^r{}_{j_1j_2}+2f_{j_1}f_{j_2}] & \mathcal{S}(f_{j_1}f_{j_2j_3}+f_{j_1r}r^r{}_{j_2j_3}) \\ 0 & 0 & 0 & \mathcal{S}\sum r^i{}_{j_1k}r^k{}_{j_2j_3}\end{pmatrix}.$$

PROOF. A calculation.

LEMMA 55. *The equation* $\theta^3(\psi)-\varphi\in k^3$ *is equivalent to the following set of equations, if*

$$\varphi+k^3=\begin{pmatrix} e & e_i & e_{ij} & e_{ijk} \\ 0 & 0 & 0 & h^i{}_{jks}\end{pmatrix},\qquad \theta^2(f)=\begin{pmatrix} f & f_i & f_{ij} \\ 0 & 0 & r^i{}_{jk}\end{pmatrix}$$

(A) $2\dfrac{\partial^3\psi^1}{\partial x_1{}^3}=(3f^2+e)\Big(\dfrac{\partial\psi^1}{\partial x_1}\Big)^3$

(B) $2\dfrac{\partial^3\psi^1}{\partial x_1{}^2\partial x_j}=(3f^2+e)\Big(\dfrac{\partial\psi^1}{\partial x_1}\Big)^2\Big(\dfrac{\partial\psi^1}{\partial x_j}\Big)+\sum_r(3ff_r+e_r)\Big(\dfrac{\partial\psi^1}{\partial x_1}\Big)^2\Big(\dfrac{\partial\psi^r}{\partial x_j}\Big)$

(C) $2\dfrac{\partial^3\psi^1}{\partial x_1\partial x_{j_1}\partial x_{j_2}}$

$$=(3f^2+e)\Big(\frac{\partial\psi^1}{\partial x_1}\Big)\Big(\frac{\partial\psi^1}{\partial x_{j_1}}\Big)\Big(\frac{\partial\psi^1}{\partial x_{j_2}}\Big)$$

$$+\Sigma(3ff_k+e_k)\Big[\Big(\frac{\partial\phi^1}{\partial x_1}\Big)\Big(\frac{\partial\phi^1}{\partial x_{j_1}}\Big)\Big(\frac{\partial\phi^k}{\partial x_{j_2}}\Big)+\Big(\frac{\partial\phi^1}{\partial x_1}\Big)\Big(\frac{\partial\phi^1}{\partial x_{j_2}}\Big)\Big(\frac{\partial\phi^k}{\partial x_{j_1}}\Big)\Big]$$

$$+\Sigma(ff_{k_1k_2}+\Sigma f_s r^s{}_{k_1k_2}+2f_{k_1}f_{k_2}+e_{k_1k_2})\Big(\frac{\partial\phi^1}{\partial x_1}\Big)\Big(\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\Big)\Big(\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\Big)$$

(D) $$2\frac{\partial^3\phi^1}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}$$

$$=(3f^2+e)\Big(\frac{\partial\phi^1}{\partial x_{j_1}}\Big)\Big(\frac{\partial\phi^1}{\partial x_{j_2}}\Big)\Big(\frac{\partial\phi^1}{\partial x_{j_3}}\Big)+\Sigma(3ff_k+e_k)\mathcal{S}\Big[\Big(\frac{\partial\phi^1}{\partial x_{j_1}}\Big)\Big(\frac{\partial\phi^1}{\partial x_{j_2}}\Big)\Big(\frac{\partial\phi^k}{\partial x_{j_3}}\Big)\Big]$$

$$+\Sigma(ff_{k_1k_2}+\Sigma f_s r^s{}_{k_1k_2}+2f_{k_1}f_{k_2}+e_{k_1k_2})\mathcal{S}\Big[\Big(\frac{\partial\phi^1}{\partial x_{j_1}}\Big)\Big(\frac{\partial\phi^{k_1}}{\partial x_{j_2}}\Big)\Big(\frac{\partial\phi^{k_2}}{\partial x_{j_3}}\Big)\Big]$$

$$+\Sigma[\mathcal{S}\{f_{k_1}f_{k_2k_3}+\Sigma f_{k_1s}r^s{}_{k_2k_3}\}+e_{k_1k_2k_3}]\Big(\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\Big)\Big(\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\Big)\Big(\frac{\partial\phi^{k_3}}{\partial x_{j_3}}\Big)$$

(E) $$2\frac{\partial^3\phi^i}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}=[\mathcal{S}\Sigma r^i{}_{k_1s}r^s{}_{k_2k_3}+h^i{}_{k_1k_2k_3}]\Big(\frac{\partial\phi^{k_1}}{\partial x_{j_1}}\Big)\Big(\frac{\partial\phi^{k_2}}{\partial x_{j_2}}\Big)\Big(\frac{\partial\phi^{k_3}}{\partial x_{j_3}}\Big)$$

PROOF. A calculation based on lemmas 53 and 54.

LEMMA 56. *If e is arbitrary in k^8, (A) is irrelevant; if e_i is arbitrary, (B) is irrelevant; if e_{ij} is arbitrary, (C) is irrelevant; if e_{ijk} is arbitrary, (D) is irrelevant; if $h^i{}_{jks}$ is arbitrary, (E) is irrelevant.*

PROOF. The proof is exactly the same as the proof of lemma 49.

LEMMA 57.

$$\theta^2(\phi)=\begin{pmatrix} f & f_i & f_{ij} \\ 0 & 0 & r^i{}_{jk}\end{pmatrix}$$

$$=\left(\begin{matrix} \dfrac{\partial^2\phi^1}{\partial x_1{}^2}\Big\{\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_1\Big\}^2 \\ 0 \end{matrix}\right.$$

$$\begin{matrix} \dfrac{\partial^2\phi^1}{\partial x_1{}^2}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_1\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_j+\Sigma\dfrac{\partial^2\phi^1}{\partial x_1\partial x_k}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^k_j\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_1 \\ 0 \end{matrix}$$

$$\left.\begin{matrix} \dfrac{\partial^2\phi^1}{\partial x_1{}^2}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_{j_1}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_{j_2}+\Sigma\dfrac{\partial^2\phi^1}{\partial x_1\partial x_k}\Big\{\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^k_{j_1}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_{j_2} \\ \Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^k_{j_2}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_{j_1}\Big\}+\Sigma\dfrac{\partial^2\phi^i}{\partial x_{k_1}\partial x_{k_2}}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{k_1}_{j_1}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{k_2}_{j_2} \\ \dfrac{\partial^2\phi^i}{\partial x_{k_1}\partial x_{k_2}}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{k_1}_{j_1}\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^{k_2}_{j_2} \end{matrix}\right)$$

PROOF. A consequence of lemma 47.

Now we begin a step by step calculation; we examine those kernel sequences on our list not of the form k^2, $\Lambda^3 k^2$, $\cdots$.

The first occurs in (2). Here $\varphi=\begin{pmatrix}0&0&0&\varphi_{j_1j_2j_3}\\0&0&0&0\end{pmatrix}$, $k^3=0$. Moreover, the G^2 pseudogroup is

$$\phi^1=cx_1+g(x_2,\cdots,x_n)$$
$$\phi^i=\sum a_{ij}x_j+b_i\,.$$

Hence $\dfrac{\partial^2\phi^1}{\partial x_1{}^2}=\dfrac{\partial^2\phi^1}{\partial x_1\partial x_j}=\dfrac{\partial^2\phi^i}{\partial x_{j_1}\partial x_{j_2}}=0$, so

$$\begin{pmatrix}f & f_i & f_{ij}\\ 0 & 0 & r^i{}_{jk}\end{pmatrix}=\begin{pmatrix}0 & 0 & \sum\dfrac{\partial^2 g}{\partial x_{k_1}\partial x_{k_2}}(a^{-1})^{k_1}{}_{j_1}(a^{-1})^{k_2}{}_{j_2}\\ 0 & 0 & 0\end{pmatrix}.$$

Equations (A), (B), (C), and (E) are trivial, and (D) becomes

$$2\frac{\partial^3 g}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}=\sum\varphi_{k_1k_2k_3}a^{k_1}{}_{j_1}a^{k_2}{}_{j_2}a^{k_3}{}_{j_3}\,.$$

But we can pick

$$\phi^1=x_1+g(x_2,\cdots,x_n)$$
$$\phi^i=x_i$$

and it suffices if $\dfrac{\partial^3 g}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}=\varphi_{j_1j_2j_3}$ which is easily done. The pscudogroup is clearly

$$\begin{cases} y_1=cx_1+P_2(x_2,\cdots,x_n)\\ y_i=\sum a_{ij}x_j+b_i\,.\end{cases}$$

Next, examine (3); here $k^3=\begin{pmatrix}0&0&0&a\\0&0&0&0\end{pmatrix}$ and $\varphi=\begin{pmatrix}0&0&0&0\\0&0&0&s\end{pmatrix}$. The G^2 pseudogroup is

$$\begin{cases} y_1=cx_1+g(x_2)\\ y_i=f(x_2)\,.\end{cases}$$

Then

$$\begin{pmatrix}f & f_i & f_{ij}\\ 0 & 0 & r^i{}_{jk}\end{pmatrix}=\begin{pmatrix}0 & 0 & \dfrac{\partial^2 g}{\partial x_2{}^2}\left(\dfrac{\partial f}{\partial x_2}\right)^{-2}\\ 0 & 0 & \dfrac{\partial^2 f}{\partial x_2{}^2}\left(\dfrac{\partial f}{\partial x_2}\right)^{-2}\end{pmatrix};$$

equations (A), (B), and (C) are trivial, and (D) is trivial by lemma 56, so we must solve

$$2\frac{\partial^3 f}{\partial x_2{}^3}=\left\{\left[3\frac{\partial^2 f}{\partial x_2{}^2}\left(\frac{\partial f}{\partial x_2}\right)^{-2}\right]^2+s\right\}\left(\frac{\partial f}{\partial x_2}\right)^3$$
$$=3\left(\frac{\partial^2 f}{\partial x_2{}^2}\right)^2\left(\frac{\partial f}{\partial x_2}\right)^{-1}+s\left(\frac{\partial f}{\partial x_2}\right)^3$$

or

$$\frac{d^3f}{dx_2^3}\frac{df}{dx_2}-\frac{3}{2}\left(\frac{d^2f}{dx_2^2}\right)^2=\frac{s}{2}\left(\frac{df}{dx_2}\right)^4.$$

But this can be solved and if $s=0$, it is the Schwarzian differential equation; the pseudogroup is

$$\begin{cases} y_1=Ax_1+g(x_2) \\ y_2=\dfrac{ax_2+b}{cx_2+d} \end{cases}$$

which is (3) on the list of pseudogroups.

The next kernel sequence is (7). Here $k^3=\begin{pmatrix}0 & 0 & 0 & e_{jkr} \\ 0 & 0 & 0 & 0\end{pmatrix}$, $\varphi=\begin{pmatrix}0 & 0 & \varphi_{ij} & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}$, and the G^2 pseudogroup is

$$\begin{cases} y_1=x_1g(x_2,\cdots,x_n)+k(x_2,\cdots,x_n) \\ y_i=\sum a_{ij}x_j+b_i\,. \end{cases}$$

Then

$$\begin{pmatrix} f & f_i & f_{ij} \\ 0 & 0 & r^i{}_{jk}\end{pmatrix}=\begin{pmatrix} 0 & \dfrac{\partial g}{\partial x_k}(a^{-1})^k{}_j g^{-1} & * \\ 0 & 0 & 0\end{pmatrix}$$

($*$ is immaterial).

Equations (A), (B), and (E) are trivial, and (D) is immaterial by lemma 56, so we only have

$$2\frac{\partial^2 g}{\partial x_{j_1}\partial x_{j_2}}=2\left[\sum\frac{\dfrac{\partial g}{\partial x_{k_1}}}{g}\frac{\dfrac{\partial g}{\partial x_{k_2}}}{g}(a^{-1})^{k_1}{}_{r_1}(a^{-1})^{k_2}{}_{r_2}a_{r_1j_1}a_{r_2j_2}g+\sum\varphi_{r_1r_2}a_{r_1j_1}a_{r_2j_2}g\right]$$

$$=2\left[g\frac{\dfrac{\partial g}{\partial x_{j_1}}}{g}\frac{\dfrac{\partial g}{\partial x_{j_2}}}{g}+g\sum\varphi_{r_1r_2}a_{r_1j_1}a_{r_2j_2}\right].$$

Letting $g=e^h$, this reads

$$2\left[e^h\frac{\partial^2 h}{\partial x_{j_1}\partial x_{j_2}}+e^h\frac{\partial h}{\partial x_{j_1}}\frac{\partial h}{\partial x_{j_2}}\right]=2\left[e^h\frac{\partial h}{\partial x_{j_1}}\frac{\partial h}{\partial x_{j_2}}+e^h\sum\varphi_{r_1r_2}a_{r_1j_1}a_{r_2j_2}\right]$$

or

$$\frac{\partial^2 h}{\partial x_{j_1}\partial x_{j_2}}=\frac{1}{2}\sum\varphi_{r_1r_2}a_{r_1j_1}a_{r_2j_2}$$

so to solve the equivalence problem we let $\begin{cases}\phi^1=x_1e^h \\ \phi^i=x_i\end{cases}$ with $\dfrac{\partial^2 h}{\partial x_{j_1}\partial x_{j_2}}=\dfrac{1}{2}\varphi_{j_1j_2}$.

Similarly the pseudogroup is

$$\begin{cases} y_1=e^{p_1(x_2,\cdots,x_n)}+k(x_2,\cdots,x_n) \\ y_i=\sum a_{ij}x_j+b_i \end{cases}$$

which is 6_1 on the list of pseudogroups.

Next consider (8) on the list of kernels; $k^2=\begin{pmatrix}0&0&a&b\\0&0&0&0\end{pmatrix}$, $\varphi=\begin{pmatrix}0&0&0&0\\0&0&0&s\end{pmatrix}$; the G^2 pseudogroup is

$$\begin{cases} y_1=x_1g(x_2)+h(x_2)\\ y_2=f(x_2)\,. \end{cases}$$

But

$$\begin{pmatrix} f & f_i & f_{ij}\\ 0 & 0 & r^i{}_{jk}\end{pmatrix}=\begin{pmatrix} 0 & * & *\\ 0 & 0 & \dfrac{d^2f}{dx_2{}^2}\Big(\dfrac{df}{dx_2}\Big)^{-2}\end{pmatrix}$$

so (using lemma 56), (A), (B), (C), and (D) are trivial, and

$$2\frac{d^3f}{dx_2{}^3}=3\Big[\frac{d^2f}{dx_2{}^2}\Big(\frac{df}{dx_2}\Big)^{-2}\Big]^2\Big(\frac{df}{dx_2}\Big)^3+s\Big(\frac{df}{dx_2}\Big)^3,$$

an equation that has already occurred. The pseudogroup is (7) on the list of pseudogroups.

In (12), $k^3=\begin{pmatrix}0&e_j&e_{jk}&e_{jkr}\\0&0&0&0\end{pmatrix}$, $\varphi=\begin{pmatrix}s&0&0&0\\0&0&0&0\end{pmatrix}$ and the G^2 pseudogroup is $\begin{cases}\phi^1=f(x_1,\cdots,x_n)\\ \phi^i=\sum a_{ij}x_j+b_i\end{cases}$. Then

$$\begin{pmatrix} f & f_i & f_{ij}\\ 0 & 0 & r^i{}_{jk}\end{pmatrix}=\begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1{}^2}\Big\{\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_1\Big\}^2 & * & *\\ 0 & 0 & 0\end{pmatrix}$$

and

$$2\frac{\partial^3 f}{\partial x_1{}^3}=\Big\{3\Big[\frac{\partial^2 f}{\partial x_1{}^2}\Big\{\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_1\Big\}^2\Big]^2+s\Big\}\Big(\frac{\partial f}{\partial x_1}\Big)^3.$$

If $a_{ij}=\delta^i{}_j$, $b_i=0$, $f(x_1,\cdots,x_n)=f(x_1)$, we have our old familiar equation, which can be solved. To find the pseudogroup, let $s=0$:

$$\frac{\partial^3 f}{\partial x_1{}^3}-\frac{3}{2}\Big(\frac{\partial^2 f}{\partial x_1}\Big)^2\Big\{\Big[\Big(\frac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_1\Big\}^4\Big(\frac{\partial f}{\partial x_1}\Big)^3=0\,.$$

But clearly $\Big[\Big(\dfrac{\partial\phi}{\partial x}\Big)^{-1}\Big]^1_1=\Big(\dfrac{\partial f}{\partial x_1}\Big)^{-1}$, so this says that for each $x_2,\cdots,x_n$, f is linear fractional; hence we have (12) on the list of pseudogroups.

In (13), (14), (15), and (16), we clearly have the same situation, since the equation on $\dfrac{\partial^3\phi^i}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}$ is a consequence of the G^2 equations.

Next consider (18). $k^3=\begin{pmatrix}0&0&0&e\\0&0&0&0\end{pmatrix}$, $\varphi=\begin{pmatrix}0&0&\lambda s&0\\0&0&0&s\end{pmatrix}$, and the pseudogroup given by G^2 is

$$\begin{cases} y_1=Cx_1\Big(\dfrac{\partial f}{\partial x_2}\Big)^{\lambda}+h(x_2)\\ y_2=f(x_2)\,. \end{cases}$$

Then

$$\begin{pmatrix} f & f_i & f_{ij} \\ 0 & 0 & r^i{}_{jk} \end{pmatrix} = \begin{pmatrix} 0 & \frac{d}{dx_2}\Big[C\Big(\frac{df}{dx_2}\Big)^{\lambda}\Big]\Big(\frac{df}{dx_2}\Big)^{-1} & \frac{1}{C\Big(\frac{df}{dx_2}\Big)^{\lambda}} & * \\ 0 & 0 & & \frac{d^2f}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-2} \end{pmatrix}$$

so equations (A), (B), and (D) are trivial. We have

$$\text{(C)}\quad 2\frac{d^2}{dx_2^2}\Big[C\Big(\frac{df}{dx_2}\Big)^{\lambda}\Big] = \Big[\lambda\Big(\frac{df}{dx_2}\Big)^{-4}\Big(\frac{d^2f}{dx_2^2}\Big)^{2}\Big(\frac{df}{dx_2}\Big)^{2} + 2\lambda^2\Big(\frac{df}{dx_2}\Big)^{-4}\Big(\frac{d^2f}{dx_2^2}\Big)^{2}\Big(\frac{df}{dx_2}\Big)^{2} + s\lambda\Big(\frac{df}{dx_2}\Big)^{2}\Big]C\Big(\frac{df}{dx_2}\Big)^{\lambda}$$

$$\text{(E)}\quad 2\frac{d^3f}{dx_2^3} = 3\Big[\frac{d^2f}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-2}\Big]^{2}\Big(\frac{df}{dx_2}\Big)^{-3} + s\Big(\frac{df}{dx_2}\Big)^{-3}.$$

This last equation is already famous. The first equation may be rewritten

$$2\lambda(\lambda-1)\Big(\frac{df}{dx_2}\Big)^{-2}\Big(\frac{d^2f}{dx_2^2}\Big)^{2} + 2\lambda\Big(\frac{df}{dx_2}\Big)^{-1}\frac{d^3f}{dx_2^3}$$
$$= \lambda\Big(\frac{d^2f}{dx_2^2}\Big)^{2}\Big(\frac{df}{dx_2}\Big)^{-2} + 2\lambda^2\Big(\frac{d^2f}{dx_2^2}\Big)^{2}\Big(\frac{df}{dx_2}\Big)^{-2} + \lambda s\Big(\frac{df}{dx_2}\Big)^{2}$$

or

$$\lambda\Big[\frac{d^3f}{dx_2^3}\frac{df}{dx_2} - \frac{3}{2}\Big(\frac{d^2f}{dx_2^2}\Big)^{2} - \frac{s}{2}\Big(\frac{df}{dx_2}\Big)^{4}\Big] = 0$$

and so is implied by (E). The pseudogroup is clearly 10 on the pseudogroup list.

Consider (18b); then $\lambda=1$, $k^2=0$, $\varphi=\begin{pmatrix} 0 & 0 & s & e \\ 0 & 0 & 0 & s \end{pmatrix}$ and the G^2 pseudogroup is as above. We may, in fact, use the results of the above calculation; there is one additional equation, (D), to consider. But

$$\begin{pmatrix} f & f_i & f_{ij} \\ 0 & 0 & r^i{}_{jk} \end{pmatrix} = \begin{pmatrix} 0 & \frac{d^2f}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-2} & * \\ 0 & 0 & \frac{d^2f}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-2} \end{pmatrix}$$

$$* = 2C\frac{d^2f}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-1}\Big[-x_1\frac{d^2f}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-2} - \frac{dh}{dx_2}\Big(\frac{df}{dx_2}\Big)^{-2}\frac{1}{C}\Big]$$
$$+\Big[Cx_1\frac{d^3f}{dx_2^3} + \frac{d^2h}{dx_2^2}\Big]\Big(\frac{df}{dx_2}\Big)^{-2}$$
$$= -2Cx_1\Big(\frac{d^2f}{dx_2^2}\Big)^{2}\Big(\frac{df}{dx_2}\Big)^{-3} - 2\frac{d^2f}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-3}\frac{dh}{dx_2}$$
$$+Cx_1\frac{d^3f}{dx_2^3}\Big(\frac{df}{dx_2}\Big)^{-2} + \frac{d^2h}{dx_2^2}\Big(\frac{df}{dx_2}\Big)^{-2}$$

and (D) becomes

$$2\left[Cx_1\frac{d^4f}{dx_2^4}+\frac{d^3f}{dx_2^3}\right]$$

$$=3\left[3\left(\frac{d^2f}{dx_2^2}\right)^2\left(\frac{df}{dx_2}\right)^{-4}+s\right]\left(\frac{df}{dx_2}\right)^2\left[Cx_1\frac{d^2f}{dx_2^2}+\frac{dh}{dx_2}\right]$$

$$+6\frac{d^2f}{dx_2^2}\left(\frac{df}{dx_2}\right)\left[-2Cx_1\left(\frac{d^2f}{dx_2^2}\right)^2\left(\frac{df}{dx_2}\right)^{-3}-2\frac{d^2f}{dx_2^2}\left(\frac{df}{dx_2}\right)^{-3}\frac{dh}{dx_2}\right.$$

$$\left.+Cx_1\frac{d^3f}{dx_2^3}\left(\frac{df}{dx_2}\right)^{-2}+\frac{d^2h}{dx_2^2}\left(\frac{df}{dx_2}\right)^{-2}\right]+e\left(\frac{df}{dx_2}\right)^3$$

or

$$Cx_1\left[2\frac{d^4f}{dx_2^4}+3\left(\frac{d^2f}{dx_2^2}\right)^3\left(\frac{df}{dx_2}\right)^{-2}-3s\frac{d^2f}{dx_2^2}\left(\frac{df}{dx_2}\right)^2-6\frac{d^2f}{dx_2^2}\frac{d^3f}{dx_2^3}\left(\frac{df}{dx_2}\right)^{-1}\right]$$

$$+2\frac{d^3h}{dx_2^3}-9\left(\frac{d^2f}{dx_2^2}\right)^2\left(\frac{df}{dx_2}\right)^{-2}\frac{dh}{dx_2}-3s\frac{dh}{dx_2}\left(\frac{df}{dx_2}\right)^2-e\left(\frac{df}{dx_2}\right)^3$$

$$+12\left(\frac{d^2f}{dx_2^2}\right)^2\left(\frac{df}{dx_2}\right)^{-2}\frac{dh}{dx_2}-6\frac{d^2f}{dx_2^2}\frac{d^2h}{dx_2^2}\left(\frac{df}{dx_2}\right)^{-1}=0.$$

But

$$\frac{d^3f}{dx_2^3}\frac{df}{dx_2}-\frac{3}{2}\left(\frac{d^2f}{dx_2^2}\right)^2=\frac{s}{2}\left(\frac{df}{dx_2}\right)^4$$

so the coefficient of Cx_1 is easily shown to vanish. The equivalence problem then can be solved by known facts about linear differential equations. To find the pseudogroup, we let $s=e=0$; the above equation becomes

$$2\frac{d^3h}{dx_2^3}-c\frac{d^2f}{dx_2^2}\left(\frac{df}{dx_2}\right)^{-1}\frac{d^2h}{dx_2^2}+3\left(\frac{d^2f}{dx_2^2}\right)^2\left(\frac{df}{dx_2}\right)^{-2}\frac{dh}{dx_2}=0.$$

But

$$\begin{cases} y_1=\dfrac{Ax_1}{(cx_2+d)^2}+h(x_2) \\ y_2=\dfrac{ax_2+b}{cx_2+d}, \end{cases}$$

and

$$\frac{\dfrac{d^2f}{dx_2^2}}{\dfrac{df}{dx_2}}=\frac{-2c}{cx_2+d},$$

so

$$\frac{d^3h}{dx_2^3}+\frac{6c}{cx_2+d}\frac{d^2h}{dx_2^2}+\frac{6c^2}{(cx_2+d)^2}\frac{dh}{dx_2}=0.$$

Writing $h(x_2)=\dfrac{\tilde{h}(x_2)}{(cx_2+d)^2}$, this equation reads $\dfrac{d^3\tilde{h}}{dx_2^3}=0$, so

$$\begin{cases} y_1 = \dfrac{Ax_1 + P_2(x_2)}{(cx_2+d)^2} \\ y_2 = \dfrac{ax_2+b}{cx_2+d} \end{cases}$$

which is 10_2 on the list of pseudogroups.

Finally consider (19b). Then $k^3 = 0$, $\varphi = \begin{pmatrix} 0 & 0 & 0 & \varphi_{jkr} \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and the G^2 pseudogroup is

$$\begin{cases} y_1 = \dfrac{Ax_1 + h(x_2, \cdots, x_n)}{\sum c_j x_j + d} \\ y_i = \dfrac{\sum a_{ij} x_j + b_i}{\sum c_j x_j + d} \end{cases}.$$

But if $\varphi = 0$, and $k^3 = \begin{pmatrix} 0 & 0 & 0 & e_{ijk} \\ 0 & 0 & 0 & 0 \end{pmatrix}$, this is the trivial extension of G^2, so all equations hold. The only difference occurs in equation (D); there the situation is almost exactly that of (18b), so we only sketch it.

$$2\frac{\partial^3 \phi^1}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} = 2x_1 \frac{\partial^3}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \left(\frac{A}{\sum c_j x_j + d} \right) + 2 \frac{\partial^3}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \left(\frac{h}{\sum c_j x_j + d} \right);$$

a simple calculation shows the right hand side is

$$2x_1 \frac{\partial^3}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \left(\frac{A}{\sum c_j x_j + d} \right) + 2 \frac{\partial^3}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \left(\frac{h}{\sum c_j x_j + d} \right)$$
$$-2 \frac{\dfrac{\partial^3 h}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}}}{\sum c_j x_j + d} + \sum e_{r_1 r_2 r_3} \frac{\partial \phi^{r_1}}{\partial x_{j_1}} \frac{\partial \phi^{r_2}}{\partial x_{j_2}} \frac{\partial \phi^{r_3}}{\partial x_{j_3}}.$$

Hence letting

$$\begin{cases} \phi^1 = x_1 + h(x_2, \cdots, x_n) \\ \phi^i = x_i \end{cases}$$

we must solve

$$\frac{\partial^3 h}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} = \frac{1}{2} e_{j_1 j_2 j_3}$$

and this is easily done. Similarly the pseudogroup is given by restricting h to polynomials of degree $\leqq 2$.

3. G^n

As n gets larger the calculations grow more involved; I will merely sketch the situation for these higher groups.

The general G^n is built over a G^{n-1}; by the equivalence result on this G^{n-1}, we can assume $g^{n-1} = \langle k^1, \cdots, k^{n-1} \rangle$. Then by lemma 16, g^n is determined by $\varphi \in \Lambda^n k^{n-1}/k^n$;

$$g^n = \left\{ \langle A_1, \cdots, A_n \rangle \mid \langle A_1, \cdots, A_{n-1} \rangle \in g^{n-1} \text{ and } A_n = -\frac{1}{n-1}[d\rho(A_1)\varphi] + k^n \right\}.$$

There are four remaining kernel sequences, (2), (7), (18b) and (19b).

Look first at (2); then $\varphi = \begin{pmatrix} 0 \cdots 0 & \phi_{j_1 \cdots j_n} \\ 0 \cdots 0 & 0 \end{pmatrix}$. But if $A_1 = \begin{pmatrix} a & b_i \\ 0 & d^i{}_j \end{pmatrix}$, $[A_1, \varphi]^1{}_{j_1 \cdots j_n}$ is $\sum A^i{}_k \varphi^k{}_{j_1 \cdots j_n} - \sum \varphi^i{}_{j_1 \cdots k \cdots j_n} A^k{}_{j_\lambda} = a\phi_{j_1 \cdots j_n} - \sum \phi_{j_1 \cdots k \cdots j_n} d^k{}_{j_\lambda}$. Hence

$$A^1{}_{j_1 \cdots j_n} = \frac{1}{n-1} [\sum \phi_{j_1 \cdots k \cdots j_n} d^k{}_{j_\lambda} - a\phi_{j_1 \cdots j_n}].$$

LEMMA 58.

$$G^n = \left\{ \begin{pmatrix} a & b_i \\ 0 & d^i{}_j \end{pmatrix}, \begin{pmatrix} 0 & 0 & e_{ij} \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 \cdots 0 & e_{j_1 \cdots j_{n-1}} \\ 0 & 0 \cdots 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \cdots 0 & \frac{1}{n-1}[\sum \phi_{k_1 \cdots k_n} d^{k_1}{}_{j_1} \cdots d^{k_n}{}_{j_n} - a\phi_{j_1 \cdots j_n}] \\ 0 \cdots 0 & 0 \end{pmatrix} \right\}.$$

PROOF. This clearly has the correct Lie algebra; it suffices if it is a group. But $(\xi \circ \eta)^1{}_{j_1 \cdots j_n} = \xi^1{}_1 \eta^1{}_{j_1 \cdots j_n} + \sum \xi^1{}_{k_1 \cdots k_n} \eta^{k_1}{}_{j_1} \cdots \eta^{k_n}{}_{j_n}$ in this case, since all other terms in the formula in theorem 1 clearly vanish. Hence if ξ is represented as above, and η is the same with $\sim$ over each element, we have

$$\begin{aligned}(\xi \circ \eta)^1{}_{j_1 \cdots j_n} &= \frac{1}{n-1} [a(\sum \phi_{k_1 \cdots k_n} \tilde{d}^{k_1}{}_{j_1} \ldots \tilde{d}^{k_n}{}_{j_n} - \tilde{a}\phi_{j_1 \cdots j_n}) \\ &\quad + \sum (\phi_{k_1 \cdots k_n} d^{k_1}{}_{r_1} \cdots d^{k_n}{}_{r_n} - a\phi_{r_1 \cdots r_n}) \tilde{d}^{r_1}{}_{j_1} \cdots \tilde{d}^{r_n}{}_{j_n}] \\ &= \frac{1}{n-1} [\sum \phi_{k_1 \cdots k_n} (d\tilde{d})^{k_1}{}_{j_1} \cdots (d\tilde{d})^{k_n}{}_{j_n} - a\tilde{a}\phi_{j_1 \cdots j_n}]\end{aligned}$$

and so a group.

The resulting pseudogroup for φ is

$$\begin{cases} y_1 = cx_1 + g(x_2, \cdots, x_n) \\ y_i = \sum a_{ij} x_j + b_i \end{cases}$$

where

$$\frac{\partial^n g}{\partial x_{j_1} \cdots \partial x_{j_n}} = \frac{1}{n-1} [\sum \phi_{k_1 \cdots k_n} a_{k_1 j_1} \cdots a_{k_n j_n} - c\phi_{j_1 \cdots j_n}].$$

In particular, for $\phi = 0$, this is a polynomial of degree $n-1$.

Now choose h with $\dfrac{\partial^n h}{\partial x_{j_1} \cdots \partial x_{j_n}} = \dfrac{1}{n-1} \phi_{j_1 \cdots j_n}$. Consider

$$\vartheta = \begin{cases} y_1 = x_1 + h(x_2, \cdots, x_n) \\ y_i = x_i. \end{cases}$$

Then for any f in the above pseudogroup of the form

$$f=\begin{cases} y_1=cx_1+h(x_2,\cdots,x_n) \\ y_i=\sum a_{ij}x_j+b_i \end{cases}$$

we have

$$\vartheta^{-1}\circ f\circ\vartheta=\begin{cases} y_1=cx_1+ch(x_2,\cdots,x_n)+g(x_2,\cdots,x_n)-h(\sum a_{ij}x_j+b_i) \\ y_i=\sum a_{ij}x_j+b_i\,. \end{cases}$$

I claim this is in the pseudogroup for $\psi=0 \Leftrightarrow f$ is in the original pseudogroup. Indeed,

$$\frac{\partial^n}{\partial x_{j_1}\cdots\partial x_{j_n}}[ch(x_2,\cdots,x_n)+g(x_2,\cdots,x_n)-h(\sum a_{ij}x_j+b_i)]$$
$$=\frac{1}{n-1}c\psi_{j_1\cdots j_n}+\frac{\partial^n g}{\partial x_{j_1}\cdots\partial x_{j_n}}-\frac{1}{n-1}\sum\psi_{k_1\cdots k_n}a_{k_1j_1}\cdots a_{k_nj_n}$$

so this is zero if and only if

$$\frac{\partial^n g}{\partial x_{j_1}\cdots\partial x_{j_n}}=\frac{1}{n-1}[\sum\psi_{k_1\cdots k_n}a_{k_1j_1}\cdots a_{k_nj_n}-c\psi_{j_1\cdots j_n}]\,.$$

This solves the equivalence problem and yields 2_n in the list of pseudogroups.

Consider (7) in the list of kernels. We can choose $\varphi=\begin{pmatrix}0\cdots 0 & \psi_{j_1\cdots j_{n-1}} & 0\\ 0\cdots 0 & 0 & 0\end{pmatrix}$.

LEMMA 59.

$$g^n=\left\{\begin{pmatrix} a & b_i \\ 0 & d^i{}_j\end{pmatrix},\begin{pmatrix} 0 & e_i & e_{ij} \\ 0 & 0 & 0\end{pmatrix},\cdots,\begin{pmatrix} 0\cdots 0 & e_{j_1\cdots j_{n-1}}e_{j_1\cdots j_n} \\ 0\cdots 0 & 0 \quad\quad 0\end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 0\cdots 0 & \frac{1}{n-1}\sum\psi_{j_1\cdots k\cdots j_{n-1}}d^k{}_{j_\lambda} & e_{j_1\cdots j_n} \\ 0\cdots 0 & 0 & 0\end{pmatrix}\right\}.$$

PROOF. A simple calculation.

We now sketch the construction of G^n. If $e_{j_1\cdots j_{n-1}}$ were arbitrary, the pseudogroup would be

$$\begin{cases} y_1=x_1g(x_2,\cdots,x_n)+h(x_2,\cdots,x_n) \\ y_i=\sum a_{ij}x_j+b_i\,. \end{cases}$$

I claim that if $\psi=0$, the pseudogroup is as above with the restriction $g(x_2,\cdots,x_n)=e^{P_{n-1}(x_2,\cdots,x_n)}$, P_{n-1} a polynomial of degree $\leqq n-1$. Let us use this to construct G^n. Since $g\neq 0$, we can write $g(x_2,\cdots,x_n)=e^{l(x_2,\cdots,x_n)}$. Then

$$\frac{\partial^n y_1}{\partial x_{j_1}\cdots\partial x_{j_{n-1}}\partial x_1}=\frac{\partial^{n-1}}{\partial x_{j_1}\cdots\partial x_{j_{n-1}}}e^{l(x_2,\cdots,x_n)}$$

$$= e^{l(x_2,\cdots,x_n)} \frac{\partial^{n-1} l}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}}$$

$$+ \text{terms of lower order in } l.$$

These terms of lower order in l can in turn be expressed in terms of $\frac{\partial^{n-k} y_1}{\partial x_1 \partial x_{j_1} \cdots \partial x_{j_{n-k-1}}}$. Hence $e_{1j_1\cdots j_{n-1}} = e^l \frac{\partial^{n-1} l}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}} +$ expression in $e_{j_1\cdots j_k}$ $(k < n-1)$. Call this last expression σ. Then let

$$G^n = \left\{ \begin{pmatrix} a & b_i \\ 0 & d^i{}_j \end{pmatrix}, \begin{pmatrix} 0 & e_i & e_{ij} \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \cdots 0 & \sigma & e_{j_1\cdots j_n} \\ 0 \cdots 0 & 0 & 0 \end{pmatrix} \right\}.$$

This would produce the above pseudogroup, as $e_{1j_1\cdots j_{n-1}} = \sigma$ implies $\frac{\partial^{n-1} l}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}} = 0$.

Now presumably we must show that this is a group with the correct algebra; these calculations I skip.

Suppose we let ψ be arbitrary.

LEMMA 60.

$$G^n = \left\{ \begin{pmatrix} a & b_i \\ 0 & d^i{}_j \end{pmatrix}, \begin{pmatrix} 0 & e_i & e_{ij} \\ 0 & 0 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \cdots 0 & e_{j_1\cdots j_{n-2}} e_{j_1\cdots j_{n-1}} \\ 0 \cdots 0 & 0 \quad\quad 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 \cdots 0 & \sigma + \frac{1}{n-1} [a \sum \psi_{k_1\cdots k_{n-1}} d^{k_1}{}_{j_1} \cdots d^{k_{n-1}}{}_{j_{n-1}} - a \psi_{j_1\cdots j_{n-1}} & e_{j_1\cdots j_n} \\ 0 \cdots 0 & 0 & 0 \end{pmatrix} \right\}.$$

PROOF. This clearly has the correct algebra (assuming G^n for $\psi = 0$ does).

But if ξ is as above, η the same with $\sim$ over each element, a short calculation shows

$$(\xi \circ \eta)^1{}_{1j_1\cdots j_{n-1}} = (\text{composition of these two for } \psi = 0)$$
$$+ \frac{1}{n-1} \sum [a\tilde{a} \psi_{k_1\cdots k_{n-1}} \tilde{d}^{k_1}{}_{j_1} \cdots \tilde{d}^{k_{n-1}}{}_{j_{n-1}}$$
$$+ a\tilde{a} \psi_{k_1\cdots k_n} d^{k_1}{}_{l_1} \cdots d^{k_{n-1}}{}_{l_{n-1}} \tilde{d}^{l_1}{}_{j_1} \cdots \tilde{d}^{l_{n-1}}{}_{j_{n-1}}$$
$$- a\tilde{a} \psi_{j_1\cdots j_{n-1}} - \tilde{a} a \psi_{k_1\cdots k_{n-1}} \tilde{d}^{k_1}{}_{j_1} \cdots \tilde{d}^{k_{n-1}}{}_{j_{n-1}}]$$

and thus G^n is a group. The resulting pseudogroup is then

$$\begin{cases} y_1 = x_1 e^{l(x_2,\cdots,x_n)} + h(x_2, \cdots, x_n) \\ y_i = \sum a_{ij} x_j + b_i \end{cases}$$

where

$$e^l \frac{\partial^{n-1} l}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}} = \frac{1}{n-1} e^l \sum \psi_{k_1\cdots k_{n-1}} a^{k_1}{}_{j_1} \cdots a^{k_{n-1}}{}_{j_{n-1}}$$

or

$$\frac{\partial^{n-1} l}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}} = \frac{1}{n-1} \sum \psi_{k_1 \cdots k_{n-1}} a^{k_1}{}_{j_1} \cdots a^{k_{n-1}}{}_{j_{n-1}} .$$

Now pick $R(x_2, \cdots, x_n)$, $\dfrac{\partial^{n-1} R}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}} = \dfrac{1}{n-1} \psi_{j_1 \cdots j_{n-1}}$. Consider

$$\vartheta = \begin{cases} y_1 = x_1 e^{R(x_2, \cdots, x_n)} \\ y_i = x_i . \end{cases}$$

Then if

$$f = \begin{cases} y_1 = x_1 e^{l(x_2, \cdots, x_n)} + h(x_2, \cdots, x_n) \\ y_i = \sum a_{ij} x_j + b_i \end{cases}$$

$$\vartheta^{-1} \circ f \circ \vartheta = \begin{cases} y_1 = x_1 e^{[R(x_2, \cdots, x_n) + h(x_2, \cdots, x_n) - R(\sum a_{ij} x_j)]} + h(x_2, \cdots, x_n) e^{-R} \\ y_i = \sum a_{ij} x_j + b_i . \end{cases}$$

I claim $\vartheta^{-1} \circ f \circ \vartheta$ is in the pseudogroup for $\psi = 0$ if and only if f is in the original pseudogroup. Indeed

$$\frac{\partial^{n-1}(R + h - R)}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}}$$

$$= \frac{1}{n-1} \psi_{j_1 \cdots j_{n-1}} + \frac{\partial^{n-1} h}{\partial x_{j_1} \cdots \partial x_{j_{n-1}}} - \frac{1}{n-1} \sum \psi_{k_1 \cdots k_{n-1}} a^{k_1}{}_{j_1} \cdots a^{k_{n-1}}{}_{j_{n-1}}$$

and the result is clear. We get ($6n$) on the list of pseudogroups.

Consider finally (18b) and (19b). Then $\varphi = \begin{pmatrix} 0 \cdots 0 & \psi_{j_1 \cdots j_n} \\ 0 \cdots 0 & 0 \end{pmatrix}$ and

$$g^n = \left\{ \begin{pmatrix} a & b_i \\ 0 & d^i{}_j \end{pmatrix}, \begin{pmatrix} 0 & \frac{n-1}{2} d^i{}_{ii} & e_{ij} \\ 0 & 0 & \Omega(T^2) \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & e_{ijk} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \cdots, \begin{pmatrix} 0 \cdots 0 & \frac{1}{n-1} (\sum \psi_{j_1 \cdots k \cdots j_n} d^k{}_{j_\lambda} - a \psi_{j_1 \cdots j_n}) \\ 0 \cdots 0 & 0 \end{pmatrix} \right\} .$$

The pseudogroup, if this last term were arbitrary, is

$$\begin{cases} y_1 = \dfrac{A x_1 + h(x_2, \cdots, x_n)}{(\sum c_j x_j + d)^{n-1}} \\ y_i = \dfrac{\sum a_{ij} x_j + b_i}{\sum c_j x_j + d} \end{cases} .$$

I claim that if $\psi = 0$, the pseudogroup is defined by $h = P_{n-1}$, a polynomial of degree $\leqq n-1$.

We use the same reasoning here as in the above example; namely

$$\frac{\partial^n y_1}{\partial x_{j_1}\cdots\partial x_{j_n}}=\frac{\dfrac{\partial^n h}{\partial x_{j_1}\cdots\partial x_{j_n}}}{(\sum c_j x_j+d)^{n-1}}+(\text{terms of lower order in } h)$$

so calling these terms σ we expect the group G^n to be the same as the group for the full pseudogroup except that $e_{j_1\cdots j_n}=\sigma$. (Again, we omit the check that this is correct.)

A calculation similar to those above shows that if $\phi\neq 0$, we must add

$$\frac{1}{n-1}[\sum \phi_{k_1\cdots k_n} d^{k_1}{}_{j_1}\cdots d^{k_n}{}_{j_n}-a\phi_{j_1\cdots j_n}]$$

to the $e_{j_1\cdots j_n}$ term.

Let

$$\vartheta=\begin{cases} y_1=x_1+k(x_2,\cdots,x_n) \\ y_i=x_i\,. \end{cases}$$

Then if

$$f=\begin{cases} y_1=\dfrac{Ax_1+h(x_2,\cdots,x_n)}{(\sum c_j x_j+d)^{n-1}} \\ y_i=\dfrac{\sum a_{ij}x_j+b_i}{\sum c_j x_j+d} \end{cases}$$

$$\vartheta^{-1}\circ f\circ\vartheta=\begin{cases} y_1=Ax_1+Ak(x_2,\cdots,x_n) \\ \qquad -\dfrac{(\sum c_j x_j+d)^{n-1}k\Big(\dfrac{\sum a_{ij}x_j+b_i}{\sum c_k x_k+d}\Big)+h(x_2,\cdots,x_n)}{(\sum c_j x_j+d)^{n-1}} \\ y_i=\dfrac{\sum a_{ij}x_j+b_i}{\sum c_j x_j+d}\,. \end{cases}$$

To solve the equivalence problem, we simply need choose k so that when $\dfrac{\partial^n h}{\partial x_{j_1}\cdots\partial x_{j_n}}=0$,

$$(\sum c_j x_j+d)^{-(n-1)}\frac{\partial^n}{\partial x_{j_1}\cdots\partial x_{j_n}}\Big[Ak(x_2,\cdots,x_n)+h(x_2,\cdots,x_n) -(\sum c_j x_j+d)^{n-1}k\Big(\frac{\sum a_{ij}x_j+b_i}{\sum c_j x_j+d}\Big)\Big]$$
$$=\frac{1}{n-1}\Big[\sum \phi_{k_1\cdots k_n}\frac{\partial}{\partial x_{j_1}}\Big(\frac{\sum a_{k_1 j}x_j+b_{k_1}}{\sum c_j x_j+d}\Big)\cdots\frac{\partial}{\partial x_{j_n}}\Big(\frac{\sum a_{k_n j}x_j+b_{k_n}}{\sum c_j x_j+d}\Big) -\frac{A}{(\sum c_j x_j+d)^{n-1}}\phi_{j_1\cdots j_n}\Big].$$

If $\dfrac{\partial^n k}{\partial x_{j_1}\cdots\partial x_{j_n}}=-\dfrac{1}{n-1}\phi_{j_1\cdots j_n}$, we are done provided:

LEMMA 61. *Given an arbitrary C^∞ function $k(x_1,\cdots,x_n)$,*

$$\frac{\dfrac{\partial^n}{\partial x_{j_1}\cdots\partial x_{j_n}}}{(\sum c_j x_j+d)^{n-1}}\left[(\sum c_j x_j+d)^{n-1}k\left(\frac{\sum a_{ij}x_j+b_i}{\sum c_j x_j+d}\right)\right]$$
$$=\frac{\partial^n k}{\partial x_{k_1}\cdots\partial x_{k_n}}\Bigg|_{\left(\frac{\sum a_{ij}x_j+b_i}{\sum c_j x_j+d}\right)}\frac{\partial}{\partial x_{j_1}}\left(\frac{\sum a_{k_1 j}x_j+b_{k_1}}{\sum c_j x_j+d}\right)\cdots\frac{\partial}{\partial x_{j_n}}\left(\frac{\sum a_{k_n j}x_j+b_{k_n}}{\sum c_j x_j+d}\right).$$

PROOF. The lemma is obvious if k is a polynomial P_{n-1} of degree $\leqq n-1$. It is also obvious (at a fixed point p) if $\frac{\partial k}{\partial x_j}(p), \cdots, \frac{\partial^{n-1}k}{\partial x_{j_1}\cdots\partial x_{j_{n-1}}}(p)$ are all zero. But any ψ may be written $\psi=P_{n-1}+\psi_0$ where ψ_0 has this property at p; consequently it is true of any ψ at p, and as p is arbitrary, true in general.

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