J. Math. Soc. Japan
Vol. 23, No. 1, 1971

Fuchsian groups contained in $SL_2(\boldsymbol{Q})$

By Kisao TAKEUCHI

(Received April 10, 1970)

§1. Let G be the special linear group $SL_2(\boldsymbol{R})$ and Γ a discrete subgroup of G such that the quotient space G/Γ is compact. The group G operates on the upper half plane $\mathfrak{H}=\{z\in\boldsymbol{C}\mid \operatorname{Im} z>0\}$ in the following way:

$$\rho(T):\ z\longmapsto \frac{az+b}{cz+d}\quad \text{where } z\in\mathfrak{H}\ \text{ and } T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G.$$

ρ is a homomorphism of G onto the group of all analytic automorphisms of $\mathfrak{H}$. The kernel of ρ is $\{\pm E\}$. The image $\rho(\Gamma)$ of Γ under the homomorphism ρ is a properly discontinuous group called the Fuchsian group of the first kind. $\rho(\Gamma)$ is generated by $2g$ hyperbolic elements $\rho(A_1), \rho(B_1), \cdots, \rho(A_g), \rho(B_g)$ and n elliptic elements $\rho(C_1), \cdots, \rho(C_n)$. There are following $n+1$ fundamental relations among these generators.

$$\rho(A_1)\cdot\rho(B_1)\cdot\rho(A_1)^{-1}\cdot\rho(B_1)^{-1}$$
$$\cdots\rho(A_g)\cdot\rho(B_g)\cdot\rho(A_g)^{-1}\cdot\rho(B_g)^{-1}\cdot\rho(C_1)\cdots\rho(C_n)=\rho(E) \tag{1}$$

$$\rho(C_j)^{e_j}=\rho(E)\qquad (1\leqq j\leqq n). \tag{2}$$

$(g;\ e_1, \cdots, e_n)$ is called the signature of Γ. These numbers satisfy the inequality

$$2g-2+\sum_{j=1}^{n}\left(1-\frac{1}{e_j}\right)>0. \tag{3}$$

Now, we shall consider the following problem: *Is there any group Γ of the above type, that is contained in $SL_2(\boldsymbol{Q})$ and that has a given signature $(g;\ e_1, \cdots, e_n)$?* The first thing to be remarked here is that if Γ is arithmetic, i.e., if it is derived from some quaternion algebras Φ in the customary way, then Γ cannot be realized in $SL(\boldsymbol{Q})$. In fact, since we assume that G/Γ is compact, Φ must be a division algebra. But then, Γ can be realized only in $SL_2(k)$, k being some splitting field of Φ, and not in $SL_2(\boldsymbol{Q})$. The second remark is that if Γ is contained in $SL_2(\boldsymbol{Q})$, then e_j $(1\leqq j\leqq n)$ must be either 2 or 3; cf. Lemma 1, §3. So, this is a necessary condition for the existence of the solution Γ.

Now, the purpose of the present paper is to prove that *if this condition on the signature is satisfied, then there exist infinitely many non-conjugate (in G)*

solutions Γ. Moreover, for a given Γ the set $\mathfrak{R}_Q(\Gamma)$ of all solutions which are isomorphic to Γ forms a dense subset in the space $\mathfrak{R}(\Gamma)$ of all "deformations" of Γ; cf. Theorem in §2. The proof depends on the two known results, namely, the result on the explicit generators and relations of $\rho(\Gamma)$, and the result on the deformations of Γ; cf. Weil [2].

§**2.** We shall first determine the generators and the fundamental relations of Γ. By (1) and (2) we have

$$A_1B_1A_1^{-1}B_1^{-1}\cdots A_gB_gA_g^{-1}B_g^{-1}C_1\cdots C_nI^{d_0}=E\,, \tag{4}$$

$$I^{d_j}C_j^{e_j}=E\,, \qquad (1\leqq j\leqq n) \tag{5}$$

where $I=-E$ and $d_j=0$ or 1 $(0\leqq j\leqq n)$.

PROPOSITION 1. (i) *If Γ does not contain the element $I=-E$, then Γ is generated by the $2g+n$ elements $A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_n$ and the fundamental relations among these generators are* (4) *and* (5). *In this case $d_j=0$ $(0\leqq j\leqq n)$.*

(ii) *If Γ contains the element $I=-E$, then Γ is generated by the $2g+n+1$ elements $A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_n, I$ and the fundamental relations are* (4), (5) *and the following relations* (6), (7).

$$I^2=E\,, \tag{6}$$

$$A_iIA_i^{-1}I^{-1}=E\,,\quad B_iIB_i^{-1}I^{-1}=E\,,\quad C_jIC_j^{-1}I^{-1}=E \qquad (1\leqq i\leqq g,\ 1\leqq j\leqq n)\,. \tag{7}$$

PROOF. In the case (i), $\rho(\Gamma)$ is isomorphic to $\Gamma/\Gamma\cap\{E, I\}$ which is equal to Γ. Moreover we have $d_j=0$ $(0\leqq j\leqq n)$. This settles Proposition 1 in the case (i).

In the case (ii) we have the isomorphism

$$\Gamma/\{E, I\}\cong\rho(\Gamma)\,. \tag{8}$$

It is easy to show that Γ is generated by $2g+n+1$ elements $A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_n, I$. Let $\tilde{\Gamma}$ be the free group generated by $2g+n+1$ letters $\tilde{A}_1, \tilde{B}_1, \cdots, \tilde{A}_g, \tilde{B}_g, \tilde{C}_1, \cdots, \tilde{C}_n, \tilde{I}$. We have a homomorphism η of $\tilde{\Gamma}$ onto Γ such that $\eta(\tilde{A}_i)=A_i$, $\eta(\tilde{B}_i)=B_i$, $\eta(\tilde{C}_j)=C_j$, $\eta(\tilde{I})=I$. Let N and $\bar{N}$ be the kernels of η and $\rho\circ\eta$ respectively. By (8), the group N is of index 2 in $\bar{N}$. By the relations (1) and (2), we see that $\bar{N}$ is a normal subgroup of $\tilde{\Gamma}$ generated by the finite set of words

$$\{\tilde{A}_1\tilde{B}_1\tilde{A}_1^{-1}\tilde{B}_1^{-1}\cdots\tilde{A}_g\tilde{B}_g\tilde{A}_g^{-1}\tilde{B}_g^{-1}\tilde{C}_1\cdots\tilde{C}_n,\ \tilde{I},\ \tilde{C}_j^{e_j}\ (1\leqq j\leqq n)\}\,.$$

Put

$$\bar{\mathfrak{M}}=\{\tilde{A}_1\tilde{B}_1\tilde{A}_1^{-1}\tilde{B}_1^{-1}\cdots\tilde{A}_g\tilde{B}_g\tilde{A}_g^{-1}\tilde{B}_g^{-1}\tilde{C}_1\cdots\tilde{C}_n\tilde{I}^{d_0},\ \tilde{I}^{d_j}\tilde{C}_j^{e_j},\ \tilde{I},\ (1\leqq j\leqq n)\}\,,$$

$$\begin{aligned}\mathfrak{M}=\{&\tilde{A}_1\tilde{B}_1\tilde{A}_1^{-1}\tilde{B}_1^{-1}\cdots\tilde{A}_g\tilde{B}_g\tilde{A}_g^{-1}\tilde{B}_g^{-1}\tilde{C}_1\cdots\tilde{C}_n\tilde{I}^{d_0},\ \tilde{I}^{d_j}\tilde{C}_j^{e_j},\ \tilde{I}^2,\\ &\tilde{A}_i\tilde{I}\tilde{A}_i^{-1}\tilde{I}^{-1},\ \tilde{B}_i\tilde{I}\tilde{B}_i^{-1}\tilde{I}^{-1},\ \tilde{C}_j\tilde{I}\tilde{C}_j^{-1}\tilde{I}^{-1}\ (1\leqq i\leqq g,\ 1\leqq j\leqq n)\}\,.\end{aligned}$$

Then N contains the normal subgroup of $\tilde{\Gamma}$ generated by the elements of $\mathfrak{M}$. As $\bar{N}$ is the normal subgroup of $\tilde{\Gamma}$ generated by $\bar{\mathfrak{M}}$, we see easily that the normal subgroup of $\tilde{\Gamma}$ generated by $\mathfrak{M}$ is of index 2 in $\bar{N}$. Therefore, N is generated by $\mathfrak{M}$. Q. E. D.

Let φ be a representation of Γ into G, i. e., a homomorphism (as abstract groups) of Γ into G. Then φ is determined by the images of generators of Γ. In the case where Γ does not contain I, φ is determined by $(\varphi(A_1), \varphi(B_1), \cdots, \varphi(A_g), \varphi(B_g), \varphi(C_1), \cdots, \varphi(C_n))$. Consider the case where Γ contains I. Then φ is determined by $(\varphi(A_1), \varphi(B_1), \cdots, \varphi(A_g), \varphi(B_g), \varphi(C_1), \cdots, \varphi(C_n), \varphi(I))$. As $\varphi(I)$ must be at most of order 2 in G by (6), $\varphi(I)$ is equal to E or I. Hence the relations in (7) are satisfied automatically. In either case, we see that the set of all representations are in one-to-one correspondence with that of all elements $(A_1^*, B_1^*, \cdots, C_1^*, \cdots, C_n^*)$ of $G^{(2g+n)}$ (resp. $(A_1^*, B_1^*, \cdots, C_1^*, \cdots, C_n^*, I^*)$ of $G^{(2g+n+1)}$ if Γ contains I) satisfying

$$A_1^*B_1^*A_1^{*-1}B_1^{*-1} \cdots A_g^*B_g^*A_g^{*-1}B_g^{*-1}C_1^* \cdots C_n^* I^{*d_0} = E \tag{4*}$$

$$I^{*d_j}C_j^{*e_j} = E \qquad (1 \leqq j \leqq n) \tag{5*}$$

where $I^* = E$ or I.

Let $\mathfrak{R}'(\Gamma)$ be the set of all representations of Γ into G. We shall identify $\mathfrak{R}'(\Gamma)$ with a closed subset of $G^{(2g+n)}$ (resp. $G^{(2g+n+1)}$) by the above correspondence. Thus, $\mathfrak{R}'(\Gamma)$ is provided with the relative topology induced by that of $G^{(2g+n)}$ (resp. $G^{(2g+n+1)}$). Let $\mathfrak{R}(\Gamma)$ be the subset of $\mathfrak{R}'(\Gamma)$ consisting of all representations φ which are injective and such that $\varphi(\Gamma)$ is discrete in G with compact quotient space $G/\varphi(\Gamma)$, and let $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma)$ be the subset of $\mathfrak{R}(\Gamma)$ consisting of all φ such that $\varphi(\Gamma)$ is contained in $SL_2(\boldsymbol{Q})$.

We shall prove the following theorem.

THEOREM. *Let Γ be a discrete subgroup of G with compact quotient space G/Γ, and let $(g; e_1, \cdots, e_n)$ be its signature.*

(i) *If $e_j > 3$ for some index j, then $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma)$ is empty.*

(ii) *Otherwise, $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma)$ is everywhere dense in $\mathfrak{R}(\Gamma)$.*

More accurately, for any element φ of $\mathfrak{R}(\Gamma)$, we can find a sequence $\{\varphi_m\}$ converging to φ such that φ_m belong to $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma)$ and that $\varphi_m(\Gamma)$ $(m = 1, 2, \cdots)$ are not G-conjugate to one another.

§ 3. We shall prove this Theorem in § 3—§ 7. We make use of the following theorem which was proved by A. Weil in [2], in the more general situation.

THEOREM (A. Weil). *$\mathfrak{R}(\Gamma)$ is an open subset of $\mathfrak{R}'(\Gamma)$.*

LEMMA 1. *If an element A of $SL_2(\boldsymbol{Q})$ other than $\pm E$ is of finite order, then its order as a transformation of $\mathfrak{H}$ is equal to 2 or 3, according as its trace*

$\operatorname{tr}(A)$ *is equal to* 0 *or* ± 1 *respectively.*

PROOF. By the assumption, the eigenvalues of A are roots of unity. Hence $\operatorname{tr}(A)$ is a rational integer whose absolute value is smaller than 2. Hence $\operatorname{tr}(A)$ is equal to 0 or ± 1. Now cosider the characteristic polynomial of A. We have $A^2 - \operatorname{tr}(A)A + E = 0$. From this we have $A^3 + \{1 - (\operatorname{tr}(A))^2\}A + \operatorname{tr}(A)E = 0$. The first equality implies that $A^2 = -E$ if $\operatorname{tr}(A) = 0$, and the second implies that $A^3 = \mp E$ if $\operatorname{tr}(A) = \pm 1$. This proves Lemma 1. Q. E. D.

By the above lemma the case (i) in our theorem is proved.

§ 4. From now on, we may assume that e_j is equal to 2 or 3 for all j $(1 \leqq j \leqq n)$ if $n \geqq 1$.

LEMMA 2. (i) $SL_2(\boldsymbol{Q})$ *is dense in* $SL_2(\boldsymbol{R})$.

(ii) *Let t be an arbitrary rational number. The set* $\{A \in SL_2(\boldsymbol{Q}) \mid \operatorname{tr}(A) = t\}$ *is everywhere dense in the set* $\{A \in SL_2(\boldsymbol{R}) \mid \operatorname{tr}(A) = t\}$.

(iii) *The set* $\{A \in SL_2(\boldsymbol{Q}) \mid (\operatorname{tr}(A))^2 - 4$ *is a square in* $\boldsymbol{Q}\}$ *is everywhere dense in the set of all hyperbolic elements of G.*

Since the proof of this lemma is easy, we omit it here.

Now we distinguish the two cases of $g \geqq 1$ and $g = 0$. Let us consider first the case $g \geqq 1$.

PROPOSITION 2. *Suppose that an element φ' of $\mathfrak{R}(\Gamma)$ differs from another φ only by an inner automorphism of G. Then the assertion* (ii) *of our theorem is true for φ' if and only if it is true for φ.*

PROOF. By the assumption, there exists an element A of G such that $\varphi' = \operatorname{Int}(A) \circ \varphi$, where $\operatorname{Int}(A)$ denotes the inner automorphism of G defined by A. Let $\{\varphi_m\}$ be a sequence converging to φ such that φ_m $(m = 1, 2, \cdots)$ belong to $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma)$ and that $\varphi_m(\Gamma)$ $(m = 1, 2, \cdots)$ are not G-conjugate to one another. Then the sequence $\{\operatorname{Int}(A) \circ \varphi_m\}$ converges to φ'. Take a sequence $\{A_m\}$ in $SL_2(\boldsymbol{Q})$ converging to A. Then $\{\operatorname{Int}(A_m) \circ \varphi_m\}$ converges to φ'. The converse part is obtained merely by changing φ and φ'. Q. E. D.

To prove the theorem, we may assume that φ is the identity map (since we may replace $\varphi(\Gamma)$ by Γ). And we may assume by Proposition 2 that $B_1 = (b_{ij})$ is the diagonal matrix i. e. $b_{11} = 1/b_{22} = b$ $(b^2 \neq 1)$, $b_{12} = b_{21} = 0$. Now put

$$D = I^{d_0} C_n^{-1} C_{n-1}^{-1} \cdots C_1^{-1} B_g A_g B_g^{-1} A_g^{-1} \cdots B_2 A_2 B_2^{-1} A_2^{-1} . \tag{9}$$

Then we have

$$A_1 B_1 A_1^{-1} B_1^{-1} = D . \tag{10}$$

PROPOSITION 3. *Let* $A_1 = (a_{ij})$, $B_1 = (b_{ij})$ *and* $D = (d_{ij}) = A_1 B_1 A_1^{-1} B_1^{-1}$ *be as above. Then none of* $a_{12}, a_{21}, d_{12}, d_{21}$ *are equal to* 0.

PROOF. First suppose that $a_{12} = 0$. Then A_1 fixes the origin of the real axis. By our assumption, B_1 fixes the origin and the point at infinity. If $a_{21}\neq 0$, then $A_1B_1A_1^{-1}B_1^{-1}$ is a parabolic element of G, fixing the origin. This is impossible because Γ contains no parabolic elements. Therefore a_{21} must be equal to 0. Hence A_1 commutes with B_1. It follows from this that for any element φ of $\mathfrak{R}(\Gamma)$, $\varphi(A_1)$ commutes with $\varphi(B_1)$. This is a contradiction because we can easily construct a Fuchsian group $\varphi(\Gamma)$ with signature $(g;\ e_1, \cdots, e_n)$ such that $\varphi(A_1)$ does not commute with $\varphi(B_1)$. (cf. [1] pp. 234—239). In the case $a_{21}=0$ we are led to a contradiction in the same way as above.

Now suppose that $d_{12}=0$. Then D must be a diagonal matrix by applying the above argument for the matrix $DB_1D^{-1}B_1^{-1}$. But by the relation $D=A_1B_1A_1^{-1}B_1^{-1}$ we have $a_{11}a_{12}=0$ and $a_{21}a_{22}=0$. Since $a_{12}a_{21}\neq 0$, we obtain $a_{11}=a_{22}=0$. Hence $\operatorname{tr}(A_1)=0$. This shows that A_1 is an elliptic element of G, which is impossible. Q.E.D.

Let $X=(x_{ij})$, $Y=(y_{ij})$ and $Z=(z_{ij})$ be variable matrices defined on the neighbourhoods of A_1, B_1 and D respectively. Consider the relation

$$XYX^{-1}Y^{-1}=Z, \tag{11}$$

where Y is a lower triangular matrix: $y_{12}=0$.

Now we shall show that all coefficients of X and Y can be expressed as rational functions of x_{12}, y_{11} and z_{ij} $(1\leqq i, j\leqq 4)$. If we fix Y and Z, (11) is equivalent to the relations

$$XY-ZYX=0, \tag{12}$$

$$x_{12}x_{22}-x_{12}x_{21}=1. \tag{13}$$

Furthermore, (12) can be expressed as a linear equation:

$$(E\otimes {}^tY-ZY\otimes E)\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{14}$$

PROPOSITION 4. *If Y and Z belong to sufficiently small neighbourhoods of B_1 and D respectively in G, satisfying*

$$\operatorname{tr}(Y)=\operatorname{tr}(ZY), \tag{15}$$

then the matrix $E\otimes {}^tY-ZY\otimes E$ is of rank 2.

PROOF. We have

$$y_{11}y_{22}=1, \tag{16}$$

$$z_{11}z_{22}-z_{12}z_{21}=1. \tag{17}$$

The condition (15) is equivalent to

$$(z_{11}-1)y_{11}+z_{12}y_{21}+(z_{22}-1)y_{22}=0\,. \tag{18}$$

Since none of $b_{11}, b_{22}, b_{11}-b_{22}, d_{12}$ and d_{21} are equal to 0, we may assume that none of $y_{11}, y_{22}, y_{11}-y_{22}, z_{12}$ and z_{21} are equal to 0. By (18), the matrix $E\otimes{}^tY-ZY\otimes E$ is explicitly given by

$$\begin{pmatrix} (z_{22}-1)y_{22}\,, & y_{21}\,, & -z_{12}y_{22}\,, & 0 \\ 0\,, & -y_{11}+z_{22}y_{22}\,, & 0\,, & -z_{12}y_{22} \\ -z_{21}y_{11}-z_{22}y_{21}, & 0\,, & y_{11}-z_{22}y_{22}, & y_{21} \\ 0\,, & -z_{21}y_{11}-z_{22}y_{21}, & 0\,, & (1-z_{22})y_{22} \end{pmatrix}. \tag{19}$$

Let $\boldsymbol{a}_i$ $(1\leqq i\leqq 4)$ be the row vectors of the matrix of (19). As $z_{12}y_{22}\neq 0$, two vectors $\boldsymbol{a}_1, \boldsymbol{a}_2$ are linearly independent. By using (17) and (18), we obtain the following expressions,

$$\boldsymbol{a}_3=\frac{y_{11}-z_{22}y_{22}}{-z_{12}y_{22}}\boldsymbol{a}_1+\frac{y_{21}}{-z_{12}y_{22}}\boldsymbol{a}_2\,,$$

$$\boldsymbol{a}_4=\frac{1-z_{22}}{-z_{12}}\boldsymbol{a}_2\,.$$

This proves Proposition 4. Q. E. D.

By (14) and (19) we obtain

$$x_{21}=\frac{z_{22}-1}{z_{12}}x_{11}+\frac{y_{11}y_{21}}{z_{12}}x_{12} \tag{20}$$

$$x_{22}=\frac{-y_{11}^2+z_{22}}{z_{12}}x_{12}\,. \tag{21}$$

Using the relations (13), (20) and (21), we obtain

$$x_{11}=\frac{y_{11}y_{21}}{1-y_{11}^2}x_{12}+\frac{z_{12}}{1-y_{11}^2}\frac{1}{x_{12}} \tag{22}$$

where x_{12} varies on some neighbourhood of a_{12} which is different from 0 by Proposition 3.

On the other hand, by (16) and (18) we obtain the following expressions;

$$Y=\begin{pmatrix} y_{11}\,, & 0 \\ \dfrac{1-z_{11}}{z_{12}}y_{11}+\dfrac{1-z_{22}}{z_{12}}\dfrac{1}{y_{11}}, & \dfrac{1}{y_{11}} \end{pmatrix}=g(y_{11}, Z)\,, \tag{23}$$

and

$$X=\begin{pmatrix} \dfrac{y_{11}y_{21}}{1-y_{11}^2}x_{12}+\dfrac{z_{12}}{1-y_{11}^2}\dfrac{1}{x_{12}}\,, & x_{12} \\ \dfrac{(z_{22}-y_{11}^2)y_{11}y_{21}}{z_{12}(1-y_{11}^2)}x_{12}+\dfrac{z_{22}-1}{1-y_{11}^2}\dfrac{1}{x_{12}}, & \dfrac{z_{22}-y_{11}^2}{z_{12}}x_{12} \end{pmatrix}=f(x_{12}, y_{11}, Z)\,. \tag{24}$$

Now we turn to the proof of our theorem. Let $\mathfrak{A}_i, \mathfrak{B}_i$ $(1\leqq i\leqq g)$ and

$\mathfrak{C}_j$ $(1 \leqq j \leqq n)$ be arbitrarily given open neighbourhoods of A_i, B_i, C_j respectively. We must prove that the intersection $\mathfrak{R}_Q(\Gamma) \cap \mathfrak{A}_1 \times \mathfrak{B}_1 \times \cdots \times \mathfrak{A}_g \times \mathfrak{B}_g \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n \times \{I\}$ is non-empty in the case where Γ contains I (resp. the intersection $\mathfrak{R}_Q(\Gamma) \cap \mathfrak{A}_1 \times \mathfrak{B}_1 \times \cdots \times \mathfrak{A}_g \times \mathfrak{B}_g \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n$ is non-empty in the case where Γ does not contain I). By the above quoted Theorem (A. Weil), we may assume that

$$\mathfrak{R}'(\Gamma) \cap \mathfrak{A}_1 \times \mathfrak{B}_1 \times \cdots \times \mathfrak{A}_g \times \mathfrak{B}_g \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n \times \{I\}$$
$$(\text{resp. } \mathfrak{R}'(\Gamma) \cap \mathfrak{A}_1 \times \mathfrak{B}_1 \times \cdots \times \mathfrak{A}_g \times \mathfrak{B}_g \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n)$$

is contained in $\mathfrak{R}(\Gamma)$. By (23) and (24) there exist neighbourhoods $\mathfrak{a}_{12}$, $\mathfrak{b}$ and $\mathfrak{D}$ of a_{12}, b_{11} and D respectively such that $f(\mathfrak{a}_{12}, \mathfrak{b}, \mathfrak{D}) \subset \mathfrak{A}_1$ and $g(\mathfrak{b}, \mathfrak{D}) \subset \mathfrak{B}_1$. On the other hand, if we consider the following map h defined on a neighbourhood of $(A_2, B_2, \cdots, A_g, B_g, C_1, \cdots, C_n)$:

$$Z = I^{d_0} W_n^{-1} \cdots W_1^{-1} Y_g X_g Y_g^{-1} x_g^{-1} \cdots Y_2 X_2 Y_2^{-1} X_2^{-1} = h(X_2, Y_2, \cdots, W_1, \cdots, W_n), \tag{25}$$

we can find a neighbourhood $\mathfrak{A}'_2 \times \mathfrak{B}'_2 \times \cdots \times \mathfrak{A}'_g \times \mathfrak{B}'_g \times \mathfrak{C}'_1 \times \cdots \times \mathfrak{C}'_n$ of $(A_2, B_2, \cdots, A_g, B_g, C_1, \cdots, C_n)$ contained in $\mathfrak{A}_2 \times \mathfrak{B}_2 \times \cdots \times \mathfrak{A}_g \times \mathfrak{B}_g \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n$ such that $h(\mathfrak{A}'_2, \mathfrak{B}'_2, \cdots, \mathfrak{A}'_g, \mathfrak{B}'_g, \mathfrak{C}'_1, \cdots, \mathfrak{C}'_n) \subset \mathfrak{D}$. Take arbitrary elements $A_i^{(0)}$, $B_i^{(0)}$ and $C_j^{(0)}$ from the intersection $\mathfrak{A}'_i \cap SL(\boldsymbol{Q})$, $\mathfrak{B}'_i \cap SL(\boldsymbol{Q})$ and $\mathfrak{C}'_j \cap SL(\boldsymbol{Q})$ such that $\mathrm{tr}(C_j^{(0)}) = \mathrm{tr}(C_j)$ respectively $(2 \leqq i \leqq g,\ 1 \leqq j \leqq n)$. This is possible by Lemma 2. Furthermore, take rational numbers $a_{12}^{(0)}$ and $b_{11}^{(0)}$ from $\mathfrak{a}_{12}$ and $\mathfrak{b}$ respectively, and put

$$D^{(0)} = h(A_2^{(0)}, B_2^{(0)}, \cdots, A_g^{(0)}, B_g^{(0)}, C_1^{(0)}, \cdots, C_n^{(0)}),$$
$$A_1^{(0)} = f(a_{12}^{(0)}, b_{11}^{(0)}, D^{(0)}),$$
$$B_1^{(0)} = g(b_{11}^{(0)}, D^{(0)}).$$

Then the representation φ_0 of Γ defined by $(A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_n, I) \mapsto (A_1^{(0)}, B_1^{(0)}, \cdots, A_g^{(0)}, B_g^{(0)}, C_1^{(0)}, \cdots, C_n^{(0)}, I)$ in the case where Γ contains I (resp. $(A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_n) \mapsto (A_1^{(0)}, B_1^{(0)}, \cdots, A_g^{(0)}, B_g^{(0)}, C_1^{(0)}, \cdots, C_n^{(0)})$ in the case where Γ does not contain I) is contained in $\mathfrak{R}_Q(\Gamma) \cap \mathfrak{A}_1 \times \mathfrak{B}_1 \times \cdots \times \mathfrak{A}_g \times \mathfrak{B}_g \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n \times \{I\}$ (resp. $\mathfrak{R}_Q(\Gamma) \cap \mathfrak{A}_1 \times \mathfrak{B}_1 \times \cdots \times \mathfrak{A}_g \times \mathfrak{B}_g \times \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n$). This proves that $\mathfrak{R}_Q(\Gamma)$ is everywhere dense in $\mathfrak{R}(\Gamma)$, in the case of $g \geqq 1$.

§5. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two elements of G. Consider variable matrices $X = (x_{ij})$ and $Y = (y_{ij})$ of G defined on some neighbourhoods of A and B respectively. We impose the condition,

$$\mathrm{tr}(Y) = \mathrm{tr}(B), \tag{26}$$

on Y. Put $v_0 = \mathrm{tr}(B)$, $w_0 = \mathrm{tr}(AB)$ and $w = \mathrm{tr}(XY)$. Then we have

$$y_{11}+y_{22}=v_0\,, \tag{27}$$

$$x_{11}y_{11}+x_{12}y_{21}+x_{21}y_{12}+x_{22}y_{22}=w\,, \tag{28}$$

$$y_{11}y_{22}-y_{12}y_{21}=1\,. \tag{29}$$

By (27), (28) and (29), we have

$$y_{22}=v_0-y_{11}\,, \tag{30}$$

and

$$x_{12}y_{21}+x_{21}y_{12}=(x_{22}-x_{11})y_{11}+w-x_{22}v_0\,, \tag{31}$$

$$x_{12}y_{21}\cdot x_{21}y_{12}=(1-x_{11}x_{22})(y_{11}^2-v_0y_{11}+1)\,. \tag{32}$$

The discriminant of the quadratic equation whose roots are $x_{12}y_{21}$ and $x_{21}y_{12}$, is given by the following polynomial,

$$\begin{aligned} d(y_{11}, w, X)= &\{(x_{11}+x_{22})^2-4\}y_{11}^2 \\ &+2\{(x_{22}-x_{11})(w-x_{22}v_0)+2(1-x_{11}x_{22})v_0\}y_{11} \\ &+(w-x_{22}v_0)^2+4(x_{11}x_{22}-1)\,. \end{aligned} \tag{33}$$

Now assume that

$$a_{12}a_{21}\neq 0\,, \tag{34}$$

$$a_{12}b_{21}\neq a_{21}b_{12}\,. \tag{35}$$

Then we have $d(b_{11}, w_0, A)>0$. Hence we have $d(y_{11}, w, X)>0$ on some neighbourhood of (b_{11}, w_0, A).

Let (y_{11}, w, X) be sufficiently near (b_{11}, w_0, A) so that $x_{12}x_{21}\neq 0$ and that $d(y_{11}, w, X)>0$. Then we have the following expression,

$$y_{12}=\frac{(x_{22}-x_{11})y_{11}+w-x_{22}v_0\pm\sqrt{d(y_{11}, w, X)}}{2x_{21}}\,, \tag{36}$$

$$y_{21}=\frac{(x_{22}-x_{11})y_{11}+w-x_{22}v_0\mp\sqrt{d(y_{11}, w, X)}}{2x_{12}}\,, \tag{37}$$

where the sign $\pm$ in (36) and (37) is determined by the one at (b_{11}, w_0, A).

Therefore, under the assumptions (34) and (35), we obtain the following expression,

$$Y=f_{A,B}(y_{11}, w, X) \tag{38}$$

where $f_{A,B}$ is a matrix valued function given explicitly by (30), (36) and (37), which is defined on some neighbourhood of (b_{11}, w_0, A).

REMARK. Let $A=(a_{ij})$ and $B=(b_{ij})$ be two elements of G which are elliptic or hyperbolic. Let $\{\xi_A, \eta_A\}$ and $\{\xi_B, \eta_B\}$ be the sets of the fixed points of A and B respectively. Assume that

$$\{\xi_A, \eta_A\}\neq\{\xi_B, \eta_B\}\,. \tag{39}$$

Now we shall show that we can find an element Q of G such that the conjugate matrices $A'=QAQ^{-1}$, $B'=QBQ^{-1}$ satisfy the conditions (34) and (35). Conjugating A and B by a suitable element of G, we may assume that $a_{12}a_{21}\neq 0$, $b_{12}b_{21}\neq 0$.

Put $A'=TAT^{-1}=(a'_{ij})$, $B'=TBT^{-1}=(b'_{ij})$, where $T=(t_{ij})$; $t_{11}=t_{22}=1$, $t_{21}=0$, $t_{12}=\alpha$. Then, for the sets $\{\xi_{A'}, \eta_{A'}\}$, $\{\xi_{B'}, \eta_{B'}\}$ of the fixed points of A', B' respectively, we have

$$\xi_{A'}=\xi_A+\alpha\,, \qquad \eta_{A'}=\eta_A+\alpha\,,$$

$$\xi_{B'}=\xi_B+\alpha\,, \qquad \eta_{B'}=\eta_B+\alpha\,.$$

Since $a_{12}/a_{21}=-\xi_A\cdot\eta_A$, and $b_{12}/b_{21}=-\xi_B\cdot\eta_B$, in view of (39) we can find a real number α such that $a'_{12}a'_{21}\neq 0$, $b'_{12}b'_{21}\neq 0$, $a'_{12}/a'_{21}\neq b'_{12}/b'_{21}$.

§ **6.** Let us consider the case of $g=0$. By the inequality (3) we see that $n\geqq 3$. If $n=3$, then (3) is equivalent to the following inequality,

$$1/e_1+1/e_2+1/e_3<1\,.$$

Hence $e_j>3$ for some j $(1\leqq j\leqq 3)$. In view of Lemma 1, we see that there exist no triangular groups Γ contained in $SL_2(\boldsymbol{Q})$ with compact quotient space G/Γ. Hence we may assume that $n\geqq 4$.

Let us note again the relations (4) and (5):

$$C_1C_2\cdots C_nI^{d_0}=E\,, \tag{40}$$

$$C_j^{e_j}=I^{d_j} \qquad (1\leqq j\leqq n)\,, \tag{41}$$

where $I=-E$, $d_j=0$ or 1, $e_j=2$ or 3.

Put

$$D_j=C_1C_2\cdots C_j \qquad (2\leqq j\leqq n-2)\,. \tag{42}$$

Then we have

$$C_1C_2=D_2\,, \tag{43}$$

$$D_{j-1}C_j=D_j \qquad (3\leqq j\leqq n-2)\,, \tag{44}$$

$$D_{n-2}C_{n-1}C_n=I^{d_0}\,. \tag{45}$$

PROPOSITION 5. *The notations being as above, the matrices D_j $(2\leqq j\leqq n-2)$ are hyperbolic.*

PROOF. Since C_j $(1\leqq j\leqq n)$ have the different fixed points, we see that $D_2\neq\pm E$, $D_{n-2}\neq\pm E$. Now suppose that D_2 is elliptic. Then D_2 is Γ-conjugate to $\pm C_k^{\nu}$ for some index k where $\nu=1$ or -1. Now we shall show that $\{D_{n-2}, C_{n-1}\}$ satisfy the condition (39) in § 5. If D_{n-2} is hyperbolic, this is obvious. Suppose that D_{n-2} is elliptic and that the fixed point of D_{n-2} coincide with the fixed point of C_{n-1}. Then we have

$$D_{n-2}=\pm C_{n-1}^{\lambda}, \qquad \text{where } \lambda=1 \text{ or } -1.$$

Hence we have $C_n=\pm C_{n-1}^{-\lambda-1}$, which is impossible.

$\{C_1, C_2\}$ also satisfy the condition (39) in § 5. Therefore, by the remark in § 5, taking a conjugate of Γ, we may assume that $\{D_{n-2}, C_{n-1}\}$ and $\{C_1, C_2\}$ satisfy the conditions (34) and (35) for $\{A, B\}$ in § 5. By applying the argument in § 5, we can find three elements C_j' $(j=2, n-1, n)$ sufficiently near C_j $(j=2, n-1, n)$ respectively such that

$$\mathrm{tr}(C_j')=\mathrm{tr}(C_j) \qquad (j=2, n-2, n),$$

and that

$$\mathrm{tr}(C_1C_2')\neq\pm\mathrm{tr}(C_1C_2), \tag{46}$$

and that

$$C_1C_2'C_3\cdots C_{n-2}C_{n-1}'C_n'I^{d_0}=E.$$

By the Theorem (A. Weil), the representation φ determined by $(C_1, C_2, C_3, \cdots, C_{n-2}, C_{n-1}, C_n)\mapsto(C_1, C_2', C_3, \cdots, C_{n-2}, C_{n-1}', C_n')$ can be taken to be contained in $\mathfrak{R}(\Gamma)$. Put $D_2'=C_1C_2'$. Then D_2' is the image $\varphi(D_2)$ of D_2 under φ. Hence, D_2' is $\varphi(\Gamma)$-conjugate to $\pm\varphi(C_k)^{\nu}$. Therefore, we have $\mathrm{tr}(D_2')=\pm\mathrm{tr}(\varphi(C_k)^{\nu})$ $=\pm\mathrm{tr}(C_k^{\nu})=\pm\mathrm{tr}(D_2)$, which is impossible by (46). This proves that D_2 is hyperbolic. In the same way, we see that D_{n-2} is also hyperbolic.

Assume now that $D_2, D_3, \cdots, D_{j-1}$ are hyperbolic. We shall show that D_j is also hyperbolic. Since $D_{j-1}=D_jC_j^{-1}$ is hyperbolic, we see that $D_j\neq\pm E$. Suppose that D_j is elliptic. Since $\{D_{j-1}, C_j\}$ and $\{D_{n-2}, C_{n-1}\}$ satisfy the condition (39) in § 5, by taking a conjugate of Γ, we may assume that $\{D_{j-1}, C_j\}$ and $\{D_{n-2}, C_{n-1}\}$ satisfy the conditions (34) and (35). Therefore, by the argument in § 5, we can find three elements C_i' $(i=j, n-1, n)$ sufficiently near C_i such that $\mathrm{tr}(C_i')=\mathrm{tr}(C_i)$, $\mathrm{tr}(D_{j-1}C_j')\neq\pm\mathrm{tr}(D_{j-1}C_j)$ and that

$$C_1C_2\cdots C_{j-1}C_j'C_{j+1}\cdots C_{n-2}C_{n-1}'C_n'I^{d_0}=E.$$

We are led to the contradiction by the same argument as in the case of D_2. This proves that D_j is hyperbolic. Q. E. D.

PROPOSITION 6. *Let $A=(a_{ij})$ be a hyperbolic element of G and let $B=(b_{ij})$ be an elliptic element such that* $\mathrm{tr}(B)=0$ *or* ±1. *Assume that $\{A, B\}$ satisfy the conditions* (34) *and* (35) *in* § 5. *Then, for an arbitrary neighbourhood $\mathfrak{B}$ of B, there exist a neighbourhood $W\times\mathfrak{A}$ of* $(\mathrm{tr}(AB), A)$ *satisfying the following condition,*

(C): *For any point $(r, A^{(0)})$ of $(W\times\mathfrak{A})\cap(\boldsymbol{Q}\times SL_2(\boldsymbol{Q}))$ such that* $\mathrm{tr}(A^{(0)})^2-4$ *is a non-zero square in $\boldsymbol{Q}$, we can find an element $B^{(0)}$ in $\mathfrak{B}\cap SL_2(\boldsymbol{Q})$ such that* $\mathrm{tr}(B^{(0)})=\mathrm{tr}(B)$ *and that* $\mathrm{tr}(A^{(0)}B^{(0)})=r$.

PROOF. We may use the notations in § 5 and we can apply the argument there. Since $f_{A,B}(y_{11}, w, X)$ and $d(y_{11}, w, X)$ are continuous, we can take a neighbourhood $\mathfrak{b}_{11}\times W\times\mathfrak{A}$ of (b_{11}, w_0, A) such that $f_{A,B}(\mathfrak{b}_{11}, W, \mathfrak{A})\subset\mathfrak{B}$ and that

$$d(y_{11}, w, X)>0 \quad \text{on} \quad \mathfrak{b}_{11}\times W\times\mathfrak{A}.$$

We may assume that $\mathrm{tr}(X)^2-4>0$ for any $X\in\mathfrak{A}$. Now, take any rational number r in W and any matrix $A^{(0)}$ in $\mathfrak{A}\cap SL_2(\boldsymbol{Q})$ such that $\mathrm{tr}(A^{(0)})^2-4$ is a square in $\boldsymbol{Q}$. Then, in view of (33), $d(y_{11}, r, A^{(0)})$ is a polynomial of degree 2 with rational coefficients. Moreover, by the assumption on $A^{(0)}$, the coefficient of the highest term is a non-zero square in $\boldsymbol{Q}$. Since $d(y_{11}, r, A^{(0)})>0$ on $\mathfrak{b}_{11}$, we can find a rational number $b_{11}^{(0)}$ in $\mathfrak{b}_{11}$ such that $d(b_{11}^{(0)}, r, A^{(0)})$ is a square in $\boldsymbol{Q}$. Put $B^{(0)}=f_{A,B}(b_{11}^{(0)}, r, A^{(0)})$. Then by (36) and (37) we see that $B^{(0)}$ is contained in $\mathfrak{B}\cap SL_2(\boldsymbol{Q})$ and that $\mathrm{tr}(B^{(0)})=\mathrm{tr}(B)$, $\mathrm{tr}(A^{(0)}B^{(0)})=r$. Q. E. D.

Now we turn to the proof of our theorem. Suppose that an arbitrary neighbourhood $\mathfrak{C}_1\times\cdots\times\mathfrak{C}_n$ of $(C_1,\cdots,C_n)$ is given. We may assume that $\mathfrak{R}'(\Gamma)\cap(\mathfrak{C}_1\times\cdots\times\mathfrak{C}_n\times\{I\})\subset\mathfrak{R}(\Gamma)$ if Γ contains I (resp. $\mathfrak{R}'(\Gamma)\cap(\mathfrak{C}_1\times\mathfrak{C}_2\times\cdots\times\mathfrak{C}_n)\subset\mathfrak{R}(\Gamma)$ if Γ does not contain I).

Since we have shown that D_j $(2\leqq j\leqq n-2)$ are hyperbolic, we see that $\{D_2, C_2^{-1}\}$, $\{D_2, C_3\}$, $\cdots$, $\{D_{n-2}, C_{n-1}\}$ satisfy the condition (39) in § 5. Therefore, by the remark of § 5, taking a conjugate of Γ, we may assume that $\{D_2, C_2^{-1}\}$, $\{D_2, C_3\}$, $\cdots$, $\{D_{n-2}, C_{n-1}\}$ satisfy the conditions (34) and (35). Now we can apply Proposition 6 to these pairs of matrices.

Let $\mathfrak{D}_{n-2}$ and $\mathfrak{C}'_{n-1}$ be the neighbourhoods of D_{n-2} and C_{n-1} respectively such that $\mathfrak{D}_{n-2}\cdot\mathfrak{C}'_{n-1}\subset\mathfrak{C}_n^{-1}I^{d_0}$ and that $\mathfrak{C}'_{n-1}\subset\mathfrak{C}_{n-1}$. Applying Proposition 6 to $\{D_{n-2}, C_{n-1}; \mathfrak{C}'_{n-1}\}$, we can find a neighbourhood $W_n\times\mathfrak{D}'_{n-2}$ of $(\mathrm{tr}(C_n^{-1}I^{d_0}), D_{n-2})$ satisfying the condition (C) for $\mathfrak{C}'_{n-1}$ in Proposition 6. Moreover we may take $\mathfrak{D}'_{n-2}$ so that $\mathfrak{D}'_{n-2}\subset\mathfrak{D}_{n-2}$. Hence we have

$$\mathfrak{D}'_{n-2}\cdot\mathfrak{C}'_{n-1}\subset\mathfrak{C}_n^{-1}I^{d_0}. \tag{47}$$

Let $\mathfrak{D}_{n-3}$ and $\mathfrak{C}'_{n-2}$ be the neighbourhoods of D_{n-3} and C_{n-2} respectively such that $\mathfrak{D}_{n-3}\cdot\mathfrak{C}'_{n-2}\subset\mathfrak{D}'_{n-2}$ and that $\mathfrak{C}'_{n-2}\subset\mathfrak{C}_{n-2}$. Applying Proposition 6 to $\{D_{n-3}, C_{n-2}; \mathfrak{C}'_{n-2}\}$, we can find a neighbourhood $W_{n-2}\times\mathfrak{D}'_{n-3}$ of $(\mathrm{tr}(D_{n-2}), D_{n-3})$ satisfying the condition (C) for $\mathfrak{C}'_{n-2}$ in Proposition 6. We may take $\mathfrak{D}'_{n-3}$ so that

$$\mathfrak{D}'_{n-3}\cdot\mathfrak{C}'_{n-2}\subset\mathfrak{D}'_{n-2}. \tag{48}$$

Repeating the above argument, we can find the neighbourhoods $\mathfrak{C}'_{j+1}$, $W_{j+1}\times\mathfrak{D}'_j$ of C_{j+1}, $(\mathrm{tr}(D_{j+1}), D_j)$ respectively such that $W_{j+1}\times\mathfrak{D}'_j$ satisfy the condition (C) for $\mathfrak{C}_{j+1}$ in Proposition 6, and that

$$\mathfrak{D}'_j\cdot\mathfrak{C}'_{j+1}\subset\mathfrak{D}'_{j+1},\ \mathfrak{C}'_{j+1}\subset\mathfrak{C}_{j+1}\ (3\leqq j\leqq n-2). \tag{49}$$

Finally, let $\mathfrak{D}_2$, $\mathfrak{C}'_2$ and $\mathfrak{C}'_3$ be the neighbourhoods of D_2, C_2 and C_3 respectively such that

$$\mathfrak{D}_2 \cdot \mathfrak{C}_2'^{-1} \subset \mathfrak{C}_1, \qquad \mathfrak{D}_2 \cdot \mathfrak{C}_3' \subset \mathfrak{D}_3',$$

$$\mathfrak{C}_2' \subset \mathfrak{C}_2, \qquad \mathfrak{C}_3' \subset \mathfrak{C}_3.$$

Then, by Proposition 6 we can find the neighbourhoods $\mathfrak{D}_2'$, W_1 and W_3 of D_2, $\mathrm{tr}(C_1)$ and $\mathrm{tr}(D_3)$ respectively such that

$$\mathfrak{D}_2' \cdot \mathfrak{C}_2'^{-1} \subset \mathfrak{C}_1, \qquad \mathfrak{D}_2' \cdot \mathfrak{C}_3' \subset \mathfrak{D}_3', \tag{50}$$

and that $W_1 \times \mathfrak{D}_2'$ and $W_3 \times \mathfrak{D}_2'$ satisfy the condition (C) for $\mathfrak{C}_2'^{-1}$ and $\mathfrak{C}_3'$ respectively in Proposition 6.

Now, take an element $D_2^{(0)}$ in $\mathfrak{D}_2' \cap SL_2(\boldsymbol{Q})$ such that $\mathrm{tr}(D_2^{(0)})^2-4$ is a non-zero square in $\boldsymbol{Q}$. Then by the choice of $W_1 \times \mathfrak{D}_2'$, we can find an element $C_2^{(0)}$ in $\mathfrak{C}_2' \cap SL_2(\boldsymbol{Q})$ such that $\mathrm{tr}(C_2^{(0)}) = \mathrm{tr}(C_2)$, $\mathrm{tr}(D_2^{(0)} C_2^{(0)-1}) = \mathrm{tr}(C_1)$. Put $C_1^{(0)} = D_2^{(0)} C_2^{(0)-1}$. Then by (50) we see that $C_1^{(0)}$ is contained in $\mathfrak{C}_1 \cap SL_2(\boldsymbol{Q})$ and that $\mathrm{tr}(C_1^{(0)}) = \mathrm{tr}(C_1)$.

Take a rational number r_3 in W_3 such that r_3^2-4 is a non-zero square in $\boldsymbol{Q}$. Then by the choice of $W_3 \times \mathfrak{D}_2'$, we can find an element $C_3^{(0)}$ in $\mathfrak{C}_3' \cap SL_2(\boldsymbol{Q})$ such that $\mathrm{tr}(C_3^{(0)}) = \mathrm{tr}(C_3)$ and that $\mathrm{tr}(D_2^{(0)} C_3^{(0)}) = r_3$. Put $D_3^{(0)} = D_2^{(0)} C_2^{(0)}$. Then by (50) we see that $D_3^{(0)}$ is contained in $\mathfrak{D}_3' \cap SL_2(\boldsymbol{Q})$ and that $\mathrm{tr}(D_3^{(0)})^2-4$ is a non-zero square in $\boldsymbol{Q}$ by the choice of r_3. Repeating the above argument, we can find inductively $C_j^{(0)}$, $D_j^{(0)}$ in $\mathfrak{C}_j' \cap SL_2(\boldsymbol{Q})$, $\mathfrak{D}_j' \cap SL_2(\boldsymbol{Q})$ respectively such that

$$\mathrm{tr}(C_j^{(0)}) = \mathrm{tr}(C_j), \qquad D_{j-1}^{(0)} C_j^{(0)} = D_j^{(0)}, \qquad (3 \leqq j \leqq n-2)$$

and that $\mathrm{tr}(D_j^{(0)})^2-4$ is a non-zero square in $\boldsymbol{Q}$.

Finally, we can find an element $C_{n-1}^{(0)}$ in $\mathfrak{C}_{n-1}' \cap SL_2(\boldsymbol{Q})$ such that

$$\mathrm{tr}(C_{n-1}^{(0)}) = \mathrm{tr}(C_{n-1}), \qquad \mathrm{tr}(D_{n-2}^{(0)} C_{n-1}^{(0)}) = \mathrm{tr}(C_n^{-1} I^{d_0}).$$

Put $C_n^{(0)} = D_{n-2}^{(0)} C_{n-1}^{(0)-1} I^{d_0}$. Then we see that $\mathrm{tr}(C_n^{(0)}) = \mathrm{tr}(C_n)$ and that $C_n^{(0)}$ is contained in $\mathfrak{C}_n \cap SL_2(\boldsymbol{Q})$ by (47).

The representation φ_0 of Γ defined by $(C_1, \cdots, C_n, I) \mapsto (C_1^{(0)}, \cdots, C_n^{(0)}, I)$ in the case where Γ contains I (resp. $(C_1, \cdots, C_n) \mapsto (C_1^{(0)}, \cdots, C_n^{(0)})$ in the case where Γ does not contain I) is contained in $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma) \cap \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n \times \{I\}$ (resp. $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma) \cap \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n$). This shows that $\mathfrak{R}_{\boldsymbol{Q}}(\Gamma)$ is everywhere dense in $\mathfrak{R}(\Gamma)$, in the case of $g=0$.

§7. In order to complete the proof of our theorem, we need the following proposition.

PROPOSITION 7. *Let Γ be a discrete subgroup of G such that the quotient space G/Γ is compact. Then the set* $\mathrm{tr}(\Gamma)$ *cosisting of* $\mathrm{tr}(A)$ *for all elements A of Γ is discrete in $\boldsymbol{R}$.*

The proof of this proposition is given in the book of Gel'fand-Graev-

Pyatetskii-Shapiro ([3] p. 88). By using Proposition 7, we shall make a sequence $\{\varphi_m\}$ converging to an arbitrarily given φ of $\mathfrak{R}(\Gamma)$ such that φ_m $(m=1, 2, \cdots)$ are contained in $\mathfrak{R}_Q(\Gamma)$ and that the set $\mathrm{tr}(\varphi_m(\Gamma))$ is different from one another. First, consider the case $g \geqq 1$. We may assume that φ is the identity representation of Γ. Fix a bounded neighbourhood U of $\mathrm{tr}(B_1)$ in $\boldsymbol{R}$. Take an element φ_1 of $\mathfrak{R}_Q(\Gamma)$. Then by Proposition 7, the intersection $\mathrm{tr}(\varphi_1(\Gamma)) \cap U$ is a finite set. As y_{11} is a variable in (23), the element φ_2 of $\mathfrak{R}_Q(\Gamma)$ can be taken such that $\mathrm{tr}(\varphi_2(B_1))$ is contained in $U - \mathrm{tr}(\varphi_1(\Gamma))$. In the same way, we can determine inductively the element φ_m of $\mathfrak{R}_Q(\Gamma)$ such that $\mathrm{tr}(\varphi_m(B_1))$ is contained in $U - \bigcup_{i=1}^{m-1} \mathrm{tr}(\varphi_i(\Gamma))$. Of course we take the sequence $\{\varphi_m\}$ so as to converge to the identity representation.

Next consider the case $g=0$. Fix a bounded neighbourhood V of $\mathrm{tr}(D_2)$ in $\boldsymbol{R}$. In view of the choice of $\varphi_m(D_2)$ in § 6, we see that the element φ_m of $\mathfrak{R}_Q(\Gamma)$ can be taken such that $\mathrm{tr}(\varphi_m(D_2))$ is contained in $V - \bigcup_{i=1}^{m-1} \mathrm{tr}(\varphi_i(\Gamma))$. This completes our theorem.

§ 8. Let us note about the generalization of our theorem. Let Γ be a discrete subgroup of G such that the quotient space G/Γ is of finite volume with respect to the invariant measure. We define $\mathfrak{R}'(\Gamma)$, $\mathfrak{R}(\Gamma)$ and $\mathfrak{R}_Q(\Gamma)$ in the same way as in § 2. If the Theorem (A. Weil) can be proved in this case, then our method used in this paper is valid, and we can generalize our theorem to the non-compact quotient case. We note here that Proposition 7 is valid in this case, although we do not give the proof.

Saitama University

References

[1] J. Lehner, Discontinuous groups and automorphic functions, AMS Mathematical Surveys, N°VIII, 1964.

[2] A. Weil, On discrete subgroups of Lie groups, Ann. of Math., **72** (1960), 369–384.

[3] Gel'fand, Graev, Pyatetskii-Shapiro, Representation theory and automorphic function, Acad. Sci. USSR, 1966, Translated from the Russian by K. A. Hirsch, W. B. Saunders Company, 1969.

[4] Kisao Takeuchi, On some discrete subgroups of $SL_2(\boldsymbol{R})$, J. Fac. Sci. Univ. Tokyo, Sec. I, **16** (1969), 97–100.