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Radicals of gamma rings

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§1. Introduction

Let M and Γ be additive abelian groups. If for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following conditions are satisfied,

(1) $a\alpha b \in M$

(2) $(a+b)\alpha c = a\alpha c + b\alpha c$
$a(\alpha+\beta)b = a\alpha b + a\beta b$
$a\alpha(b+c) = a\alpha b + a\alpha c$

(3) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then, following Barnes [**1**], M is called a Γ-ring. If these conditions are strengthened to,

(1′) $a\alpha b \in M$, $\alpha a\beta \in \Gamma$

(2′) same as (2)

(3′) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$

(4′) $x\gamma y = 0$ for all $x, y \in M$ implies $\gamma = 0$,

then M is called a Γ-ring in the sense of Nobusawa [**5**].

Any ring can be regarded as a Γ-ring by suitably choosing Γ. Many fundamental results in ring theory have been extended to Γ-rings: Nobusawa [**5**] proved the analogues of the Wedderburn-Artin theorems for simple Γ-rings and for semi-simple Γ-rings (but see [**4**]); Barnes [**1**] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings; Luh [**3, 4**] gave a generalization of the Jacobson structure theorem for primitive Γ-rings having minimum one-sided ideals, and obtained several other structure theorems for simple Γ-rings.

In this paper the notions of Jacobson radical, Levitzki nil radical, nil radical and strongly nilpotent radical for Γ-rings are introduced, and Barnes' [**1**] prime radical is studied further. Inclusion relations for these radicals are obtained, and it is shown that the radicals all coincide in the case of a Γ-ring which satisfies the descending chain condition on one-sided ideals. The other usual radical properties from ring theory are similarly considered.

For all notions relevant to ring theory we refer to [**2**].

§2. Preliminaries

If A and B are subsets of a Γ-ring M and $\Theta, \Phi \subseteq \Gamma$, then we denote by $A\Theta B$, the subset of M consisting of all finite sums of the form $\sum_i a_i\alpha_i b_i$, where $a_i \in A$, $b_i \in B$, and $\alpha_i \in \Theta$. We define $\Theta A\Phi$ analogously in case M is a Γ-ring in the sense of Nobusawa. For singleton subsets we abbreviate these notations to, for example, $\{a\}\Theta B = a\Theta B$.

A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M \subseteq I$ ($M\Gamma I \subseteq I$). If I is both a right ideal and a left ideal then we say that I is an ideal, or redundantly, a two-sided ideal, of M.

For each a of a Γ-ring M, the smallest right ideal containing a is called the principal right ideal generated by a and is denoted by $|a\rangle$. We similarly define $\langle a|$ and $\langle a\rangle$, the principal left and two-sided (respectively) ideals generated by a. We have $|a\rangle = Za + a\Gamma M$, $\langle a| = Za + M\Gamma a$, and $\langle a\rangle = Za + a\Gamma M + M\Gamma a + M\Gamma a\Gamma M$, where $Za = \{na\colon n \text{ is an integer}\}$.

Let I be an ideal of Γ-ring M. If for each $a+I$, $b+I$ in the factor group M/I, and each $\gamma \in \Gamma$, we define $(a+I)\gamma(b+I) = a\gamma b + I$, then M/I is a Γ-ring which we shall call the difference Γ-ring of M with respect to I.

Let M be a Γ-ring and F the free abelian group generated by $\Gamma \times M$. Then

$$A = \{\sum_i n_i(\gamma_i, x_i) \in F\colon a \in M \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$$

is a subgroup of F. Let $R = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified easily that $[\alpha, x] + [\beta, x] = [\alpha+\beta, x]$ and $[\alpha, x] + [\alpha, y] = [\alpha, x+y]$ for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication in R by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i\beta_j y_j].$$

Then R forms a ring. If we define a composition on $M \times R$ into M by $a\sum_i [\alpha_i, x_i] = \sum_i a\alpha_i x_i$ for $a \in M$, $\sum_i [\alpha_i, x_i] \in R$, then M is a right R-module, and we call R the right operator ring of the Γ-ring M. Similarly, we may construct a left operator ring L of M so that M is a left L-module. Clearly I is a right (left) ideal of M if and only if I is a right R-module (left L-module) of M. Also if A is a right (left) ideal of $R(L)$ then $MA(AM)$ is an ideal of M. For subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i [\gamma_i, x_i]$ in R, where $\gamma_i \in \Phi$, $x_i \in N$, and we denote by $[(\Phi, N)]$ the set of all elements $[\varphi, x]$ in R, where $\varphi \in \Phi$, $x \in N$. Thus, in particular, $R = [\Gamma, M]$.

A Γ-ring M is said to be simple if $M\Gamma M \neq 0$ and 0 and M are the only ideals of M. M is said to be right primitive if R is a right primitive ring and $M\Gamma x=0 \Rightarrow x=0$ (see [**3**, **4**]). M is said to be completely prime if $a\Gamma b=0$, with $a, b \in M$ implies $a=0$ or $b=0$. Following Nobusawa [**5**], M is semi-simple if $a\Gamma a=0$, with $a \in M$, implies $a=0$.

For $S \subseteqq R$ we define $S^*=\{a \in M: [\Gamma, a]=[\Gamma, \{a\}] \subseteqq S\}$. It then follows that if S is a right (left) ideal of R, then S^* is a right (left) ideal of M. Also for any collection C of sets in R, $\bigcap_{S \in C} S^*=(\bigcap_{S \in C} S)^*$.

If M_i is a Γ_i-ring for $i=1, 2$, then an ordered pair (θ, ϕ) of mappings is called a homomorphism of M_1 onto M_2 if it satisfies the following properties:

(i) θ is a group homomorphism from M_1 onto M_2.

(ii) ϕ is a group isomorphism from Γ_1 onto Γ_2.

(iii) For every $x, y \in M_1$, $\gamma \in \Gamma_1$,

$$(x\gamma y)\theta=(x\theta)(\gamma\phi)(y\theta).$$

This concept is a generalization of the definition of homomorphism for Γ-rings given by Barnes [**1**]. The kernel of the homomorphism (θ, ϕ) is defined to be $K=\{x \in M: x\theta=0\}$. Clearly K is an ideal of M. If θ is a group isomorphism, i.e., if $K=0$, then (θ, ϕ) is called an isomorphism from the Γ_1-ring M_1 onto the Γ_2-ring M_2.

Let I be an ideal of the Γ-ring M. Then the ordered pair (ρ, ι) of mappings, where $\rho: M \rightarrow M/I$ is defined by $x\rho=x+I$, and ι is the identity mapping of Γ, is a homomorphism called the natural homomorphism from M onto M/I.

We omit the proof, which is precisely analogous to that for rings, of the following fundamental theorem of homomorphism for Γ-rings.

Theorem 2.1. *If (θ, ϕ) is a homomorphism from the Γ_1-ring M_1 onto the Γ_2-ring M_2 with kernel K, then M_1/K and M_2 are isomorphic.*

Finally, we remark that the analogues of the other homomorphism theorems (Theorems 2 and 3 in Barnes [**1**]) remain true under the modified definition of homomorphism for Γ-rings.

§ 3. Γ-rings in the sense of Nobusawa

Every ring A is a Γ-ring if we take $\Gamma=A$ and interpret the ternary operation in the natural way; but A may not be a Γ-ring in the sense of Nobusawa. It is of interest to know if every ring is a Γ-ring in the sense of Nobusawa for *some* choice of Γ. In this section we establish an affirmative answer to this question by proving

Theorem 3.1. *Every Γ-ring M is a Γ'-ring in the sense of Nobusawa for some abelian group Γ'.*

PROOF. We first construct $\Gamma' = \Phi/K$, where Φ is the free abelian group generated by $\Gamma \times M \times \Gamma$ and K is the subgroup consisting of all elements $\sum_i n_i(\alpha_i, a_i, \beta_i)$ of Φ with the property that $\sum_i n_i(x\alpha_i a_i)\beta_i y = 0$ for every $x, y \in M$.

We write $[\alpha, a, \beta]$ for the coset $(\alpha, a, \beta)+K$. For subsets $\Theta, \Phi \subseteq \Gamma$, $N \subseteq M$, we define $[(\Theta, N, \Phi)] = \{[\theta, x, \varphi] \in \Gamma' : \theta \in \Theta,\ x \in N,\ \varphi \in \Phi\}$. Then for $\sum_i [\alpha_i, a_i, \beta_i]$ and $\sum_j [\gamma_j, b_i, \delta_j]$ in Γ' and $x, y \in M$, we define $x(\sum_i [\alpha_i, a_i, \beta_i])y = \sum_i (x\alpha_i a_i)\beta_i y$ and $(\sum_i [\alpha_i, a_i, \beta_i])x(\sum_j [\gamma_j, b_j, \delta_j]) = \sum_{i,j} [\alpha_i, (a_i\beta_i x)\gamma_j b_j, \delta_j]$. These two compositions are well-defined and M is a Γ'-ring in the sense of Nobusawa. Note in passing that for subsets A, B of M, $A\Gamma' B = A\Gamma M\Gamma B$. Also, if M is already a Γ-ring in the sense of Nobusawa, then the Γ'-ring M which we have constructed is isomorphic to M considered as a $(\Gamma M\Gamma)$-ring.

It can be shown that complete primeness, simplicity, semi-simplicity and primitivity are hereditary under the transition of M to a Γ'-ring in the sense of Nobusawa.

§ 4. The Prime Radical

Following Barnes [1], an ideal P of a Γ-ring M is prime if for any ideals A, $B \subseteq M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. A subset S of M is an m-system in M if $S = \phi$ or if $a, b \in S$ implies $\langle a \rangle \Gamma \langle b \rangle \cap S \neq \phi$. The prime radical of M, which we denote by $\mathcal{P}(M)$, is defined as the set of elements x in M such that every m-system containing x contains 0. Barnes [1] has characterized $\mathcal{P}(M)$ as the intersection of all prime ideals of M, has shown that an ideal P is a prime if and only if its complement P^c is an m-system, and that an ideal P of a Γ-ring M in the sense of Nobusawa is prime if and only if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$.

THEOREM 4.1. *If $\mathcal{P}(R)$ is the prime radical of the right operator ring R of the Γ-ring M, then $\mathcal{P}(M) = \mathcal{P}(R)^*$.*

Our proof requires a lemma which is of interest in its own right:

LEMMA 4.1. *If P is a prime ideal of R then P^* is a prime ideal of M.*

PROOF OF LEMMA. Suppose $A\Gamma B \subseteq P^*$ where A and B are ideals of M. Then $[\Gamma, A][\Gamma, B] = [\Gamma, A\Gamma B] \subseteq P$. By the primeness of P, either $[\Gamma, A] \subseteq P$ or $[\Gamma, B] \subseteq P$. This means that either $A \subseteq P^*$ or $B \subseteq P^*$.

PROOF OF THEOREM. If Q is an ideal of M then

$$P = \{\sum_i [\alpha_i, a_i] \in R : M(\sum_i [\alpha_i, a_i]) \subseteq Q\}$$

is an ideal of R. If Q is prime and A, B are ideals of R such that $AB \subseteq P$ then also $ARB \subseteq P$, hence $MA\Gamma MB \subseteq MP \subseteq Q$. Since MA and MB are ideals of M, it follows that $MA \subseteqq Q$ or $MB \subseteqq Q$. Thus $A \subseteqq P$ or $B \subseteqq P$ and we may conclude that P is prime. Note also that $P^* = \{x \in M : [\Gamma, x] \subseteqq P\} = \{x \in M : M\Gamma x \subseteqq Q\}$. Thus if Q is a prime ideal of M then $Q = P^*$. It follows that $\mathcal{P}(M)$, which is the intersection of all prime ideals of M, contains $\bigcap_{P \in \mathcal{D}} P^* = (\bigcap_{P \in \mathcal{D}} P)^*$, where $\mathcal{D}$ is a certain collection of prime ideals of R. But $(\bigcap_{P \in \mathcal{D}} P)^* \supseteqq \mathcal{P}(R)^*$ so we may conclude that $\mathcal{P}(M) \supseteqq \mathcal{P}(R)^*$.

On the other hand, $\mathcal{P}(R^*) = (\cap P)^* = (\cap P^*)$, where the intersection is taken over all prime ideals of R. Since, by Lemma 4.1., each P^* is a prime ideal of M, and since $\mathcal{P}(M)$ is the intersection of all prime ideals of M, it follows that $\mathcal{P}(M) \subseteqq \mathcal{P}(R)^*$.

THEOREM 4.2. *If I is an ideal of the Γ-ring M then $\mathcal{P}(I) = I \cap \mathcal{P}(M)$, where $\mathcal{P}(I)$ denotes the prime radical of I considered as a Γ-ring.*

We begin by proving

LEMMA 4.2. *If P is a prime ideal of M then $P \cap I$ is a prime ideal of I.*

PROOF OF LEMMA. Let A, B be ideals of I such that $A\Gamma B \subseteqq P \cap I$. If $\langle A \rangle = A + A\Gamma M + M\Gamma A + M\Gamma A\Gamma M$ and $\langle B \rangle = B + B\Gamma M + M\Gamma B + M\Gamma B\Gamma M$, then $I\Gamma\langle A \rangle \Gamma I \subseteqq A$ and $\langle A \rangle \subseteqq I$. Thus, and similarly,

$$(\langle A \rangle \Gamma \langle A \rangle \Gamma \langle A \rangle)\Gamma(\langle B \rangle \Gamma \langle B \rangle \Gamma \langle B \rangle) \subseteqq A\Gamma B \subseteqq P .$$

Since P is prime in M and $\langle A \rangle \Gamma \langle A \rangle \Gamma \langle A \rangle$, $\langle B \rangle \Gamma \langle B \rangle \Gamma \langle B \rangle$ are ideals of M, we conclude that $\langle A \rangle \Gamma \langle A \rangle \Gamma \langle A \rangle \subseteqq P$ or $\langle B \rangle \Gamma \langle B \rangle \Gamma \langle B \rangle \subseteqq P$. By repeated use of the primeness of P we get $\langle A \rangle \subseteqq P$ or $\langle B \rangle \subseteqq P$, hence $A \subseteqq P$ or $B \subseteqq P$. Therefore either $A \subseteqq P \cap I$ or $B \subseteqq P \cap I$ and $P \cap I$ is a prime ideal of I.

PROOF OF THEOREM. $\mathcal{P}(I)$ is the set of all elements x in I such that every m-system of I which contains x contains 0. Every m-system of I is certainly also an m-system of M. It follows that $\mathcal{P}(I) \supseteqq I \cap \mathcal{P}(M)$. By Lemma 4.2, $\mathcal{P}(I) \subseteqq I \cap \mathcal{P}(M)$. Thus $\mathcal{P}(I) = I \cap \mathcal{P}(M)$.

§ 5. The Strongly Nilpotent Radical

An element a of a Γ-ring M is strongly nilpotent if there exists a positive integer n such that $(a\Gamma)^n a = (a\Gamma a\Gamma a\Gamma \cdots a\Gamma)a = 0$. A subset S of M is strongly nil if each of its elements is strongly nilpotent. S is strongly nilpotent if there exists a positive integer n such that $(S\Gamma)^n S = (S\Gamma S\Gamma \cdots S\Gamma)S = 0$. Clearly a strongly nilpotent set is also strongly nil.

THEOREM 5.1. *If M is a Γ-ring in the sense of Nobusawa and $a \in M$, then the following are equivalent:*

(i) *a is strongly nilpotent*

(ii) *$\langle a \rangle$ is strongly nil*

(iii) *$\langle a \rangle$ is strongly nilpotent*

PROOF. That (iii) implies (ii) and (ii) implies (i) is trivial. The proof that (i) implies (iii) is left to the reader.

We define the strongly nilpotent radical, $\mathfrak{S}(M)$, of the Γ-ring M to be the sum of all strongly nilpotent ideals of M.

THEOREM 5.2. *If A and B are strongly nilpotent ideals of a Γ-ring M, then their sum is a strongly nilpotent ideal of M.*

PROOF. If $(A\Gamma)^n A=0$ then $((A+B)\Gamma)^n(A+B)=(A\Gamma)^n A+B_1=B_1$, where $B_1\subseteqq B$. If $(B\Gamma)^m B=0$ then $((A+B)\Gamma)^{mn+m+n}(A+B)=(((A+B)\Gamma)^n(A+B)\Gamma)^m((A+B)\Gamma)^n(A+B)=(B_1\Gamma)^m B_1=0$, hence $A+B$ is strongly nilpotent.

THEOREM 5.3. *If M is a Γ-ring then $\mathfrak{S}(M)$ is a strongly nil ideal of M.*

PROOF. Each element x of $\mathfrak{S}(M)$ is in a finite sum of strongly nilpotent ideals of M, which, by Theorem 5.2, is strongly nilpotent. Therefore x is strongly nilpotent, whence $\mathfrak{S}(M)$ is strongly nil.

THEOREM 5.4. *If A and B are strongly nil ideals of a Γ-ring M, then their sum is a strongly nil ideal of M.*

PROOF. The proof parallels that of Theorem 5.2 and is left to the reader.

THEOREM 5.5. *If M is a Γ-ring in the sense of Nobusawa then $\mathfrak{S}(M)$ is the sum, $\mathcal{S}$, of all strongly nil ideals of M.*

PROOF. By Theorem 5.3, $\mathfrak{S}(M)\subseteqq\mathcal{S}$. On the other hand, if $a\in\mathcal{S}$ then a belongs to a finite sum of strongly nil ideals of M, which, by Theorem 5.4, is a strongly nil ideal of M. By Theorem 5.1, $\langle a\rangle$ is strongly nilpotent. Therefore $\langle a\rangle\subseteqq\mathfrak{S}(M)$, hence $a\in\mathfrak{S}(M)$, whence $\mathcal{S}\subseteqq\mathfrak{S}(M)$.

THEOREM 5.6. *If M is a semi-simple Γ-ring then $\mathfrak{S}(M)=0$.*

PROOF. Let $a\in\mathfrak{S}(M)$ and $(a\Gamma)^n a=0$. We may assume that $n=2^m-1$ where m is a positive integer. If $A=(a\Gamma)^{2^{m-1}-1}a$ then $A\Gamma A\subseteqq(a\Gamma)^n a=0$. Because M is semi-simple, $A=0$; i.e., $(a\Gamma)^{2^{m-1}-1}a=0$. Continuing this argument we finally obtain $a\Gamma a=0$, hence $a=0$.

THEOREM 5.7. *If M is a Γ-ring in the sense of Nobusawa, then M is semi-simple if and only if $\mathfrak{S}(M)=0$.*

PROOF. If M is not semi-simple then there exists $0\neq a\in M$ such that $a\Gamma a=0$. But then $\langle a\rangle\Gamma\langle a\rangle=0$ so $\langle a\rangle$ is strongly nilpotent and therefore $\mathfrak{S}(M)\neq 0$.

The necessity follows from Theorem 5.6.

THEOREM 5.8. *If I is an ideal of the Γ-ring M then $\mathfrak{S}(I)=I\cap\mathfrak{S}(M)$.*

PROOF. If S is a strongly nilpotent ideal of I with $(S\Gamma)^n S=0$, then $T=S+M\Gamma S+S\Gamma M+M\Gamma S\Gamma M$ is an ideal of M and $(T\Gamma)^2 T\subseteqq S$. Hence $(T\Gamma)^{3n+2}T=0$ and T is a strongly nilpotent ideal of M. It follows that $T\subseteqq\mathfrak{S}(M)$, hence $S\subseteqq I\cap\mathfrak{S}(M)$. Thus $\mathfrak{S}(I)\subseteqq I\cap\mathfrak{S}(M)$.

On the other hand, if $a\in I\cap\mathfrak{S}(M)$ then $\langle a\rangle$ is a strongly nilpotent ideal of M. Since the principal ideal (of I) generated by a in I is contained in

$\langle a\rangle$, $a\in\mathfrak{S}(I)$. Thus $I\cap\mathfrak{S}(M)\subseteq\mathfrak{S}(I)$.

§ 6. The Nil Radical

An element x of a Γ-ring M is nilpotent if for any $\gamma\in\Gamma$ there exists a positive integer $n=n(\gamma)$ such that $(x\gamma)^n x=(x\gamma)(x\gamma)\cdots(x\gamma)x=0$. A subset S of M is nil if each element of S is nilpotent. The nil radical of M is defined as the sum of all nil ideals of M, and is denoted by $\mathcal{N}(M)$.

THEOREM 6.1. *If A and B are nil ideals of the Γ-ring M, then their sum is a nil ideal of M.*

PROOF. The proof parallels that of Theorem 5.2 and is left to the reader.

THEOREM 6.2. *If M is a Γ-ring then $\mathcal{N}(M)$ contains $\mathcal{N}(R)^*$, where $\mathcal{N}(R)$ denotes the upper nil radical of R.*

PROOF. Let $a\in\mathcal{N}(R)^*$. If $b\in\langle a\rangle$ and $\gamma\in\Gamma$ then $[\gamma,b]\in\mathcal{N}(R)$, so there exists a positive integer n such that $[\gamma,b]^n=0$. Hence $(b\gamma)^n b=0$, whence b is nilpotent and consequently $\langle a\rangle$ is nil. Therefore $a\in\mathcal{N}(M)$.

THEOREM 6.3. *If M is a Γ-ring then $\mathcal{N}(M/\mathcal{N}(M))=\mathcal{N}(M)$, the zero ideal of $M/\mathcal{N}(M)$.*

PROOF. Let $a+\mathcal{N}(M)\in\mathcal{N}(M/\mathcal{N}(M))$ and let $b\in\langle a\rangle$. Then $b+\mathcal{N}(M)$ is in the nil principal ideal of $M/\mathcal{N}(M)$ generated by $a+\mathcal{N}(M)$. Hence for any $\gamma\in\Gamma$ there exists a positive integer n such that $((b+\mathcal{N}(M))\gamma)^n(b+\mathcal{N}(M))=\mathcal{N}(M)$; i.e., $(b\gamma)^n b\in\mathcal{N}(M)$. Since $\mathcal{N}(M)$ is a nil ideal of M, there exists a positive integer m such that $((b\gamma)^n b\gamma)^m((b\gamma)^n b)=0$, or $(b\gamma)^{nm+m+n}b=0$. Hence b is nilpotent, whence $\langle a\rangle$ is nil and $a\in\mathcal{N}(M)$.

THEOREM 6.4. *If I is an ideal of the Γ-ring M then $\mathcal{N}(I)=I\cap\mathcal{N}(M)$.*

PROOF. Every principal ideal generated in I by a is contained in the principal ideal generated in M by a, so $I\cap\mathcal{N}(M)\subseteq\mathcal{N}(I)$.

To show $\mathcal{N}(I)\subseteq I\cap\mathcal{N}(M)$, let $a\in\mathcal{N}(I)$ and $b\in\langle a\rangle$, the principal ideal generated in M by a. For any $\gamma\in\Gamma$, $(b\gamma)^2 b$ belongs to the nil principal ideal generated in I by a, so $((b\gamma)^2 b\gamma)^n(b\gamma)^2 b=0$, or $(b\gamma)^{3n+2}b=0$ for some $n=n(\gamma)$. Thus $\langle a\rangle$ is nil and $a\in\mathcal{N}(M)$. Clearly $a\in I$, so $a\in I\cap\mathcal{N}(M)$.

§ 7. The Levitzki Nil Radical

A subset S of a Γ-ring M is locally nilpotent if for any finite set $F\subseteq S$ and any finite set $\Phi\subseteq\Gamma$, there exists a positive integer n such that $(F\Phi)^n F=0$. By taking $F=\{x\}$ and $\Phi=\{\gamma\}$ we see that a locally nilpotent set is nil. The Levitzki nil radical of M is the sum of all locally nilpotent ideals of M, and is denoted by $\mathcal{L}(M)$.

LEMMA 7.1. *If A_1 and A_2 are locally nilpotent ideals of a Γ-ring M then their sum is a locally nilpotent ideal of M.*

PROOF. If F, Φ are finite subsets of A_1+A_2, Γ, respectively, then there exist finite subsets F_1 of A_1 and F_2 of A_2 such that $F\subseteqq F_1+F_2$. Since A_1 is locally nilpotent, there exists $n=n(F_1, \Phi)$ such that $(F_1\Phi)^nF_1=0$. It follows that $((F_1+F_2)\Phi)^n(F_1+F_2)\subseteqq(F_1\Phi)^nF_1+F_2\subseteqq F_2$. There exists $m=m(F_2, \Phi)$ such that $(F_2\Phi)^mF_2=0$. It follows that $((F_1+F_2)\Phi)^{nm+n+m}(F_1+F_2)=0$.

THEOREM 7.1. *If M is a Γ-ring then $\mathcal{L}(M)$ is a locally nilpotent ideal.*

PROOF. It suffices to note that each element of a finite subset F of $\mathcal{L}(M)$ lies in a finite sum of locally nilpotent ideals of M, hence F lies in a finite sum of locally nilpotent ideals of M, which by an extension of Lemma 7.1 is a locally nilpotent ideal of M.

LEMMA 7.2. *If I is a locally nilpotent ideal of the right operator ring R of a Γ-ring M, then I^* is a locally nilpotent ideal of M.*

PROOF. Let F and Φ be finite subsets of I^* and Γ respectively. Then $[(\Phi, F)]$ is a finite subset of I, hence there exists n such that $[(\Phi, F)]^n=0$, so $[\Phi, F]^n=0$. It follows that $(F\Phi)^nF=0$ so I^* is locally nilpotent.

LEMMA 7.3. *If I is a locally nilpotent (right) ideal of a Γ-ring M, then there exists a locally nilpotent (right) ideal J of R, the right operator ring of M, such that $I\subseteqq J^*$.*

PROOF. If $J=[\Gamma, I]$ then clearly J is an (a right) ideal of R and $I\subseteqq J^*$. To show that J is locally nilpotent let F be a finite subset of J. Then there are finite subsets $F_1\subseteqq I$, $\Phi_1\subseteqq\Gamma$, such that $F\subseteqq[\Phi_1, F_1]$. Since I is locally nilpotent, $(F_1\Phi_1)^nF_1=0$ for some n. It follows that $MF^{n+1}\subseteqq M[\Phi_1, F_1]^{n+1}=M\Phi_1(F_1\Phi_1)^nF_1=0$. Hence $F^{n+1}=0$ and J is locally nilpotent.

THEOREM 7.2. *If M is a Γ-ring then $\mathcal{L}(M)=\mathcal{L}(R)^*$, where $\mathcal{L}(R)$ is the Levitzki nil radical of the right operator ring R of M.*

PROOF. Since $\mathcal{L}(R)$ is locally nilpotent, $\mathcal{L}(R)^*\subseteqq\mathcal{L}(M)$ by Lemma 7.2. $\mathcal{L}(M)\subseteqq\mathcal{L}(R)^*$ by Theorem 7.1 and Lemma 7.3.

REMARK. Since $\mathcal{L}(R)$ contains all locally nilpotent right ideals of R, Theorem 7.2 implies that $\mathcal{L}(M)$ contains all locally nilpotent right ideals of M. Since $\mathcal{L}(M)$ is itself a locally nilpotent right ideal of M, we see that $\mathcal{L}(M)$ can be characterized as the sum of all locally nilpotent right ideals of M. By the left-right symmetry of the definition of local nilpotency, $\mathcal{L}(M)$ may also be characterized as the sum of all locally nilpotent left ideals of M.

THEOREM 7.3. *If I is an ideal of the Γ-ring M then $\mathcal{L}(I)=I\cap\mathcal{L}(M)$.*

PROOF. $I\cap\mathcal{L}(M)\subseteqq\mathcal{L}(I)$ because $I\cap\mathcal{L}(M)$ is a locally nilpotent ideal of I as a Γ-ring.

To see that $\mathcal{L}(I)\subseteqq I\cap\mathcal{L}(M)$ we consider an arbitrary locally nilpotent ideal S of I. $T=S+S\Gamma M$ is a right ideal of M containing S. Since $T\subseteqq I$ we are done if we show $T\subseteqq\mathcal{L}(M)$. Let F and Φ be finite subsets of T and Γ respectively. Then $F\Phi F$ is contained in a subgroup of M generated by a finite subset, F_1, of S, hence there exists $n=n(F_1, \Phi)$ such that $(F_1\Phi)^n F_1=0$ so $(F\Phi)^{2n+1}F=0$. Thus T is locally nilpotent and by the remark preceding the theorem, $T\subseteq \mathcal{L}(M)$.

THEOREM 7.4. *If M is a Γ-ring then $\mathcal{L}(M/\mathcal{L}(M))=\mathcal{L}(M)$, the zero ideal of $M/\mathcal{L}(M)$.*

PROOF. It suffices to show that for $a+\mathcal{L}(M)\in \mathcal{L}(M/\mathcal{L}(M))$ $|a\rangle$ is locally nilpotent, hence $a\in \mathcal{L}(M)$.

Let F and Φ be finite subsets of $|a\rangle$ and Γ respectively. Let $\bar{F}=\{\bar{x}=x+\mathcal{L}(M): x\in F\}$. Then $\bar{F}$ is a finite subset of the principal right ideal generated by $a+\mathcal{L}(M)$ in $M/\mathcal{L}(M)$, hence $(\bar{F}\Phi)^n\bar{F}=0$ or $(F\Phi)^nF\subseteq \mathcal{L}(M)$ for some n. Since $(F\Phi)^nF$ is contained in a subgroup of M generated by a finite set, F_1, and since $\mathcal{L}(M)$ is locally nilpotent, there exists m such that $(F_1\Phi)^mF_1=0$. Thus $(F\Phi)^{mn+m+n}F=0$, proving that $|a\rangle$ is locally nilpotent as desired.

§ 8. The Jacobson Radical

An element a of a Γ-ring M is right quasi-regular (abbreviated *rqr*) if, for any $\gamma\in\Gamma$, there exist $\eta_i\in\Gamma$, $x_i\in M$, $i=1, 2, \cdots, n$ such that

$$x\gamma a+\sum_{i=1}^{n} x\eta_i x_i-\sum_{i=1}^{n} x\gamma a\eta_i x_i=0 \qquad \text{for all } x\in M.$$

A subset S of M is *rqr* if every element in S is *rqr*. $\mathcal{J}(M)=\{a\in M: \langle a\rangle$ is *rqr*$\}$ is the right Jacobson radical of M.

THEOREM 8.1. *Every nilpotent element in a Γ-ring M is rqr.*

PROOF. If $a\in M$ is nilpotent and $\gamma\in\Gamma$, then $(a\gamma)^na=0$ for some n. Let $\eta_1=\eta_2=\cdots=\eta_n=\gamma$ and let $x_1=-a$, $x_i=-(a\gamma)^{i-1}a$ for $i=2, 3, \cdots, n$. Then

$$x\gamma a+\sum_{i=1}^{n} x\eta_i x_i-\sum_{i=1}^{n} x\gamma a\eta_i x_i=x\gamma(a\gamma)^na=0 \qquad \text{for all } x\in M.$$

Hence a is *rqr*.

LEMMA 8.1. *An element a of a Γ-ring M is rqr if and only if, for all $\gamma\in\Gamma$, $[\gamma, a]$ is rqr in the right operator ring R of M.*

PROOF. Left to the reader.

THEOREM 8.2. *If M is a Γ-ring then $\mathcal{J}(M)=\mathcal{J}(R)^*$, where $\mathcal{J}(R)$ denotes the Jacobson radical of the right operator ring R of M.*

PROOF. In R, $|[\gamma, a]\rangle=\{[\gamma, b]\in R: b\in|a\rangle\}$. If $a\in\mathcal{J}(M)$ then $\langle a\rangle$ is *rqr*, hence $|a\rangle$ is *rqr*. Thus by Lemma 8.1, $|[\gamma, a]\rangle$ is *rqr* in R for all γ, and therefore $a\in\mathcal{J}(R)^*$.

If $a\in\mathcal{J}(R)^*$ then $\langle[\gamma, a]\rangle$ is *rqr* in R for all $\gamma\in\Gamma$, hence $[\gamma, b]$ is *rqr* in R for all $\gamma\in\Gamma$, $b\in\langle a\rangle$. Thus by Lemma 8.1, $\langle a\rangle$ is *rqr*, hence $a\in\mathcal{J}(M)$ proving that $\mathcal{J}(R)^*\subseteq\mathcal{J}(M)$.

It follows from Theorem 8.2 that $\mathcal{J}(M)$ is an ideal of M and contains all *rqr* right ideals of M. Thus $\mathcal{J}(M)$ may be characterized as the sum of all *rqr* right ideals of M.

THEOREM 8.3. *If M is a Γ-ring in the sense of Nobusawa then $\mathcal{J}(M)$ is the sum of all rqr left ideals of M.*

PROOF. It suffices to show that every *rqr* principal left ideal of M is contained in $\mathcal{J}(M)$. Let $\gamma\in\Gamma$ and let $b\in\langle a|$ where $\langle a|$ is *rqr*. Since M is a Γ-ring in the sense of Nobusawa, every element in $\langle[\gamma, b]|$ can be expressed as $n[\gamma, b]+\sum_i[\lambda_i, x_i][\gamma, b]=[n\gamma+\sum_i\lambda_i x_i\gamma, b]$, where n is an integer. By Lemma 8.1 every element in $\langle[\gamma, b]|$ is *rqr* in R so $\langle[\gamma, b]|\subseteqq\mathcal{J}(R)$. Since γ was arbitrary, $b\in\mathcal{J}(R)^*=\mathcal{J}(M)$.

THEOREM 8.4. *If I is an ideal of a Γ-ring M then $\mathcal{J}(I)=I\cap\mathcal{J}(M)$.*

PROOF. To show that $I\cap\mathcal{J}(M)\subseteqq\mathcal{J}(I)$, we prove that $I\cap\mathcal{J}(M)$ is a *rqr* ideal of I. Let $a\in I\cap\mathcal{J}(M)$ and $\gamma\in\Gamma$. Since $a\in\mathcal{J}(M)$ there exist $x_i\in M$, $\eta_i\in\Gamma$, such that $x\gamma a+\sum x\eta_i x_i-\sum x\gamma a\eta_i x_i=0$ for all $x\in M$. Then $x\gamma a\gamma a+\sum x\eta_i x_i\gamma a-\sum x\gamma a\eta_i x_i\gamma a=0$ and $x\gamma a+(\sum x\eta_i(x_i\gamma a)-x\gamma a)-(\sum x\gamma a\eta_i(x_i\gamma a)-x\gamma a\gamma a)=0$. Since $a\in I$ and each $x_i\gamma a\in I$, we see that a is *rqr* in I.

To prove that $\mathcal{J}(I)\subseteqq I\cap\mathcal{J}(M)$, let $a\in\mathcal{J}(I)$ and $b\in|a\rangle$. Then for any $\gamma\in\Gamma$, $(b\gamma)^2b$ is in the principal right ideal in I generated by a. Hence $(b\gamma)^2b$ is *rqr* in I, say $y\gamma(b\gamma)^2b+\sum y\delta_j y_j-\sum y\gamma(b\gamma)^2b\delta_j y_j=0$ for all $y\in I$, where $\delta_j\in\Gamma$, $y_j\in I$. If $x\in M$ then $x\gamma b\in I$, so $(x\gamma b)\gamma(b\gamma)^2b+\sum x\gamma b\delta_j y_j-\sum x\gamma b\gamma(b\gamma)^2b\delta_j y_j=0$ or $x(\gamma b)^4+\sum x\gamma b\delta_j y_j-\sum x(\gamma b)^4\delta_j y_j=0$. This may be written as $x\gamma b+(\sum x(\gamma b)^3\delta_j y_j+\sum x(\gamma b)^2\delta_j y_j+\sum x(\gamma b)\delta_j y_j-x(\gamma b)^3-x(\gamma b)^2-x\gamma b)-(x(\gamma b)^4\delta_j y_j+\sum x(\gamma b)^3\delta_j y_j+\sum x(\gamma b)^2\delta_j y_j-x(\gamma b)^4-x(\gamma b)^3-x(\gamma b)^2)=0$, which is of the form

$$x\gamma b+\sum x\lambda_k z_k-\sum x\gamma b\lambda_k z_k=0\,.$$

Hence b is *rqr* in M, whence $|a\rangle$ is *rqr* in M, thence $a\in\mathcal{J}(M)$.

THEOREM 8.5. *If M is a Γ-ring then $\mathcal{J}(M/\mathcal{J}(M))=\mathcal{J}(M)$, the zero ideal of $M/\mathcal{J}(M)$.*

PROOF. If $a+\mathcal{J}(M)\in\mathcal{J}(M/\mathcal{J}(M))$ and $b\in|a\rangle$, $\gamma\in\Gamma$, then $b+\mathcal{J}(M)$ belongs to the *rqr* principal right ideal generated in $M/\mathcal{J}(M)$ by $a+\mathcal{J}(M)$, hence $b+\mathcal{J}(M)$ is *rqr* in $M/\mathcal{J}(M)$. It follows that there exist $\eta_i\in\Gamma$, $x_i\in M$, $i=1, 2,\cdots, n$, such that $x\gamma b+\sum x\eta_i x_i-\sum x\gamma b\eta_i x_i\in\mathcal{J}(M)$ for all $x\in M$. Put $x=b\gamma b$. Then $c=b(\gamma b)^2+\sum_i b\gamma b\eta_i x_i-\sum_i b(\gamma b)^2\eta_i x_i\in\mathcal{J}(M)$. If $y\in M$ then $y\gamma b\in M$ and hence $(y\gamma b)\gamma c+\sum_j(y\gamma b)\lambda_j z_j-\sum_j(y\gamma b)\gamma c\lambda_j z_j=0$. Substituting for c and rearranging terms, we obtain $y\gamma b+(-y\gamma b-y(\gamma b)^2-y(\gamma b)^3+\sum_i y(\gamma b)^3\eta_i x_i-\sum_{i,j}y(\gamma b)^3\eta_i x_i\lambda_j z_j+\sum_j y\gamma b\lambda_j z_j+\sum_j y(\gamma b)^2\lambda_j z_j+\sum_j y(\gamma b)^3\lambda_j z_j)-(-y(\gamma b)^2-y(\gamma b)^3-y(\gamma b)^4+\sum_i y(\gamma b)^4\eta_i x_i-\sum_{i,j}y(\gamma b)^4\eta_i x_i\lambda_j z_j+\sum_j y(\gamma b)^2\lambda_j z_j+\sum_j y(\gamma b)^3\lambda_j z_j+\sum_j y(\gamma b)^4\lambda_j z_j)=0$, hence b is *rqr*. Therefore $|a\rangle$ is *rqr* and $a\in\mathcal{J}(M)$.

We note in passing that we can also define left quasi-regularity and the left Jacobson radical for Γ-rings. It is unlikely that the left Jacobson radical is equal to $\mathcal{J}(M)$.

§ 9. Relations among the Radicals

We will prove:

THEOREM 9.1. *If M is a Γ-ring then $\mathfrak{S}(M) \subseteqq \mathcal{P}(M) \subseteqq \mathcal{L}(M) \subseteqq \mathcal{N}(M) \subseteqq \mathcal{J}(M)$.*

THEOREM 9.2. *If M is a Γ-ring which satisfies the descending chain condition on right ideals, then $\mathfrak{S}(M) = \mathcal{P}(M) = \mathcal{L}(M) = \mathcal{N}(M) = \mathcal{J}(M)$.*

PROOF OF THEOREM 9.1. From ring theory it is known that $\mathcal{P}(R) \subseteqq \mathcal{L}(R) \subseteqq \mathcal{J}(R)$. By Theorems 4.1, 7.2, and 8.2, it follows that $\mathcal{P}(M) \subseteqq \mathcal{L}(M) \subseteqq \mathcal{J}(M)$.

Evidently, every strongly nilpotent ideal is contained in any prime ideal, so $\mathfrak{S}(M) \subseteqq \mathcal{P}(M)$.

It is also clear that every locally nilpotent ideal is nil, so $\mathcal{L}(M) \subseteqq \mathcal{N}(M)$. By Theorem 8.1, every nil ideal is *rqr*, hence $\mathcal{N}(M) \subseteqq \mathcal{J}(M)$.

PROOF OF THEOREM 9.2. It suffices to show $\mathcal{J}(M) \subseteqq \mathfrak{S}(M)$. For convenience, let $J = \mathcal{J}(M)$. Consider the chain $J \supseteqq J\Gamma J \supseteqq (J\Gamma)^2 J \supseteqq \cdots$ of ideals. By the descending chain condition, $(J\Gamma)^n J = (J\Gamma)^{n+1} J = \cdots$ for some n. Denote $(J\Gamma)^n J$ by I. Clearly $I\Gamma I = I$.

If $I \neq 0$ then the set $\mathcal{R}$, of all right ideals A of M contained in I such that $A\Gamma I \neq 0$, is non-empty. By the descending chain condition, $\mathcal{R}$ contains a minimal element, B. Then there exist $b \in B$, $\delta \in \Gamma$ such that $b\delta I \neq 0$. Thus $(b\delta I)\Gamma I = b\delta I \neq 0$, and $b\delta I \subseteqq B \in \mathcal{R}$. Consequently $b\delta I = B$, and there exists $a \in I$ such that $b\delta a = b$. But $a \in J$ is *rqr* so there exist $\eta_i \in \Gamma$, $x_i \in M$ such that $x\delta a + \sum x\eta_i x_i - \sum x\delta a\eta_i x_i = 0$ for all $x \in M$. Putting $x = b$ we obtain $b + \sum b\eta_i x_i - \sum b\eta_i x_i = 0$, or $b = 0$, a contradiction. Hence $I = 0$; i.e., $(J\Gamma)^n J = 0$. Therefore $J = \mathcal{J}(M)$ is strongly nilpotent and $\mathcal{J}(M) \subseteqq \mathfrak{S}(M)$.

It can be shown that $\mathcal{P}(M)$, $\mathfrak{S}(M)$, $\mathcal{L}(M)$, and $\mathcal{J}(M)$ are invariant under the transition of M to a Γ'-ring in the sense of Nobusawa. Moreover, $\mathcal{N}(M)$ contains $\mathcal{N}'(M)$, the nil radical of M as a Γ'-ring; and if M is already a Γ-ring in the sense of Nobusawa, then $\mathcal{N}(M) = \mathcal{N}'(M)$.

Finally, we remark that Theorem 9.1 remains true if we replace $\mathcal{J}(M)$ by the left Jacobson radical of M. Moreover if M satisfies the descending chain condition on left ideals, then the left Jacobson radical of M coincides with $\mathfrak{S}(M)$. Hence if M satisfies the descending chain conditions on both left ideals and right ideals then the right Jacobson radical and the left Jacobson radical coincide.

§ 10. Concluding Remarks

By virtue of Theorems 8.5, 7.4 and 6.3 every Γ-ring M has a homomorphic image with zero radical, where radical can be taken as $\mathcal{J}(M)$, $\mathcal{L}(M)$ or $\mathcal{N}(M)$. Barnes [1] established this fact for $\mathcal{P}(M)$.

Although it is true that any ring M can be regarded as a Γ-ring by taking $\Gamma = M$, it is not necessarily true that M can be regarded as a Γ-ring in the sense of Nobusawa by taking $\Gamma = M$. But if M is a simple ring then $M^2 = M$, and considered as a Γ-ring with $\Gamma = M$, M is simple. Also if M is a semi-simple ring and $a\Gamma a = 0$ with $\Gamma = M$, then $(a)^3 = 0$, where (a) denotes the principal ideal generated in the ring M by a. This says (a) is nilpotent; but in a semi-simple ring there are no nonzero nilpotent ideals. Therefore $a = 0$, and M is semi-simple when regarded as a Γ-ring. Finally we note that if the ring M satisfies the descending chain condition on one-sided ideals, then regarded as a Γ-ring with $\Gamma = M$, M also satisfies the descending chain condition on one-sided ideals. Thus the analogues of the Wedderburn-Artin Theorems for Γ-rings obtained by Nobusawa [5] are indeed generalizations of the corresponding theorems for rings.

Nobusawa [5] defined a Γ-ring M to be semi-simple if $a\Gamma a = 0$ for $a \in M$ implies $a = 0$, and this is the definition of semi-simplicity used in this paper. However, a ring M regarded as a Γ-ring with $\Gamma = M$ which is semi-simple in the sense of Nobusawa may not have zero Jacobson radical. The simple radical rings due to Sasiada [6] are such examples. Therefore it would seem preferable to define a Γ-ring M to be semi-simple if $\mathcal{J}(M) = 0$. Since $\mathfrak{S}(M) \subseteq \mathcal{J}(M)$, a Γ-ring in the sense of Nobusawa with the property that $\mathcal{J}(M) = 0$ would be semi-simple in the sense of Nobusawa, hence Nobusawa's proof of the analogue of the Wedderburn-Artin Theorem would apply. Further justification for redefining semi-simplicity by $\mathcal{J}(M) = 0$ comes from the following

THEOREM 10.1. *If M is a ring with Jacobson radical J, then regarded as a Γ-ring with $\Gamma = M$, $\mathcal{J}(M) = J$.*

PROOF. J is an ideal of the ring M, hence is an ideal of the Γ-ring M with $\Gamma = M$. If $a \in J$ and $g \in M$ then $ga \in J$, hence $ga + y - gay = 0$ and therefore $xga + xy - xgay = 0$ for all $x \in M$. Since $y = (ga)y - ga \in M^2$, we see that a is *rqr* in M as a Γ-ring with $\Gamma = M$. Thus $J \subseteq \mathcal{J}(M)$.

For the opposite inclusion it suffices to show that $\mathcal{J}(M)$ is a *rqr* left ideal of M. Consider $|ba\rangle$, where $a \in \mathcal{J}(M)$, $b \in M$. Every element of $|ba\rangle$ can be written as be, where $e = na + \sum au_j z_j \in \mathcal{J}(M) + \mathcal{J}(M)\Gamma M \subseteq \mathcal{J}(M)$. Let $g \in \Gamma = M$. Then $gb \in \Gamma$ also, and since e is *rqr*, there exist $v_i \in \Gamma$, $y_i \in M$ such that $x(gb)e + \sum xv_i y_i - \sum x(gb)ev_i y_i = 0$ for all $x \in M$. But this may also be interpreted as $xg(be) + \sum xv_i y_i - \sum xg(be)v_i y_i = 0$ for all $x \in M$, hence be is

rqr in M as a Γ-ring with $\Gamma=M$. Therefore $|ba\rangle$ is rqr and $ba\in\mathcal{J}(M)$, proving that $\mathcal{J}(M)$ is a left ideal of M.

If $a\in\mathcal{J}(M)$ then there exist $p_i\in\Gamma$, $w_i\in M$, such that

$$xaa+\sum_i xp_iw_i-\sum_i xaap_iw_i=0 \qquad \text{for every } x \text{ in } M.$$

Letting $\sum_i p_iw_i=c$ for convenience, we see that a^2+c-a^2c belongs to the right annihilator of M, which is a nilpotent ideal of index two; hence $a^2+c-a^2c\in J$. But if $a^2\circ c\in J$ then there exists d such that $a^2\circ c\circ d=0$; i.e., a^2 is rqr in M. This implies that a is rqr in M, hence $\mathcal{J}(M)$ is a rqr left ideal of M and we are done.

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