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Topological entropy of distal affine transformations on compact abelian groups

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§ 1. Introduction.

The object of this paper is to calculate the topological entropy for a distal affine transformation on a compact abelian group.

§ 2. Definitions and Preliminaries.

Let X be a set and let $\boldsymbol{T}[X]$ be a family of subsets which satisfies the three conditions: the intersection of any two members of $\boldsymbol{T}[X]$ is a member of $\boldsymbol{T}[X]$, and the union of the members of each sub-family of $\boldsymbol{T}[X]$ is a member of $\boldsymbol{T}[X]$, and $\boldsymbol{T}[X]$ contains the whole space X and the empty set. We say that such a family $\boldsymbol{T}[X]$ is a topology of X. As in [**1**], we write $\mathfrak{A}\vee\mathfrak{B}=\{A\cap B: A\in\mathfrak{A}, B\in\mathfrak{B}\}$ for two covers $\mathfrak{A}$ and $\mathfrak{B}$ of X. A cover $\mathfrak{B}$ is said to be a *refinement* of a cover $\mathfrak{A}$ if every member of $\mathfrak{B}$ is a subset of some member of $\mathfrak{A}$, and we write $\mathfrak{A}\prec\mathfrak{B}$. Let X be a compact topological space. Topological entropy is given by [**1**] as follows: for any open cover $\mathfrak{A}$ we denote the topological entropy $H(\mathfrak{A})$ of $\mathfrak{A}$ by $H(\mathfrak{A})=\log N(\mathfrak{A})$ where $N(\mathfrak{A})$ is the number of sets in a subcover of minimal cardinality, and the topological entropy $h(\mathfrak{A}, S)$ of a continuous mapping S of X onto itself with respect to $\mathfrak{A}$ is defined by

$$h(\mathfrak{A}, S)=\lim_{n\to\infty} 1/n\, H(\bigvee_{j=0}^{n-1} S^{-j}\mathfrak{A}),$$

and the topological entropy $h(S)$ of S is defined as

$$h(S)=\sup h(\mathfrak{A}, S)$$

where the supremum is taken over all open covers. If $\mathfrak{A}\prec\mathfrak{B}$, then $h(\mathfrak{A}, S)\leqq h(\mathfrak{B}, S)$. It is easily seen that if ϕ is a homomorphism of X onto some Y

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The standard facts concerning topological groups used in this work are contained in [**10**].

then $h(S) \geqq h(\phi S \phi^{-1})$. A sequence $\{\mathfrak{A}_n\}$ of open covers is said to be *refining* if $\mathfrak{A}_n \prec \mathfrak{A}_{n+1}$ and for every open cover $\mathfrak{B}$ there exists $\mathfrak{A}_n$ such that $\mathfrak{B} \prec \mathfrak{A}_n$. Let X be a compact metric space with metric d. The diameter $d(\mathfrak{A})$ of a cover $\mathfrak{A}$ is defined by

$$d(\mathfrak{A}) = \sup_{A \in \mathfrak{A}} d(A)$$

where $d(A)$ is the diameter of the set A. If $\{\mathfrak{A}_n\}$ is a sequence of open covers such that $\mathfrak{A}_n \prec \mathfrak{A}_{n+1}$, and $d(\mathfrak{A}_n) \to 0$ as $n \to \infty$, then $\{\mathfrak{A}_n\}$ is a refining sequence [**1**]. A homeomorphism S of a compact Hausdorff space X onto itself is said to be *distal* if for any net of integers $\{n_j : j \in \Delta\}$ (Δ a directed system) and points $x, y, z \in X$ the relation

$$\lim_{\Delta} S^{n_j}(x) = \lim_{\Delta} S^{n_j}(y) = z$$

implies $x = y$. A homeomorphism S of X onto itself is called *minimal* if no non-empty proper closed subset of X is invariant under S. S is called *totally minimal* if for every non-zero integer n, the homeomorphism S^n is minimal.

Let X be a measure space with a probability measure m and let T be a m-probability measure preserving transformation on X. For a finite measurable partition $\bar{\mathfrak{A}}$ of X, we write, as in [**5**] and [**9**],

$$H_m(\bar{\mathfrak{A}}) = -\sum_{A \in \bar{\mathfrak{A}}} m(A) \log m(A)\,,$$

and

$$h_m(\bar{\mathfrak{A}}, T) = \lim_{n \to \infty} 1/n\, H_m\Big(\bigvee_{i=0}^{n-1} T^{-i} \bar{\mathfrak{A}}\Big)\,.$$

The measure-theoretic entropy of T is defined as

$$h_m(T) = \sup h_m(\bar{\mathfrak{A}}, T)$$

where the supremum is taken over all finite measurable partitions of X. Abramov [**2**] has shown that if T is a totally ergodic measure preserving transformation and T has quasi-discrete spectrum, then $h_m(T) = 0$. Recently, Parry [**7**] has proved the following theorem: let X be a compact metric space and let T be a distal homeomorphism of X onto itself and preserves a probability measure m, then $h_m(T) = 0$.

By an affine transformation on a compact abelian group X, we mean a transformation of the form $T(x) = aW(x)$, where a is an element of X and W is a continuous automorphism of X. If T is a totally minimal affine transformation, then T has quasi-discrete spectrum [**7**] and moreover T is distal [**6**].

Throughout this paper, for any subset E of a topological group X, we denote by $gp[W, E]$ the smallest W-invariant closed subgroup of X containing E. Continuous automorphisms and their duals are denoted by the same symbol.

§ 3. The theorem.

As before, X is a compact abelian group with character group Γ, and T is a distal affine transformation on X onto itself.

THEOREM. *The topological entropy of T is zero.*

PROOF. Let Θ be the family of all sets of finite members of Γ. For each $\alpha \in \Theta$, the subgroup $gp[W, \alpha]$ is finitely generated. We show this statement.

We denote by $\mathrm{ann}(gp[W, \alpha])$ the annihilator of $gp[W, \alpha]$, and denote by X_α the factor group $X/\mathrm{ann}(gp[W, \alpha])$. The factor group X_α is compact and metrizable. Let T_α be the affine transformation on X_α induced by T. Then T_α is distal. Thus the measure-theoretic entropy of T_α with respect to the Haar measure m on X_α is zero by [8]. For each $f \in \alpha$, we denote by $G(f)$ the subgroup of $gp[W, \alpha]$ generated by the set

$$\{Wf, W^2f, \cdots, W^nf, \cdots\}.$$

If the group $W(G(f))$ is a proper subgroup of $G(f)$ then we obtain $h_m(T_\alpha)>0$, because the Borel field of all measurable sets of the space X_α in which T_α acts contains a sub σ-field $\mathcal{F}$ such that $T_\alpha(\mathcal{F}) \subset \mathcal{F}$ and $T_\alpha(\mathcal{F}) \neq \mathcal{F}$. But this is impossible. Thus we obtain $f = \prod_{j=1}^{r} W^{n_j} f^{p_j}$ where n_j and p_j, $j=1, 2, \cdots, r$, are integers. If $G'(f)$ is a subgroup generated by

$$\{Wf, W^2f, \cdots, W^{r'}f\}$$

where $r' = \max\{n_1, n_2, \cdots, n_r\}$, we see, from the argument above, that $G'(f)$ is invariant under W. Since the set α is finite,

$$gp[W, \alpha] = \prod_{f \in \alpha} G'(f)$$

is finitely generated.

We consider the following two cases: either the compact group X is connected or X is disconnected.

FIRST CASE. Let X be connected, then the group $gp[W, \alpha]$ is torsion free. Thus the factor group X_α is a finite-dimensional torus with character group $gp[W, \alpha]$. We may suppose that X_α is an n-dimensional torus

$$X_\alpha = X_1 \otimes X_2 \otimes \cdots \otimes X_n,$$

and

$$X_j = X_1, \quad j = 2, 3, \cdots, n$$

where X_1 is the unit interval $[0, 1)$. Let H denote the projection mapping of X onto X_α. The factor transformation T_α of X_α is an affine transformation of the form $T_\alpha(\dot{x}) = \dot{a} + W_\alpha(\dot{x})$, $\dot{x} \in X_\alpha$, where $\dot{a} \in X_\alpha$ and W_α is the continuous automorphism on X_α induced by W, such that $T_\alpha = H \circ T \circ H^{-1}$. For the corresponding matrix $[W_\alpha]$ of W_α, we put

$$[W_\alpha] = [n_{ij} : i, j = 1, 2, \cdots, n],$$

$$\dot{a} = (a_1, a_2, \cdots, a_n).$$

Then there exists a unique element $(y_1, y_2, \cdots, y_n) \in X_\alpha$ such that

$$\sum_{j=1}^{n} n_{kj} y_j = a_k, \qquad k = 1, 2, \cdots, n.$$

For each $\dot{x} = (x_1, x_2, \cdots, x_n) \in X_\alpha$ we have

$$\begin{aligned} T_\alpha(x_1, x_2, \cdots, x_n) &= \dot{a} + W_\alpha(x_1, x_2, \cdots, x_n) \\ &= W_\alpha(x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) \quad \text{(additions mod 1)}. \end{aligned}$$

Let $\mathfrak{A}^p$ be a family of all open sets in X_1 of diameter less than $1/p$. Such a family enjoys the properties that $\mathfrak{A}^p \vee \mathfrak{A}^p = \mathfrak{A}^p$ and $\mathfrak{A}^p$ is invariant under an isometry. Since every open subset of X_α is a union of rectangles $A_1 \otimes A_2 \otimes \cdots \otimes A_n$, A_i open subset of X_i for $i = 1, 2, \cdots, n$, we see that for an arbitrary open cover $\mathfrak{D}$ of X there exists a refinement of the form $\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \cdots \otimes \mathfrak{A}_n$ where $\mathfrak{A}_i$ is an open cover of X_i for $i = 1, 2, \cdots, n$. Since the sequence $\{\mathfrak{A}^p : p = 1, 2, \cdots\}$ of open covers of X_1 is refining, there exist integers $p_1, p_2, \cdots, p_n$ such that

$$\mathfrak{D} \prec \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \cdots \otimes \mathfrak{A}_n \prec \mathfrak{A}^{p_1} \otimes \mathfrak{A}^{p_2} \otimes \cdots \otimes \mathfrak{A}^{p_n}.$$

Thus we have the following relation

$$\begin{aligned} &T_\alpha(\mathfrak{A}^{p_1} \otimes \mathfrak{A}^{p_2} \otimes \cdots \otimes \mathfrak{A}^{p_n}) \\ &\quad = W_\alpha\{(A_1 + y_1) \otimes (A_2 + y_2) \otimes \cdots \otimes (A_n + y_n) : \\ &\qquad\qquad A_i \in \mathfrak{A}^{p_i},\ i = 1, 2, \cdots, n\} \\ &\quad = W_\alpha(\mathfrak{A}^{p_1} \otimes \mathfrak{A}^{p_2} \otimes \cdots \otimes \mathfrak{A}^{p_n}), \end{aligned}$$

because the translation $T_{y_i}(x_i) = x_i + y_i$, $x_i \in X_i$, is an isometry on X_i for $i = 1, 2, \cdots, n$. This implies $h(T_\alpha) = h(W_\alpha)$. It follows that the transformation T_α is distal on X_α if and only if the automorphism W_α is so. Thus we see by [8] that $h_m(W_\alpha) = 0$ since X_α is metrizable. Since X_α is actually an n-dimensional torus and W_α a continuous automorphism on X_α, the measure-theoretic entropy and the topological entropy of W_α are equal by [4]. Thus we obtain

$$h(T_\alpha)=h(W_\alpha)=h_m(W_\alpha)=0\,.$$

We denote by $\boldsymbol{T}'[X_\alpha]$ a sub-topology $H^{-1}(\boldsymbol{T}[X_\alpha])$ of X, and denote by $\bigvee_{\alpha\in\Theta}\boldsymbol{T}'[X_\alpha]$ a topology of X generated by $\bigcup_{\alpha\in\Theta}\boldsymbol{T}'[X_\alpha]$. A topology $\bigvee_{\alpha\in\Theta}\boldsymbol{T}'[X_\alpha]$ must coincide with the topology $\boldsymbol{T}[X]$, because each member of α in Θ is continuous with respect to the topology $\boldsymbol{T}'[X_\alpha]$. Thus we see that every open subset of X is a union of sets of finite intersection of sets in $\bigcup_{\alpha\in\Theta}\boldsymbol{T}'[X_\alpha]$. For an arbitrary open cover $\mathfrak{B}$ of X, choose a minimal subcover $\mathfrak{B}'$ of $\mathfrak{B}$. Then we obtain a refinement $\mathfrak{C}$ of $\mathfrak{B}'$ consisting only of open sets in the topology $\boldsymbol{T}'[X_\alpha]$ for some $\alpha\in\Theta$. We may assume that the projection mapping H is a set mapping of the sub-topology $\boldsymbol{T}'[X_\alpha]$ onto the topology $\boldsymbol{T}[X_\alpha]$. This establishes the following relation

$$\sup\{h(\mathfrak{A},T):\ \mathfrak{A}\text{ an open cover of }X,\ \mathfrak{A}\subset\boldsymbol{T}'[X_\alpha]\}$$
$$=\sup\{h(\mathfrak{A}',T_\alpha):\ \mathfrak{A}'\text{ an open cover of }X_\alpha\}\,,$$

and thus

$$h(\mathfrak{B},T)\leqq h(T_\alpha)=0\,.$$

It follows that $h(T)=0$, because $\mathfrak{B}$ is an arbitrary open cover of X.

SECOND CASE. Let X be disconnected. Since $gp[W,\alpha]$ is finitely generated, $gp[W,\alpha]$ is decomposable into the direct product of cyclic subgroups

$$U_1,U_2,\cdots,U_m\,;\qquad V_1,V_2,\cdots,V_n$$

where U_i, $i=1,2,\cdots,m$, are free cyclic groups and V_i, $i=1,2,\cdots,n$, are cyclic groups of finite order, that is to say,

$$gp[W,\alpha]=G_u\otimes G_v$$

where $G_u=U_1\otimes U_2\otimes\cdots\otimes U_m$ and $G_v=V_1\otimes V_2\otimes\cdots\otimes V_n$. We denote by X_α the dual space of $gp[W,\alpha]$, and by T_α the affine transformation on X_α induced by T. Let X_v be the factor group $X/\mathrm{ann}(G_v)$, and let X_u be the factor group $X/\mathrm{ann}(G_u)$. The factor group X_v is a compact totally disconnected abelian group with character group G_v. However, X_v is actually discrete. The factor group X_u is the finite-dimensional torus. We may assume that the group X_α is equal to the product group $X_u\otimes X_v$ with the product topology $\boldsymbol{T}[X_u]\otimes\boldsymbol{T}[X_v]$. Since the transformation T_α is a distal affine transformation on the group $X_u\otimes X_v$, the transformation T_α induces a distal affine transformation T' on the factor group $(X_u\otimes X_v)/\{e_u\}\otimes X_v$ where e_u is the identity of X_u. Clearly the group $(X_u\otimes X_v)/\{e_u\}\otimes X_v$ is a finite-dimensional torus. Thus we conclude as in the first case that the topological entropy of the transformation T' is zero, in other words, $h(T')=0$. The transformation T_α induces a distal affine transformation T'' on the factor group $(X_u\otimes X_v)/X_u\otimes\{e_v\}$ where e_v is the identity of X_v. Since the factor group is discrete, it is clearly that $h(T'')=0$. Let $\mathfrak{B}$ be an arbitrary open cover of $X_u\otimes X_v$. Then there exist open covers $\mathfrak{A}_u\subset \boldsymbol{T}[X_u]$ and $\mathfrak{A}_v\subset \boldsymbol{T}[X_v]$ such that $\mathfrak{B}\prec\mathfrak{A}_u\otimes\mathfrak{A}_v$. So we obtain

$$\bigvee_{j=0}^{n-1}T^{-j}(\mathfrak{A}_u\otimes\mathfrak{A}_v)=\bigvee_{j=0}^{n-1}T^{-j}((\mathfrak{A}_u\otimes\{X_v\})\vee(\{X_u\}\otimes\mathfrak{A}_v)).$$

Thus it is easy to see that

$$h(\mathfrak{B}, T_\alpha)\leqq h(\mathfrak{A}_u\otimes\mathfrak{A}_v, T_\alpha)\leqq h(\mathfrak{A}_u\otimes\{X_v\}, T_\alpha)+h(\{X_u\}\otimes\mathfrak{A}_v, T_\alpha).$$

Since the transformations T' and T'' are continuous images of T_α, we see that

$$h(\mathfrak{A}_u\otimes\{X_v\}, T_\alpha)\leqq h(T'),\qquad h(\{X_u\}\otimes\mathfrak{A}_v, T_\alpha)\leqq h(T'').$$

Thus we obtain

$$h(\mathfrak{B}, T_\alpha)\leqq h(T')+h(T'')=0$$

and since $\mathfrak{B}$ is arbitrary, $h(T_\alpha)=0$. By the technique of the first case, we have the desired result, i.e., $h(T)=0$.

REMARK 1. Let T be a totally minimal affine transformation on a compact abelian group. Then $h(T)=0$, because T is distal by [6].

REMARK 2. Let X be the unit interval [0, 1) imposed the usual topology and let S be a translation mapping on X onto itself. A transformation T defined by

$$T(x, y)=(Sx, y+nx)\quad(\text{addition mod }1),$$

where n is an integer, is called a skew product transformation. As in the theorem above, we obtain $h(T)=0$.

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