J. Math. Soc. Japan
Vol. 61, No. 4 (2009) pp. 1097–1110
doi: 10.2969/jmsj/06141097

Extension dimension of a wide class of spaces

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(Received Aug. 21, 2008)

Abstract. We prove the existence of extension dimension for a much expanded class of spaces. First we obtain several theorems which state conditions on a polyhedron or CW-complex K and a space X in order that X be an absolute co-extensor for K. Then we prove the existence of and describe a wedge representative of extension dimension for spaces in a wide class relative to polyhedra or CW-complexes. We also obtain a result on the existence of a "countable" representative of the extension dimension of a Hausdorff compactum.

1. Introduction.

Extension theory, which was introduced by A. Dranishnikov in [**2**], is based on the following notion. If K is a CW-complex and X is a space, then one says that K is an *absolute extensor* for X, $K \in \mathrm{AE}(X)$, or X is an *absolute co-extensor* for K, $X \tau K$, if for each closed subset A of X and map (i.e. continuous function) $f : A \to K$, there exists a map $F : X \to K$ such that F is an extension of f. For example, X is a normal space if and only if $X \tau \boldsymbol{R}$. (We note that spaces in this paper are not assumed to be Hausdorff.)

In [**2**], Dranishnikov defined *extension dimension*. Given a class $\mathscr{C}$ of spaces and a class $\mathscr{T}$ of CW-complexes, there is a certain equivalence relation $\sim_{(\mathscr{C},\mathscr{T})}$ on $\mathscr{T}$. The equivalence class of K under this relation, denoted $[K]_{(\mathscr{C},\mathscr{T})}$, is called the *extension type* of K relative to $(\mathscr{C},\mathscr{T})$. For $X \in \mathscr{C}$, one defines the *extension dimension* relative to $(\mathscr{C},\mathscr{T})$, which exists under certain conditions. When it exists (Section 5) it is a uniquely determined extension type. We shall define *weak extension dimension* for a space X that might fall out of the class $\mathscr{C}$. It agrees with extension dimension when $X \in \mathscr{C}$.

The existence of extension dimension for certain cases has been treated in [**2**], [**4**], [**6**], and [**5**]. The notion of a dd-space was introduced in [**6**] and proved useful in that work. We shall define in Section 2 a wider class, the ddP-spaces. This will allow us to consolidate most of the previous ideas. Our Main Theorem is

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2000 *Mathematics Subject Classification.* Primary 54C55, 54C20.
Key Words and Phrases. absolute co-extensor, absolute extensor, anti-basis, cardinality of a complex, CW-complex, dd-space, ddP-space, extension dimension, extension theory, extension type, Hausdorff σ-compactum, polyhedron, pseudo-compact, σ-pseudo-compactum, σ-compactum, weak extension dimension, weight.

Theorem 5.5. We prove statements (1)–(5) on the existence of weak extension dimension for a ddP-space with some additional properties. In (1)–(3) of that theorem, the representative we obtain for the weak extension dimension of the given space is a wedge of polyhedra whose number of summands depends on a certain infinite cardinal β, and similarly the cardinality of each summand depends on β. We generalize the notion of a σ-compact space or a compact Hausdorff space to that of a σ-pseudo-compact space (Section 4). With a space of this type, parts (4,5) of Theorem 5.5 provide a representative that is a wedge of at most $2^{\aleph_0}$ summands each of which has cardinality at most $\aleph_0$. This generalizes Theorem 13 of [**3**].

Section 6 visits the question of whether a "better" representative of the extension dimension of a compact Hausdorff space exists. Dranishnikov and Dydak asked in [**4**] (Problems 6.1 below, 2.19.2 of [**3**], 2.1 of [**1**], and 5.4 of [**9**]) whether with respect to the classes $\mathscr{K}$ of compact Hausdorff spaces and $\mathscr{T}$ of CW-complexes, the extension dimension of every metrizable compactum has a "countable" polyhedral representative. We have shown in Theorem 5.10 of [**9**] that under certain conditions it would. In Theorem 6.5 we give a sufficient condition that a polyhedron $|K|$ which represents the extension dimension of a Hausdorff compactum X relative to $(\mathscr{K}, \mathscr{T})$, contains a countable subcomplex M such that $|M|$ represents the extension dimension of X relative to $(\mathscr{K}, \mathscr{T})$.

2. dd- and ddP-spaces.

Recall [**6**] that a space X is called a dd-space if $X\tau K$ for every contractible CW-complex K. Such spaces, and the larger class of ddP-spaces (Definition 2.1), will play a prominent role in the sequel. Sometimes we wish to consider only polyhedra instead of arbitrary CW-complexes; therefore we make the next definition.

Definition 2.1. A space X will be called a ddP-space if $X\tau P$ for every contractible polyhedron P.

A check of the proof of the "wedge" theorem, Theorem 2.6 of [**6**], shows that it can be generalized as follows.

Theorem 2.2. *Let X be a* ddP*-space and $\{K_\alpha \mid \alpha \in \Gamma\}$ be a collection of nonempty simplicial complexes. Put $K = \bigvee_v \{K_\alpha \mid \alpha \in \Gamma\}$, where say v is a vertex common to K_α for all $\alpha \in \Gamma$. Suppose that for each $\alpha \in \Gamma$, $X\tau|K_\alpha|$. Then $X\tau|K|$. Conversely, for any space X, if $X\tau|K|$, then $X\tau|K_\alpha|$ for all $\alpha \in \Gamma$.*

Lemma 2.3. *Let X be a space.*

(1) *If X is a* dd-*space, then it is a* ddP-*space.*
(2) *If X is a* ddP-*space, then it is normal.*
(3) *If X has the homotopy extension property with respect to* CW-*complexes, then X is a* dd-*space and for every* CW-*complex K, $K \in \mathrm{ANE}(X)$.*
(4) *If X has the homotopy extension property with respect to polyhedra, then X is a* ddP-*space and for every simplicial complex K, $|K| \in \mathrm{ANE}(X)$.*

PROOF. We leave (1) and (2) to the reader. We shall prove only (3), since a proof of (4) is similar. Let K be a contractible CW-complex, A a closed subset of X, and $f : A \to K$ a map. Then f is null homotopic, so it is homotopic to a map that extends to a map of X to K. The homotopy extension property shows that f extends to a map of X to K, so $X\tau K$.

For the second part, let K be an arbitrary CW-complex, A closed in X, and $f : A \to K$ a map. The cone on K, say $v * K$, is a contractible CW-complex and we treat $K \subset v * K$ canonically. So $f : A \to K \subset v * K$ extends to a map $G : X \to v * K$. Put $U = (v * K) \setminus \{v\}$ and $r : U \to K$ the obvious retraction. Then $G^{-1}(U)$ is a neighborhood of A in X. Define $h = G|G^{-1}(U) : G^{-1}(U) \to U$. Then $r \circ h : G^{-1}(U) \to K$ is a map that extends f. □

LEMMA 2.4. *Let K be a* CW-*complex. Suppose that X is a normal space and $K \in \mathrm{ANE}(X)$. Assume that $X = \bigcup\{X_n \mid n \in \boldsymbol{N}\}$ where for each $n \in \boldsymbol{N}$, X_n is closed and $X_n\tau K$. Then $X\tau K$.*

PROOF. Let A be a closed subspace of X and $f : A \to K$ a map. We shall proceed with an induction argument.

Since $X_1\tau K$, then we may choose a map $g_1 : A \cup X_1 \to K$ such that $g_1|A = f$. Using the ANE property of K and the fact that X is normal, there exists a closed neighborhood D_1 of $A \cup X_1$ in X and a map $h_1 : D_1 \to K$ that extends g_1.

Suppose that $k \in \boldsymbol{N}$ and we have found $D_1, \dots, D_k$, and $h_1, \dots, h_k$ such that for $1 \le i \le k$:

(a) D_i is a closed neighborhood of $A \cup X_i$ in X,
(b) h_i is a map of D_i to K,
(c) $h_i|A = f$, and
(d) if $1 \le i < j \le k$, then $D_i \subset D_j$ and $h_j|D_i = h_i$.

Choose a map $g_{k+1} : D_k \cup X_{k+1} \to K$ such that $g_{k+1}|D_k = h_k$. There exists a closed neighborhood D_{k+1} of $D_k \cup X_{k+1}$ in X and a map $h_{k+1} : D_{k+1} \to K$ that extends g_{k+1}.

This completes the induction. Observe that $\bigcup\{\operatorname{int} D_k \mid k \in \boldsymbol{N}\} = X$. Define a function $F : X \to K$ to be $\bigcup\{h_k \mid k \in \boldsymbol{N}\}$. Clearly F is a map, and $F|A = f$. □

LEMMA 2.5. *Suppose that X is a normal space and every contractible* CW*-complex is an* ANE *for X. Assume that X is the union of a countable collection $\{X_n \mid n \in \boldsymbol{N}\}$ of closed subspaces each of which is a* dd*-space. Then X is a* dd*-space.*

PROOF. Let K be a contractible CW-complex. Then by the definition of a dd-space, for each $n \in \boldsymbol{N}$, $X_n \tau K$. An application of Lemma 2.4 shows that $X \tau K$. □

One similarly has,

LEMMA 2.6. *Suppose that X is a normal space and every contractible polyhedron is an* ANE *for X. Assume that X is the union of a countable collection $\{X_n \mid n \in \boldsymbol{N}\}$ of closed subspaces each of which is a* ddP*-space. Then X is a* ddP*-space.*

When a cover such as that in Lemma 2.4 is not countable, the situation is quite different. In [**11**], K. Morita defined *the weak topology with respect to* a collection of subsets of a given space. This terminology conflicts with current usage, so let us make the following definition.

DEFINITION 2.7. Let X be a space and $\mathscr{F}$ a collection of closed subspaces of X. Then we shall say that X satisfies the Morita conditions with respect to $\mathscr{F}$ if for each $\mathscr{G} \subset \mathscr{F}$

(1) $\bigcup \mathscr{G}$ is closed in X, and
(2) as a subspace of X, $\bigcup \mathscr{G}$ has the weak topology with respect to $\mathscr{G}$.

By virtue of the proof of Theorem 2 of [**11**], we have the next fact.

THEOREM 2.8. *Let K be a* CW*-complex, X a space, and $\mathscr{F}$ a closed cover of X so that X satisfies the Morita conditions with respect to $\mathscr{F}$. If for all $F \in \mathscr{F}$, $F \tau K$, then $X \tau K$.*

Applying the definition of a dd-space, or that of a ddP-space, and Theorem 2.8, we obtain:

COROLLARY 2.9. *Let X be a space and $\mathscr{F}$ a closed cover of X so that X satisfies the Morita conditions with respect to $\mathscr{F}$. Suppose that for all $F \in \mathscr{F}$, F is a* dd*-space. Then X is a* dd*-space. The same is true with* dd *replaced by* ddP.

The next result is the same as Theorem 2.2 of [**8**].

COROLLARY 2.10. *Let K be a* CW*-complex and X a paracompactum that is a local absolute co-extensor for K. Then $X\tau K$.*

If we apply the definition of a dd-space, or that of a ddP-space, and Corollary 2.10, we obtain:

COROLLARY 2.11. *Let X be a paracompactum that is a local* dd*-space. Then X is a* dd*-space. The same is true with* dd *replaced by* ddP.

3. Extension theorems.

This section contains several theorems providing conditions for $X\tau|K|$ when X is a space and K is a simplicial complex.

Let us first state a version of Lemma 3.4 of [**6**] which is true simply by requiring $\mathscr{U}$ to be a cover of X, not necessarily open.

LEMMA 3.1. *Let $\mathscr{U} = \{U_G \mid G \in \Gamma\}$ be an indexed cover of a topological space X and $\mathscr{B}$ be a locally finite cover of X by nonempty closed subsets of X such that for each $B \in \mathscr{B}$, $B \cap U_G \neq \emptyset$ for at most finitely many $G \in \Gamma$. For each finite subset T of Γ, let $B_T = \bigcup\{B \in \mathscr{B} \mid B \cap U_G \neq \emptyset \iff G \in T\}$. Then,*

(1) *for each finite subset T of Γ, B_T is closed, and $B_T \subset \bigcup\{U_G \mid G \in T\}$,*
(2) *if G_1, $G_2 \in \Gamma$ where $G_1 \neq G_2$, then $B_{\{G_1\}} \cap B_{\{G_2\}} = \emptyset$,*
(3) *if T_1, T_2 are finite subsets of Γ, $G \in \Gamma$, and $(B_{T_1} \cap B_{T_2}) \cap U_G \neq \emptyset$, then $G \in T_1 \cap T_2$, i.e., $B_{T_1} \cap B_{T_2} \subset \bigcup\{U_G \mid G \in T_1 \cap T_2\}$, and*
(4) $\{B_T \mid T \text{ a finite subset of } \Gamma\}$ *is a locally finite closed cover of X.*

THEOREM 3.2. *Let K be a simplicial complex and X a normal space such that $|K| \in \mathrm{ANE}(X)$. Suppose that Γ is a collection of subcomplexes of K such that*:

(1) *Γ is directed by inclusion*;
(2) $\bigcap\{|G| \mid G \in \Gamma\} \neq \emptyset$;
(3) *$X\tau|G|$ for all $G \in \Gamma$*;
(4) *and* $|K| = \bigcup\{\operatorname{int}|G| \mid G \in \Gamma\}$.

Then $X\tau|K|$.

PROOF. Let A be a closed subspace of X and $g : A \to |K|$ a map. We may as well assume that g is defined on an open neighborhood W of A. Now we choose a neighborhood U of A such that $\overline{U} \subset W$. We are going to show that there exists a map $F : \overline{U} \to |K|$ having the property that $F|A = g$ and $F|\operatorname{bd} U$ is constant. Our proof will be completed by extending F to the complement of U by that constant map.

Put $f = g|\overline{U} : \overline{U} \to |K|$. Select $v \in \bigcap\{|G| \mid G \in \Gamma\}$. Let $\widetilde{A} = A \cup \operatorname{bd} U$. Define $\tilde{f} : \widetilde{A} \to |K|$ by setting $\tilde{f}|A = f$ and putting $\tilde{f}(d) = v$ for all $d \in \operatorname{bd} U$.

Choose a locally finite cover $\mathscr{E} = \{E_G \mid G \in \Gamma\}$ of $|K|$ so that $E_G \subset \operatorname{int}|G|$ for all $G \in \Gamma$. For each $G \in \Gamma$ define, $U_G = f^{-1}(E_G)$. Then $\mathscr{U} = \{U_G \mid G \in \Gamma\}$ is a cover of $\overline{U}$.

Let $\mathscr{Q}$ be a locally finite closed cover of $|K|$ such that for each $Q \in \mathscr{Q}$, $\{G \in \Gamma \mid E_G \cap Q \neq \emptyset\}$ is finite. Then $\mathscr{B} = \{f^{-1}(Q) \mid Q \in \mathscr{Q}\}$ is a locally finite closed cover of $\overline{U}$ such that for each $B \in \mathscr{B}$, $\{G \in \Gamma \mid B \cap U_G \neq \emptyset\}$ is finite.

Thus $\mathscr{U}$ and $\mathscr{B}$ satisfy the hypotheses of Lemma 3.1 in the space $\overline{U}$. For each finite subset T of Γ, let B_T be as in Lemma 3.1. Using (2) and (4) of that lemma, one sees that $\{B_{\{G\}} \mid G \in \Gamma\}$ is a discrete closed collection in $\overline{U}$. Because of Lemma 3.1(1) and the definition of $\tilde{f}$, one has that $\tilde{f}(B_{\{G\}} \cap \widetilde{A}) = \tilde{f}(B_{\{G\}} \cap A) \cup \tilde{f}(B_{\{G\}} \cap \operatorname{bd} U) \subset |G|$.

The remainder of our proof is the same as the part of the proof of Lemma 3.5 of [**6**] beginning with the sentence just before (8). One only replaces A there by $\widetilde{A}$ and f by $\tilde{f}$. □

COROLLARY 3.3. *Let K be a simplicial complex and X a space having the homotopy extension property with respect to polyhedra. Suppose that Γ is a collection of subcomplexes of K such that*:

(1) Γ *is directed by inclusion*;
(2) $\bigcap\{|G| \mid G \in \Gamma\} \neq \emptyset$;
(3) $X\tau|G|$ *for all* $G \in \Gamma$;
(4) *and* $|K| = \bigcup\{|G| \mid G \in \Gamma\}$.

Then $X\tau|K|$.

PROOF. Since X has the homotopy extension property with respect to polyhedra, then Lemma 2.3(3) shows that X is a ddP-space and $|K| \in \mathrm{ANE}(X)$. By this and Lemma 2.3(1,2), X is normal.

Let K'' be the second barycentric subdivision of K. For each $G \in \Gamma$, let G^* denote the simplicial neighborhood of G with respect to K''. This means that G^* is the subcomplex of K'' consisting of all simplexes σ such that $\sigma \subset |G|$ or σ is a face of a simplex τ of K'' with $\tau \cap |G| \neq \emptyset$. It is well known that $|G|$ is a strong deformation retract of $|G^*|$, so since X has the homotopy extension property with respect to G^*, then $X\tau|G^*|$. Now just replace Γ by $\Gamma^* = \{G^* \mid G \in \Gamma\}$ and apply Theorem 3.2. □

The next result is a slight variation of Theorem 3.3 of [**6**]. The reader will easily see that if one fixes in advance a vertex v of K, then one can replace L by

$L \cup \{v\}$ (see also the proof of Theorem 4.12) in the proof to obtain statement (5) below, so we will not repeat that proof.

For a simplicial complex K and cardinal number α, let

$$K_{\leq \alpha} = \{L \mid L \text{ is a subcomplex of } K \text{ and } \operatorname{card} L \leq \alpha\}.$$

THEOREM 3.4. *Let α be an infinite cardinal, X a space such that $\operatorname{wt} X \leq \alpha$, and K a simplicial complex with $X\tau|K|$. Then there exists a collection $\mathscr{F} = \{F_T \mid T \in K_{\leq \alpha}\}$ of subcomplexes of K so that for each $T \in K_{\leq \alpha}$:*

(1) $T \subset F_T$;
(2) $X\tau|F_T|$;
(3) $\operatorname{card} F_T \leq 2^\alpha$;
(4) *for each $T_0 \in K_{\leq \alpha}$ with $T \subset T_0$, $F_T \subset F_{T_0}$; and*
(5) $\bigcap\{F_T \mid T \in K_{\leq \alpha}\} \neq \emptyset$.

4. Extension properties of pseudo-compact spaces.

Recall that a space X is called *pseudo-compact* if for every map $f : X \to \boldsymbol{R}$, $f(X)$ is contained in a compact subset of $\boldsymbol{R}$. The prototypical pseudo-compact non compact space is the first uncountable ordinal space, $[0, \Omega)$ with the order topology. It is known that $[0, \Omega)$ is binormal, i.e., $[0, \Omega) \times I$ is normal. In this section we prove some extension-theoretic properties of pseudo-compact spaces, which in the realm of extension theory, behave in many ways exactly like compact spaces.

LEMMA 4.1. *Let X be a space. Then X is pseudo-compact if and only if for each* CW*-complex K and map $f : X \to K$, $f(X)$ is contained in a compact subset of K.*

PROOF. Suppose that X is pseudo-compact, K is a CW-complex, $f : X \to K$ is a map, and $f(X)$ is not contained in a compact subset of K. Then there exists a countably infinite closed discrete subspace A of K such that $A \subset f(X)$. Let $g : A \to \boldsymbol{R}$ be a function such that $g(A) = \boldsymbol{N}$. Then g is a map, and since K is normal, there exists a map $h : K \to \boldsymbol{R}$ such that $h|A = g$.

Define $F = h \circ f : X \to \boldsymbol{R}$. Then F is a map of X to $\boldsymbol{R}$. But $\boldsymbol{N} \subset F(X)$, which implies that X is not pseudo-compact, a contradiction.

The converse follows from the fact that $\boldsymbol{R}$ may be given the structure of a CW-complex. □

LEMMA 4.2. *Let X be a normal pseudo-compact space and A a closed subset of X. Then A is pseudo-compact.*

LEMMA 4.3. *Let X be a normal pseudo-compact space and K a* CW*-complex. Then $K \in \mathrm{ANE}(X)$.*

PROOF. Let $A \subset X$ be closed and $f : A \to K$ a map. By Lemma 4.2, A is pseudo-compact. So by Lemma 4.1, there exists a finite subcomplex L of K with $f(A) \subset L$. Since X is normal and L is finite, then $L \in \mathrm{ANE}(X)$. So we may extend the map f to a map of a neighborhood of A in X with values in $L \subset K$. $\square$

PROPOSITION 4.4. *Let X be a binormal pseudo-compact space. Then for each* CW*-complex K, X has the homotopy extension property with respect to K.*

PROOF. Let $A \subset X$ be closed, $h : X \times \{0\} \to K$, and $H : A \times I \to K$ be maps, and suppose that $H(a,0) = h(a,0)$ for all $a \in A$. Put $D = (A \times I) \cup (X \times \{0\})$ and $F = H \cup h : D \to K$. Now $X \times I$ is normal and pseudo-compact. By Lemma 4.2, its closed subspace D is pseudo-compact. So there exists a finite subcomplex L of K with $F(D) \subset L$. Since $X \times I$ is normal, then $L \in \mathrm{ANE}(X \times I)$. Hence there exists a neighborhood W of D in $X \times I$ and a map $F^* : W \to L \subset K$ that extends F. We leave it to the reader to apply from this the standard argument that F extends to a map of $X \times I$ to K. $\square$

Let us recall Definition 4.2 of [**9**].

DEFINITION 4.5. A space X is called a Hausdorff σ-compactum if X is a normal Hausdorff space, every CW-complex is an absolute neighborhood extensor for X, and X can be written as a countable union of compact Hausdorff subspaces.

The next definition extends Definition 4.5.

DEFINITION 4.6. A space X will be called a σ-pseudo-compactum if it is a normal Hausdorff space, every CW-complex is an absolute neighborhood extensor for X, and X can be written as a countable union of closed subspaces each of which is a pseudo-compactum.

LEMMA 4.7. *Every Hausdorff σ-compactum is a σ-pseudo-compactum.*

LEMMA 4.8. *Let X be a σ-pseudo-compactum, K a* CW*-complex, and $f : X \to K$ a map. Then $f(X)$ is contained in a countable subcomplex of K.*

Proposition 4.3 of [**9**] goes through with σ-pseudo-compacta in place of Hausdorff σ-compacta. Therefore, Corollary 4.5 of [**9**] can be stated as follows.

PROPOSITION 4.9. *Let K be a simplicial complex, X a σ-pseudo-compactum, and $X\tau|K|$. Then for every subcomplex L of K, $X\tau|\Psi^\infty(L)|$.*

The Ψ^∞ in Proposition 4.9 is an operator on the subcomplexes of K. The important properties of this operator for us come from its definition and from Lemma 3.2(2,3) of [**9**]. Let us state these here.

LEMMA 4.10. *Let K be a simplicial complex and $L \subset L_0$ be subcomplexes of K. Then,*

(1) $L \subset \Psi^\infty(L)$,
(2) *if L is countable, then $\Psi^\infty(L)$ is countable, and*
(3) $\Psi^\infty(L) \subset \Psi^\infty(L_0)$.

Let us repeat Lemma 3.1 of [**6**].

LEMMA 4.11. *Let X be a space of weight $\leq \alpha$ for some infinite cardinal α. Suppose that $f : X \to |K|$ is a map where K is a simplicial complex. Then $f(X) \subset |L|$ for some subcomplex L of K with* $\operatorname{card} L \leq \alpha$.

The next result is similar to Theorem 3.3 of [**6**] (see also Theorem 3.4 above).

THEOREM 4.12. *Let X be a σ-pseudo-compactum and K a nonempty simplicial complex with $X\tau|K|$. Then there exists a collection $\mathscr{F} = \{F_T \mid T \in K_{\leq \aleph_0}\}$ of subcomplexes of K so that for each $T \in K_{\leq \aleph_0}$,*

(1) $T \subset F_T$;
(2) $X\tau|F_T|$;
(3) $\operatorname{card} F_T \leq \aleph_0$;
(4) *for each $T_0 \in K_{\leq \aleph_0}$ with $T \subset T_0$, $F_T \subset F_{T_0}$; and*
(5) $\bigcap\{F_T \mid T \in K_{\leq \aleph_0}\} \neq \emptyset$.

PROOF. Let v be a vertex of K. For each $L \in K_{\leq \aleph_0}$, let $F_L = \Psi^\infty(L \cup \{v\})$. Now just apply Proposition 4.9 and Lemma 4.10. $\square$

5. Extension Dimension.

Let $\mathscr{C}$ be a class of spaces, $\mathscr{T}$ a class of CW-complexes, and $K, K' \in \mathscr{T}$. If it is true that for all $X \in \mathscr{C}$, $X\tau K$ implies $X\tau K'$, then we write $K \leq_{(\mathscr{C},\mathscr{T})} K'$, (see [**4**]). This defines a preorder on $\mathscr{T}$. One specifies $K \sim_{(\mathscr{C},\mathscr{T})} K'$ if and only if $K \leq_{(\mathscr{C},\mathscr{T})} K'$ and $K' \leq_{(\mathscr{C},\mathscr{T})} K$; then $\sim_{(\mathscr{C},\mathscr{T})}$ is an equivalence relation on $\mathscr{T}$. The equivalence class of K under this relation is called the *extension type* of K relative to $(\mathscr{C},\mathscr{T})$. By $\mathrm{ET}_{(\mathscr{C},\mathscr{T})}$ we mean the class of extension types relative to $(\mathscr{C},\mathscr{T})$. The relation $\leq_{(\mathscr{C},\mathscr{T})}$ induces a partial order, also denoted $\leq_{(\mathscr{C},\mathscr{T})}$, on the extension types $\mathrm{ET}_{(\mathscr{C},\mathscr{T})}$. Let $D \in \mathrm{ET}_{(\mathscr{C},\mathscr{T})}$ and $X \in \mathscr{C}$. In [**6**] we have the notion that $X\tau D$ when $X \in \mathscr{C}$, and this means that $X\tau L$ for all $L \in D$. If $X \in \mathscr{C}$, then the *extension* *dimension* relative to $(\mathscr{C}, \mathscr{T})$ of X, $\operatorname{extdim}_{(\mathscr{C},\mathscr{T})} X$, is the initial element[1], if it exists, of the following class of extension types:

$$\{D \in \mathrm{ET}_{(\mathscr{C},\mathscr{T})} | X\tau D\}.$$

If the space X falls out of the class $\mathscr{C}$, then we are going to propose two definitions of its extension dimension relative to $(\mathscr{C}, \mathscr{T})$.

DEFINITION 5.1. Let $\mathscr{C}$ be a class of spaces, $\mathscr{T}$ a class of CW-complexes, X a space, and $D \in \mathrm{ET}_{(\mathscr{C},\mathscr{T})}$. Then denote $X\tau_w D$ to mean that for some $L \in D$, $X\tau L$, and $X\tau D$ to mean that for all $L \in D$, $X\tau L$.

DEFINITION 5.2. Let $\mathscr{C}$ be a class of spaces, $\mathscr{T}$ a class of CW-complexes, and X a space. Define $\mathscr{D}(X) = \{D \in \mathrm{ET}_{(\mathscr{C},\mathscr{T})} \mid X\tau D\}$ and $\mathscr{D}_w(X) = \{D \in \mathrm{ET}_{(\mathscr{C},\mathscr{T})} \mid X\tau_w D\}$.

If there is an initial element $P \in \mathscr{D}(X)$, then P is called the extension dimension of X relative to $(\mathscr{C}, \mathscr{T})$, $\operatorname{extdim}_{(\mathscr{C},\mathscr{T})} X$. If there is an initial element $P \in \mathscr{D}_w(X)$, then P is called the weak extension dimension of X relative to $(\mathscr{C}, \mathscr{T})$, $\operatorname{wextdim}_{(\mathscr{C},\mathscr{T})} X$.

REMARK 5.3. If $X \in \mathscr{C}$, then $\mathscr{D}(X) = \mathscr{D}_w(X)$. So if $X \in \mathscr{C}$ and one of the extension dimensions in Definition 5.2 exists, then so does the other and they are the same.

Let us repeat Lemma 3.1 of [**6**].

LEMMA 5.4. *Let X be a space of weight $\leq \alpha$ for some infinite cardinal α. Suppose that $f : X \to |K|$ is a map where K is a simplicial complex. Then $f(X) \subset |L|$ for some subcomplex L of K with* $\operatorname{card} L \leq \alpha$.

Next is our Main Theorem on the existence of extension dimension.

THEOREM 5.5. *Suppose that X is a space.*

(1) *Let $\mathscr{S}$ be the class of polyhedra, α an infinite cardinal, and $\mathscr{C}$ a class of spaces of* $\mathrm{wt} \leq \alpha$. *If X is a* ddP*-space, then* $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X$ *exists.*
(2) *Let $\mathscr{S}$ be the class of polyhedra and $\mathscr{C}$ a class of spaces each having the homotopy extension property with respect to $\mathscr{S}$. If X is a* ddP*-space, then* $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X$ *exists.*
(3) *Let $\mathscr{S}$ be the class of* CW*-complexes, α an infinite cardinal, and $\mathscr{C}$ a class*

[1] By an *initial element* of S, we mean $s_0 \in S$ having the property that $s_0 \leq_{(\mathscr{C},\mathscr{T})} s$ for all $s \in S$. If such s_0 exists, it is unique.

of spaces of wt $\leq \alpha$ *each having the homotopy extension property with respect to* $\mathscr{S}$*. If* X *is a* ddP*-space, then* $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X$ *exists.*

(4) *Let* $\mathscr{S}$ *be the class of polyhedra and* $\mathscr{C}$ *a class of* σ*-pseudo-compacta. Suppose that* X *is a* ddP*-space and a* σ*-pseudo-compactum. Then* $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X$ *exists.*

(5) *Let* $\mathscr{S}$ *be the class of* CW*-complexes and* $\mathscr{C}$ *a class of* σ*-pseudo-compacta each having the homotopy extension property with respect to* $\mathscr{S}$*. Suppose that* X *is a* ddP*-space and a* σ*-pseudo-compactum. Then* $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X$ *exists.*

Indeed, in (1)–(3) *we may represent* $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X$ *by a wedge of at most* 2^ρ *polyhedra each having triangulation with at most* $\rho = 2^\beta$ *elements where in cases* (1) *and* (3) $\beta = \max\{\alpha, \mathrm{wt} X\}$ *and in case* (2) $\beta = \max\{\aleph_0, \mathrm{wt}\, X\}$*. In cases* (4), (5) *we may represent* $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X$ *by a wedge of at most* $2^{\aleph_0}$ *polyhedra each having triangulation with at most* $\aleph_0$ *elements.*

Proof. We need to prepare some notation. Let β be an infinite cardinal and denote $\rho = 2^\beta$. Choose a collection $\mathscr{U}$ of triangulated polyhedra $|M|$, each M having cardinality $\leq \rho$, so that $\mathscr{U}$ enjoys the property that if L is a simplicial complex with $\operatorname{card} L \leq \rho$, then for some $|M| \in \mathscr{U}$, L is simplicially isomorphic to M, and if $|M|$, $|N| \in \mathscr{U}$ with M simplicially isomorphic to N, then $M = N$. Then $\operatorname{card} \mathscr{U} \leq 2^\rho$. We may assume that there is a fixed 0-simplex v such that for each $|M| \in \mathscr{U}$, $v \in M$.

For (1) and (3), put $\beta = \max\{\alpha, \mathrm{wt}\, X\}$, for (2), put $\beta = \max\{\aleph_0, \mathrm{wt}\, X\}$, and use ρ, $\mathscr{U}$ as in the preceding paragraph. Let $K = \bigvee_v \{M \mid |M| \in \mathscr{U} \text{ and } X\tau|M|\}$. Since $\operatorname{card} \mathscr{U} \leq 2^\rho$, then the number of summands in K is at most 2^ρ. In both cases X is a ddP-space, so by Theorem 2.2, $X\tau|K|$. We claim that $\operatorname{wextdim}_{(\mathscr{C},\mathscr{S})} X = [|K|]_{(\mathscr{C},\mathscr{S})}$. Let $|L| \in \mathscr{S}$, $X\tau|L|$, and $Y \in \mathscr{C}$. We must show that $Y\tau|K|$ implies that $Y\tau|L|$.

Noting that $\mathrm{wt} X \leq \beta$ and $X\tau|K|$, apply Theorem 3.4 to X, the simplicial complex L, and $\alpha = \beta$. Using (3) of Theorem 3.4, for all $T \in L_{\leq\beta}$, there is an isomorphic copy of F_T in $\mathscr{U}$. Therefore because of (2) of Theorem 3.4, we may as well assume that $|F_T|$ is a summand in $|K|$. By Theorem 2.2, $Y\tau|F_T|$.

In case of (1), let $A \subset Y$ be closed and $f : A \to |L|$ be a map. Since $\mathrm{wt} Y \leq \beta$, then $\mathrm{wt}\, A \leq \beta$, so by Lemma 5.4, there exists $T \in L_{\leq\beta}$ with $f(A) \subset |T|$. By (1) of Theorem 3.4, $|T| \subset |F_T|$. So there exists a map $F : Y \to |F_T| \subset |L|$ such that $F|A = f$.

Now to prove (2). Put $\Gamma = L_{\leq\beta}$. Then from Theorem 3.4 we see that for L and Y, the hypotheses of Corollary 3.3 have been satisfied, so $Y\tau|L|$.

For (3), recall that if B, $C \in \mathscr{S}$ are homotopy equivalent and Z is a space having the homotopy extension property with respect to $\mathscr{S}$, then $Z\tau B$ if and only if $Z\tau C$. In the proof of (2) where $|L|$ arises, in the setting of (3) one would have an arbitrary CW-complex B. But Y has the homotopy extension property with respect to $\mathscr{S}$, so we may replace B by a polyhedron and proceed with the rest of the proof of (2).

For (4) or (5), construct $\mathscr{U}$ with $\rho = \aleph_0$, i.e., ignore β. Then card $\mathscr{U} \leq 2^{\aleph_0}$, so the number of summands in K is at most $2^{\aleph_0}$, and each summand has cardinality at most $\aleph_0$. The proof of (4) goes as above for (1); this time we do not need any information about $\operatorname{wt} X$ because we may use Theorem 4.12 in place of Theorem 3.4. The proof of (5) just employs the notions we used in (3). □

This theorem and Remark 5.3 show the following.

COROLLARY 5.6. *If in any part of Theorem* 5.5 *the space X lies in the class $\mathscr{C}$, then* wextdim *may be replaced by* extdim.

With the help of Corollary 5.6, part (5) (as noted in Section 1) generalizes Theorem 13 of [**3**]. Since stratifiable spaces have the homotopy extension property with respect to CW-complexes, then part (4) includes Theorem 4.4 of [**6**].

6. Countable representatives.

In this section $\mathscr{K}$ denotes the class of Hausdorff compacta and $\mathscr{T}$ the class of CW-complexes.

PROBLEM 6.1. Determine whether for each compact metrizable space X, there is a countable CW-complex M such that $\operatorname{extdim}_{(\mathscr{K},\mathscr{T})} X = [M]_{(\mathscr{K},\mathscr{T})}$.

We give a partial affirmative solution to this problem in Theorem 6.5 (see also the remarks after Proposition 6.2). The next fact is immediate from Corollary 1.3 of [**7**].

PROPOSITION 6.2. *Let K be a countable* CW-*complex and α an infinite ordinal. Suppose that X is a compact Hausdorff space with* $\operatorname{wt} X \leq \alpha$ *having the property that $X\tau K$ and each compact Hausdorff space Y with $Y\tau K$ and* $\operatorname{wt} Y \leq \alpha$ *embeds in X. Then* $\operatorname{extdim}_{(\mathscr{K},\mathscr{T})} X = [K]_{(\mathscr{K},\mathscr{T})}$.

This provides many examples of compact Hausdorff spaces with "countable" extension dimension, since by Corollary 1.9 of [**10**], every finite CW-complex K admits a universal Hausdorff compactum X of a given weight, i.e., X meets the requirements set forth in Proposition 6.2.

Now we state Definition 5.8 of [**9**].

Definition 6.3. Let $\mathscr{K}^*$ be a class of spaces, K be a simplicial complex, and $\mathscr{F}$ a collection of subcomplexes of K having the property that whenever $Y \in \mathscr{K}^*$ and $|K|$ is not an absolute extensor for Y, then there exist a closed subspace A of Y, $F \in \mathscr{F}$, and map $f : A \to |F|$ that does not extend to a map of Y into $|K|$. Then we shall call $\mathscr{F}$ an anti-basis for K relative to $\mathscr{K}^*$.

Next is a slight variation of Theorem 5.10 of [**9**]. The addition that $M \subset K$ comes from the proof given there.

Theorem 6.4. *Let $\mathscr{K}^*$ be a class of Hausdorff σ-compacta, $X \in \mathscr{K}^*$, and K a simplicial complex. Suppose that* $\operatorname{extdim}_{(\mathscr{K}^*,\mathscr{T})} X$ *exists and equals* $[|K|]_{(\mathscr{K}^*,\mathscr{T})}$. *If K has a countable anti-basis $\mathscr{F}$ relative to $\mathscr{K}^*$ such that $\mathscr{F}$ consists of finite subcomplexes of K, then there is a countable subcomplex M of K such that* $\operatorname{extdim}_{(\mathscr{K}^*,\mathscr{T})} X = [|M|]_{(\mathscr{K}^*,\mathscr{T})}$.

Let $\mathscr{G}$ be a collection of finite simplicial complexes having the property that:

(1) if G_0 is a finite simplicial complex, then there exists $G \in \mathscr{G}$ and a simplicial isomorphism from G to G_0, and

(2) if G, $G' \in \mathscr{G}$ where G is simplicially isomorphic to G', then $G = G'$.

Then $\mathscr{G}$ is a countably infinite set. Let K be a simplicial complex. For each $G \in \mathscr{G}$, let $\mathscr{M}_G$ be the set of maps of $|G|$ to $|K|$ that are induced by simplicial injections of G to K. Define $\mathscr{M}_{G,\simeq}$ to be the set of $[h] \in [|G|, |K|]$ as h varies in $\mathscr{M}_G$.

Theorem 6.5. *Let $\mathscr{K}^*$ be a subclass of $\mathscr{K}$, $X \in \mathscr{K}^*$, K a simplicial complex, and* $[|K|]_{(\mathscr{K}^*,\mathscr{T})} = \operatorname{extdim}_{(\mathscr{K}^*,\mathscr{T})} X$. *Suppose that for all $G \in \mathscr{G}$, $\mathscr{M}_{G,\simeq}$ is countable. Then K contains a countable subcomplex M so that* $[|M|]_{(\mathscr{K}^*,\mathscr{T})} = [|K|]_{(\mathscr{K}^*,\mathscr{T})}$.

Proof. We will show that there is a countable set $\mathscr{F}$ of finite subcomplexes of K such that $\mathscr{F}$ is an anti-basis for K relative to $\mathscr{K}^*$. Then Theorem 6.4 will yield our result.

For each $G \in \mathscr{G}$, select a countable set R_G consisting of one element from each class in $\mathscr{M}_{G,\simeq}$. For each $[g] \in R_G$ let $g^s : G \to K$ be a simplicial injection so that $g^s \in [g]$. Define $L_G = \{g^s(G) \mid g \in R_G\}$. Since R_G is countable, then L_G is a countable collection of finite subcomplexes of K. Hence $\mathscr{F} = \bigcup\{L_G \mid G \in \mathscr{G}\}$ is a countable collection of finite subcomplexes of K. We shall show that $\mathscr{F}$ is as stated above.

Let $Y \in \mathscr{K}^*$ and suppose that $Y\tau|K|$ is false. Choose a closed subset A of Y and a map $f : A \to |K|$ that does not extend to a map of Y to $|K|$. There exist a finite subcomplex H of K such that $f(A) \subset |H|$, $G \in \mathscr{G}$, and a simplicial isomorphism $\sigma : G \to H$. Let $j : H \to K$ be the inclusion. Then $g_0 = j \circ \sigma : G \to$

K is a simplicial injection. Above we have chosen a simplicial injection $g_0^s : G \to K$ so that $j \circ \sigma \simeq g_0^s$. Note that $g_0^s(G) \in L_G$. Also note that since σ is a homeomorphism, then $g_0^s \circ \sigma^{-1} \simeq j$. Thus $g_0^s \circ \sigma^{-1} \circ f \simeq j \circ f$ as maps of A to $|K|$. Since $j \circ f$ does not extend to a map of Y to $|K|$, then the homotopy extension property implies that $g_0^s \circ \sigma^{-1} \circ f$ does not extend to a map of Y to $|K|$. We finally observe that $g_0^s \circ \sigma^{-1} \circ f(A) \subset |g_0^s(G)|$ and $g_0^s(G) \in L_G \subset \mathscr{F}$. $\square$

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