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On the L_p analytic semigroup associated with the linear thermoelastic plate equations in the half-space

By Yuka Naito and Yoshihiro Shibata

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Abstract. The paper is concerned with linear thermoelastic plate equations in the half-space $\boldsymbol{R}^n_+ = \{x = (x_1, \ldots, x_n) \mid x_n > 0\}$:

$$u_{tt} + \Delta^2 u + \Delta\theta = 0 \text{ and } \theta_t - \Delta\theta - \Delta u_t = 0 \quad \text{in } \boldsymbol{R}^n_+ \times (0, \infty),$$

subject to the boundary condition: $u|_{x_n=0} = D_n u|_{x_n=0} = \theta|_{x_n=0} = 0$ and initial condition: $(u, D_t u, \theta)|_{t=0} = (u_0, v_0, \theta_0) \in \mathscr{H}_p = W^2_{p,D} \times L_p \times L_p$, where $W^2_{p,D} = \{u \in W^2_p \mid u|_{x_n=0} = D_n u|_{x_n=0} = 0\}$. We show that for any $p \in (1, \infty)$, the associated semigroup $\{T(t)\}_{t\geq 0}$ is analytic in the underlying space $\mathscr{H}_p$. Moreover, a solution (u, θ) satisfies the estimates:

$$\|\nabla^j(\nabla^2 u(\cdot, t), u_t(\cdot, t), \theta(\cdot, t))\|_{L_q(\boldsymbol{R}^n_+)} \leq C_{p,q} t^{-\frac{j}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla^2 u_0, v_0, \theta_0)\|_{L_p(\boldsymbol{R}^n_+)} \quad (t > 0)$$

for $j = 0, 1, 2$ provided that $1 < p \leq q \leq \infty$ when $j = 0$, 1 and that $1 < p \leq q < \infty$ when $j = 2$, where ∇^j stands for space gradient of order j.

1. Introduction.

In this paper, we shall consider the following equations:

$$u_{tt} + \Delta^2 u + \Delta\theta = 0 \text{ and } \theta_t - \Delta\theta - \Delta u_t = 0 \quad \text{in } \Omega \times \boldsymbol{R}_+. \tag{1.1}$$

These equations describe a linear thermoelastic plate and they are derived in [**6**]. In (1.1), u denotes a mechanical variable denoting the vertical displacement of the plate, while θ denotes a thermal variable describing the temperature relative to a constant reference temperature $\bar{\theta}$. Since the equations (1.1) represent the transfer of the mechanical energy to the thermal energy through coupling, we expect that total energy of the system decays, because of the thermal damping. In fact, when Ω is a bounded reference configuration, the exponential stability of the associated semigroup under several different kind of boundary conditions have been proved by Kim [**7**], Munõz Rivera and Racke [**17**], Liu and Zheng [**15**], Avalos and

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Lasiecka [3], Lasiecka and Triggiani [8], [9], [10], [11] and Shibata [19]. But, more significant aspect that the equations (1.1) have is that the associated semigroup is analytic. Namely, although the first equation in (1.1) is a simply dispersive equation (the product of two Schrödinger equations), the effect from the heat equation through coupling is strong enough to have analyticity of the total system. This fact was first proved by Liu and Renardy [13] and then it has been studied by Liu and Liu [12], Liu and Yong [14] with different dampings in the L_2 framework when Ω is also a bounded reference configuration (see a book due to Liu and Zheng [16] for a survey).

The original equations describing the motion and transfer of the energy of thermo-elastic plate is non-linear and it is widely accepted that the L_p approach is more relevant to handle with the non-linear problem under less regularity assumption on initial data. Therefore, it is worth while studying the equations (1.1) in the L_p settings. In this respect, recently Denk and Racke [5] proved the generation of analytic semigroup and its decay property of the Cauchy problem for equations (1.1) in the L_p framework. In fact, they studied more general system, so called α-β system, consisting of the following equations:

$$u_{tt} + Su - S^\beta\theta = 0 \text{ and } \theta_t + S^\alpha\theta + S^\beta u_t = 0 \quad \text{in } \Omega \times \boldsymbol{R}_+. \tag{1.2}$$

where $S = (-\Delta)^\eta$ ($\eta > 0$) and α, $\beta \in [0,1]$ are parameters. They proved that the region of analyticity is the set $\boldsymbol{U} = \{(\alpha,\beta) \mid \alpha \geq \beta, \alpha \leq 2\beta - (1/2)\}$. In proving resolvent estimates in L_p spaces they used the theory of parameter-elliptic mixed-order systems by Denk, Mennicken and Volevich [4]. Moreover, they proved decay rates for $\|(S^{1/2}u(\cdot,t), u_t(\cdot,t), \theta(\cdot,t))\|_{L_q(\boldsymbol{R}^n)}$ if $2 \leq q \leq \infty$ and (α,β) is the analyticity region $\boldsymbol{U}$, but also if $1/4 \leq \beta \leq 3/4$ while $\alpha = 1/2$ (exemplarily). The equations (1.1) are obtained, setting $\eta = 2$ and $\alpha = \beta = 1/2$, and therefore the semigroup associated with (1.1) is analytic and $\|(\Delta u(\cdot,t), u_t(\cdot,t), \theta(\cdot,t))\|_{L_q(\boldsymbol{R}^n)}$ ($2 \leq q \leq \infty$) decays polynomially.

Before Denk and Racke [5] the α-β system was independently introduced by Ammar Khodja and Benabdallah [2] and Munõz Rivera and Racke [18], and the region of parameters was classified by smoothing property, decay property and analyticity of associated semigroup by [2], [18], Liu and Liu [12] and Liu and Yong [14] in the L_2 or Hilbert space setting.

In this paper, we shall consider the initial boundary value problem for equations (1.1) in the half-space $\boldsymbol{R}^n_+ = \{x = (x_1,\ldots,x_n) \in \boldsymbol{R}^n \mid x_n > 0\}$ subject to the initial condition:

$$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x),\ \theta(x,0) = \theta_0(x), \tag{1.3}$$

and boundary condition:

$$u|_{x_n=0} = D_n u|_{x_n=0} = \theta|_{x_n=0} = 0. \tag{1.4}$$

We shall show that initial boundary value problem (1.1), (1.3) and (1.4) generates an analytic semigroup in the $L_p(\boldsymbol{R}^n_+)$ framework and we shall show its decay property. To state our result precisely, introducing the unknown function $v = u_t$ we rewrite (1.1) in the matrix form:

$$U_t = AU \quad \text{in } \Omega \times \boldsymbol{R}_+,\ U|_{t=0} = U_0 \tag{1.5}$$

where we have set

$$U = \begin{pmatrix} u \\ v \\ \theta \end{pmatrix},\ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ \theta_0 \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{pmatrix}. \tag{1.6}$$

To solve the initial boundary value problem (1.5) with (1.4), we consider the corresponding resolvent problem:

$$(\lambda I - A)U = F \quad \text{in } \boldsymbol{R}^n_+ \tag{1.7}$$

subject to the boundary condition (1.4), where I denotes the $n \times n$ unit matrix. To state our main result concerning the resolvent problem, we introduce several spaces and some symbols. $L_p(\Omega)$ and $W^m_p(\Omega)$ stand for the usual Lebesgue space and Sobolev space, respectively. Let $\|\ \|_{L_p(\Omega)}$ and $\|\ \|_{W^m_p(\Omega)}$ denote their norms. For $1 < p < \infty$ the spaces $W^2_{p,0}(\boldsymbol{R}^n_+)$, $W^2_{p,D}(\boldsymbol{R}^n_+)$ and $W^4_{p,D}(\boldsymbol{R}^n_+)$ are defined by the formulas:

$$\begin{aligned} W^2_{p,0}(\boldsymbol{R}^n_+) &= \{u \in W^2_p(\boldsymbol{R}^n_+) \mid u|_{x_n=0} = 0\}, \\ W^m_{p,D}(\boldsymbol{R}^n_+) &= \{u \in W^m_p(\boldsymbol{R}^n_+) \mid u|_{x_n=0} = D_n u|_{x_n=0} = 0\}\ (m = 2, 4). \end{aligned} \tag{1.8}$$

The space $\mathscr{H}_p(\boldsymbol{R}^n_+)$ for right member F in (1.7) and the space $\mathscr{D}_p(\boldsymbol{R}^n_+)$ for solution U in (1.7) are defined by the formulas:

$$\begin{aligned} \mathscr{H}_p(\boldsymbol{R}^n_+) &= \{F = {}^T(f, g, h) \mid f \in W^2_{p,D}(\boldsymbol{R}^n_+),\ g \in L_p(\boldsymbol{R}^n_+),\ h \in L_p(\boldsymbol{R}^n_+)\}, \\ \mathscr{D}_p(\boldsymbol{R}^n_+) &= \{U = {}^T(u, v, \theta) \mid u \in W^4_{p,D}(\boldsymbol{R}^n_+),\ v \in W^2_{p,D}(\boldsymbol{R}^n_+),\ \theta \in W^2_{p,0}(\boldsymbol{R}^n_+)\}. \end{aligned} \tag{1.9}$$

Here and hereafter, ${}^T M$ denotes the transposed M. For differentiation we use the following symbols:

$$D_t = \partial/\partial t,\ D_j u = \partial u/\partial x_j,\ D_x^\alpha u = D_1^{\alpha_1} \cdots D_n^{\alpha_n} u\ (\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{N}_0^n),$$
$$\nabla^0 u = u, \nabla^1 u = \nabla u = (D_1 u, \dots, D_n u),\ \nabla^j u = (D_x^\alpha u \mid |\alpha| = \alpha_1 + \cdots + \alpha_n = j)\ (j \geq 2),$$

where $\boldsymbol{N}$ denotes the set of all natural numbers, $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$, and $\boldsymbol{N}_0^n = \underbrace{\boldsymbol{N}_0 \times \cdots \times \boldsymbol{N}_0}_{n\text{-times}}$. Then, we have the following theorem.

Theorem 1.1. *Let $1 < p < \infty$ and set $\boldsymbol{C}_+ = \{\lambda \in \boldsymbol{C} \setminus \{0\} \mid \mathrm{Re}\,\lambda \geq 0\}$. Then, for any $\lambda \in \boldsymbol{C}_+$ and $F = {}^T(f, g, h) \in \mathscr{H}_p(\boldsymbol{R}^n)$ resolvent equation* (1.7) *admits a unique solution $U = {}^T(u, v, \theta) \in \mathscr{D}_p(\boldsymbol{R}_+^n)$ which satisfies the estimate*:

$$\begin{aligned}\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|(\nabla^{j+2} u, \nabla^j v, \nabla^j \theta)\|_{L_p(\boldsymbol{R}_+^n)} &\leq C\|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}_+^n)}, \\ \sum_{j=0}^{1} |\lambda|^{\frac{4-j}{2}} \|\nabla^j u\|_{L_p(\boldsymbol{R}_+^n)} &\leq C\|(|\lambda| f, g, h)\|_{L_p(\boldsymbol{R}_+^n)}.\end{aligned} \tag{1.10}$$

Concerning the evolution equation (1.5) with boundary condition (1.4), we introduce the operator $\mathscr{A}_p$ which is defined by the operation:

$$\mathscr{A}_p U = AU \quad \text{for } U \in \mathscr{D}_p(\boldsymbol{R}_+^n). \tag{1.11}$$

Then, we have the following theorem.

Theorem 1.2. *Let $1 < p < \infty$. Let $\rho(\mathscr{A}_p)$ be the resolvent set of $\mathscr{A}_p$. Then, $\boldsymbol{C}_+ \subset \rho(\mathscr{A}_p)$ and $\mathscr{A}_p$ generates an analytic semigroup $\{T_p(t)\}_{t \geq 0}$ on $\mathscr{H}_p(\boldsymbol{R}_+^n)$.*

Remark 1.3. Theorem 1.2 tells us that if we set $U(t) = T_p(t)F$ for $F \in \mathscr{H}_p(\boldsymbol{R}_+^n)$, then $U(t) \in \mathscr{D}_p(\boldsymbol{R}_+^n)$ for $t > 0$, $U(t)$ satisfies the equation (1.5) and attains the initial data in the following way:

$$\lim_{t \to 0+} \|T_p(t)U - F\|_{W_p^2(\boldsymbol{R}_+^n) \times L_p(\boldsymbol{R}_+^n) \times L_p(\boldsymbol{R}_+^n)} = 0.$$

To show some asymptotic behaviour of solutions to (1.5) with boundary condition (1.4), we use the homogeneous space $\dot{W}_{p,D}^m(\boldsymbol{R}_+^n)$ instead of $W_{p,D}^m(\boldsymbol{R}_+^n)$ for $m = 2, 4$, which are defined by the following formulas:

$$\dot W^2_{p,D}(\boldsymbol{R}^n_+) = \{f \in W^2_{p,\mathrm{loc}}(\overline{\boldsymbol{R}^n_+}) \mid D^\alpha_x u \in L_p(\boldsymbol{R}^n_+)\ (|\alpha| = 2),\ u|_{x_n=0} = D_n u|_{x_n=0} = 0\},$$
$$\dot W^4_{p,D}(\boldsymbol{R}^n_+) = \{f \in W^4_{p,\mathrm{loc}}(\overline{\boldsymbol{R}^n_+}) \cap \dot W^2_{p,D}(\boldsymbol{R}^n_+) \mid D^\alpha_x u \in L_p(\boldsymbol{R}^n_+)\ (2 \le |\alpha| \le 4)\}. \tag{1.12}$$

Let $\|u\|_{\dot W^2_{p,D}(\boldsymbol{R}^n_+)}$ denote the seminorm defined by $\|\nabla^2 u\|_{L_p(\boldsymbol{R}^n_+)}$. For $u \in \dot W^2_{p,D}(\boldsymbol{R}^n_+)$, what $\nabla^2 u = 0$ implies that $u = 0$, because $u|_{x_n=0} = D_n u|_{x_n=0} = 0$. Therefore, $\dot W^2_{p,D}(\boldsymbol{R}^n_+)$ is a Banach space equipped with norm $\|\ \|_{\dot W^2_{p,D}(\boldsymbol{R}^n_+)}$. Moreover, $C^\infty_0(\boldsymbol{R}^n_+)$ is dense in $\dot W^2_{p,D}(\boldsymbol{R}^n_+)$. We introduce the spaces $\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$, $\dot{\mathscr{D}}_p(\mathscr{H})$, norm $\|\ \|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}$ and the operator $\dot{\mathscr{A}}_p$ by the formulas:

$$\begin{aligned}
\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+) &= \{F = {}^T(f, g, h) \mid f \in \dot W^2_{p,D}(\boldsymbol{R}^n_+),\ g \in L_p(\boldsymbol{R}^n_+),\ h \in L_p(\boldsymbol{R}^n_+)\},\\
\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} &= \|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)},\\
\dot{\mathscr{D}}_p(\boldsymbol{R}^n_+) &= \{U = {}^T(u, v, \theta) \mid u \in \dot W^4_{p,D}(\boldsymbol{R}^n_+),\ v \in W^2_{p,D}(\boldsymbol{R}^n_+),\ \theta \in W^2_{p,0}(\boldsymbol{R}^n_+)\},\\
\dot{\mathscr{A}}_p U &= AU \quad \text{for } U \in \dot{\mathscr{D}}_p(\boldsymbol{R}^n_+).
\end{aligned} \tag{1.13}$$

Concerning the asymptotic behaviour of solutions to equations (1.5) with boundary condition (1.4), we have the following theorem.

THEOREM 1.4. *Let $1 < p < \infty$. Then, $\dot{\mathscr{A}}_p$ generates an analytic semigroup $\{\dot T(t)\}_{t\ge0}$ on $\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$ which satisfies so called L_p-L_q estimates. Namely, for any q with $p \le q$ and $j = 0, 1, 2$ there exists a constant $C_{p,q}$ such that there hold the estimates*:

$$\|\nabla^j \dot T_p(t) F\|_{\dot{\mathscr{H}}_q(\boldsymbol{R}^n_+)} \le C_{p,q} t^{-\frac{j}{2} - \frac{n}{2}\left(\frac1p - \frac1q\right)} \|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \quad (t > 0,\ F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)) \tag{1.14}$$

provided that $q \le \infty$ when $j = 0$, 1 and $q < \infty$ when $j = 2$.

REMARK 1.5. (1) Since $\mathscr{H}_p(\boldsymbol{R}^n_+) \subset \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$, $T(t)F = \dot T(t)F$ for $F = {}^T(u_0, v_0, \theta_0) \in \mathscr{H}_p(\boldsymbol{R}^n_+)$, and therefore Theorem 1.4 tells us that solution (u, θ) to the initial boundary value problem (1.1) with (1.3) and (1.4) satisfies the estimates:

$$\|\nabla^j (\nabla^2 u(\cdot, t), D_t u(\cdot, t), \theta(\cdot, t))\|_{L_q(\boldsymbol{R}^n_+)} \le C_{p,q} t^{-\frac{j}{2} - \frac{n}{2}\left(\frac1p - \frac1q\right)} \|(\nabla^2 u_0, v_0, \theta_0)\|_{L_p(\boldsymbol{R}^n_+)} \quad (t > 0)$$

provided that $1 < p \leq q \leq \infty$ when $j = 0$, 1 and $1 < p \leq q < \infty$ when $j = 2$.
(2) Our assumptions on the exponents p and q are optimal from a view point of Gagliardo-Nirenberg-Sobolev inquality, while the exponents should satisfy stronger restrictions in Denk and Racke [**5**] like $p \geq 2$ and $1/p + 1/q = 1$.

Remark 1.6. To make our results clear, we consider (1.1) subject to the cramped boundary conditions:

$$u|_{x_n=0} = \Delta u|_{x_n} = \theta|_{x_n=0} = 0. \tag{1.15}$$

As was observed in Liu and Renardy [**13**], introducing the new variables $w = \Delta u$ and $v = u_t$, equations (1.1) are rewritten in the matrix form with new unknown functions: $U = {}^T(w, v, \theta)$ as follows:

$$U_t = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \Delta U, \ U|_{t=0} = (\, \Delta u_0 \quad v_0 \quad \theta_0 \,),$$

subject to the boundary condition: $w|_{x_n=0} = v|_{x_n=0} = \theta|_{x_n=0} = 0$. Since the characteristic equation for the matrix: $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ is $\lambda^3 - \lambda^2 + 2\lambda - 1$, which has three different roots $-\alpha$, $-\beta$ and $-\bar{\beta}$ (cf. Lemma 3.2), there exists a non-singular matrix B such that

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \bar{\beta} \end{pmatrix} = B \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} B^{-1}.$$

Introducing new unknown $V = BU = {}^T(v_1, v_2, v_3)$, finally we have

$$D_t V = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \bar{\beta} \end{pmatrix} \Delta V \ (t > 0), \ V|_{x_n=0} = 0, \ V|_{t=0} = B^T(\Delta u_0, v_0, \theta_0).$$

Namely, we can factorize the system (1.1) with initial condition and boundary condition (1.15), and therefore we have the theorems corresponding to Theorems 1.1 and 1.2 by using known results for the heat equation. But, in case of

(1.4), we do not have any nice transformations unlike the case of (1.15), and therefore we have to treat the essential difficulty arising from the boundry conditions.

The paper is orginized as follows: In section 2, we shall discuss the resolvent problem: $\lambda I - A = F$ in the whole space $\boldsymbol{R}^n$. We give an exact formula of $(\lambda I - A)^{-1}$ by using the Fourier transform and drive the optimal resolvent estimate by applying the Fourier multiplier theorem, although the resolvent estimates were obtained by Denk and Racke [**5**] for the general α-β system given in (1.3) by using the Newton polygon method (cf. also Agranovich and Vishik [**1**] and Denk, Mennicken and Volevich [**4**], Volevich [**28**]). Because, the results obtained in Section 2 are used to derive the representation formula of solutions to equation (1.7) as well as to obtain estimate (1.10). In section 3 we shall derive a solution formula to equation (1.7) and prepare several technical lemmas to estimate solutions. In section 4 we prove Theorem 1.1. In section 5 we prove Theorems 1.2 and 1.4. In section 6, we make a remark about the extension of $\{T_p(t)\}_{t\geq 0}$ to $\dot{\mathscr{H}}_1(\boldsymbol{R}^n_+)$.

2. Analysis in $\boldsymbol{R}^n$.

In this section, we consider the resolvent problem:

$$(\lambda I - A)U = F \quad \text{in } \boldsymbol{R}^n \tag{2.1}$$

and we shall prove the following theorem.

THEOREM 2.1. *Let $1 < p < \infty$. Set*

$$\begin{aligned} \mathscr{H}_p(\boldsymbol{R}^n) &= \{F = {}^T(f,g,h) \mid f \in W_p^2(\boldsymbol{R}^n),\ g \in L_p(\boldsymbol{R}^n),\ h \in L_p(\boldsymbol{R}^n)\}, \\ \mathscr{D}_p(\boldsymbol{R}^n) &= \{U = {}^T(u,v,\theta) \mid u \in W_p^4(\boldsymbol{R}^n),\ v \in W_p^2(\boldsymbol{R}^n),\ \theta \in W_p^2(\boldsymbol{R}^n)\}, \\ \Sigma_\epsilon &= \{\lambda \in \boldsymbol{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon\}. \end{aligned} \tag{2.2}$$

Then, there exists an ϵ $(0 < \epsilon < \pi/2)$ such that for any $\lambda \in \Sigma_\epsilon$ and $F = {}^T(f,g,h) \in \mathscr{H}_p(\boldsymbol{R}^n)$ there exists a $U = {}^T(u,v,\theta) \in \mathscr{W}_p(\boldsymbol{R}^n)$ which solves resolvent problem (2.1) *uniquely and satisfies the estimates*:

$$\begin{aligned} \sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|(\nabla^{j+2}u, \nabla^j v, \nabla^j\theta)\|_{L_p(\boldsymbol{R}^n)} &\leq C\|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}^n)} \\ \sum_{j=0}^{1} |\lambda|^{\frac{4-j}{2}} \|\nabla^j u\|_{L_p(\boldsymbol{R}^n)} &\leq C\|(|\lambda| f, g, h)\|_{L_p(\boldsymbol{R}^n)}. \end{aligned} \tag{2.3}$$

Remark 2.2. ϵ will be given in Lemma 2.4, below.

For $a(x) = a(x_1, \dots, x_n)$ its Fourier transform is defined by the formula:

$$\hat{a}(\xi) = \mathscr{F}[a](\xi) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} a(x)\,dx \quad (\xi = (\xi_1, \dots, \xi_n)).$$

Applying the Fourier transform to (2.1) we have

$$\lambda \hat{U}(\xi) - \hat{A}(\xi)\hat{U}(\xi) = \hat{F}(\xi) \quad \text{in } \boldsymbol{R}^n, \tag{2.4}$$

where we have set

$$\hat{U}(\xi) = \begin{pmatrix} \hat{u}(\xi) \\ \hat{v}(\xi) \\ \hat{\theta}(\xi) \end{pmatrix}, \ \hat{F}(\xi) = \begin{pmatrix} \hat{f}(\xi) \\ \hat{g}(\xi) \\ \hat{\theta}(\xi) \end{pmatrix}, \ \hat{A}(\xi) = \begin{pmatrix} 0 & 1 & 0 \\ -|\xi|^4 & 0 & |\xi|^2 \\ 0 & -|\xi|^2 & -|\xi|^2 \end{pmatrix}.$$

If we write

$$\lambda I - \hat{A}(\xi) = \hat{A}_\lambda(\xi) = \begin{pmatrix} \lambda & -1 & 0 \\ |\xi|^4 & \lambda & -|\xi|^2 \\ 0 & |\xi|^2 & \lambda + |\xi|^2 \end{pmatrix}$$

then (2.4) is written in the form:

$$\hat{A}_\lambda(\xi)\, {}^T(\hat{u}(\xi), \hat{v}(\xi), \hat{\theta}(\xi)) = {}^T(\hat{f}(\xi), \hat{g}(\xi), \hat{h}(\xi)). \tag{2.5}$$

To solve (2.5), we have to investigate some property of the inverse operator $\hat{A}_\lambda(\xi)^{-1}$. For this purpose we consider the determinant of $\hat{A}_\lambda(\xi)$, which is given by the formula:

$$\det \hat{A}_\lambda(\xi) = \lambda^3 + \lambda^2|\xi|^2 + 2\lambda|\xi|^4 + |\xi|^6. \tag{2.6}$$

Lemma 2.3. *Let us define a polynominal $p(t)$ by the formula: $p(t) = t^3 + t^2 + 2t + 1$. Then, there exist a real number α $(0 < \alpha < 1)$ and a complex number β $(\operatorname{Re}\beta = \frac{1-\alpha}{2} > 0)$ such that $p(t) = (t+\alpha)(t+\beta)(t+\bar{\beta})$, where $\bar{\beta}$ denotes the complex conjugate of β. Moreover, we have*

$$\det \hat{A}_\lambda(\xi) = (\lambda + \alpha|\xi|^2)(\lambda + \beta|\xi|^2)(\lambda + \bar{\beta}|\xi|^2) = |\xi|^6 p(\lambda/|\xi|^2).$$

PROOF. In view of (2.6), obviously $\det \hat{A}_\lambda(\xi) = |\xi|^6 p(\lambda/|\xi|^2)$. Concerning the roots of the polynomial $p(t)$, there exists a unique real number α with $0 < \alpha < 1$ such that $p(-\alpha) = 0$, because $p(0) = 1 > 0$, $f(-1) = -1 < 0$ and $p'(t) > 0$ for all $t \in \boldsymbol{R}$. Since p is a polynomial with real coefficients, there exists a complex number β such that $p(t) = (t + \alpha)(t + \beta)(t + \bar{\beta})$. In particular, we have $\alpha + \beta + \bar{\beta} = 1$, which implies that $\operatorname{Re} \beta = (1 - \alpha)/2 > 0$. This completes the proof of the lemma. □

LEMMA 2.4. *Let α and β be the same numbers as in Lemma 2.3. Let θ be the argument of β, that is $\beta = |\beta|e^{i\theta}$. Let ϵ be a small number such that $0 < \epsilon < (\pi/2) - \theta$. Then, we have the estimates*:

$$|\lambda + \kappa|\xi|^2| \geq \sin\frac{\epsilon}{2}\,(|\lambda| + |\kappa||\xi|^2)\ (\kappa = \alpha, \beta, \bar{\beta}) \tag{2.7}$$

for any $\lambda \in \Sigma_\epsilon$, where Σ_ϵ is the same set as in (2.2).

REMARK 2.5. Since $\operatorname{Re}\beta > 0$, we see that $0 < \arg\beta < \pi/2$ or $3\pi/2 < \arg\beta < 2\pi$. We consider β and $\bar{\beta}$ at the same time, so that we may assume that $0 < \arg\beta < \pi/2$ without loss of generality.

PROOF. Set $\beta = |\beta|e^{i\theta}$ and $\lambda = |\lambda|e^{i\tau}$. If $\lambda \in \Sigma_\epsilon$, then $-\pi + \theta + \epsilon < \tau < \pi + \theta - \epsilon$. Observe that

$$\begin{aligned}|\lambda + \beta|\xi|^2|^2 &= |\beta|^2|\lambda\beta^{-1} + |\xi|^2|^2 = |\beta|^2||\lambda||\beta|^{-1}e^{i(\tau-\theta)} + |\xi|^2|^2 \\ &= |\beta|^2[(|\lambda||\beta|^{-1})^2 + |\xi|^4 + 2\cos(\tau - \theta)|\lambda||\beta|^{-1}|\xi|^2]\end{aligned}$$

The condition that $-\pi + \epsilon < \tau - \theta < \pi - 2\theta - \epsilon < \pi$ implies that $\cos(\tau - \theta) \geq \cos(-\pi + \epsilon) = -\cos\epsilon$, and therefore

$$\begin{aligned}&|\lambda + \beta|\xi|^2|^2 \geq |\beta|^2[(|\lambda||\beta|^{-1})^2 + |\xi|^4 - 2\cos\epsilon|\lambda||\beta|^{-1}|\xi|^2] \\ &\quad = \cos\epsilon|\beta|^2[(|\lambda||\beta|^{-1})^2 - 2|\lambda||\beta|^{-1}|\xi|^2 + |\xi|^4] + (1 - \cos\epsilon)|\beta|^2[(|\lambda||\beta|^{-1})^2 + |\xi|^4] \\ &\quad \geq 2\sin^2\frac{\epsilon}{2}\,|\beta|^2((|\lambda||\beta|^{-1})^2 + |\xi|^4) \geq \left[\sin\frac{\epsilon}{2}\,|\beta|(|\lambda||\beta|^{-1} + |\xi|^2)\right]^2,\end{aligned}$$

which implies (2.7). Other two cases in (2.7) are obtained in the similar manner. □

Calculating the cofactor matrix of $\hat{A}_\lambda(\xi)$, we have

$$\hat{A}_\lambda(\xi)^{-1} = \frac{\begin{pmatrix} \lambda(\lambda+|\xi|^2)+|\xi|^4 & \lambda+|\xi|^2 & |\xi|^2 \\ -(\lambda+|\xi|^2)|\xi|^4 & \lambda(\lambda+|\xi|^2) & \lambda|\xi|^2 \\ |\xi|^6 & -\lambda|\xi|^2 & \lambda^2+|\xi|^4 \end{pmatrix}}{(\lambda+\alpha|\xi|^2)(\lambda+\beta|\xi|^2)(\lambda+\bar{\beta}|\xi|^2)}$$

and therefore we have

$$\begin{aligned} \hat{u}(\lambda,\xi) &= \frac{(\lambda^2+\lambda|\xi|^2+|\xi|^4)\hat{f}(\xi)+(\lambda+|\xi|^2)\hat{g}(\xi)+|\xi|^2\hat{h}(\xi)}{(\lambda+\alpha|\xi|^2)(\lambda+\beta|\xi|^2)(\lambda+\bar{\beta}|\xi|^2)}, \\ \hat{v}(\lambda,\xi) &= \frac{-(\lambda+|\xi|^2)|\xi|^4\hat{f}(\xi)+\lambda(\lambda+|\xi|^2)\hat{g}(\xi)+\lambda|\xi|^2\hat{h}(\xi)}{(\lambda+\alpha|\xi|^2)(\lambda+\beta|\xi|^2)(\lambda+\bar{\beta}|\xi|^2)}, \\ \hat{\theta}(\lambda,\xi) &= \frac{|\xi|^6\hat{f}(\xi)-\lambda|\xi|^2\hat{g}(\xi)+(\lambda^2+|\xi|^4)\hat{h}(\xi)}{(\lambda+\alpha|\xi|^2)(\lambda+\beta|\xi|^2)(\lambda+\bar{\beta}|\xi|^2)}. \end{aligned} \tag{2.8}$$

To derive a slightly simpler formula than that in (2.8), we use the following formulas:

$$\frac{\lambda^k}{(\lambda+\alpha|\xi|^2)(\lambda+\beta|\xi|^2)(\lambda+\bar{\beta}|\xi|^2)} = \sum_{j=1}^{3} \frac{A_{j,k}}{(\lambda+\gamma_j|\xi|^2)|\xi|^{4-2k}} \tag{2.9}$$

for $k = 0, 1, 2$. Here and hereafter, we use the following symbols:

$$\begin{aligned} &\gamma_1 = \alpha, && \gamma_2 = \beta, && \gamma_3 = \bar{\beta} \\ &A_{1,0} = A_\alpha, && A_{2,0} = -A_\alpha \frac{\alpha-\bar{\beta}}{\beta-\bar{\beta}}, && A_{3,0} = A_\alpha \frac{\alpha-\beta}{\beta-\bar{\beta}} \\ &A_{1,1} = -A_\alpha \alpha, && A_{2,1} = A_\alpha \frac{(\alpha-\bar{\beta})\beta}{\beta-\bar{\beta}}, && A_{3,1} = -A_\alpha \frac{(\alpha-\beta)\bar{\beta}}{\beta-\bar{\beta}} \\ &A_{1,2} = A_\alpha \alpha^2, && A_{2,2} = -A_\alpha \frac{(\alpha-\bar{\beta})\beta^2}{\beta-\bar{\beta}}, && A_{3,2} = A_\alpha \frac{(\alpha-\beta)\bar{\beta}^2}{\beta-\bar{\beta}} \end{aligned} \tag{2.10}$$

where we have set $A_\alpha = \frac{\alpha}{(\alpha-1)(\alpha-3)}$. We see easily that

$$A_{1,0}+A_{2,0}+A_{3,0} = 0, \quad A_{1,1}+A_{2,1}+A_{3,1} = 0, \quad A_{1,2}+A_{2,2}+A_{3,2} = 1 \tag{2.11}$$

Combining (2.8) and (2.9), we have

$$\begin{aligned}
\hat{u}(\xi) &= \sum_{j=1}^{3}\left[\frac{A_{j,0}+A_{j,1}+A_{j,2}}{\lambda+\gamma_j|\xi|^2}\,\hat{f}(\xi)+\frac{A_{j,0}+A_{j,1}}{(\lambda+\gamma_j|\xi|^2)|\xi|^2}\,\hat{g}(\xi)+\frac{A_{j,0}}{(\lambda+\gamma_j|\xi|^2)|\xi|^2}\,\hat{h}(\xi)\right]\\
\hat{v}(\xi) &= \sum_{j=1}^{3}\left[-\frac{(A_{j,0}+A_{j,1})|\xi|^2}{\lambda+\gamma_j|\xi|^2}\,\hat{f}(\xi)+\frac{A_{j,1}+A_{j,2}}{\lambda+\gamma_j|\xi|^2}\,\hat{g}(\xi)+\frac{A_{j,1}}{\lambda+\gamma_j|\xi|^2}\,\hat{h}(\xi)\right]\\
\hat{\theta}(\xi) &= \sum_{j=1}^{3}\left[\frac{A_{j,0}|\xi|^2}{\lambda+\gamma_j|\xi|^2}\,\hat{f}(\xi)-\frac{A_{j,1}}{\lambda+\gamma_j|\xi|^2}\,\hat{g}(\xi)+\frac{A_{j,2}+A_{j,0}}{\lambda+\gamma_j|\xi|^2}\,\hat{h}(\xi)\right]
\end{aligned} \tag{2.12}$$

Using (2.11) and the formula:

$$\frac{1}{(\lambda+\gamma_j|\xi|^2)|\xi|^2}=\frac{1}{\lambda}\left(\frac{1}{|\xi|^2}-\frac{1}{\gamma_j^{-1}\lambda+|\xi|^2}\right), \tag{2.13}$$

we have also the representation formula for $\hat{u}(\xi)$ as follows:

$$\hat{u}(\xi)=\sum_{j=1}^{3}\left[\frac{A_{j,0}+A_{j,1}+A_{j,2}}{\lambda+\gamma_j|\xi|^2}\,\hat{f}(\xi)-\frac{\gamma_j(A_{j,0}+A_{j,1})}{\lambda(\lambda+\gamma_j|\xi|^2)}\,\hat{g}(\xi)-\frac{\gamma_j A_{j,0}}{\lambda(\lambda+\gamma_j|\xi|^2)}\,\hat{h}(\xi)\right]. \tag{2.14}$$

Set

$$u(x)=\mathscr{F}_\xi^{-1}[\hat{u}(\xi)](x),\quad v(x)=\mathscr{F}_\xi^{-1}[\hat{v}(\xi)](x),\quad \theta(x)=\mathscr{F}_\xi^{-1}[\hat{\theta}(\xi)](x).$$

Here and hereafter, $\mathscr{F}_\xi^{-1}[a(\xi)](x)$ denotes the Fourier inverse transform of $a(\xi)$ which is defined by the formula:

$$\mathscr{F}_\xi^{-1}[a(\xi)](x)=\frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n}e^{ix\cdot\xi}a(\xi)\,d\xi.$$

To estimate u, v and θ, we use the Fourier multiplier theorem (cf. [**20**], [**24**], [**25**]).

THEOREM 2.6 (Fourier multiplier theorem). *Let $1<p<\infty$, let $\mathscr{S}(\boldsymbol{R}^n)$ denote the Schwartz space of rapidly decreasing funtions on $\boldsymbol{R}^n$, and let $m(\xi)\in C^\infty(\boldsymbol{R}^n\setminus\{0\})$ satisfy the multiplier condition*:

$$|D_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \text{for any multi-index } \alpha \in \boldsymbol{N}_0^n.$$

Then, the operator T defined on $\mathscr{S}(\boldsymbol{R}^n)$ by $Tf = \mathscr{F}_\xi^{-1}[m(\xi)\hat{f}(\xi)](x)$ admits an extension to a bounded linear operator $T : L_p(\boldsymbol{R}^n) \to L_p(\boldsymbol{R}^n)$. Furthermore, the norm of the operator T is estimated by $c(p,n)\max\{C_\alpha \mid |\alpha| < n/2\}$ with some absolute constant $c(p,n)$ depending only on n and p.

By (2.7) we have

$$|D_\xi^\alpha((\lambda + \gamma_j|\xi|^2)^{-1})| \leq C_{\alpha,\epsilon}(|\lambda| + |\xi|^2)^{-1}|\xi|^{-|\alpha|}$$

for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}_0^n$ and $(\lambda, \xi) \in \Sigma_\epsilon \times (\boldsymbol{R}^n \setminus \{0\})$. And also,

$$|D_\xi^\alpha(\xi_j|\xi|^{-1})| \leq C_\alpha |\xi|^{-|\alpha|}.$$

Applying Fourier multiplier theorem to the solution formulas (2.12) and (2.14), we have immediately the estimates:

$$\begin{aligned}
\|(|\lambda|\nabla^2 u, |\lambda|^{\frac{1}{2}}\nabla^3 u, \nabla^4 u)\|_{L_p(\boldsymbol{R}^n)} &\leq C\|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}^n)} \\
\|(|\lambda| v, |\lambda|^{\frac{1}{2}}\nabla v, \nabla^2 v)\|_{L_p(\boldsymbol{R}^n)} &\leq C\|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}^n)} \\
\|(|\lambda|\theta, |\lambda|^{\frac{1}{2}}\nabla\theta, \nabla^2\theta)\|_{L_p(\boldsymbol{R}^n)} &\leq C\|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}^n)} \\
\|(|\lambda|^2 u, |\lambda|^{\frac{3}{2}}\nabla u)\|_{L_p(\boldsymbol{R}^n)} &\leq C\|(|\lambda| f, g, h)\|_{L_p(\boldsymbol{R}^n)}
\end{aligned}$$

for any $\lambda \in \Sigma_\epsilon$. This completes the proof of Theorem 2.1.

3. Solution formula and some technical lemmas.

To prove Theorem 1.1, first of all we reduce equation (1.7) to the case where $F = 0$. For this purpose, we make the odd extension of $F = (f, g, h) \in \mathscr{H}_{p,K}(\boldsymbol{R}_+^n)$ which is defined as follows: Given k defined on $\boldsymbol{R}_+^n$, k^o and k^e denote its odd and even extension to $\boldsymbol{R}^n$, that is

$$k^o(x) = \begin{cases} k(x', x_n) & (x_n > 0) \\ -k(x', -x_n) & (x_n < 0) \end{cases}, \quad k^e(x) = \begin{cases} k(x', x_n) & (x_n > 0) \\ k(x', -x_n) & (x_n < 0) \end{cases}$$

where $x' = (x_1, \ldots, x_{n-1})$. Using this notation, let (u_0, v_0, θ_0) be a solution to the whole space resolvent problem (2.1) with $F = {}^T(f^o, g^o, h^o)$, which are defined by using the formula (2.12) as follows:

$$\begin{aligned}
\hat{u_0}(\xi) &= \sum_{j=1}^{3}\left[\frac{A_{j,0}+A_{j,1}+A_{j,2}}{\lambda+\gamma_j|\xi|^2}\,\hat{f^o}(\xi) + \frac{A_{j,0}+A_{j,1}}{(\lambda+\gamma_j|\xi|^2)|\xi|^2}\,\hat{g^o}(\xi) + \frac{A_{j,0}}{(\lambda+\gamma_j|\xi|^2)|\xi|^2}\,\hat{h^o}(\xi)\right]\\
\hat{v_0}(\xi) &= \sum_{j=1}^{3}\left[-\frac{(A_{j,0}+A_{j,1})|\xi|^2}{\lambda+\gamma_j|\xi|^2}\,\hat{f^o}(\xi) + \frac{A_{j,1}+A_{j,2}}{\lambda+\gamma_j|\xi|^2}\,\hat{g^o}(\xi) + \frac{A_{j,1}}{\lambda+\gamma_j|\xi|^2}\,\hat{h^o}(\xi)\right]\\
\hat{\theta_0}(\xi) &= \sum_{j=1}^{3}\left[\frac{A_{j,0}|\xi|^2}{\lambda+\gamma_j|\xi|^2}\,\hat{f^o}(\xi) - \frac{A_{j,1}}{\lambda+\gamma_j|\xi|^2}\,\hat{g^o}(\xi) + \frac{A_{j,0}+A_{j,2}}{\lambda+\gamma_j|\xi|^2}\,\hat{h^o}(\xi)\right]
\end{aligned} \tag{3.1}$$

Since $f \in W^2_q(\boldsymbol{R}^n_+)$ and $f|_{x_n=0} = 0$, $f^o \in W^2_p(\boldsymbol{R}^n)$ with $D_n(f^o) = (D_n f)^e$ and $D_n^2(f^o) = (D_n^2 f)^o$, and therefore

$$\|\nabla^j f^o\|_{L_p(\boldsymbol{R}^n)} \le C\|\nabla^j f\|_{L_p(\boldsymbol{R}^n_+)} \quad (j=0,1,2), \quad \|(g^o,h^o)\|_{L_p(\boldsymbol{R}^n)} \le 2\|(g,h)\|_{L_p(\boldsymbol{R}^n_+)} \tag{3.2}$$

Applying Theorem 2.1 and using (3.2), we have

$$(\lambda I - A)^T(u_0, v_0, \theta_0) = F^o = {}^T(f^o, g^o, h^o) \quad \text{in } \boldsymbol{R}^n, \tag{3.3}$$

$$\begin{aligned}
\|(|\lambda|\nabla^2 u_0, |\lambda|^{\frac12}\nabla^3 u_0, \nabla^4 u_0)\|_{L_p(\boldsymbol{R}^n)} &\le C\|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)}\\
\|(|\lambda|(v_0,\theta_0), |\lambda|^{\frac12}\nabla(v_0,\theta_0), \nabla^2(v_0,\theta_0))\|_{L_p(\boldsymbol{R}^n)} &\le C\|(\nabla^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)}\\
\|(|\lambda|^2 u_0, |\lambda|^{\frac32}\nabla u_0)\|_{L_p(\boldsymbol{R}^n)} &\le C\|(|\lambda| f, g, h)\|_{L_p(\boldsymbol{R}^n_+)}
\end{aligned} \tag{3.4}$$

Moreover, thanks to the odd extension of (f, g, h) we have

$$u_0 = v_0 = \theta_0 = 0 \quad \text{when } x_n = 0. \tag{3.5}$$

Setting $u = u_0 + w$, $v = v_0 + z$ and $\theta = \theta_0 + \tau$, we have the equation for new unknown functions w, z, τ as follows:

$$\left.\begin{aligned}
\lambda w - z &= 0\\
\lambda z + \Delta^2 w + \Delta\tau &= 0\\
\lambda\tau - \Delta\tau - \Delta z &= 0
\end{aligned}\right\} \quad \text{in } \boldsymbol{R}^n_+ \tag{3.6}$$

subject to the boundary conditions:

$$w|_{x_n=0} = \tau|_{x_n=0} = 0, \quad D_n w|_{x_n=0} = -D_n u_0|_{x_n=0}. \tag{3.7}$$

Setting $z = \lambda w$ in (3.7), we have

$$\left.\begin{aligned} \lambda^2 w + \Delta^2 + \Delta\tau = 0 \\ \lambda\tau - \Delta\tau - \lambda\Delta w = 0 \end{aligned}\right\} \quad \text{in } \boldsymbol{R}^n_+ \tag{3.8}$$

To solve (3.8) with (3.7), we apply the partial Fourier transform with respect to x' variables to (3.8), and then we have the system of ordinary differential equations:

$$\left.\begin{aligned} \lambda^2\tilde{w} + (D_n^2 - |\xi'|^2)^2\tilde{w} + (D_n^2 - |\xi|^2)\tilde{\tau} = 0 \\ \lambda\tilde{\tau} - (D_n^2 - |\xi'|^2)\tilde{\tau} - \lambda(D_n^2 - |\xi'|^2)\tilde{w} = 0 \end{aligned}\right\} \quad \text{in } (0,\infty) \tag{3.9}$$

subject to the boundary conditions:

$$\tilde{w}|_{x_n=0} = \tilde{\tau}|_{x_n=0} = 0, \;\; D_n\tilde{w}|_{x_n=0} = \tilde{G}|_{x_n=0}. \tag{3.10}$$

where $G = -D_n u_0$. Here and hereafter, for $a(x) = a(x', x_n)$ $(x' = (x_1, \ldots, x_{n-1}))$ we define the partial Fourier transform $\tilde{a}(\xi', x_n)$ $(\xi' = (\xi_1, \ldots, \xi_{n-1}))$ by the formula:

$$\tilde{a}(\xi', x_n) = \int_{\boldsymbol{R}^{n-1}} e^{-ix'\cdot\xi'}\, a(x', x_n)\, dx'.$$

To solve (3.9) with (3.10), we consider the characteristic root of the determinant of the following matrix:

$$L(\lambda, \xi', t) = \begin{pmatrix} \lambda^2 + (t^2 - |\xi'|^2) & t^2 - |\xi'|^2 \\ -\lambda(t^2 - |\xi'|^2) & \lambda - (t^2 - |\xi'|^2) \end{pmatrix}$$

Then,

$$\det L(\lambda, \xi', t) = \prod_{j=1}^{3} (\lambda + \gamma_j(|\xi'|^2 - t^2))$$

Here and hereafter, $\gamma_1 = \alpha$, $\gamma_2 = \beta$ and $\gamma_3 = \bar{\beta}$ are the same as in Lemma 2.3 and

(2.10). Therefore, the characteristic roots for the system of ordinary differential equations (3.9) are: $\pm\sqrt{\gamma_j^{-1}\lambda+|\xi'|^2}$ $(j=1,2,3)$. In what follows, let ϵ and Σ_ϵ denote the number given in Lemma 2.4 and the set defined in (2.2), respectively. When $\lambda\in\Sigma_\epsilon$, we have

$$|\arg\gamma_j^{-1}\lambda|<\pi-\epsilon\quad(j=1,2,3)\tag{3.11}$$

In fact, what $\lambda\in\Sigma_\epsilon$ means that $-\pi+\theta+\epsilon<\arg\lambda<\pi-\theta-\epsilon$. Since $\gamma_1=\alpha\in\boldsymbol{R}$, $\arg((\gamma_1)^{-1}\lambda)=\arg\lambda$. Recall that $\beta=\gamma_2$ and that $\arg\beta=\theta$. We have $\arg(\gamma_2^{-1}\lambda)=\arg\lambda-\theta$, which implies that $-\pi+\epsilon<\arg((\gamma_2)^{-1}\lambda)<\pi-2\theta-\epsilon<\pi-\epsilon$. Recall that $\gamma_3=\bar{\beta}$, and then $\arg\gamma_3=-\theta$, $\arg((\gamma_3)^{-1}\lambda)=\arg\lambda+\theta$, which implies that $\pi-\epsilon>\arg((\gamma_3)^{-1}\lambda)>-\pi+2\theta+\epsilon>-\pi+\epsilon$. Therefore, we have (3.11). In what follows, for the notational simplicity we set

$$A_j=\sqrt{(\gamma_j)^{-1}\lambda+|\xi'|^2}\ (j=1,2,3).$$

Combining (2.7) with (3.11), we have

$$(\sin(\epsilon/2))^{\frac{3}{2}}\sqrt{|\gamma_j|^{-1}|\lambda|+|\xi'|^2}\le\operatorname{Re}A_j\le\sqrt{|\gamma_j|^{-1}|\lambda|+|\xi'|^2},\quad(j=1,2,3).\tag{3.12}$$

We shall look for the solutions $\tilde{w}$ and $\tilde{\tau}$ to equation (3.9) of the formulas:

$$\tilde{w}(\lambda,\xi',x_n)=\sum_{j=1}^3P_je^{-A_j(\lambda,\xi')x_n},\ \tilde{\tau}(\lambda,\xi',x_n)=\sum_{j=1}^3Q_je^{-A_j(\lambda,\xi')x_n}.$$

Plugging these formulas into equation (3.9) and using the formulas: $(D_n^2-|\xi'|^2)^\ell e^{-A_jx_n}=(A_j^2-|\xi'|^2)^\ell e^{-A_jx_n}$, we have

$$(\lambda^2+(A_j^2-|\xi'|^2)^2)P_j+(A_j^2-|\xi'|^2)Q_j=0\tag{3.13}$$

$$(\lambda-(A_j^2-|\xi'|^2))Q_j-\lambda(A_j^2-|\xi'|^2)P_j=0\tag{3.14}$$

From (3.13) we set

$$Q_j=-\frac{\lambda^2+(A_j^2-|\xi'|^2)^2}{A_j^2-|\xi'|^2}P_j,$$

and then using the fact that $\det L(\lambda,\xi',A_j(\lambda,\xi'))=0$, we see that (P_j,Q_j) also satisfies (3.14). From this observation, we set

$$\tilde{w}(\lambda,\xi',x_n)=\sum_{j=1}^{3}P_je^{-A_j(\lambda,\xi')x_n},\quad \tilde{\tau}(\lambda,\xi',x_n)=\sum_{j=1}^{3}\frac{\lambda^2+(|\xi'|^2-A_j(\lambda,\xi')^2)^2}{|\xi'|^2-A_j(\lambda,\xi')^2}P_je^{-A_j(\lambda,\xi')x_n}.$$

Recalling that $t^3+t^2+2t+1=(t+\gamma_1)(t+\gamma_2)(t+\gamma_3)$ (cf. Lemma 2.3 and (2.10)), we have $\gamma_j^3+2\gamma_j=\gamma_j^2+1$. Using this formula, we have

$$\frac{\lambda^2+(|\xi'|^2-A_j(\lambda,\xi')^2)^2}{|\xi'|^2-A_j(\lambda,\xi')^2}=-\frac{\lambda^2+(\gamma_j^{-1}\lambda)^2}{-\gamma_j^{-1}\lambda}=-\frac{1+\gamma_j^2}{\gamma_j}\lambda=-(\gamma_j^2+2)\lambda$$

and therefore we arrive at the formulas:

$$\tilde{w}(\lambda,\xi',x_n)=\sum_{j=1}^{3}P_je^{-A_j(\lambda,\xi')x_n},\quad \tilde{\tau}(\lambda,\xi',x_n)=\sum_{j=1}^{3}(-\lambda)(\gamma_j^2+2)P_je^{-A_j(\lambda,\xi')x_n} \tag{3.15}$$

To decide P_j, we use the boundary condition. Plugging the formulas in (3.15) into the boundary condition, we have

$$\begin{aligned} P_1+P_2+P_3&=0\\ -(A_1P_1+A_2P_2+A_3P_3)&=\tilde{G}(\xi',0)\\ -\lambda((\gamma_1^2+2)P_1+(\gamma_2^2+2)P_2+(\gamma_3^2+2)P_3)&=0 \end{aligned} \tag{3.16}$$

In view of (3.16), we define the Lopatinski matrix $\Delta(\lambda,\xi')$ by the formula:

$$\Delta(\lambda,\xi')=\begin{pmatrix} 1, & 1, & 1\\ -A_1, & -A_2, & -A_3\\ \gamma_1^2+2, & \gamma_2^2+2, & \gamma_3^2+2 \end{pmatrix} \tag{3.17}$$

and then

$$\det\Delta(\lambda,\xi')=-[(\gamma_2^2-\gamma_3^2)A_1+(\gamma_3^2-\gamma_1^2)A_2+(\gamma_1^2-\gamma_2^2)A_3] \tag{3.18}$$

$$\begin{aligned} \tilde{w}(\lambda,\xi',x_n)&=\sum_{(j,k,\ell)}\frac{\gamma_k^2-\gamma_\ell^2}{\det\Delta(\lambda,\xi')}e^{-A_j(\lambda,\xi')x_n}\tilde{G}(\xi',0)\\ \tilde{\tau}(\lambda,\xi',x_n)&=-\sum_{(j,k,\ell)}\frac{\lambda(\gamma_j^2+2)(\gamma_k^2-\gamma_\ell^2)}{\det\Delta(\lambda,\xi')}e^{-A_j(\lambda,\xi')x_n}\tilde{G}(\xi',0) \end{aligned} \tag{3.19}$$

Setting

$$w(x) = \mathscr{F}_{\xi'}^{-1}[\tilde{w}(\lambda,\xi',x_n)](x'),\ v(x) = \lambda w(x),\ \tau(x) = \mathscr{F}_{\xi'}^{-1}[\tilde{\tau}(\lambda,\xi',x_n)](x') \tag{3.20}$$

(w, v, θ) is a required solution to (3.6) with boundary condition (3.7).

In what follows, we shall estimate $w(x)$, $v(x)$, $\tau(x)$. For this purpose, we introduce some terminologies concerning the Fourier multiplier theorem.

Definition 3.1. Let $1 < p < \infty$ and Ξ be a set in $\boldsymbol{C}$. Let $m(\lambda,\xi')$ be a function defined on $\Xi \times (\boldsymbol{R}^{n-1} \setminus \{0\})$.
(1) We call $m(\lambda,\xi')$ a Fourier multiplier of first kind if it satisfies the following condition: For any multi-index $\alpha = (\alpha_1,\ldots,\alpha_{n-1}) \in \boldsymbol{N}_0^{n-1}$, there exists a constant $C_{\alpha',\epsilon}$ depending on α' and ϵ such that

$$|D_{\xi'}^{\alpha'} m(\lambda,\xi')| \le C_{\alpha',\epsilon}(|\lambda|^{\frac{1}{2}} + |\xi'|)^{-|\alpha'|} \quad ((\lambda,\xi') \in \Xi \times (\boldsymbol{R}^{n-1} \setminus \{0\})).$$

(2) We call $m(\lambda,\xi')$ a Fourier multiplier of second kind if it satisfies the following condition: For any multi-index $\alpha = (\alpha_1,\ldots,\alpha_{n-1}) \in \boldsymbol{N}_0^{n-1}$, there exists a constant $C_{\alpha',\epsilon}$ depending on α' and ϵ such that

$$|D_{\xi'}^{\alpha'} m(\lambda,\xi')| \le C_{\alpha',\epsilon}|\xi'|^{-|\alpha'|} \quad ((\lambda,\xi') \in \Xi \times (\boldsymbol{R}^{n-1} \setminus \{0\})).$$

Since $(|\lambda|^{\frac{1}{2}} + |\xi'|)^{-|\alpha'|} \le |\xi'|^{-|\alpha'|}$, a Fourier multiplier of first kind is also that of second kind. The multiplication of several multipliers of first kind and second kind becomes a Fourier multiplier of second kind. If $m(\lambda,\xi')$ is a Fourier multiplier of first kind or second kind, then setting $M_\lambda[g](y) = \mathscr{F}_{\xi'}^{-1}[m(\lambda,\xi')\tilde{g}(\xi',y_n)](y')$, by Theorem 2.6 we have

$$\|M_\lambda[g]\|_{L_p(\boldsymbol{R}_+^n)} \le C_{n,p} \sum_{|\alpha'| \le n-1} \{\max_{(\lambda,\xi') \in \Xi \times (\boldsymbol{R}^{n-1} \setminus \{0\})} |D_{\xi'}^{\alpha'} m(\lambda,\xi')|\} \|g\|_{L_p(\boldsymbol{R}_+^n)} \tag{3.21}$$

where $C_{n,p}$ is a constant depending on n and p. The following two lemmas are the bases of our estimations.

Lemma 3.2. *Let ϵ and Σ_ϵ be the same number and set as in Lemma 2.4, respectively. Then, for any real number s and multi-index $\alpha' \in \boldsymbol{N}_0^{n-1}$ we have*

$$|D_{\xi'}^{\alpha'} A_j(\lambda,\xi')^s| \le C_{\alpha',\epsilon,s}(|\lambda|^{\frac{1}{2}} + |\xi'|)^{s-|\alpha'|},$$
$$|D_{\xi'}^{\alpha'} |\xi'|^s| \le C_{\alpha',s}|\xi'|^{s-|\alpha'|},$$
$$|D_{\xi'}^{\alpha'} (A_j(\lambda,\xi') + |\xi'|)^s| \le C_{\alpha',s}|\xi'|^{s-|\alpha'|}$$

where the constants $C_{\alpha',\epsilon,s}$ and $C_{\alpha',s}$ depend on α', ϵ, s and α', s, respectively.

LEMMA 3.3. *Let $1 < p < \infty$ and let Ξ be a subset of $\boldsymbol{C}$. Let $m_1(\lambda, \xi')$ and $m_2(\lambda, \xi')$ be a Fourier multiplier of first kind multiplier and that of second kind defined on $\Xi \times (\boldsymbol{R}^{n-1} \setminus \{0\})$, respectively. Let us define the operators $K_{j\ell}(\lambda)$ by the formulas*:

$$[K_{j1}(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}[m_1(\lambda, \xi')|\lambda|^{\frac{1}{2}} e^{-A_j(\lambda,\xi')(x_n+y_n)} \tilde{g}(\xi', y_n)](x')\, dy_n$$
$$[K_{j2}(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}[m_2(\lambda, \xi')|\xi'| e^{-A_j(\lambda,\xi')(x_n+y_n)} \tilde{g}(\xi', y_n)](x')\, dy_n$$
$$[K_{j3}(\lambda)g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}[m_1(\lambda, \xi')|\xi'|^2 \mathscr{M}(\lambda, \xi', x_n + y_n) \tilde{g}(\xi', y_n)](x')\, dy_n$$

where we have set

$$\mathscr{M}_j(\lambda, \xi', x_n) = \frac{e^{-A_j(\lambda,\xi')x_n} - e^{-|\xi'|x_n}}{A_j(\lambda, \xi') - |\xi'|}\,.$$

Then, $K_{j\ell}(\lambda)$ is a bounded linear operator on $L_p(\boldsymbol{R}^n_+)$ and

$$\|K_{j\ell}(\lambda)g\|_{L_p(\boldsymbol{R}^n_+)} \leq C_{n,p,\epsilon} \|g\|_{L_p(\boldsymbol{R}^n_+)}$$

for any $g \in L_p(\boldsymbol{R}^n_+)$ with some constant $C_{n,p,\Xi}$ depending on n, p and Ξ.

Lemma 3.2 was proved in [**22**, Lemma 4.4] and [**23**, Lemma 5.4] and Lemma 3.3 can be proved by the same argument as in the proof of Lemma 3.4 in [**21**].

In what follows, we use the symbol: $A_j(\lambda, D')^a |D'|^b$ defined by the formula:

$$[A_j(\lambda, D')^a |D'|^b g](x) = \mathscr{F}_{\xi'}^{-1}[A_j(\lambda, \xi')^a |\xi'|^b \tilde{g}(\xi', x_n)](x').$$

In particuler, we have

$$\|A_j(\lambda, D')^a |D'|^b g\|_{L_p(\boldsymbol{R}^n_+)} \leq C_{n,p,a,\epsilon} \sum_{0 \leq c \leq a} |\lambda|^{\frac{a-c}{2}} \sum_{|\alpha'|=b+c} \|D_{x'}^{\alpha'} g\|_{L_p(\boldsymbol{R}^n_+)}, \tag{3.22}$$

where $D_{x'}^{\alpha'} = D_1^{\alpha_1} \cdots D_{n-1}^{\alpha_{n-1}}$ and $\alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \in \boldsymbol{N}_0^{n-1}$. In fact, writing

$$A_j(\lambda,\xi') = \frac{A_j(\lambda,\xi')^2}{A_j(\lambda,\xi')} = \frac{\lambda}{\gamma_j A_j(\lambda,\xi')} - \sum_{k=1}^{n-1}\frac{i\xi_k}{A_j(\lambda,\xi')}i\xi_k,\ |\xi'| = -\sum_{k=1}^{n-1}\frac{i\xi_k}{|\xi'|}i\xi_k, \tag{3.23}$$

and noting that $\lambda/(\gamma_j A_j(\lambda,\xi')|\lambda|^{\frac{1}{2}})$, $i\xi_k/A_j(\lambda,\xi')$, $i\xi_k/|\xi'|$ are Fourier multipliers of second kind, by (3.21) we have (3.22).

We shall prepare two lemmas for estimations of w, v and τ defined in (3.20).

LEMMA 3.4. *Let $1 < p < \infty$ and let $m(\lambda,\xi')$ be a Fourier multiplier defined on $\Sigma_\epsilon \times (\boldsymbol{R}^{n-1}\setminus\{0\})$ of first kind. Let $\mathscr{B}_{j,k}(\lambda)$ $(j,k=1,2,3)$ be an operator defined by the formula*:

$$[\mathscr{B}_{j,k}(\lambda)g](x) = \int_0^\infty \mathscr{F}^{-1}[m(\lambda,\xi')A_k(\lambda,\xi')e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde{g}(\xi',y_n)](x')\,dy_n.$$

Then, for any $\lambda\in\Sigma_\epsilon$ we have

$$\|\mathscr{B}_{j,k}(\lambda)g\|_{L_p(\boldsymbol{R}^n_+)} \le C_{p,\epsilon}\|g\|_{L_p(\boldsymbol{R}^n_+)}.$$

PROOF. Using (3.23), we write

$$\begin{aligned}[\mathscr{B}_{j,k}(\lambda)g](x) &= \int_0^\infty \mathscr{F}^{-1}[m_1(\lambda,\xi')|\lambda|^{\frac{1}{2}}e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde{g}(\xi',y_n)](x')\,dy_n \\ &\quad + \int_0^\infty \mathscr{F}^{-1}[m_2(\lambda,\xi')|\xi'|e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde{g}(\xi',y_n)](x')\,dy_n\end{aligned}$$

where we have set

$$m_1(\lambda,\xi') = (m(\lambda,\xi')\lambda)/(\gamma_k A_k(\lambda,\xi')|\lambda|^{1/2}),\ m_2(\lambda,\xi') = (m(\lambda,\xi')|\xi'|)/A_k(\lambda,\xi').$$

By Lemma 3.2 we see that $m_1(\lambda,\xi')$ and $m_2(\lambda,\xi')$ are Fourier multipliers of first kind and of second kind, respectively. Therefore, applying Lemma 3.3 we have the lemma immediately. □

LEMMA 3.5. *Let $1 < p < \infty$ and $\boldsymbol{C}_+ = \{\lambda\in\boldsymbol{C} \mid \operatorname{Re}\lambda \ge 0\}$. Let $\psi_0(s)$ be a function in $C_0^\infty(\boldsymbol{R})$ such that $\psi_0(s)=1$ for $|s|\le r_0$ and $\psi_0(s)=0$ for $|s|\ge r_1$ with some positive numbers r_0 and r_1 such that $r_0<r_1$. For $\lambda\in\boldsymbol{C}_+$ let $m(\lambda,\xi')$ be a function defined on $\operatorname{supp}\psi_0(|\lambda|/|\xi'|^2)$. Assume that*

$$|D_{\xi'}^{\alpha'}m(\lambda,\xi')| \le C_{\alpha',\Xi}|\xi'|^{-|\alpha'|} \tag{3.24}$$

for any $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \boldsymbol{C}_+$ *and* $\xi' \in \operatorname{supp}\psi_0(|\lambda|/|\xi'|^2)$.
(1) *Given* $\ell \in \boldsymbol{N}_0$, *we set*

$$\mathscr{C}_{j0}^{\ell}(\lambda)[g](x) =$$
$$\lambda^{-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_0(|\lambda|/|\xi'|^2)|\xi'|^{-\ell}m(\lambda,\xi')\{e^{-A_j(\lambda,\xi')(x_n+y_n)} - e^{-|\xi'|(x_n+y_n)}\}\tilde{g}(\xi',y_n)](x')\,dy_n$$

Then, for any $(a,b,c,d) \in \boldsymbol{N}_0^4$ *with* $a+b+c+d = \ell+3$, $\lambda \in \boldsymbol{C}_+$ *and* $j,k = 1,2,3$, *we have*

$$|\lambda|^{\frac{a}{2}}\|D_n^b A_k(\lambda,D')^c|D'|^d\mathscr{C}_{j0}^{\ell}(\lambda)[g]\|_{L_p(\boldsymbol{R}_+^n)} \le C_{\ell,p}\|g\|_{L_p(\boldsymbol{R}_+^n)}. \tag{3.25}$$

(2) *Given* $\ell \in \boldsymbol{N}_0$, *we set*

$$\mathscr{D}_{j0}^{\ell}(\lambda)[g](x) =$$
$$\lambda^{-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_0(|\lambda|/|\xi'|^2)|\xi'|^{-\ell}m(\lambda,\xi')(A_j(\lambda,\xi') - |\xi'|)e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde{g}(\xi',y_n)](x')\,dy_n$$

Then, for any $(a,b,c,d) \in \boldsymbol{N}_0^4$ *with* $a+b+c+d = \ell+2$, $\lambda \in \boldsymbol{C}_+$ *and* $j,k = 1,2,3$, *we have*

$$|\lambda|^{\frac{a}{2}}\|D_n^b A_k(\lambda,D')^c|D'|^d\mathscr{D}_{j0}^{\ell}(\lambda)[g]\|_{L_p(\boldsymbol{R}_+^n)} \le C_{\ell,p}\|g\|_{L_p(\boldsymbol{R}_+^n)}. \tag{3.26}$$

Remark 3.6. We note that $\boldsymbol{C}_+ \subset \Sigma_\epsilon$.

Proof. (1) We write

$$\begin{aligned}
&|\lambda|^{\frac{a}{2}}D_n^b A_k(\lambda,D')^c|D'|^d\mathscr{C}_{j0}^{\ell}(\lambda)[g](x)\\
&= \lambda^{-1}(-1)^b\int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_0(|\lambda|/|\xi'|^2)|\lambda|^{\frac{a}{2}}|\xi'|^{-\ell+d}m(\lambda,\xi')A_k(\lambda,\xi')^c\\
&\quad\cdot(A_j(\lambda,\xi')^b - |\xi'|^b)e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde{g}(\xi',y_n)](x')\,dy_n\\
&\quad+ \lambda^{-1}(-1)^b\int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_0(|\lambda|/|\xi'|^2)|\lambda|^{\frac{a}{2}}|\xi'|^{-\ell+b+d}m(\lambda,\xi')A_k(\lambda,\xi')^c\\
&\quad\cdot\{e^{-A_j(\lambda,\xi')(x_n+y_n)} - e^{-|\xi'|(x_n+y_n)}\}e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde{g}(\xi',y_n)](x')\,dy_n\\
&= I_b + II_b
\end{aligned}$$

where $I_b = 0$ when $b = 0$. To estimate I_b, setting

$$n_1(\lambda, \xi') = (-1)^b |\xi'|^{-\ell-1+d} A_k(\lambda, \xi')^c \lambda^{-1}(A_j(\lambda, \xi')^b - |\xi'|^b) m(\lambda, \xi'),$$

we write

$$I_b = \int_0^\infty \mathscr{F}^{-1}_{\xi'}[\psi_0(|\lambda|/|\xi'|^2) n_1(\lambda, \xi')|\xi'| e^{-A_j(\lambda,\xi')(x_n+y_n)} \tilde{g}(\xi', y_n)](x')\, dy_n.$$

Using the identity:

$$\lambda = (A_j(\lambda, \xi') - |\xi'|)(A_j(\lambda, \xi') + |\xi'|)\gamma_j \tag{3.27}$$

we have

$$\frac{A_j(\lambda, \xi')^b - |\xi'|^b}{\lambda} = \frac{\sum_{m=1}^{b-1} A_j(\lambda, \xi')^{b-1-m} |\xi'|^m}{\gamma_j (A_j(\lambda, \xi') + |\xi'|)}.$$

Therefore, by Lemma 3.2 and (3.24) we have

$$|D^{\alpha'}_{\xi'} n_1(\lambda, \xi')| \leq C_{\alpha'} |\lambda|^{\frac{a}{2}} |\xi'|^{-\ell-1+d} (|\lambda|^{\frac{1}{2}} + |\xi'|)^{c+b-2} |\xi'|^{-|\alpha'|} \leq C_{\alpha'} |\xi'|^{-|\alpha'|} \tag{3.28}$$

when $\xi' \in \operatorname{supp} \psi_0(|\lambda|/|\xi'|^2)$, because $a + b + c + d = \ell + 3$.

On the other hand,

$$|D^{\alpha'}_{\xi'} \psi_0(|\lambda|/|\xi'|^2)| \leq C_{\alpha'} (|\lambda|^{\frac{1}{2}} + |\xi'|)^{-|\alpha'|} \tag{3.29}$$

for any $\alpha' \in \boldsymbol{N}_0^{n-1}$ and $(\lambda, \xi') \in \boldsymbol{C} \times (\boldsymbol{R}^{n-1} \setminus \{0\})$. In fact, by the Bell formula we have

$$\begin{aligned} &D^{\alpha'}_{\xi'} \psi_0(|\lambda|/|\xi'|^2) \\ &\quad = \sum_{\ell=1}^{|\alpha'|} (D^\ell_s \psi_0)(|\lambda|/|\xi'|^2) \sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha' \\ |\alpha'_i|\geq 1}} \Gamma^{\alpha',\ell}_{\alpha'_1,\ldots,\alpha'_\ell} D^{\alpha'_1}_{\xi'}(|\lambda||\xi'|^{-2}) \cdots D^{\alpha'_\ell}_{\xi'}(|\lambda||\xi'|^{-2}) \end{aligned}$$

with suitable coefficients $\Gamma^{\alpha',\ell}_{\alpha'_1,\ldots,\alpha'_\ell}$. Noting that $r_0 \leq |\lambda|/|\xi'|^2 \leq r_1$ on $\operatorname{supp} \chi^{(\ell)}(|\lambda|/|\xi'|^2)$ $(\ell \geq 1)$ and using Lemma 3.2 we have

$$|D_{\xi'}^{\alpha'}\psi_0(|\lambda|/|\xi'|^2)| \leq C_{\alpha'}\sum_{\ell=1}^{|\alpha'|}|\chi^{(\ell)}(|\lambda|/|\xi'|^2)|(|\lambda|/|\xi'|^2)^{\ell}|\xi'|^{-|\alpha'|} \leq C(|\lambda|^{\frac{1}{2}}+|\xi'|)^{-|\alpha'|},$$

which shows (3.29).

Combining (3.28) and (3.29), we see that $\psi_0(|\lambda|/|\xi'|^2)n_1(\lambda,\xi')$ is a Fourier multiplier of second kind. Therefore, applying Lemma 3.3 we have

$$\|I_b\|_{L_p(\boldsymbol{R}^n_+)} \leq C_p\|g\|_{L_p(\boldsymbol{R}^n_+)}. \tag{3.30}$$

Using (3.27) and setting

$$n_2(\lambda,\xi') = (-1)^b|\lambda|^{\frac{a}{2}}|\xi'|^{-\ell-2+b+d}m(\lambda,\xi')A_k(\lambda,\xi')^c(\gamma_j(A_j(\lambda,\xi')+|\xi'|))^{-1},$$
$$\mathscr{M}_j(\lambda,\xi',x_n) = \frac{e^{-A_j(\lambda,\xi')x_n}-e^{-|\xi'|x_n}}{A_j(\lambda,\xi')-|\xi'|},$$

we have

$$II_b = \int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_0(|\lambda|/|\xi'|^2)n_2(\lambda,\xi')|\xi'|^2\mathscr{M}_j(\lambda,\xi',x_n+y_n)\tilde{g}(\xi',y_n)](x')\,dy_n.$$

By Lemma 3.2, (3.24) and the assumption: $a+b+c+d=\ell+3$ we have

$$|D_{\xi'}^{\alpha'}n_2(\lambda,\xi')| \leq C_{\alpha'}|\lambda|^{\frac{a}{2}}(|\lambda|^{\frac{1}{2}}+|\xi'|)^{c-1}|\xi'|^{-\ell-2+b+d}|\xi'|^{-|\alpha'|} \leq C_\alpha|\xi'|^{-|\alpha'|}$$

when $\xi \in \operatorname{supp}\psi_0(|\lambda|/|\xi'|^2)$, which combined with (3.29) implies that $n_2(\lambda,\xi')$ $\psi_0(|\lambda|/|\xi'|^2)$ is a Fourier multiplier of second kind. Therefore, applying Lemma 3.3 we have

$$\|II_b\|_{L_p(\boldsymbol{R}^n_+)} \leq C_p\|g\|_{L_p(\boldsymbol{R}^n_+)},$$

which combined with (3.30) implies (3.25). Employing the same argument as in proving (3.30) with $b=1$, we have (3.26), which completes the proof of the lemma. □

LEMMA 3.7. *Let $1<p<\infty$. Let $\psi_\infty(s)$ be a function in $C^\infty(\boldsymbol{R})$ such that $\psi_\infty(s)=1$ for $|s|\geq r_1$ and $\psi_\infty(s)=0$ for $|s|\leq r_0$ with some positive numbers r_0 and r_1 such that $r_0<r_1$. For $\lambda\in\boldsymbol{C}_+$ let $m(\lambda,\xi')$ be a function defined on*

$\operatorname{supp}\psi_\infty(|\lambda|/|\xi'|^2)$. *Assume that*

$$|D_{\xi'}^{\alpha'} m(\lambda,\xi')| \leq C_{\alpha'}(|\lambda|^{\frac12}+|\xi'|)^{-|\alpha'|} \tag{3.31}$$

for any $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \boldsymbol{C}_+$ *and* $\xi' \in \operatorname{supp}\psi_\infty(|\lambda|/|\xi'|^2)$. *Given* $\ell \in \boldsymbol{N}_0$ *we set*

$$\mathscr{C}^\ell_{j\infty}(\lambda)[g](x) = \lambda^{-1}\int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_\infty(|\lambda|/|\xi'|^2)|\lambda|^{-\frac{\ell}{2}} m(\lambda,\xi')e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde g(\xi',y_n)](x')\,dy_n$$

Then, for any $(a,b,c,d)\in\boldsymbol{N}_0^4$ *with* $a+b+c+d=\ell+3$, $\lambda\in\boldsymbol{C}_+$ *and* $j,k=1,2,3$, *we have*

$$|\lambda|^{\frac a2}\|D_n^b A_k(\lambda,D')^c|D'|^d\mathscr{C}^\ell_{j,\infty}(\lambda)[g]\|_{L_p(\boldsymbol{R}^n_+)} \leq C_{\ell,p}\|g\|_{L_p(\boldsymbol{R}^n_+)}. \tag{3.32}$$

Proof. Employing the same argument as in proving (3.29), we have

$$|D_{\xi'}^{\beta'}\psi_\infty(|\lambda|/|\xi'|^2)| \leq C_{\beta'}(|\lambda|^{\frac12}+|\xi|)^{-|\beta'|}, \tag{3.33}$$

for any $\beta'\in\boldsymbol{N}_0^{n-1}$ and $(\lambda,\xi')\in\boldsymbol{C}\times(\boldsymbol{R}^{n-1}\setminus\{0\})$. First, we consider the case where $d=0$. In this case, we write

$$\begin{aligned}&|\lambda|^{\frac a2}D_n^bA_k(\lambda,D')^c\mathscr{C}^\ell_{j\infty}(\lambda)[g](x)\\&\quad=\int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_\infty(|\lambda|/|\xi'|^2)n(\lambda,\xi')|\lambda|^{\frac12}e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde g(\xi',y_n)](x')\,dy_n\end{aligned}$$

where we have set $n(\lambda,\xi')=\lambda^{-1}|\lambda|^{-\frac{\ell}{2}-\frac12}|\lambda|^{\frac a2}(-1)^bA_j(\lambda,\xi')^bA_k(\lambda,\xi')^cm(\lambda,\xi')$. By Lemma 3.2 and (3.31) we have

$$|D_{\xi'}^{\alpha'}n(\lambda,\xi')|\leq C_{\alpha'}|\lambda|^{-\frac32-\frac{\ell}{2}+\frac a2}(|\lambda|^{\frac12}+|\xi'|)^{-(b+c)}(|\lambda|^{\frac12}+|\xi'|)^{-|\alpha'|}\leq C_{\alpha'}(|\lambda|^{\frac12}+|\xi'|)^{-|\alpha'|}$$

when $\xi'\in\operatorname{supp}\psi_\infty(|\lambda|/|\xi'|^2)$, because $a+b+c=\ell+3$. Combining this with (3.33) implies that $\psi_\infty(|\lambda|/|\xi'|^2)n(\lambda,\xi')$ is a Fourier multiplier of first kind. Applying Lemma 3.3, we have (3.32).

When $d\geq 1$, we write

$$\begin{aligned}&|\lambda|^{\frac a2}D_n^bA_k(\lambda,D')^c|D'|^d\mathscr{C}^\ell_{j\infty}(\lambda)[g](x)\\&\quad=\int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_\infty(|\lambda|/|\xi'|^2)\tilde n(\lambda,\xi')|\xi'|e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde g(\xi',y_n)](x')\,dy_n\end{aligned}$$

where we have set $\tilde{n}(\lambda,\xi')=\lambda^{-1}|\lambda|^{-\frac{\ell}{2}}|\lambda|^{\frac{a}{2}}(-1)^{b}A_j(\lambda,\xi')^{b}A_k(\lambda,\xi')^{c}|\xi'|^{d-1}m(\lambda,\xi')$. By Lemma 3.2 and (3.31) we have

$$|D_{\xi'}^{\alpha'}n(\lambda,\xi')|\le C_{\alpha'}|\lambda|^{-1-\frac{\ell}{2}+\frac{a}{2}}(|\lambda|^{\frac{1}{2}}+|\xi'|)^{-(b+c)}|\xi'|^{d-1-|\alpha'|}\le C_{\alpha'}|\xi'|^{-|\alpha'|}$$

when $\xi'\in\operatorname{supp}\psi_\infty(|\lambda|/|\xi'|^2)$, because $a+b+c+d=\ell+3$ and $d\ge 1$. Combining this with (3.33) implies that $\psi_\infty(|\lambda|/|\xi'|^2)\tilde{n}(\lambda,\xi')$ is a Fourier multiplier of second kind. Therefore, applying Lemma 3.3 we have (3.32), which completes the proof of Lemma 3.7. □

4. A proof of Theorem 1.1.

In this section, we shall prove Theorem 1.1. First of all, we shall examine the behaviour of $\det\Delta(\lambda,\xi')$.

Lemma 4.1. *Let $\Delta(\lambda,\xi')$ be the same matrix as in* (3.17). *Assume that* $\operatorname{Re}\lambda\ge 0$. *Then, there exist positive numbers σ_0 and σ_1 such that*

$$\begin{aligned}&|\det\Delta(\lambda,\xi')|\ge\sigma_0|\lambda||\xi'|^{-1} &&\text{when } |\lambda|/|\xi'|^2\le\sigma_1,\\ &|\det\Delta(\lambda,\xi')|\ge\sigma_0(|\lambda|^{\frac{1}{2}}+|\xi'|) &&\text{when } |\lambda|/|\xi'|^2\ge\sigma_1/2.\end{aligned}\tag{4.1}$$

Moreover, for any multi-index $\alpha'\in\boldsymbol{N}_0^{n-1}$ we have

$$|D_{\xi'}^{\alpha'}(\det\Delta(\lambda,\xi'))^{-1}|\le C_{\alpha'}|\lambda|^{-1}|\xi'|^{1-|\alpha'|}\qquad\text{when } |\lambda|/|\xi'|^2\le\sigma_1,\tag{4.2}$$

$$|D_{\xi'}^{\alpha'}(\det\Delta(\lambda,\xi'))^{-1}|\le C_{\alpha'}(|\lambda|^{\frac{1}{2}}+|\xi'|)^{-1-|\alpha'|}\qquad\text{when } |\lambda|/|\xi'|^2\ge\sigma_1/2,\tag{4.3}$$

Proof. Let σ_1 be a small positive number determined later and we first consider the case where $|\lambda|/|\xi'|^2\le\sigma_1$. For the notational simplicity, we set

$$\delta_1=\gamma_2^2-\gamma_3^2,\ \delta_2=\gamma_3^2-\gamma_1^2,\ \delta_3=\gamma_1^2-\gamma_2^2,\tag{4.4}$$

and then by (3.18) we have

$$\det\Delta(\lambda,\xi')=-\sum_{j=1}^{3}\delta_jA_j(\lambda,\xi')=-\sum_{j=1}^{3}\delta_j\sqrt{\gamma_j^{-1}\lambda+|\xi'|^2}.\tag{4.5}$$

Since $\sqrt{\gamma_j^{-1}t+1}=1+(2\gamma_j)^{-1}t+O(t^2)$ as $t\to 0$, it follows from (4.5) that

$$\det \Delta(\lambda, \xi') = -|\xi'| \sum_{j=1}^{3} \delta_j \{(1 + (2\gamma_j)^{-1} t + O(t^2))\},$$

where $t = \lambda/|\xi'|^2$. Since

$$\sum_{j=1}^{3} \delta_j = 0, \quad \sum_{j=1}^{3} \gamma_j^{-1} \delta_j = (\beta - \bar{\beta})|\alpha - \beta|^2 \tag{4.6}$$

we have

$$|\det \Delta(\lambda, \xi')| \geq (1/2)|\xi'||t|(|\beta - \bar{\beta}||\alpha - \beta|^2 - C|t|)$$

with some constant $C > 0$. Choose σ_1 in such a way that $C\sigma_1 \leq (1/2)|\beta - \bar{\beta}||\alpha - \beta|^2$, we have

$$|\det \Delta(\lambda, \xi')| \geq (1/4)|\beta - \bar{\beta}||\alpha - \beta|^2 |\lambda||\xi'|^{-1} \quad \text{when } |\lambda|/|\xi'|^2 \leq \sigma_1. \tag{4.7}$$

Now, we consider the case where $|\lambda|/|\xi'|^2 \geq (\sigma_1/2)$. Set

$$\tilde{\lambda} = \lambda(|\lambda| + |\xi'|^2)^{-1}, \quad \tilde{\xi}_j = \xi_j(|\lambda| + |\xi'|^2)^{-\frac{1}{2}} \quad (j = 1, \ldots, n-1).$$

By (3.18) we have

$$\det \Delta(\lambda, \xi') = -\sqrt{|\lambda| + |\xi'|^2} D(\tilde{\lambda}, \tilde{\xi}) \tag{4.8}$$

where

$$D(\tilde{\lambda}, \tilde{\xi}) = \sqrt{\gamma_1^{-1}\tilde{\lambda} + |\tilde{\xi}'|^2}\,(\gamma_2^2 - \gamma_3^2) + \sqrt{\gamma_2^{-1}\tilde{\lambda} + |\tilde{\xi}'|^2}\,(\gamma_3^2 - \gamma_1^2) + \sqrt{\gamma_3^{-1}\tilde{\lambda} + |\tilde{\xi}'|^2}\,(\gamma_1^2 - \gamma_2^2)$$

What $|\lambda|/|\xi'|^2 \geq \sigma_1/2$ implies that

$$|\tilde{\lambda}| = (1 + |\xi'|^2/|\lambda|)^{-1} \geq (1 + (2/\sigma_1))^{-1},$$

and therefore the range of $(\tilde{\lambda}, \tilde{\xi}')$ is in the following set:

$$\Omega = \{(\tilde{\lambda}, \tilde{\xi}') \in \boldsymbol{C} \times \boldsymbol{R}^{n-1} \mid (1 + (2/\sigma_1))^{-1} \leq |\tilde{\lambda}| \leq 1, \quad |\tilde{\lambda}| + |\tilde{\xi}'|^2 = 1, \quad \operatorname{Re}\tilde{\lambda} \geq 0\} \tag{4.9}$$

If we show that

$$\det \Delta(\lambda, \xi') \neq 0 \quad \text{when } \operatorname{Re}\lambda \geq 0 \text{ and } \lambda \neq 0, \tag{4.10}$$

then from (4.8) and the fact that Ω is compact it follows that $\inf_{(\tilde\lambda,\tilde\xi')\in\Omega} |D(\tilde\lambda, \tilde\xi')| > 0$, which combined with (4.8) and (4.7) implies (4.1).

Therefore, we shall prove (4.10) finally. Suppose that there exists a $(\lambda, \xi') \in \boldsymbol{C} \times \boldsymbol{R}^{n-1}$ such that $\operatorname{Re}\lambda \geq 0$ and $\det \Delta(\lambda, \xi') = 0$. Then, by (3.17) there exists a $(P_1, P_2, P_3) \neq (0, 0, 0)$ such that

$$\begin{aligned} P_1 + P_2 + P_3 &= 0, \\ A_1(\lambda, \xi')P_1 + A_2(\lambda, \xi')P_2 + A_3(\lambda, \xi')P_3 &= 0, \\ (\gamma_1^2 + 2)P_1 + (\gamma_2^2 + 2)P_2 + (\gamma_3^2 + 2)P_3 &= 0. \end{aligned} \tag{4.11}$$

Set

$$w(x_n) = \sum_{j=1}^{3} P_j e^{-A_j(\lambda,\xi')x_n}, \quad \tau(x_n) = -\lambda \sum_{j=1}^{3} (\gamma_j^2 + 2)P_j e^{-A_j(\lambda,\xi')x_n}.$$

Then, by (3.15) and (4.11) we see that $w(x_n)$ and $\tau(x_n)$ satisfy the homogeneous system of ordinary differential equations:

$$\begin{aligned} \lambda^2 w + (D_n^2 - |\xi'|^2)^2 w + (D_n^2 - |\xi'|^2)\tau = 0 \quad (x_n > 0), \\ \lambda\tau - (D_n^2 - |\xi'|^2)\tau - \lambda(D_n^2 - |\xi'|^2)w = 0 \quad (x_n > 0), \\ w(0) = (D_n w)(0) = \tau(0) = 0. \end{aligned} \tag{4.12}$$

Multiplying the first equation of (4.12) by $\bar\lambda\bar w$ and the second equation of (4.12) by $\bar\tau$ and integrating the resultant formulas, by integration by parts we have

$$\begin{aligned} |\lambda|^2\lambda\|w\|^2 + \bar\lambda\|D_n^2 w\|^2 + 2\bar\lambda|\xi'|^2\|D_n w\|^2 + \bar\lambda|\xi'|^4\|w\|^2 - \bar\lambda(D_n\tau, D_n w) - \bar\lambda|\xi'|^2(\tau, w) \\ + \lambda\|\tau\|^2 + \|D_n\tau\|^2 + |\xi'|^2\|\tau\|^2 + \lambda(D_n w, D_n\tau) + \lambda|\xi'|^2(w, \tau) = 0 \end{aligned}$$

where $(u, v) = \int_0^\infty u(x)\overline{v(x)}\,dx$ and $\|u\|^2 = (u, u)$. Taking the real part of the above formula, we have

$$(\mathrm{Re}\,\lambda)\,[|\lambda|^2\|w\|^2+\|D_n^2w\|^2+2|\xi'|^2\|D_nw\|^2 \\ +|\xi'|^4\|w\|^2+\|\tau\|^2]+\|D_n\tau\|^2+|\xi'|^2\|\tau\|^2=0$$

Since $\mathrm{Re}\,\lambda\geq 0$, we have $\|D_n\tau\|^2=0$, which implies that τ is a constant. But, $\tau(0)=0$, and therefore $\tau=0$, which implies that $P_1=P_2=P_3=0$ because $\lambda\neq 0$ and $\{e^{-A_j(\lambda,\xi')x_n}\}_{j=1,2,3}$ are linearly independent functions. This leads a contradiction. Therefore, (4.10) holds.

Now, we shall prove (4.2) and (4.3). To prove (4.2), first we observe that

$$|D_{\xi'}^{\alpha'}\det\Delta(\lambda,\xi')|\leq C_{\alpha'}|\lambda||\xi|^{-1-|\alpha'|} \tag{4.13}$$

for any $\alpha'\in\boldsymbol{N}_0^{n-1}$ when $|\lambda|/|\xi'|^2\leq\sigma_1$. In fact, recalling (4.5) and taking $f(t)=t^{\frac12}$, by the Bell formula we have

$$D_{\xi'}^{\alpha'}(\det\Delta(\lambda,\xi')) \\ =\sum_{j=1}^3\delta_j\left\{\sum_{\ell=1}^{|\alpha'|}f^{(\ell)}(\gamma_j^{-1}\lambda+|\xi'|^2)\sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha'\\|\alpha'_i|\geq 1}}\Gamma^{\alpha',\ell}_{\alpha'_1,\ldots,\alpha'_\ell}(D_{\xi'}^{\alpha'_1}|\xi'|^2)\cdots(D_{\xi'}^{\alpha'_\ell}|\xi'|^2)\right\}$$

where $\Gamma^{\alpha',\ell}_{\alpha'_1,\ldots,\alpha'_\ell}$ are suitable constants. By (4.6) we have

$$\sum_{j=1}^3\delta_jf^{(\ell)}(\gamma_j^{-1}\lambda+|\xi'|^2)=\frac12\cdots\left(\frac12-\ell+1\right)\sum_{j=1}^3\delta_j(\gamma_j^{-1}\lambda+|\xi'|^2)^{\frac12-\ell} \\ =\frac12\cdots\left(\frac12-\ell+1\right)|\xi'|^{1-2\ell}\left\{\sum_{j=1}^3\delta_j+O(t)\right\}\quad(t=\gamma_j^{-1}\lambda/|\xi'|^2\text{ and }|t|<1).$$

We may assume that what $|\lambda|/|\xi'|^2\leq\sigma_1$ implies that $|\gamma_j^{-1}\lambda/|\xi'|^2|\leq 1/2$. Since $\sum_{j=1}^3\delta_j=0$ as follows from (4.4), we have

$$\left|\sum_{j=1}^3\delta_jf^{(\ell)}(\gamma_j^{-1}\lambda+|\xi'|^2)\right|\leq C_\ell|\lambda||\xi'|^{-1-2\ell}.$$

On the other hand, by Lemma 3.2 we have

$$|D^{\alpha'_1}|\xi'|^2|\cdots|D^{\alpha'_\ell}|\xi'|^2|\leq C_\ell|\xi'|^{2\ell-|\alpha'|}$$

because $|\alpha'_1|+\cdots+|\alpha'_\ell|=|\alpha'|$. Combining these two inequalities implies (4.13).

To show (4.2), taking $f(t)=t^{-1}$, by the Bell formula we have

$$D^{\alpha'}_{\xi'}(\det\Delta(\lambda,\xi'))^{-1}=\sum_{j=1}^{3}\delta_j\Biggl\{\sum_{\ell=1}^{|\alpha'|}f^{(\ell)}(\det\Delta(\lambda,\xi'))\sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha'\\ |\alpha'_i|\geq 1}}\Gamma^{\alpha',\ell}_{\alpha'_1,\ldots,\alpha'_\ell}(D^{\alpha'_1}_{\xi'}\det\Delta(\lambda,\xi'))\cdots(D^{\alpha'_\ell}_{\xi'}\det\Delta(\lambda,\xi'))\Biggr\}, \tag{4.14}$$

and therefore using (4.13) and (4.1) we have

$$|D^{\alpha'}_{\xi'}(\det\Delta(\lambda,\xi'))^{-1}|\leq C_{\alpha'}\sum_{\ell=1}^{|\alpha'|}(\sigma_0|\lambda||\xi'|^{-1})^{-\ell-1}(|\lambda||\xi'|^{-1})^{\ell}|\xi'|^{-|\alpha'|}\leq C_{\alpha'}|\lambda|^{-1}|\xi'|^{1-|\alpha'|}$$

which shows (4.2).

To prove (4.3), first we observe that

$$|D^{\alpha'}_{\xi'}\det\Delta(\lambda,\xi')|\leq C_{\alpha'}(|\lambda|^{\frac12}+|\xi'|)^{1-|\alpha'|} \tag{4.15}$$

for any $\alpha'\in \boldsymbol{N}_0^{n-1}$ and $(\lambda,\xi')\in \boldsymbol{C}_+\times(\boldsymbol{R}^{n-1}\setminus\{0\})$. In fact, applying Lemma 3.2 to the formula (3.18), we have (4.15). Combining (4.14), (4.15) and (4.1), when $|\lambda/|\xi'|^2|\geq\sigma_1/2$ we have

$$|D^{\alpha'}_{\xi'}(\det\Delta(\lambda,\xi'))^{-1}|\leq C_{\alpha'}\sum_{\ell=1}^{|\alpha'|}(\det\Delta(\lambda,\xi'))^{-\ell-1}(|\lambda|^{\frac12}+|\xi'|)^{\ell-|\alpha'|}\leq C_{\alpha'}(|\lambda|^{\frac12}+|\xi'|)^{-1-|\alpha'|}$$

which shows (4.3). □

Now, we shall discuss the estimation of w, v and τ defined by (3.19) and (3.20). Before starting estimations of these function, we give a lemma to estimate $G=-D_nu_0$.

Lemma 4.2. *Let $1<p<\infty$ and $F=(f,g,h)\in\mathscr{H}_p(\boldsymbol{R}^n_+)$. Let u_0 be the function defined in* (3.1). *Then, there exists a $W(x)\in W^3_p(\boldsymbol{R}^n_+)$ such that $W|_{x_n=0}=D_nu_0|_{x_n=0}$ and*

$$|\lambda|^{\frac{a}{2}}\|D_n^b A_k(\lambda, D')^c |D'|^d W\|_{L_p(\boldsymbol{R}^n_+)} \leq C_{p,\epsilon}\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)}$$

for any $(a, b, c, d) \in \boldsymbol{N}_0^4$ *with* $a + b + c + d = 3$, $\lambda \in \Sigma_\epsilon$ *and* $k = 1, 2, 3$.

PROOF. In view of (3.1) and (2.14), we have

$$D_n \tilde{u_0}(\xi', 0) = \sum_{j=1}^{3} \int_{-\infty}^{\infty} \left\{ \frac{\kappa_j^0 i\xi_n}{\lambda + \gamma_j|\xi|^2} \hat{f}^o(\xi) - \sum_{k=1}^{2} \frac{\gamma_j \kappa_j^k i\xi_n}{\lambda(\lambda + \gamma_j|\xi'|^2)} \hat{g}^k(\xi) \right\} d\xi_n$$

where we have set $\hat{g}^1(\xi) = \hat{g}^o(\xi)$, $\tilde{g}^2(\xi) = \tilde{h}^o(\xi)$, $\kappa_j^0 = A_{j,0} + A_{j,1} + A_{j,2}$, $\kappa_j^1 = A_{j,0} + A_{j,1}$ and $\kappa_2 = A_{j,0}$. Changing the order of the integrations by Fubini's theorem, we have

$$\begin{aligned} &D_n \tilde{u_0}(\xi', 0) \\ &\quad = \sum_{j=1}^{3} \int_0^{\infty} (\tilde{f}(\xi', y_n)\kappa_j^0 - \sum_{k=1}^{2} \lambda^{-1}\gamma_j \kappa_j^k \tilde{g}^k(\xi', y_n)) \left(\int_{-\infty}^{\infty} \frac{(e^{-iy_n\xi_n} - e^{iy_n\xi_n}) i\xi_n}{\lambda + \gamma_j|\xi|^2} \, d\xi_n \right) dy_n \end{aligned}$$

If we write $\lambda + \gamma_j|\xi|^2 = \gamma_j(\xi_n + iA_j(\lambda, \xi'))(\xi_n - iA_j(\lambda, \xi'))$, by the residue theorem we have

$$\int_{-\infty}^{\infty} \frac{(e^{-iy_n\xi_n} - e^{iy_n\xi_n}) i\xi_n}{\lambda + \gamma_j|\xi|^2} \, d\xi_n = \frac{2\pi}{\gamma_j} e^{-A_j(\lambda,\xi')y_n},$$

and therefore we have

$$D_n \tilde{u_0}(\xi', 0) = 2\pi \sum_{j=1}^{3} \int_0^{\infty} e^{-A_j(\lambda,\xi')y_n} (\kappa_j^0 \gamma_j^{-1} \tilde{f}(\xi', y_n) - \sum_{k=1}^{2} \lambda^{-1}\kappa_j^k \tilde{g}^k(\xi', y_n)) \, dy_n.$$

Since $f(x', 0) = D_n f(x', 0) = 0$, we have

$$\int_0^{\infty} e^{-A_j(\lambda,\xi')y_n} \tilde{f}(\xi', y_n) \, dy_n = A_j(\lambda, \xi')^{-2} \int_0^{\infty} e^{-A_j(\lambda,\xi')y_n} D_n^2 \tilde{f}(\xi', y_n) \, dy_n,$$

and therefore we define W by the formula: $W(\xi', x_n) = 2\pi(W_1(\xi', x_n) - W_2(\xi', x_n))$, where we have set

$$W_1(x) = \sum_{j=1}^{3} \kappa_j^0 \int_0^\infty \mathscr{F}_{\xi'}^{-1}[A_j(\lambda,\xi')^{-2} e^{-A_j(\lambda,\xi')(x_n+y_n)} D_n^2 \tilde{f}(\xi', y_n)](x')\,dy_n$$

$$W_2(x) = \sum_{j=1}^{3}\sum_{k=1}^{2} \kappa_j^k \lambda^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}[e^{-A_j(\lambda,\xi')(x_n+y_n)} \tilde{g}^k(\xi', y_n)](x')\,dy_n.$$

Obviously, we have $W(x',0) = D_n u_0(x',0)$. Since $\sum_{j=1}^3 \kappa_j^k = 0$ for $k = 1,2$ as follows from (2.11), we can write

$$\begin{aligned} W_2(x) = &\sum_{j=1}^{3}\sum_{k=1}^{2} \kappa_j^k \lambda^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_0(|\lambda|/|\xi'|^2)(e^{-A_j(\lambda,\xi')(x_n+y_n)} \\ &- e^{-|\xi'|(x_n+y_n)})\tilde{g}^k(\xi',y_n)](x')\,dy_n \\ &+ \sum_{j=1}^{3}\sum_{k=1}^{2} \kappa_j^k \lambda^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}[\psi_\infty(|\lambda|/|\xi'|^2) e^{-A_j(\lambda,\xi')(x_n+y_n)} \tilde{g}^k(\xi',y_n)](x')\,dy_n \end{aligned}$$

where we defined ψ_0 and ψ_∞ by the formulas : $\psi_0 = \chi$ and $\psi_\infty = 1-\chi$ with function $\chi \in C_0^\infty(\boldsymbol{R})$ such that $\chi(s) = 1$ for $|s| \le 1/2$ and $\chi(s) = 0$ for $|s| \ge 1$. Applying Lemmas 3.5 and 3.7 with $\ell = 0$ we have

$$|\lambda|^{\frac{a}{2}} \| D_n^b A_k(\lambda, D')^c |D'|^d W_2 \|_{L_p(\boldsymbol{R}_+^n)} \le C_{p,\epsilon} \|(g,h)\|_{L_p(\boldsymbol{R}_+^n)}$$

for any $(a,b,c,d) \in \boldsymbol{N}_0^4$ with $a+b+c+d = 3$ and $\lambda \in \Sigma_\epsilon$.

On the other hand, given $(a,b,c,d) \in \boldsymbol{N}_0^4$, studying the cases where $d = 0$ and $d \ge 1$, by Lemmas 3.2, 3.3 and 3.4 we see easily that

$$|\lambda|^{\frac{a}{2}} \| D_n^b A_k(\lambda, D')^c |D'|^d W_1 \|_{L_p(\boldsymbol{R}_+^n)} \le C_{p,\epsilon} \|D_n^2 f\|_{L_p(\boldsymbol{R}_+^n)}$$

for any $\lambda \in \Sigma_\epsilon$, which completes the proof of Lemma 4.2. □

Under above preparations, we shall estimate w, v and τ. Let W be a function constructed in Lemma 4.2 and set $H(x) = -W(x)$, and then $H(x',0) = -(D_n u_0)(x',0)$. We start with the estimate of $\lambda w = v$ and τ. For this purpose, setting

$$z_j(x) = \mathscr{F}_{\xi'}^{-1}\left[\frac{\lambda e^{-A_j(\lambda,\xi')x_n}}{\det \Delta(\lambda,\xi')} \tilde{H}(\xi',0)\right](x') \tag{4.16}$$

we shall show that

$$\|(|\lambda|z_j, |\lambda|^{\frac{1}{2}}\nabla z_j, \nabla^2 z_j)\|_{L_p(\boldsymbol{R}^n_+)} \le C\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)} \tag{4.17}$$

for any $\lambda \in \boldsymbol{C}_+ = \{\lambda \in \boldsymbol{C} \mid \operatorname{Re}\lambda \ge 0\}$. In fact, using the symbols z_j and (3.19) we have

$$\lambda w(x) = v(x) = \sum_{j=1}^{3} \delta_j z_j(x), \ \ \tau(x) = -\sum_{j=1}^{3} \delta'_j z_j \tag{4.18}$$

where δ_j $(j = 1, 2, 3)$ are the same as in (4.4) and $\delta'_j = (\gamma_j^2 + 2)\delta_j$ $(j = 1, 2, 3)$. Therefore, (4.17) and (4.18) imply that

$$\sum_{j=0}^{2} |\lambda|^{\frac{2-j}{2}} \|\nabla^j(|\lambda|w, v, \tau)\|_{L_p(\boldsymbol{R}^n_+)} \le C_p\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)} \tag{4.19}$$

for any $\lambda \in \boldsymbol{C}_+$.

To prove (4.17), using the Volevich trick [**27**], we write

$$z_j(x) = \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\lambda e^{-A_j(\lambda,\xi')(x_n+y_n)}}{\det\Delta(\lambda,\xi')A_j(\lambda,\xi')^2}\,\tilde{K}_{j\lambda}(\xi', y_n)\right](x')\,dy_n \tag{4.20}$$

where $K_{j\lambda}(x) = A_j(\lambda, D')^3 H(x) - A_j(\lambda, D')^2 D_n H(x)$. Recalling that $H = -W$ and using Lemma 4.2 and (3.22) we have

$$\|K_{j\lambda}\|_{L_p(\boldsymbol{R}^n_+)} \le C_p\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)} \quad (\lambda \in \Sigma_\epsilon). \tag{4.21}$$

Using a function $\chi(s) \in C_0^\infty(\boldsymbol{R})$ such that $\chi(s) = 1$ for $|s| \le \sigma_1/2$ and $\chi(s) = 0$ for $|s| \ge \sigma_1$, we devide z_j into the following two parts:

$$z_{j,N}(x) = \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\chi_N(|\lambda|/|\xi'|^2)\,\frac{\lambda e^{-A_j(\lambda,\xi')(x_n+y_n)}}{\det\Delta(\lambda,\xi')A_j(\lambda,\xi')^2}\,\tilde{K}_{j\lambda}(\xi', y_n)\right](x')\,dy_n \tag{4.22}$$

for $N = 0$ and ∞, where $\chi_0 = \chi$, $\chi_\infty = 1 - \chi$ and σ_1 is the same constant as in Lemma 4.1. To prove (4.17), for $a = 0$, 1, 2 and $\alpha' \in \boldsymbol{N}^{n-1}$ with $|\alpha'| \le 2 - a$ we write

$$|\lambda|^{\frac{a}{2}} D_{x'}^{\alpha'} D_n^{2-a-|\alpha'|} z_{j,N}(x) = (-1)^{2-a-|\alpha'|} \int_0^\infty \mathscr{F}_{\xi'}^{-1}[\chi_N(|\lambda|/|\xi'|^2) m_{a,\alpha'}(\lambda,\xi') \\ |\xi'| e^{-A_j(\lambda,\xi')(x_n+y_n)} \tilde{K}_{j\lambda}(\xi',y_n)](x')\,dy_n$$

where we have set $m_{a,\alpha'}(\lambda,\xi') = \lambda|\xi'|^{-1}(\det\Delta(\lambda,\xi'))^{-1}|\lambda|^{\frac{a}{2}}(i\xi')^{\alpha'}A_j(\lambda,\xi')^{-a-|\alpha'|}$. If $N=\infty$, then we assume that $d \geq 1$. By (4.2), (4.3) and Lemma 3.2 we have

$$|D_{\xi'}^{\beta'} m_{a,\alpha'}(\lambda,\xi')| \leq C_{\beta'}|\xi'|^{-|\beta'|} \tag{4.23}$$

for any $\beta' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \boldsymbol{C}_+$ and $\xi' \in \operatorname{supp}\chi_N(|\lambda|/|\xi'|^2)$. On the other hand, employing the same argument as in the proof of (3.29), we have

$$|D_{\xi'}^{\alpha'}\chi_N(|\lambda|/|\xi'|^2)| \leq C_{\alpha'}(|\lambda|^{\frac{1}{2}} + |\xi'|)^{-|\alpha'|} \tag{4.24}$$

which combined with (4.23) implies that $m_{a,\alpha'}(\lambda,\xi')\chi_N(|\lambda|/|\xi'|^2)$ is a Fourier multiplier of second kind. Therefore, applying Lemma 3.3 and using (4.21) we have

$$|\lambda|^{\frac{a}{2}} \|D_{x'}^{\alpha'} D_n^{2-a-|\alpha'|} z_{j,N}\|_{L_p(\boldsymbol{R}_+^n)} \leq C_p \|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}_+^n)}. \tag{4.25}$$

for any $(a,\alpha') \in \boldsymbol{N}_0^n$ with $a + |\alpha'| \leq 2$, where we assume that $|\alpha'| \geq 1$ when $N=\infty$.

When $N=\infty$ and $|\alpha'|=0$, we write

$$|\lambda|^{\frac{a}{2}} D_n^{2-a} z_{j,\infty}(x) = (-1)^{2-a} \int_0^\infty \mathscr{F}_{\xi'}^{-1}[\chi_\infty(|\lambda|/|\xi'|^2) m_{a,0}(\lambda,\xi') \\ |\lambda|^{\frac{1}{2}} e^{-A_j(\lambda,\xi')(x_n+y_n)} \tilde{K}_{j\lambda}(\xi',y_n)](x')\,dy_n$$

where we have set $m_{a,0}(\lambda,\xi') = \lambda|\lambda|^{-\frac{1}{2}}(\det\Delta(\lambda,\xi'))^{-1}|\lambda|^{\frac{a}{2}}A_j(\lambda,\xi')^{-a}$. By (4.3) and Lemma 3.2 we have

$$|D_{\xi'}^{\beta'} m_{a,0}(\lambda,\xi')| \leq C_{\beta'}(|\lambda|^{\frac{1}{2}} + |\xi'|)^{-|\beta'|}$$

for any $\beta' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \boldsymbol{C}_+$ and $\xi' \in \operatorname{supp}\chi_\infty(|\lambda|/|\xi'|^2)$, which combined with (4.24) implies that $\chi_\infty(|\lambda|/|\xi'|^2)m_{a,0}(\lambda,\xi')$ is a Fourier multiplier of first kind. Therefore applying Lemma 3.3 and using (4.21), we have

$$|\lambda|^{\frac{a}{2}}\|D_n^{2-a}z_{j,\infty}\|_{L_p(\boldsymbol{R}^n_+)} \le C_p\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)}$$

for any $\lambda \in \boldsymbol{C}_+$, which combined with (4.25) implies (4.17), because $z_j = z_{j,0} + z_{j,\infty}$.

Now, we shall show that

$$\|(|\lambda|^{\frac{1}{2}}\nabla^3 w, \nabla^4 w)\|_{L_p(\boldsymbol{R}^n_+)} \le C\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)}. \tag{4.26}$$

Since $\sum_{j=1}^3 \delta_j = 0$ as follows from (4.4), by (4.16) and (4.18) we devide $w(x)$ into the following two parts:

$$w_0(x) = \sum_{j=1}^3 \delta_j \mathscr{F}^{-1}_{\xi'}\left[\frac{(e^{-A_j(\lambda,\xi')x_n} - e^{-|\xi'|x_n})\chi_0(|\lambda|/|\xi'|^2)}{\det\Delta(\lambda,\xi')}\tilde{H}(\xi',0)\right](x'),$$
$$w_\infty(x) = \sum_{j=1}^3 \delta_j \mathscr{F}^{-1}_{\xi'}\left[\frac{e^{-A_j(\lambda,\xi')x_n}\chi_\infty(|\lambda|/|\xi'|^2)}{\det\Delta(\lambda,\xi')}\tilde{H}(\xi',0)\right](x').$$

where χ_0 and χ_∞ are the same functions as in (4.22). First we consider $w_0(x)$. Using the Volevich trick, we write

$$\begin{aligned} w_0(x) &= \sum_{j=1}^3 \delta_j \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{(e^{-A_j(\lambda,\xi')(x_n+y_n)} - e^{-|\xi'|(x_n+y_n)})\chi_0(|\lambda|/|\xi'|^2)}{(\det\Delta(\lambda,\xi'))A_j(\lambda,\xi')^2}\tilde{K}_{j\lambda}(\xi',y_n)\right](x')\,dy_n \\ &+ \sum_{j=1}^3 \delta_j \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{(A_j(\lambda,\xi') - |\xi'|)e^{-A_j(\lambda,\xi')(x_n+y_n)}\chi_0(|\lambda|/|\xi'|^2)}{(\det\Delta(\lambda,\xi'))A_j(\lambda,\xi')^3}\tilde{N}_{j\lambda}(\xi',y_n)\right](x')dy_n \\ &= w_0^0(x) + w_0^1(x) \end{aligned}$$

where we have set $K_{j\lambda} = -(A_j(\lambda,D')^3 W - A_j(\lambda,D')^2 D_n W)$ and $N_{j\lambda} = -A_j(\lambda,D')^3 W$. By (4.21), Lemma 4.2 and (3.22) we have

$$\|(K_{j\lambda}, N_{j\lambda})\|_{L_p(\boldsymbol{R}^n_+)} \le C_p\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}^n_+)} \tag{4.27}$$

for any $\lambda \in \boldsymbol{C}_+$ and $j = 1, 2, 3$. If we set

$$m_j^k(\lambda,\xi') = \lambda(\det\Delta(\lambda,\xi'))^{-1}|\xi'|^{1+k}A_j(\lambda,\xi')^{-(2+k)}$$

for $k = 0, 1$, then we have

$$w_0^0(x) = \sum_{j=1}^3 \delta_j \lambda^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\chi_0(|\lambda|/|\xi'|^2)|\xi'|^{-1} m_j^0(\lambda,\xi') \\ (e^{-A_j(\lambda,\xi')(x_n+y_n)} - e^{-|\xi'|(x_n+y_n)})\tilde K_{j\lambda}(\xi',y_n)\Big](x')\,dy_n,$$

$$w_0^1(x) = \sum_{j=1}^3 \delta_j \lambda^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}\Big[\chi_0(|\lambda|/|\xi'|^2)|\xi'|^{-2} m_j^1(\lambda,\xi') \\ (A_j(\lambda,\xi') - |\xi'|)e^{-A_j(\lambda,\xi')(x_n+y_n)}\tilde N_{j\lambda}(\xi',y_n)\Big](x')\,dy_n.$$

By (4.2) and Lemm 3.2 we have

$$|D_{\xi'}^{\alpha'} m_j^k(\lambda,\xi')| \le C_{\alpha'}|\xi'|^{-|\alpha'|}$$

for any $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \boldsymbol{C}_+$ and $\xi' \in \operatorname{supp}\chi_0(|\lambda|/|\xi'|^2)$. Therefore, applying Lemma 3.5 and using (4.27) we have

$$\|(|\lambda|^{\frac12}\nabla^3 w_0, \nabla^4 w_0)\|_{L_p(\boldsymbol{R}_+^n)} \le C_p\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}_+^n)} \tag{4.28}$$

for any $\lambda \in \boldsymbol{C}_+$.

Finally, we consider $w_\infty(x)$. Using the Volevich trick, we write

$$w_\infty(x) \\ = \sum_{j=1}^3 \tilde\delta_j \lambda^{-1} \int_0^\infty \mathscr{F}_{\xi'}^{-1}[e^{-A_j(\lambda,\xi')(x_n+y_n)}\chi_\infty(|\lambda|/|\xi'|^2)|\lambda|^{-\frac12} m_\infty(\lambda,\xi')\tilde K_{j\lambda}(\xi',y_n)](x')\,dy_n$$

where we have set $m_\infty(\lambda,\xi') = \lambda|\lambda|^{\frac12}(\det\Delta(\lambda,\xi'))^{-1}A_j(\lambda,\xi')^{-2}$. By (4.3) and Lemma 3.2 we have

$$|D_{\xi'}^{\alpha'} m_\infty(\lambda,\xi')| \le C_{\alpha'}(|\lambda|^{\frac12} + |\xi'|)^{-|\alpha'|}$$

for any $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \boldsymbol{C}_+$ and $\xi' \in \operatorname{supp}\chi_\infty(|\lambda|/|\xi'|^2)$. By Lemma 3.7 and (4.27) we have

$$\|(|\lambda|^{\frac12}\nabla^3 w_\infty, \nabla^4 w_\infty)\|_{L_p(\boldsymbol{R}_+^n)} \le C_p\|(D_n^2 f, g, h)\|_{L_p(\boldsymbol{R}_+^n)}$$

for any $\lambda \in \boldsymbol{C}_+$, which combined with (4.28) implies Theorem 1.1.

5. Generation of analytic semigroup and its asymptotic behaviour.

In this section, we shall show Theorems 1.2 and 1.4. First, we shall give

A PROOF OF THEOREM 1.2 Let $\mathscr{A}_p$ be the operator defined in (1.9). By Theorem 1.1 $\mathscr{A}_p$ is densely defined, closed operator on $\mathscr{H}_p(\boldsymbol{R}^n_+)$. Let $\rho(\mathscr{A}_p)$ and $(\lambda I - \mathscr{A}_p)^{-1}$ be the resolvent set and resolvent operator of $\mathscr{A}_p$, respectively. Set $\|F\|_{\mathscr{H}_p(\boldsymbol{R}^n_+)} = \|f\|_{W^2_p(\boldsymbol{R}^n_+)} + \|(g,h)\|_{L_p(\boldsymbol{R}^n_+)}$ for $F = {}^T(f,g,h) \in \mathscr{H}_p(\boldsymbol{R}^n_+)$. Then, by Theorem 1.1 we see that $\boldsymbol{C}_+ \subset \rho(\mathscr{A}_p)$ and that there exists a constant $M > 0$ such that

$$|\lambda| \|(\lambda I - \mathscr{A}_p)^{-1} F\|_{\mathscr{H}_p(\boldsymbol{R}^n_+)} \leq M \|F\|_{\mathscr{H}_p(\boldsymbol{R}^n_+)} \tag{5.1}$$

for any $\lambda \in \boldsymbol{C}_+$ with $|\lambda| \geq 1$ and $F \in \mathscr{H}_p(\boldsymbol{R}^n_+)$. If we write $\lambda I - \mathscr{A}_p = (i\beta - \mathscr{A}_p)(I - \alpha(i\beta - \mathscr{A}_p)^{-1})$ for $\lambda = -\alpha + i\beta$ $(\alpha > 0)$, by (5.1) we see that whenever $M\alpha|\beta|^{-1} < 1$ and $\lambda = -\alpha + i\beta$, the resolvent $(\lambda I - \mathscr{A}_p)^{-1}$ exists and satisfies the estimate:

$$|\lambda| \|(\lambda I - \mathscr{A}_p)^{-1} F\|_{\mathscr{H}_p(\boldsymbol{R}^n_+)} \leq 2M \|F\|_{\mathscr{H}_p(\boldsymbol{R}^n_+)} \tag{5.2}$$

for any $F \in \mathscr{H}_p(\boldsymbol{R}^n_+)$. If we set $\Lambda_\mu = \{\lambda \in \boldsymbol{C} \mid |\arg\lambda| \leq (\pi/2) + \mu,\ |\lambda| \geq 1\}$ with $\mu = \tan^{-1}(1/M)$, then $\Lambda \subset \rho(\mathscr{A}_p)$ and the estimate (5.2) holds for any $\lambda \in \Lambda_\mu$, which shows that $\mathscr{A}_p$ generates an analytic semigroup on $\mathscr{H}_p(\boldsymbol{R}^n_+)$. This completes the proof of Theorem 1.2. □

Now, we shall prove Theorem 1.4. The idea is the same as in the proof of Theorem 1.2 as above, but we have to consider any small neighborhood of $\lambda = 0$ to get polynomial decay rate of solutions to (1.1), (1.3) and (1.4). For this purpose, we use $\dot{\mathscr{H}}(\boldsymbol{R}^n_+)$ and $\dot{\mathscr{A}}_p$ instead of $\mathscr{H}(\boldsymbol{R}^n_+)$ and $\mathscr{A}_p$.

A PROOF OF THEOREM 1.4 Let $\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$, $\dot{\mathscr{D}}_p(\mathscr{H})$, $\|\ \|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}$ and $\dot{\mathscr{A}}_p$ be the same as in (1.13). Since $C^\infty_0(\boldsymbol{R}^n_+)^3 \subset \dot{\mathscr{D}}_p(\mathscr{H}) \subset \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$ and $C^\infty_0(\boldsymbol{R}^n_+)^3$ is dense in $\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$, by Theorem 1.1 we see that $\dot{\mathscr{A}}_p$ is a densely defined, closed operator on $\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$. Let $\rho(\dot{\mathscr{A}}_p)$ and $(\lambda I - \dot{\mathscr{A}}_p)^{-1}$ be the set and the resolvent operator of $\dot{\mathscr{A}}_p$, respectively. Then, by Theorem 1.1 we see that $\boldsymbol{C}_+ \subset \rho(\dot{\mathscr{A}}_p)$ and that there exists a constant C independent of $\lambda \in \boldsymbol{C}_+$ and $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$ such that

$$|\lambda| \|(\lambda I - \dot{\mathscr{A}}_p)^{-1} F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq C \|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}. \tag{5.3}$$

Employing the same argument as in the proof of Theorem 1.2, by (5.3) we see that whenever $C\alpha|\beta|^{-1} < 1$, $\alpha > 0$ and $\lambda = -\alpha + i\beta$, the resolvent $(\lambda I - \dot{\mathscr{A}}_p)^{-1}$ exists and satisfies the estimate:

$$|\lambda|\|(\lambda I - \dot{\mathscr{A}}_p)^{-1}F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq 2C\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \tag{5.4}$$

for any $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$. Therefore, setting $\sigma = \tan^{-1}(1/C)$, we have

$$\Xi_\sigma = \left\{\lambda \setminus \{0\} \middle| \left|\arg\lambda\right| < \frac{\pi}{2} + \sigma\right\} \subset \rho(\dot{\mathscr{A}}_p) \tag{5.5}$$

and (5.4) holds for any $\lambda \in \Xi_\sigma$. It follows from these facts that $\dot{\mathscr{A}}_p$ is a sectorial operator, and therefore $\mathscr{A}_p$ generates an analytic semigroup $\{\dot{T}_p(t)\}_{t\geq 0}$ on $\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$.

Now, we shall show the estimate (1.14). First we consider the case where $p = q$. Let θ be a number such that $\pi/2 < \theta < (\pi/2) + \sigma$ and then by (5.3) and (5.4) for any $\delta > 0$ we have

$$|\lambda|\|(\lambda I - \dot{\mathscr{A}}_p)^{-1}F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq 2C\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \tag{5.6}$$

whenever $\lambda \in \delta + \Xi_\theta = \{\delta + z \mid |\arg z| \leq (\pi/2) + \theta\}$. Let $\Gamma^\pm_\delta$ be contours defined by the formulas:

$$\Gamma^+_\delta : \lambda = \delta + se^{i(\theta + \frac{\pi}{2})}\ (s : \infty \to 0);\ \Gamma^-_\delta : \lambda = \delta + se^{-i(\theta + \frac{\pi}{2})}\ (s : 0 \to \infty),$$

and set $\Gamma_\delta = \Gamma^+_\delta \cup \Gamma^-_\delta$. Then, by well-known theory of analytic semigroup (cf. I. I. Vrabie [**26**, Chapter 7, Section 7.1]) we have

$$\dot{T}_p(t) = \frac{1}{2\pi i}\int_{\Gamma_\delta} e^{\lambda t}(\lambda I - \dot{\mathscr{A}}_p)^{-1}\, d\lambda.$$

By (5.4) and Fubini's theorem, we have

$$D^j_t\dot{T}_p(t) = \frac{1}{2\pi}\int_{\Gamma_\delta} \lambda^j e^{\lambda t}(\lambda I - \dot{\mathscr{A}}_p)^{-1}\, d\lambda.$$

After this observation, by the theorem of Cauchy in the theory of functions of one complex variable we change the contour from Γ_δ to $\Gamma_{\delta,t}$ which is defined by $\Gamma_{\delta,t} = \Gamma^+_{\delta,t} \cup C_t \cup \Gamma^-_{\delta,t}$, where

$$\Gamma^+_{\delta,t} : \lambda = \delta + se^{i(\theta+\frac{\pi}{2})} \ (s : \infty \to 1/t);\ C_t : \lambda = \delta + (1/t)e^{is} \left(s : \theta + \frac{\pi}{2} \to -\left(\theta + \frac{\pi}{2}\right)\right);$$
$$\Gamma^-_{\delta,t} : \lambda = \delta + se^{-i(\theta+\frac{\pi}{2})} \ (s : 1/t \to \infty).$$

Then, using (5.6), we see easily that

$$\|D_t^j \dot{T}_p(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq C_j e^{\delta t} t^{-j} \|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}$$

for any $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$, where C_j is independent of $\delta > 0$ and $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$. Letting $\delta \to 0$ we have

$$\|D_t^j \dot{T}_p(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq C_j t^{-j} \|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \tag{5.7}$$

for any $t > 0$ and $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$. Since

$$D_t \dot{T}_p(t)F = A\dot{T}_p(t)F,\ \dot{T}_p(t)F \in \dot{\mathscr{D}}_p(\boldsymbol{R}^n_+)$$

for any $t > 0$ and $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$ as follows from theory of analytic semigroup, we have

$$\gamma\dot{T}_p(t)F - A\dot{T}_p(t)F = \gamma\dot{T}_p(t)F - D_t\dot{T}_p(t)F \quad (t > 0)$$

for any $\gamma > 0$. Therefore applying Theorem 1.1 with $\lambda = \gamma$, by (5.7) with $j = 0$ and 1 we have

$$\|\nabla^2 \dot{T}_p(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq C(\|\gamma\dot{T}_p(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} + \|D_t\dot{T}_p(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}) \leq C(\gamma + t^{-1})\|F\|_{\dot{\mathscr{H}}(\boldsymbol{R}^n_+)}.$$

Since C is independent of $\gamma > 0$, letting $\gamma \to 0$ we have

$$\|\nabla^2 \dot{T}_p(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq Ct^{-1} \|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \tag{5.8}$$

for any $t > 0$ and $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$. To obtain

$$\|\nabla \dot{T}_p(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq Ct^{-\frac{1}{2}}\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}. \tag{5.9}$$

for any $t>0$ and $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$, we use (5.7) with $j=0$, (5.8) and the interpolation inequality: $\|\nabla v\|_{L_p(\boldsymbol{R}^n_+)} \leq M\|\nabla^2\|^{\frac{1}{2}}_{L_p(\Omega)}\|v\|^{\frac{1}{2}}_{L_p(\boldsymbol{R}^n_+)}$. Combining (5.7), (5.8) and (5.9), we have the estimate (1.14) when $p=q$.

Now, we consider the case where $p<q\leq\infty$. First to consider the case where $n((1/p)-(1/q))<1$, we use the Gagliardo-Nirenberg-Sobolev inequality:

$$\|v\|_{L_q(\boldsymbol{R}^n_+)} \leq C_{p,q}\|\nabla v\|^{n\left(\frac{1}{p}-\frac{1}{q}\right)}_{L_p(\boldsymbol{R}^n_+)}\|v\|^{1-n\left(\frac{1}{p}-\frac{1}{q}\right)}_{L_p(\boldsymbol{R}^n_+)}. \tag{5.10}$$

By (5.7) with $j=0$, (5.8) and (5.9) we have

$$\|\nabla^j \dot{T}_p(t)F\|_{\dot{\mathscr{H}}_q(\boldsymbol{R}^n_+)} \leq C_{p,q}t^{-\frac{j}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \tag{5.11}$$

for $j=0$, 1 provided that $p<q\leq\infty$ and $n((1/p)-(1/q))<1$. At this point, we remark the following fact: Since $C_0^\infty(\boldsymbol{R}^n_+)^3$ is dense in $\dot{\mathscr{H}}_q(\boldsymbol{R}^n_+)$ for $1\leq q<\infty$, by (5.11) we can extend $\{\dot{T}_p(t)\}_{t\geq 0}$ to $\dot{\mathscr{H}}_q(\boldsymbol{R}^n_+)$ for any q with $p<q<\infty$, and therefore from now on we write $\{\dot{T}(t)\}_{t\geq 0}$ instead of $\{\dot{T}_p(t)\}_{t\geq 0}$. Of course, the inequalities (5.7), (5.8) and (5.9) hold, replacing $\dot{T}_p(t)$ by $\dot{T}(t)$ and exponent p by q, respectively. Moreover, inequality (5.11) holds, replacing $T_p(t)$ by $\dot{T}(t)$ and exponents p and q by q and r whenever $p\leq q\leq r\leq\infty$ and $n((1/q)-(1/r))<1$.

To prove (1.14) in the case where $j=0$ and $n((1/p)-(1/q))\geq 1$, we choose $q_0,\dots,q_\ell$ in such a way that $q_0=q>q_1>\cdots>q_{\ell-1}>q_\ell=p$ and $n((1/q_{j+1})-(1/q_j))<1$ $(j=0,1,\dots,\ell-1)$. Since $\{\dot{T}(t)\}_{t\geq 0}$ is semigroup, we have $\dot{T}(t)F=\dot{T}(t/\ell)\cdots\dot{T}(t/\ell)F$, and therefore applying (5.3) with $j=0$ ℓ-times implies that

$$\|\dot{T}(t)F\|_{\dot{\mathscr{H}}_q(\boldsymbol{R}^n_+)} \leq \prod_{j=0}^{\ell-1} C_{q_{j+1},q_j}(t/\ell)^{-\frac{n}{2}\left(\frac{1}{q_{j+1}}-\frac{1}{q_j}\right)}\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \leq C_{p,q}t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)},$$

which shows that the estimate (1.14) holds for $j=0$.

For the gradient estimate, we choose q_1 in such a way that $p<q_1<q\leq\infty$ and $n((1/q_1)-(1/q))<1$. The semigroup property implies that $\nabla\dot{T}(t)F=\nabla\dot{T}(t/2)[\dot{T}(t/2)F]$, and therefore by (5.11) with $j=1$ and (1.14) with $j=0$ we have

$$\begin{aligned}\|\nabla\dot{T}(t)F\|_{\dot{\mathscr{H}}_q(\boldsymbol{R}^n_+)} &\le C_{q_1,q}(t/2)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q_1}-\frac{1}{q}\right)}\|\dot{T}(t/2)F\|_{\dot{\mathscr{H}}_{q_1}(\boldsymbol{R}^n_+)} \\ &\le C_{q_1,q}C_{p,q_1}(t/2)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}.\end{aligned}$$

Analogously, writing $\nabla^2\dot{T}(t)F = \nabla^2\dot{T}(t/2)[\dot{T}(t/2)F]$, by (5.8) and (1.14) with $j = 0$ we have

$$\|\nabla^2\dot{T}(t)F\|_{\dot{\mathscr{H}}_q(\boldsymbol{R}^n_+)} \le C_{p,q}t^{-1-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)}$$

for any $t > 0$ and $F \in \dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)$, where to use (5.8) we needed the restriction that $p \le q < \infty$, which completes the proof of Theorem 1.2.

6. Concluding remark.

For any F, $G \in C_0^\infty(\boldsymbol{R}^n_+)$, we have

$$(\dot{T}(t)F, G) = (F, \dot{T}(t)G),$$

where $(U_1, U_2) = \sum_{j=1}^3 \int_{\boldsymbol{R}^n_+} u_{j1}(x)u_{j2}(x)\,dx$ for $U_k = {}^T(u_{1k}, u_{2k}, u_{3k})$ $(k = 1, 2)$. By (1.14) we have

$$|(\dot{T}(t)F, G)| \le \|F\|_{\dot{\mathscr{H}}_1(\boldsymbol{R}^n_+)}\|\dot{T}(t)G\|_{\dot{\mathscr{H}}_\infty(\boldsymbol{R}^n_+)} \le C_{p',\infty}t^{-\frac{n}{2p'}}\|F\|_{\dot{\mathscr{H}}_1(\boldsymbol{R}^n_+)}\|G\|_{\dot{\mathscr{H}}_{p'}(\boldsymbol{R}^n_+)}$$

where p' is a dual exponent of p, from which it follows that

$$\|\dot{T}(t)F\|_{\dot{\mathscr{H}}_p(\boldsymbol{R}^n_+)} \le C_{p',\infty}t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\|F\|_{\dot{\mathscr{H}}_1(\boldsymbol{R}^n_+)} \tag{6.1}$$

for any $F \in C_0^\infty(\boldsymbol{R}^n_+)^3$ and $t > 0$. Since $C_0^\infty(\boldsymbol{R}^n_+)^3$ is dense in $\dot{\mathscr{H}}_1(\boldsymbol{R}^n_+)$, we can extend $\{\dot{T}(t)\}_{t\ge0}$ to $\dot{\mathscr{H}}_1(\boldsymbol{R}^n_+)$ and we have (6.1) for any p with $1 < p < \infty$.

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Yuka Naito
Department of Mathematical Sciences
School of Science and Engineering
Waseda University
Ohkubo 3-4-1, Shinjuku-ku
Tokyo 169-8555, Japan

Yoshihiro Shibata
Department of Mathematical Sciences
School of Science and Engineering
Waseda University
Ohkubo 3-4-1, Shinjuku-ku
Tokyo 169-8555, Japan