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Weak extension theorem for measure-preserving homeomorphisms of noncompact manifolds

By Tatsuhiko Yagasaki

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Abstract. In this paper we deduce weak type extension theorems for the groups of measure-preserving homeomorphisms of noncompact manifolds. As an application, we show that the group of measure-preserving homeomorphisms with compact support of a noncompact connected manifold, endowed with the Whitney topology, is locally contractible.

1. Introduction.

In this paper we study some topological properties of groups of measure-preserving homeomorphisms and spaces of measure-preserving embeddings in noncompact manifolds (cf. [**4**], [**5**], [**8**], [**11**], [**12**]). Suppose M is a σ-compact topological n-manifold possibly with boundary and U is an open subset of M. Let $\mathscr{E}^*(U, M)$ denote the space of proper embeddings of U into M endowed with the compact-open topology. The local deformation lemma for $\mathscr{E}^*(U, M)$ [**6**], [**7**] asserts that for any compact subset C of U and any compact neighborhood K of C in U there exists a deformation φ_t $(t \in [0,1])$ of an open neighborhood $\mathscr{V}$ of the inclusion map $i_U : U \subset M$ in $\mathscr{E}^*(U, M)$ such that $\varphi_0(f) = f$, $\varphi_1(f)|_C = i_C$ and $\varphi_t(f)|_{U-K} = f|_{U-K}$ $(t \in [0,1])$ for each $f \in \mathscr{V}$. For a subset A of M let $\mathscr{H}_A(M)$ denote the group of homeomorphisms h of M with $h|_A = \mathrm{id}_A$ endowed with the compact-open topology. The local deformation lemma is equivalent to the following weak type extension theorem: for any compact neighborhood L of C in U there exists a neighborhood $\mathscr{V}$ of i_U in $\mathscr{E}^*(U, M)$ and a homotopy $s_t : \mathscr{U} \to \mathscr{H}_{M-L}(M)$ such that $s_0(f) = \mathrm{id}_M$ and $s_1(f)|_C = f|_C$ $(f \in \mathscr{U})$.

This result motivates the following general formulation: Suppose G is a topological group acting on M with the unit element e. Consider the subspace of $\mathscr{E}^*(U, M)$ defined by $\mathscr{E}^G(U, M) = \{\widehat{g}|_U \mid g \in G\}$, where $\widehat{g}$ denotes the homeo-

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morphism on M induced by $g \in G$. The weak extension theorem for the group action of G on M asserts that there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^G(U, M)$ and a homotopy $s_t : \mathscr{U} \to G$ such that $s_0(f) = e$ and $\widehat{s_1(f)}|_C = f|_C$ $(f \in \mathscr{U})$.

Suppose μ is a good Radon measure on M with $\mu(\partial M) = 0$. Let $\mathscr{H}(M; \mu)$ and $\mathscr{H}(M; \mu\text{-reg})$ denote the subgroups of $\mathscr{H}(M)$ consisting of μ-preserving homeomorphisms and μ-biregular homeomorphisms of M and let $\mathscr{E}^*(U, M; \mu\text{-reg})$ denote the subspace of $\mathscr{E}^*(U, M)$ consisting of μ-biregular proper embeddings of U into M. In [**8**] A. Fathi obtained a local deformation lemma for the space $\mathscr{E}^*(U, M; \mu\text{-reg})$ ([**8**, Theorem 4.1]). This is reformulated as the weak extension theorem for the group $\mathscr{H}(M; \mu\text{-reg})$ ([**8**, Corollary 4.2]). In the case M is compact and connected, he also obtained a selection theorem for μ-biregular measures on M ([**8**, Theorem 3.3]) and used these results to deduce the weak extension theorem for the group $\mathscr{H}(M; \mu)$ ([**8**, Theorem 4.12]).

In this paper we are concerned with the case where M is non-compact. In [**4**] R. Berlanga has already extended the selection theorem for μ-biregular measures to the non-compact case ([**4**, Theorem 4.1]). We combine these results to obtain the weak extension theorem for the group $\mathscr{H}(M; \mu)$ (cf. Corollary 5.1).

THEOREM 1.1. *Suppose M is an n-manifold, μ is a good Radon measure on M with $\mu(\partial M) = 0$, C is a compact subset of M, U is an open neighborhood of C in M. Then there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^{\mathscr{H}(M;\mu)}(U, M)$ and a homotopy $s : \mathscr{U} \times [0, 1] \to \mathscr{H}(M; \mu)$ such that*

(1) *for each $f \in \mathscr{U}$*
   (i) $s_0(f) = \mathrm{id}_M$, (ii) $s_1(f)|_C = f|_C$, (iii) *if* $f = \mathrm{id}$ *on* $U \cap \partial M$, *then* $s_t(f) = \mathrm{id}$ *on* ∂M $(t \in [0, 1])$,

(2) $s_t(i_U) = \mathrm{id}_M$ $(t \in [0, 1])$.

In comparison with topological or μ-biregular homeomorphisms, "μ-preserving homeomorphism" is a global property and we can not obtain a compactly supported weak extension theorem for the group $\mathscr{H}(M; \mu)$. This obstruction vanishes on the kernel of the end charge homomorphism c^μ.

In [**2**] S. R. Alpern and V. S. Prasad introduced the end charge homomorphism c^μ, which is a continuous homomorphism defined on the subgroup $\mathscr{H}_{E_M}(M; \mu)$ of μ-preserving homeomorphisms of M which fix the ends of M. The kernel of c^μ, $\ker c^\mu$, includes the subgroup $\mathscr{H}_c(M; \mu)$ of μ-preserving homeomorphisms of M with compact support. If $h \in \mathscr{H}_{E_M}(M, E; \mu)$ and $c^\mu(h) = 0$, then one can split moves of μ-volume by h. Hence, we can obtain the compactly supported weak extension theorem for the subgroup $\ker c^\mu$ (cf. Theorem 5.2).

THEOREM 1.2. *Suppose M is a connected n-manifold, μ is a good Radon measure on M with $\mu(\partial M) = 0$, C is a compact subset of M and U and V are open* *neighborhoods of C in M such that $V \cap O$ is connected for each connected component O of $M - C$. Then there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^{\ker c^\mu}(U, M)$ and a homotopy $s : \mathscr{U} \times [0,1] \to \mathscr{H}_{M-V,c}(M; \mu)$ such that*

(1) *for each $f \in \mathscr{U}$*

(i) $s_0(f) = \mathrm{id}_M$, (ii) $s_1(f)|_C = f|_C$, (iii) *if* $f = \mathrm{id}$ *on* $U \cap \partial M$, *then* $s_t(f) = \mathrm{id}$ *on* ∂M $(t \in [0,1])$,

(2) $s_t(i_U) = \mathrm{id}_M$ $(t \in [0,1])$.

We also discuss a non-ambient deformation lemma for μ-preserving embeddings (Theorem 5.3).

In the last section we study the group $\mathscr{H}_c(M; \mu)_w$ endowed with the Whitney topology (cf. [**3**]). It is known that the group $\mathscr{H}(N)$ and the subgroup $\mathscr{H}(N; \nu)$ are locally contractible for any compact n-manifold N and any good Radon measure ν on N with $\nu(\partial N) = 0$ ([**7**, Corollary 1.1], [**8**, Theorem 4.4]). In [**3**] it is shown that the group $\mathscr{H}_c(M)_w$ consisting of homeomorphisms of M with compact support, endowed with the Whitney topology, is locally contractible. In this article, as an application of the weak extension theorem for $\mathscr{H}_c(M; \mu)$, we show that the group $\mathscr{H}_c(M; \mu)_w$ is also locally contractible for any connected n-manifold M (Theorem 6.1).

This paper is organized as follows. Section 2 is devoted to the general formulations and basic properties of local weak extension property and local weak section property for group actions. Section 3 contains fundamental facts related to Radon measures on manifolds (selection theorems for measures, end charge homomorphism, etc.). In Section 4 we recall the local deformation lemma for biregular embeddings and discuss some direct consequences of this lemma. In Section 5 we obtain the weak extension theorems for the groups $\mathscr{H}(M; \mu)$, $\ker c^\mu$ and $\mathscr{H}_c(M; \mu)$ and a non-ambient deformation lemma for μ-preserving embeddings. In Section 6 we recall basic facts on the Whitney topology and show that the group $\mathscr{H}_c(M; \mu)_w$ is locally contractible for any connected n-manifold M.

2. Fundamental facts on group actions.

2.1. Conventions.

For a topological space X and a subset A of X, the symbols $\mathrm{Int}_X A$, $\mathrm{cl}_X A$ and $\mathrm{Fr}_X A$ denote the topological interior, closure and frontier of A in X. Let $\mathscr{C}(X)$ denote the collection of all connected components of X.

Suppose Y is a locally connected, locally compact Hausdorff space. Let $\mathscr{H}(Y)$ denote the group of homeomorphisms of Y endowed with the compact-open topology. For a subset A of Y, let $\mathscr{H}_A(Y) = \{h \in \mathscr{H}(Y) \mid h|_A = \mathrm{id}_A\}$ (with the subspace topology). The group $\mathscr{H}(Y)$ and the subgroup $\mathscr{H}_A(Y)$ are topological groups. In general, for any topological group G, the symbols G_0 and G_1 denote the connected component and the path-component of the unit element e in G.

For subspaces $A \subset X$ of Y let $\mathscr{E}(X, Y)$ denote the space of embeddings $f : X \hookrightarrow Y$ endowed with the compact-open topology, and let $\mathscr{E}_A(X, Y) = \{f \in \mathscr{E}(X, Y) \mid f|_A = \mathrm{id}_A\}$ (with the subspace topology). By $i_X : X \subset Y$ we denote the inclusion map of X into Y.

In this article, an n-manifold means a paracompact σ-compact (separable metrizable) topological n-manifold *possibly with boundary.* Suppose M is an n-manifold. The symbols ∂M and $\mathrm{Int}\, M$ denote the boundary and interior of M as a manifold. For a subspace X of M, an embedding $f : X \to M$ is said to be *proper* if $f^{-1}(\partial M) = X \cap \partial M$. Let $\mathscr{E}^*(X, M)$ denote the subspace of $\mathscr{E}(X, M)$ consisting of proper embeddings $f : X \to M$. For a subset A of X let $\mathscr{E}^*_A(X, M) = \mathscr{E}^*(X, M) \cap \mathscr{E}_A(X, M)$.

By an *n-submanifold* of M we mean a closed subset N of M such that N is an n-manifold and $\mathrm{Fr}_M N$ is locally flat in M and transverse to ∂M so that (i) $M - \mathrm{Int}_M N$ is an n-manifold and (ii) $\mathrm{Fr}_M N$ and $N \cap \partial M$ are $(n-1)$-manifolds with the common boundary $(\mathrm{Fr}_M N) \cap (N \cap \partial M)$. For simplicity, let $\partial_+ N = \mathrm{Fr}_M N$, $\partial_- N = N \cap \partial M$ and $N^c = M - \mathrm{Int}_M N$. More generally, for a subset U of M let $\partial_- U = U \cap \partial M$.

Suppose M is an n-manifold.

Lemma 2.1 ([**1**, Theorem 0], cf. [**9**]). *Suppose C is a compact subset of M and U is a neighborhood of C in M. Then there exists a compact n-submanifold N of M such that $C \subset \mathrm{Int}_M N$ and $N \subset U$.*

Lemma 2.2.

(1) *If M is connected and L is an n-submanifold of M such that $\partial_+ L$ is compact, then there exists a connected n-submanifold N of M such that $L \subset \mathrm{Int}_M N$ and $N \cap L^c$ is compact.*

(2) *Suppose C is a compact subset of M.*

   (i) *For any neighborhood U of C in M there exists a compact n-submanifold N of M such that $C \subset \mathrm{Int}_M N$, $N \subset U$ and $O - N$ is connected for each $O \in \mathscr{C}(M - C)$.*

   (ii) *If U is an open neighborhood of C in M such that $U \cap O$ is connected for each $O \in \mathscr{C}(M - C)$, then there exists a compact n-submanifold N of M such that $C \subset \mathrm{Int}_M N$, $N \subset U$ and $N \cap O$ is connected for each $O \in \mathscr{C}(M - C)$.*

Proof.

(1) Since M is connected and $\partial_+ L$ is compact, $\mathscr{C}(L)$ is a finite collection. Since M is connected, there exists a finite collection of disjoint arcs $\{\alpha_i\}_i$ in L^c such that

$L \cup (\bigcup_i \alpha_i)$ is connected. We apply Lemma 2.1 to $C = \partial_+ L \cup (\bigcup_i \alpha_i)$ in the n-manifold L^c in order to find a compact n-submanifold N_0 of L^c such that $C \subset \mathrm{Int}_{L^c} N_0$ and each $K \in \mathscr{C}(N_0)$ meets C. Then $N = L \cup N_0$ satisfies the required conditions.

(2) (i) We may assume that M is connected (apply the connected case to each component of M). By Lemma 2.1 there exists a compact n-submanifold N_1 of M such that $C \subset \mathrm{Int}_M N_1$ and $N_1 \subset U$. Let $\mathscr{C} = \{O \in \mathscr{C}(M - C) \mid O \not\subset N_1\}$. Since $\mathscr{C}(N_1^c)$ is a finite collection, so is $\mathscr{C}$.

For each $O \in \mathscr{C}$, it is seen that O is a connected n-manifold, $N_1^c \cap O$ is an n-submanifold of O, $(N_1^c \cap O)^c = N_1 \cap O$ in O and $\mathrm{Fr}_O(N_1^c \cap O) = (\mathrm{Fr}_M N_1) \cap O$ is compact (it is a union of components of $\mathrm{Fr}_M N_1$). Thus, by (1) we can find a connected n-submanifold L_O of O such that $N_1^c \cap O \subset \mathrm{Int}_O L_O$ and $L_O \cap (N_1 \cap O)$ is compact. Note that L_O is closed in M so that it is also a connected n-submanifold of M. Let $L = \bigcup_{O \in \mathscr{C}} L_O$. Then, $N = L^c$ satisfies the required conditions. In fact, $C \subset M - L = \mathrm{Int}_M N$, $N \subset N_1$, $\mathscr{C} = \{O \in \mathscr{C}(M - C) \mid O \not\subset N\}$ and $O - N = \mathrm{Int}_M L_O$ for each $O \in \mathscr{C}$.

(ii) Since $\mathscr{C}(U - C) = \{O \cap U \mid O \in \mathscr{C}(M - C)\}$, by replacing M by U, we may assume that $U = M$. Again we may assume that M is connected. By Lemma 2.1 there exists a compact n-submanifold N_1 of M such that $C \subset \mathrm{Int}_M N_1$. Consider the finite collection $\mathscr{C} = \{O \in \mathscr{C}(M - C) \mid O \not\subset N_1\}$. For each $O \in \mathscr{C}$, it is seen that O is a connected n-manifold, $N_1 \cap O$ is an n-submanifold of O, $(N_1 \cap O)^c = N_1^c \cap O$ in O and $\mathrm{Fr}_O(N_1 \cap O) = (\mathrm{Fr}_M N_1) \cap O$ is compact. Thus, by (1) we can find a connected n-submanifold K_O of O such that $N_1 \cap O \subset \mathrm{Int}_O K_O$ and $K_O \cap (N_1^c \cap O)$ is compact. Then, $N = N_1 \cup (\bigcup_{O \in \mathscr{C}} K_O)$ satisfies the required conditions. In fact, $\{O \in \mathscr{C}(M - C) \mid O \not\subset N\} \subset \mathscr{C}$ and $N \cap O = K_O$ for each $O \in \mathscr{C}$. □

2.2. Pull-backs.

For maps $B_1 \xrightarrow{p} B \xleftarrow{\pi} E$, we obtain the *pull-back* diagram in the category of topological spaces and continuous maps:

$$\begin{array}{ccc} p^*E & \xrightarrow{p'} & E \\ \pi' \downarrow & & \downarrow \pi \\ B_1 & \xrightarrow[p]{} & B \end{array}$$

Explicitly, the space p^*E and the maps $B_1 \xleftarrow{\pi'} p^*E \xrightarrow{p'} E$ are defined by

$$p^*E = \{(b_1, e) \in B_1 \times E \mid p(b_1) = \pi(e)\} \quad \text{and} \quad \pi'(b_1, e) = b_1, \quad p'(b_1, e) = e.$$

Suppose a topological group G acts on spaces B and B_1 transitively. Let $p : B_1 \to B$ be a G-equivariant map. Fix a point $b_1 \in B_1$ and let $b = p(b_1) \in B$ and let G_b be the stabilizer of b under the G-action on B. Consider the orbit map $\pi : G \to B$, $\pi(g) = gb$. Then the maps $B_1 \xrightarrow{p} B \xleftarrow{\pi} G$ induce the pull-back diagram:

$$\begin{array}{ccc} p^*G & \overset{p'}{\longrightarrow} & G \\ \pi' \downarrow & & \downarrow \pi \\ B_1 & \underset{p}{\longrightarrow} & B \end{array}$$

The group G_b acts freely on p^*G on the right by $(x, g) \cdot h = (x, gh)$ $((x, g) \in p^*G$, $h \in G_b)$. The induced map $p' : p^*G \to G$ admits a right inverse $r : G \to p^*G$, $r(g) = (gb_1, g)$ (i.e., $p'r = \mathrm{id}_G$).

DEFINITION 2.1. We say that the G-equivariant map $p : B_1 \to B$ has the *local section property* for G (LSP_G) at b_1 if there exists a neighborhood U_1 of b_1 in B_1 and a map $s_1 : U_1 \to G$ such that $\pi s_1 = p|_{U_1}$.

LEMMA 2.3.

(1) *The map p has LSP_G at b_1 if and only if the induced map $\pi' : p^*G \to B_1$ is a principal G_b-bundle.*

(2) *If the fiber $p^{-1}(b)$ is contractible, then the map $p' : p^*G \to G$ is a homotopy equivalence.*

PROOF.

(1) Suppose the map p has LSP_G at b_1. Take any point $b_2 \in B_1$. Since G acts on B_1 transitively, there exists a $g \in G$ with $b_2 = gb_1$. Then $U_2 = gU_1$ is a neighborhood of b_2 in B_1 and the map $s_2 : U_2 \to G$, $s_2(x) = gs_1(g^{-1}x)$ satisfies the condition $\pi s_2 = p|_{U_2}$ (i.e., $\pi s_2(x) = gs_1(g^{-1}x)b = g(p(g^{-1}x)) = p(x)$). The map $\pi' : p^*G \to B_1$ admits a local trivialization

$$\phi : U_2 \times G_b \cong (\pi')^{-1}(U_2) = \bigcup_{x \in U_2} (\{x\} \times \pi^{-1}(p(x)))$$

over U_2 defined by $\phi(x, h) = (x, s_2(x)h)$.

The converse is obvious.

(2) It remains to show that $rp' \simeq \mathrm{id}_{p^*G}$. There exists a contraction $\phi_t : p^{-1}(b) \to p^{-1}(b)$ $(t \in [0, 1])$ such that $\phi_1(p^{-1}(b)) = \{b_1\}$. If $(x, g) \in p^*G$, then $x \in$

$p^{-1}(gb) = g\,p^{-1}(b)$. Thus, we can define a homotopy

$$\Phi_t : p^*G \to p^*G \quad \text{from } \mathrm{id}_{p^*G} \text{ to } rp' \text{ by} \quad \Phi_t(x,g) = (g\phi_t(g^{-1}x), g). \qquad \square$$

2.3. Group actions and spaces of embeddings.

Suppose a topological group G acts continuously on a locally compact Hausdorff space Y. Each $g \in G$ induces $\hat{g} \in \mathscr{H}(Y)$ defined by $\hat{g}(y) = gy$ $(y \in Y)$. Let H be any subset of G. For subsets A, B of Y we have the following subsets of H:

$$H_A = \{h \in H \mid \hat{h}|_A = \mathrm{id}_A\}, \quad H(B) = H_{Y\setminus B}, \quad H_A(B) = H_A \cap H(B),$$
$$H_c = \{h \in H \mid \mathrm{supp}\ \hat{h} \text{ is compact}\}.$$

If H is a subgroup of G, then these are subgroups of H.

For subsets $X \subset C \subset U$ of Y, the group $G_X(U)$ acts continuously on the space $\mathscr{E}_X(C,U)$ by the left composition $g \cdot f = \hat{g}f$ $(g \in G_X(U),\ f \in \mathscr{E}_X(C,U))$ and we have the following subspace of $\mathscr{E}_X(C,U)$:

$$\mathscr{E}^H_X(C,U) = H_X(U)i_C = \{\hat{g}|_C \mid g \in H_X(U)\} \quad \text{(with the compact-open topology).}$$

Since $\mathscr{E}^H_X(C,U) = \mathscr{E}^{H_X}(C,U)$, by replacing H by H_X if necessary, we omit X in the subsequent statements.

Consider the pull-back diagram:

$$\begin{array}{ccc} p^*G & \xrightarrow{p'} & G \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathscr{E}^G(U,Y) & \xrightarrow[p]{} & \mathscr{E}^G(C,Y), \end{array} \qquad \text{where} \quad \pi(g) = \hat{g}|_C \quad \text{and} \quad p(f) = f|_C.$$

The group G acts on the spaces $\mathscr{E}^G(U,Y)$ and $\mathscr{E}^G(C,Y)$ transitively. The restriction map p is G-equivariant and has the fiber $p^{-1}(i_C) = \mathscr{E}^G_C(U,Y)$.

Definition 2.2. We say that the pair (U,C) has the local section property for G (LSP_G) if the G-equivariant map $p : \mathscr{E}^G(U,Y) \to \mathscr{E}^G(C,Y)$ has LSP_G at i_U.

Lemma 2.4. *The pair (U,C) has LSP_G if and only if the map $\pi' : p^*G \to \mathscr{E}^G(U,Y)$ is a principal G_C-bundle.*

This lemma follows directly from Lemma 2.3 (1).

Lemma 2.5. *Suppose there exists a path $h : [0,1] \to G$ such that $h_0 = e$, $\widehat{h_1}(U) \subset C$ and $\widehat{h_t}(U) \subset U$, $\widehat{h_t}(C) \subset C$ $(t \in [0,1])$. Then the following hold.*

(1) *The map $p : \mathscr{E}^G(U,Y) \to \mathscr{E}^G(C,Y)$ is a homotopy equivalence.*
(2) *There exists a strong deformation retraction χ_t $(t \in [0,1])$ of $\mathscr{E}^G_C(U,Y)$ onto the singleton $\{i_U\}$.*
(3) *The map $p' : p^*G \to G$ is a homotopy equivalence.*

Proof.

(1) We can define a map $p_1 : \mathscr{E}^G(C,Y) \to \mathscr{E}^G(U,Y)$ by $p_1(f) = f\,\widehat{h_1}|_U$. It follows that

(i) $p_1 p(f) = f\,\widehat{h_1}|_U$ and a homotopy $\phi_t : \mathrm{id} \simeq p_1 p$ is defined by $\phi_t(f) = f\,\widehat{h_t}|_U$, and
(ii) $pp_1(f) = f\,\widehat{h_1}|_C$ and a homotopy $\psi_t : \mathrm{id} \simeq pp_1$ is defined by $\psi_t(f) = f\,\widehat{h_t}|_C$.

(2) The contraction χ_t of $\mathscr{E}^G_C(U,Y)$ is defined by $\chi_t(f) = \widehat{h_t}^{-1} f \widehat{h_t}|_U$.

(3) The assertion follows from (2) and Lemma 2.3 (2). ☐

Lemmas 2.4 and 2.5 yield the following consequence.

Proposition 2.1. *If a subset C of Y satisfies the condition $(*)$ below, then the map*

$$G_C \subset G \xrightarrow{\pi} \mathscr{E}^G(C,Y) \quad \textit{defined by } \pi(h) = \hat{h}|_C$$

is a locally trivial bundle up to homotopy equivalences and hence has the exact sequence for homotopy groups.

$(*)$ *There exists a subset U of Y such that* (i) *$C \subset U$,* (ii) *the pair (U,C) has* LSP_G, *and* (iii) *there exists a path $h_t \in G$ $(t \in [0,1])$ such that*

$$h_0 = e, \quad \widehat{h_1}(U) = C, \quad \widehat{h_t}(U) \subset U, \quad \widehat{h_t}(C) \subset C \quad (t \in [0,1]).$$

2.4. Weak extension property.

Suppose a topological group G acts on an n-manifold M. Consider a pair (H,F) of subsets of G and a triple (V,U,C) of subsets of M such that $C \subset U \cap V$ (we do not assume that $F \subset H$ and $U \subset V$).

Definition 2.3. We say that the triple (V,U,C) has the *weak extension property* for (H,F) (abbreviated as $\mathrm{WEP}_{H,F}$ or $\mathrm{WEP}(H,F)$) if there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^H(U,M)$ and a homotopy $s : \mathscr{U} \times [0,1] \to F(V)$ such that

(1) for each $f \in \mathscr{U}$ (i) $s_0(f) = e$, (ii) $\widehat{s_1(f)}|_C = f|_C$, (iii) if $f = \mathrm{id}$ on $\partial_- U$, then $\widehat{s_t(f)} = \mathrm{id}$ on ∂M ($t \in [0,1]$),

(2) $s_t(i_U) = e$ ($t \in [0,1]$).

The map $s_t : \mathscr{U} \to F(V)$ ($t \in [0,1]$) is called the *local weak extension map* (LWE map). When $H = F$, we simply say that (V, U, C) has WEP_H. When $V = U$, we say that the pair (U, C) has $\mathrm{WEP}_{H,F}$. Note that WEP_G for (U, C) implies LSP_G for (U, C).

One of our interest is the following problem.

Problem 2.1. Given a class of triples (V, U, C) in Y and a subset F of G, determine the largest subset H of G for which each triple (V, U, C) in this class has $\mathrm{WEP}(H, F)$.

The next lemma easily follows from the definition.

Lemma 2.6. *Suppose (V, U, C) and (V', U', C') are two triples of subsets in M such that $C \subset U \cap V$ and $C' \subset U' \cap V'$ and (H, F) and (H', F') are two pairs of subsets in G. If* (i) *(V, U, C) has* $\mathrm{WEP}(H, F)$*,* (ii) *$V \subset V'$, $U \subset U'$, $C \supset C'$ and* (iii) *$H \supset H'$, $F \subset F'$, then (V', U', C') has* $\mathrm{WEP}(H', F')$.

Lemma 2.7. *Suppose F is a subgroup of G. If two triples (V_1, U_1, C_1) and (V_2, U_2, C_2) have* $\mathrm{WEP}(H, F)$ *and $V_1 \cap V_2 = \emptyset$, then the triple $(V_1 \cup V_2, U_1 \cup U_2, C_1 \cup C_2)$ also has* $\mathrm{WEP}(H, F)$.

Proof. For $i = 1, 2$ let $\mathscr{E}^H(U_i, M) \supset \mathscr{U}_i \xrightarrow{s_t^i} F(V_i)$ be the associated LWE map for (V_i, U_i, C_i). Take a neighborhood $\mathscr{U}$ of $i_{U_1 \cup U_2}$ in $\mathscr{E}^H(U_1 \cup U_2, M)$ such that $f|_{U_i} \in \mathscr{U}_i$ ($i = 1, 2$) for each $f \in \mathscr{U}$. Then the required LWE map $s_t : \mathscr{U} \to F(V_1 \cup V_2)$ for $(V_1 \cup V_2, U_1 \cup U_2, C_1 \cup C_2)$ is defined by

$$s_t(f) = s_t^1(f|_{U_1}) s_t^2(f|_{U_2}) \quad \text{(the multiplication in } G).$$

Note that $\widehat{s_t(f)} = \widehat{s_t^i(f|_{U_i})}$ on V_i and $\widehat{s_t(f)} = \mathrm{id}$ on $M - (V_1 \cup V_2)$. □

3. Spaces of Radon measures and groups of measure-preserving homeomorphisms.

3.1. Spaces of Radon measures.

Suppose Y is a locally connected, locally compact, σ-compact (separable metrizable) space. Let $\mathscr{B}(Y)$ denote the σ-algebra of Borel subsets of Y. A *Radon measure* on Y is a measure μ on the measurable space $(Y, \mathscr{B}(Y))$ such that $\mu(K) <$

∞ for any compact subset K of Y. Let $\mathscr{M}(Y)$ denote the set of Radon measures on Y. The *weak* topology w on $\mathscr{M}(Y)$ is the weakest topology such that the function

$$\Phi_f : \mathscr{M}(Y) \longrightarrow \boldsymbol{R} \;:\; \mu \longmapsto \int_Y f\, d\mu$$

is continuous for any continuous function $f : Y \to \boldsymbol{R}$ with compact support. The set $\mathscr{M}(Y)$ is endowed with the weak topology w, otherwise specified.

For $\mu \in \mathscr{M}(Y)$ and $A \in \mathscr{B}(Y)$, the restriction $\mu|_A$ is the Radon measure on A defined by $(\mu|_A)(B) = \mu(B)$ $(B \in \mathscr{B}(A))$.

Lemma 3.1 ([**4**, Lemma 2.2]). *For any closed subset A of Y, the map $\mathscr{M}(Y) \to \mathscr{M}(A) : \mu \mapsto \mu|_A$ is continuous at each $\mu \in \mathscr{M}(Y)$ with $\mu(\mathrm{Fr}_M A) = 0$.*

We say that $\mu \in \mathscr{M}(Y)$ is *good* if $\mu(p) = 0$ for any point $p \in Y$ and $\mu(U) > 0$ for any nonempty open subset U of Y. For $A \in \mathscr{B}(Y)$ let $\mathscr{M}_g^A(Y)$ denote the subspace of $\mathscr{M}(Y)$ consisting of good Radon measures μ on Y with $\mu(A) = 0$. For $\mu, \nu \in \mathscr{M}(Y)$, we say that ν is *μ-biregular* if ν and μ have same null sets (i.e., $\nu(B) = 0$ if and only if $\mu(B) = 0$ for any $B \in \mathscr{B}(Y)$). For $\mu \in \mathscr{M}_g^A(Y)$ we set

$$\mathscr{M}_g^A(Y; \mu\text{-reg}) = \{\nu \in \mathscr{M}_g^A(Y) \mid \nu \text{ is } \mu\text{-biregular}\} \quad \text{(with the weak topology).}$$

For $h \in \mathscr{H}(Y)$ and $\mu \in \mathscr{M}(Y)$, the induced measures $h_*\mu, h^*\mu \in \mathscr{M}(Y)$ are defined by

$$(h_*\mu)(B) = \mu(h^{-1}(B)) \text{ and } (h^*\mu)(B) = \mu(h(B)) \; (B \in \mathscr{B}(Y)).$$

The group $\mathscr{H}(Y)$ acts continuously on the space $\mathscr{M}(Y)$ by $h \cdot \mu = h_*\mu$. We say that $h \in \mathscr{H}(Y)$ is

(i) *μ-preserving* if $h_*\mu = \mu$ (i.e., $\mu(h(B)) = \mu(B)$ for any $B \in \mathscr{B}(Y)$) and
(ii) *μ-biregular* if $h_*\mu$ and μ have the same null sets (i.e., $\mu(h(B)) = 0$ if and only if $\mu(B) = 0$ for any $B \in \mathscr{B}(Y)$).

Let $\mathscr{H}(Y;\mu) \subset \mathscr{H}(Y;\mu\text{-reg})$ denote the subgroups of $\mathscr{H}(Y)$ consisting of μ-preserving and μ-biregular homeomorphisms of Y respectively. For a subset A of Y, the subgroups $\mathscr{H}_A(Y;\mu)$, $\mathscr{H}_A(Y;\mu)_1$, $\mathscr{H}_{A,c}(Y;\mu)$, $\mathscr{H}_A(Y;\mu\text{-reg})$, etc. are defined according to the conventions in Sections 2.1 and 2.3.

For spaces of embeddings, we use the following notations. Suppose Y is a locally compact, σ-compact (separable metrizable) space and $\mu \in \mathscr{M}(Y)$. For any $X \in \mathscr{B}(Y)$, an embedding $f : X \to Y$ is said to be

(i) Borel if $f(X) \in \mathscr{B}(Y)$,
(ii) μ-biregular provided f is Borel and $\mu(f(B)) = 0$ if and only if $\mu(B) = 0$ for any $B \in \mathscr{B}(X)$,
(iii) μ-preserving provided f is Borel and $f : (X, \mu|_X) \cong (f(X), \mu|_{f(X)})$ is a measure-preserving homeomorphism (i.e., $\mu(f(B)) = \mu(B)$ for any $B \in \mathscr{B}(X)$).

For a subset A of X, let $\mathscr{E}_A(X, Y; \mu\text{-reg})$ and $\mathscr{E}_A(X, Y; \mu)$ denote the subspaces of $\mathscr{E}_A(X, Y)$ consisting of μ-biregular embeddings and μ-preserving embeddings respectively.

Suppose M is a *compact connected* n-manifold and $\mu \in \mathscr{M}_g^{\partial}(M) (= \mathscr{M}_g^{\partial M}(M))$.

THEOREM 3.1 ([**10**]). *If $\nu \in \mathscr{M}_g^{\partial}(M)$ and $\nu(M) = \mu(M)$, then there exists $h \in \mathscr{H}_{\partial}(M)_1$ such that $h_*\mu = \nu$.*

Let $\mathscr{M}_g^{\partial}(M; \mu) = \{\nu \in \mathscr{M}_g^{\partial}(M; \mu\text{-reg}) \mid \nu(M) = \mu(M)\}$ (with the weak topology). (See Section 3.2 for the definition in the case where M is noncompact.) The group $\mathscr{H}(M; \mu\text{-reg})$ acts continuously on $\mathscr{M}_g^{\partial}(M; \mu)$ by $h \cdot \nu = h_*\nu$. This action induces the map

$$\pi : \mathscr{H}(M; \mu\text{-reg}) \longrightarrow \mathscr{M}_g^{\partial}(M; \mu) \ : \ h \longmapsto h_*\mu.$$

THEOREM 3.2 ([**8**, Theorem 3.3]). *The map π admits a section*

$$\sigma : \mathscr{M}_g^{\partial}(M; \mu) \longrightarrow \mathscr{H}_{\partial}(M; \mu\text{-reg})_1 \subset \mathscr{H}(M; \mu\text{-reg})$$

such that ($\pi\sigma = \mathrm{id}$ *and*) $\sigma(\mu) = \mathrm{id}_M$.

Next we recall basic facts on the product of measures. Suppose $(X, \mathscr{F}, \mu)$ and $(Y, \mathscr{G}, \nu)$ are σ-finite measure spaces. Let $\mathscr{F} \times \mathscr{G}$ denote the σ-algebra on $X \times Y$ generated by the family $\{A \times B \mid A \in \mathscr{F}, B \in \mathscr{G}\}$. For $G \in \mathscr{F} \times \mathscr{G}$ and $x \in X$, the slice $G_x \subset Y$ is defined by $G_x = \{y \in Y \mid (x, y) \in G\}$. It is well known that

(1) there exists a unique measure ω on the measurable space $(X \times Y, \mathscr{F} \times \mathscr{G})$ such that $\omega(A \times B) = \mu(A) \cdot \nu(B)$ $(A \in \mathscr{F}, B \in \mathscr{G})$ (we follow the convention $0 \cdot \infty = 0$),
(2) for any $G \in \mathscr{F} \times \mathscr{G}$
   (i) $\nu(G_x)$ $(x \in X)$ is an $\mathscr{F}$-measurable function on X and
   (ii) $\omega(G) = \displaystyle\int_X \nu(G_x)\, d\mu(x)$.

This result yields the following consequences on the product of Radon measures.

Proposition 3.1. *Suppose (X,μ) and (Y,ν) are locally compact separable metrizable spaces with Radon measures. Then the following hold*:

(0) $\mathscr{B}(X)\times\mathscr{B}(Y)=\mathscr{B}(X\times Y)$.

(1) *There exists a unique* $\omega\in\mathscr{M}(X\times Y)$ *such that* $\omega(A\times B)=\mu(A)\cdot\nu(B)$ $(A\in\mathscr{B}(X), B\in\mathscr{B}(Y))$.

(2) *For any* $G\in\mathscr{B}(X\times Y)$

  (i) $\nu(G_x)$ $(x\in X)$ *is a* $\mathscr{B}(X)$*-measurable function on* X *and*

  (ii) $\omega(G)=\displaystyle\int_X\nu(G_x)\,d\mu(x)$.

The measure ω is called the product of μ and ν and denoted by $\mu\times\nu$.

Proposition 3.2. *Suppose* $f:(X,\mu)\to(X_1,\mu_1)$ *and* $g:(Y,\nu)\to(Y_1,\nu_1)$ *are homeomorphisms between locally compact separable metrizable spaces with Radon measures. Then the product homeomorphism* $f\times g:(X\times Y,\mu\times\nu)\to(X_1\times Y_1,\mu_1\times\nu_1)$ *has the following properties*:

(1) *If f and g are biregular, then $f\times g$ is biregular.*

(2) *If f and g are measure-preserving, then $f\times g$ is measure-preserving.*

Proof. For $G\in\mathscr{B}(X\times Y)$, we have

$$\text{(a)}\quad (\mu\times\nu)(G)=\int_X\nu(G_x)\,d\mu(x)\ \text{and}$$

$$\text{(b)}\quad (\mu_1\times\nu_1)((f\times g)(G))=\int_{X_1}\nu_1(((f\times g)(G))_{x_1})\,d\mu_1(x_1)$$
$$=\int_{X_1}\nu_1(g(G_{f^{-1}(x_1)}))\,d\mu_1(x_1).$$

(1) Note that

(i) $(\mu\times\nu)(G)=0$ if and only if $\nu(G_x)=0$ (μ-a.e. $x\in X$)

(i.e., $\exists A\in\mathscr{B}(X)$ such that $\mu(A)=0$ and $\nu(G_x)=0$ $(x\in X-A)$),

(ii) $(\mu_1\times\nu_1)((f\times g)(G))=0$ if and only if $\nu_1(g(G_{f^{-1}(x_1)}))=0$ (μ_1-a.e. $x_1\in X_1$).

Since f and g are biregular, if (i) holds, then it follows that

$$f(A)\in\mathscr{B}(X_1),\ \mu_1(f(A))=0\ \text{ and }\ \nu_1(g(G_{f^{-1}(x_1)}))=0\ (x_1\in X_1-f(A)).$$

This implies (ii). The same argument shows the opposite implication. This means that $f \times g$ is biregular.

(2) Since f and g are measure-preserving, it follows that

$$(\mu_1 \times \nu_1)((f \times g)(G)) = \int_{X_1} \nu_1(g(G_{f^{-1}(x_1)}))\, d\mu_1(x_1) = \int_{X_1} \nu(G_{f^{-1}(x_1)})\, d\mu_1(x_1)$$
$$= \int_X \nu(G_x)\, d\mu(x) = (\mu \times \nu)(G).$$

This means that $f \times g$ is measure-preserving. We also note that $(f \times g)^*(\mu_1 \times \nu_1) \in \mathscr{M}(X \times Y)$ satisfies the condition: for any $A \in \mathscr{B}(X)$ and $B \in \mathscr{B}(Y)$

$$((f \times g)^*(\mu_1 \times \nu_1))(A \times B) = (\mu_1 \times \nu_1)((f \times g)(A \times B)) = (\mu_1 \times \nu_1)(f(A) \times g(B))$$
$$= \mu_1(f(A)) \cdot \nu_1(g(B)) = \mu(A) \cdot \nu(B).$$

By definition we have $(f \times g)^*(\mu_1 \times \nu_1) = \mu \times \nu$. This also implies the conclusion. □

We conclude this subsection with some remarks on collars of the boundary of a submanifold. Suppose M is an n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$.

REMARK 3.1. Suppose N is an n-submanifold of M such that $\partial_+ N$ is compact. Since $\mu(\partial M) = 0$, we have $\mu(\partial N) = \mu(\partial_+ N)$. Take a bicollar $\partial_+ N \times [-1, 1]$ of $\partial_+ N$ in M. Since $\partial_+ N \times [-1, 1]$ is compact, it follows that $\mu(\partial_+ N \times [-1, 1]) < \infty$ and $\{t \in [-1, 1] \mid \mu(\partial_+ N \times \{t\}) \neq 0\}$ is a countable subset of $[-1, 1]$. Hence, we can modify N by adding or subtracting a thin collar of $\partial_+ N$ so that $\mu(\partial N) = \mu(\partial_+ N) = 0$.

Let m denote the Lebesgue measure on the real line $\boldsymbol{R}$.

LEMMA 3.2. *Suppose N is an n-submanifold of M such that $\partial_+ N$ is compact and $\mu(\partial_+ N) = 0$ and suppose $\nu \in \mathscr{M}_g^\partial(\partial_+ N)$. Then, there exists a bicollar $E = \partial_+ N \times [a, b]$ $(a < 0 < b)$ of $\partial_+ N$ in M such that $\partial_+ N = \partial_+ N \times \{0\}$, $N \cap E = \partial_+ N \times [a, 0]$ and $\mu|_E = \nu \times (m|_{[a,b]})$.*

PROOF. Let $\mathscr{C}(\partial_+ N) = \{F_1, \cdots, F_m\}$. For each $i = 1, \cdots, m$, choose a small bicollar $E_i = F_i \times [a_i, b_i]$ $(a_i < 0 < b_i)$ such that $F_i = F_i \times \{0\}$, $N \cap E_i = F_i \times [a_i, 0]$, $\mu(\partial_+ E_i) = 0$, $\mu(F_i \times [a_i, 0]) = |a_i| \nu(F_i)$ and $\mu(F_i \times [0, b_i]) = b_i \nu(F_i)$. We can apply Theorem 3.1 to

$$\mu|_{F_i\times[a_i,0]}, \nu|_{F_i} \times (m|_{[a_i,0]}) \in \mathscr{M}_g^{\partial}(F_i \times [a_i,0]) \quad \text{and}$$
$$\mu|_{F_i\times[0,b_i]}, \nu|_{F_i} \times (m|_{[0,b_i]}) \in \mathscr{M}_g^{\partial}(F_i \times [0,b_i])$$

to replace the identification of the collar $E_i = F_i \times [a_i, b_i]$ so that $\mu|_{E_i} = \nu|_{F_i} \times (m|_{[a_i,b_i]})$. Finally, take a, b such that $\max_i a_i < a < 0 < b < \min_i b_i$ and set $E = \partial_+ N \times [a,b] = \bigcup_i (F_i \times [a,b])$. □

3.2. End compactification and finite-end weak topology (cf. [2], [4]). In order to extend the selection theorem 3.2 to the noncompact case, it is necessary to include the information of the ends. Suppose Y is a noncompact, connected, locally connected, locally compact, separable metrizable space. Let $\mathscr{K}(Y)$ denote the collection of all compact subsets of Y. An *end* of Y is a function e which assigns an $e(K) \in \mathscr{C}(Y - K)$ to each $K \in \mathscr{K}(Y)$ such that $e(K_1) \supset e(K_2)$ if $K_1 \subset K_2$. The set of ends of Y is denoted by E_Y. The *end compactification* of Y is the space $\overline{Y} = Y \cup E_Y$ equipped with the topology defined by the following conditions: (i) Y is an open subspace of $\overline{Y}$, (ii) the fundamental open neighborhoods of $e \in E_Y$ are given by

$$N(e,K) = e(K) \cup \{e' \in E_Y \mid e'(K) = e(K)\} \quad (K \in \mathscr{K}(Y)).$$

Then $\overline{Y}$ is a connected, locally connected, compact, metrizable space, Y is a dense open subset of $\overline{Y}$ and E_Y is a compact 0-dimensional subset of $\overline{Y}$.

For $h \in \mathscr{H}(Y)$ and $e \in E_Y$ we define $h(e) \in E_Y$ by $h(e)(K) = h(e(h^{-1}(K)))$ $(K \in \mathscr{K}(Y))$. Each $h \in \mathscr{H}(Y)$ has a unique extension $\overline{h} \in \mathscr{H}(\overline{Y})$ defined by $\overline{h}(e) = h(e)$ $(e \in E_Y)$. The map $\mathscr{H}(Y) \to \mathscr{H}(\overline{Y}) : h \mapsto \overline{h}$ is a continuous group homomorphism. For $A \subset Y$ we set $\mathscr{H}_{A\cup E_Y}(Y) = \{h \in \mathscr{H}_A(Y) \mid \overline{h}|_{E_Y} = \mathrm{id}_{E_Y}\}$. Note that $\mathscr{H}_{A\cup E_Y}(Y)_0 = \mathscr{H}_A(Y)_0$.

Let $\mu \in \mathscr{M}(Y)$. An end $e \in E_Y$ is said to be *μ-finite* if $\mu(e(K)) < \infty$ for some $K \in \mathscr{K}(Y)$. Let $E_Y^\mu = \{e \in E_Y \mid e \text{ is } \mu\text{-finite}\}$. Then $Y \cup E_Y^\mu$ is an open subset of $\overline{Y}$. For $A \in \mathscr{B}(Y)$ and $\mu \in \mathscr{M}_g^A(Y)$ we set

$$\mathscr{M}_g^A(Y;\mu\text{-e-reg}) = \{\nu \in \mathscr{M}_g^A(Y) \mid \nu \text{ is } \mu\text{-biregular},\ E_Y^\nu = E_Y^\mu\},$$
$$\mathscr{M}_g^A(Y;\mu) = \{\nu \in \mathscr{M}_g^A(Y;\mu\text{-e-reg}) \mid \nu(Y) = \mu(Y)\}.$$

The *finite-ends weak* topology ew on $\mathscr{M}_g^A(Y;\mu\text{-e-reg})$ is the weakest topology such that the function

$$\Phi_f : \mathscr{M}_g^A(Y;\mu\text{-e-reg}) \longrightarrow \boldsymbol{R} \ : \ \nu \longmapsto \int_Y f|_Y \, d\nu$$

is continuous for any continuous function $f : Y \cup E_Y^\mu \to \boldsymbol{R}$ with compact support.

There is an alternative description of this topology ([**4**, Section 3, p. 245]). Consider the space $\mathscr{M}(Y \cup E_Y^\mu)$ (with the weak topology). Each $\nu \in \mathscr{M}_g(Y; \mu\text{-e-reg})$ has a natural extension $\overline{\nu} \in \mathscr{M}_g(Y \cup E_Y^\mu)$ defined by $\overline{\nu}(B) = \nu(B \cap Y)$ ($B \in \mathscr{B}(Y \cup E_Y^\mu)$). The topology *ew* on $\mathscr{M}_g^A(Y; \mu\text{-e-reg})$ is the weakest topology for which the injection

$$\iota : \mathscr{M}_g^A(Y; \mu\text{-e-reg}) \longrightarrow \mathscr{M}(Y \cup E_Y^\mu)_w \ : \ \nu \longmapsto \overline{\nu}$$

is continuous. The symbol $\mathscr{M}_g^A(Y; \mu\text{-e-reg})_{ew}$ denotes the space $\mathscr{M}_g^A(Y; \mu\text{-e-reg})$ endowed with the topology *ew*.

We say that $h \in \mathscr{H}(Y)$ is *μ-end-biregular* if h is μ-biregular and $E_Y^{h_*\mu} = E_Y^\mu$ (i.e., $\overline{h}(E_Y^\mu) = E_Y^\mu$). Let $\mathscr{H}(Y; \mu\text{-e-reg})$ denote the subgroup of $\mathscr{H}(Y)$ consisting of μ-end-biregular homeomorphisms of Y.

Suppose M is a connected n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$. The group $\mathscr{H}(M; \mu\text{-e-reg})$ acts continuously on $\mathscr{M}_g^\partial(M; \mu)_{ew}$ by $h \cdot \nu = h_*\nu$. This action induces the map

$$\pi : \mathscr{H}(M; \mu\text{-e-reg}) \longrightarrow \mathscr{M}_g^\partial(M; \mu)_{ew} \ : \ h \longmapsto h_*\mu.$$

THEOREM 3.3 ([**4**, Theorem 4.1]). *The map π has a section*

$$\sigma : \mathscr{M}_g^\partial(M; \mu)_{ew} \longrightarrow \mathscr{H}_\partial(M; \mu\text{-e-reg})_1 = \mathscr{H}_\partial(M; \mu\text{-reg})_1$$

such that ($\pi\sigma = \mathrm{id}$ *and*) $\sigma(\mu) = \mathrm{id}_M$.

3.3. End charge homomorphism.

We recall basic properties of the end charge homomorphisms defined in [**2**, Section 14]. Suppose Y is a connected, locally connected, locally compact separable, metrizable space. Let $\mathscr{Q}(E_Y)$ denote the algebra of clopen subsets of E_Y and let $\mathscr{B}_c(Y) = \{C \in \mathscr{B}(Y) \mid \mathrm{Fr}_Y\, C \text{ is compact}\}$. For each $C \in \mathscr{B}_c(Y)$ let

$$E_C = \{e \in E_Y \mid e(K) \subset C \text{ for some } K \in \mathscr{K}(Y)\} \quad \text{and} \quad \overline{C} = C \cup E_C \subset \overline{Y}.$$

Note that (i) $E_C \in \mathscr{Q}(E_Y)$ and $\overline{C}$ is a neighborhood of E_C in $\overline{Y}$ with $\overline{C} \cap E_Y = E_C$, (ii) for $C, D \in \mathscr{B}_c(Y)$ it follows that $E_C = E_D$ if and only if $C \Delta D = (C - D) \cup (D - C)$ is relatively compact (i.e., has the compact closure) in Y, (iii) if $C \in \mathscr{B}_c(Y)$ and $h \in \mathscr{H}_{E_Y}(Y)$, then $h(C) \in \mathscr{B}_c(Y)$ and $E_{h(C)} = E_C$.

An *end charge* of Y is a finitely additive signed measure c on $\mathscr{Q}(E_Y)$, that is, a function $c : \mathscr{Q}(E_Y) \to \boldsymbol{R}$ which satisfies the following condition:

$$c(F \cup G) = c(F) + c(G) \quad \text{for} \quad F, G \in \mathscr{Q}(E_Y) \quad \text{with} \quad F \cap G = \emptyset.$$

Let $\mathscr{S}(Y)$ denote the space of end charges c of Y endowed with the *weak topology* (or the product topology). This topology is the weakest topology such that the function

$$\Psi_F : \mathscr{S}(Y) \longrightarrow \boldsymbol{R} \ : \ c \longmapsto c(F)$$

is continuous for any $F \in \mathscr{Q}(E_Y)$. For $\mu \in \mathscr{M}(Y)$ let

$$\mathscr{S}(Y, \mu) = \left\{ c \in \mathscr{S}(Y) \;\middle|\; \begin{array}{ll} \text{(i)} & c(F) = 0 \text{ for } F \in \mathscr{Q}(E_Y) \text{ with } F \subset E_Y^\mu \\ \text{(ii)} & c(E_Y) = 0 \end{array} \right\}$$

(with the weak topology). Then $\mathscr{S}(Y)$ is a topological linear space and $\mathscr{S}(Y, \mu)$ is a linear subspace.

For $h \in \mathscr{H}_{E_Y}(Y; \mu)$ the end charge $c_h^\mu \in \mathscr{S}(Y, \mu)$ is defined as follows: For any $F \in \mathscr{Q}(E_Y)$ there exists $C \in \mathscr{B}_c(Y)$ with $E_C = F$. Since $\overline{h}|_{E_Y} = \mathrm{id}$, it follows that $E_C = E_{h(C)}$ and that $C \Delta\, h(C)$ is relatively compact in Y. Thus $\mu(C - h(C))$, $\mu(h(C) - C) < \infty$ and we can define

$$c_h^\mu(F) = \mu(C - h(C)) - \mu(h(C) - C) \in \boldsymbol{R}.$$

This quantity is independent of the choice of C.

PROPOSITION 3.3. *The end charge homomorphism $c^\mu : \mathscr{H}_{E_Y}(Y; \mu) \to \mathscr{S}(Y, \mu)$ is a continuous group homomorphism* ([**2**, Section 14.9, Lemma 14.21 (iv)]).

In [**12**] we have shown that, for any connected n-manifold M and $\mu \in \mathscr{M}_g^\partial(M)$, the end charge homomorphism $c^\mu : \mathscr{H}_{E_M}(M; \mu) \to \mathscr{S}(M; \mu)$ has a (non-homomorphic) section $s : \mathscr{S}(M, \mu) \to \mathscr{H}_\partial(M; \mu)_1$.

For any subset A of Y we have the restriction of c^μ

$$c_A^\mu : \mathscr{H}_{A \cup E_Y}(Y; \mu) \longrightarrow S(Y, \mu).$$

The kernel of the homomorphism c^μ is denoted by $\ker c^\mu$. Note that $\mathscr{H}_c(M; \mu) \subset \ker c^\mu$ and $(\ker c^\mu)_A = \ker c_A^\mu$. By the definition, if $h \in \ker c^\mu$, then for any $C \in \mathscr{B}_c(Y)$ we have $\mu(C - h(C)) = \mu(h(C) - C)$.

LEMMA 3.3. *Suppose $h \in \ker c^\mu$ and $C \in \mathscr{B}_c(Y)$. If $L \in \mathscr{B}(C \cap h(C))$ and $C - L$ is relatively compact in Y, then $h(C) - L$ is also relatively compact and $\mu(h(C) - L) = \mu(C - L)$.*

PROOF. Since $\mu(C - h(C)) = \mu(h(C) - C)$, the assertion follows from the equalities:

$$h(C) - L = (h(C) - C) \cup ((C \cap h(C)) - L) \quad \text{and}$$
$$C - L = (C - h(C)) \cup ((C \cap h(C)) - L). \qquad \square$$

4. Weak extension theorem for biregular homeomorphisms.

Throughout this section M is an n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$. The weak extension theorem for the group $G = \mathscr{H}(M; \mu\text{-reg})$ is already obtained in [**8**]. In this section we discuss some consequences of this extension theorem. In Section 5 we combine the weak extension theorem for $\mathscr{H}(M; \mu\text{-reg})$ and the selection theorem for μ-biregular measures (Theorems 3.2 and 3.3) in order to obtain the weak extension theorems for the groups $\mathscr{H}(M; \mu)$ and $\ker c^\mu$.

First we recall the deformation theorem for μ-biregular embeddings [**8**, Theorem 4.1]. For $X \in \mathscr{B}(M)$ and $A \subset X$, let $\mathscr{E}^*_A(X, M; \mu\text{-reg})$ denote the space of proper μ-biregular embeddings $f : X \to M$ with $f|_A = \mathrm{id}_A$, endowed with the compact-open topology (cf. Sections 2.1 and 3.1).

Suppose C is a compact subset of M, $U \in \mathscr{B}(M)$ is a neighborhood of C in M and $D \subset E$ are two closed subsets of M such that $D \subset \mathrm{Int}_M E$.

THEOREM 4.1 ([**8**, Theorem 4.1]). *For any compact neighborhood K of C in U, there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^*_{E\cap U}(U, M; \mu\text{-reg})$ and a homotopy $\varphi : \mathscr{U} \times [0,1] \longrightarrow \mathscr{E}^*_{D\cap U}(U, M; \mu\text{-reg})$ such that*

(1) *for each $f \in \mathscr{U}$,*
 (i) $\varphi_0(f) = f$, (ii) $\varphi_1(f)|_C = i_C$, (iii) $\varphi_t(f)|_{U-K} = f|_{U-K}$ $(t \in [0,1])$,
 (iv) *if $f = \mathrm{id}$ on $\partial_- U$, then $\varphi_t(f) = \mathrm{id}$ on $\partial_- U$* $(t \in [0,1])$,

(2) $\varphi_t(i_U) = i_U$ $(t \in [0,1])$.

Theorem 4.1 is equivalent to the next weak extension theorem.

THEOREM 4.2 ([**8**, Corollary 4.2]). *For any compact neighborhood L of C in U, there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^*_{E\cap U}(U, M; \mu\text{-reg})$ and a homotopy $s : \mathscr{U} \times [0,1] \to \mathscr{H}_{D\cup(M-L)}(M; \mu\text{-reg})_1$ such that*

(1) *for each $f \in \mathscr{U}$* (i) $s_0(f) = \mathrm{id}_M$, (ii) $s_1(f)|_C = f|_C$, (iii) *if $f = \mathrm{id}$ on $\partial_- U$, then $s_t(f) = \mathrm{id}$ on ∂M,*

(2) $s_t(i_U) = \mathrm{id}_M$ $(t \in [0,1])$.

(In [**8**, Corollary 4.2] the map s_1 alone is mentioned.)

Now we discuss some consequences of Theorem 4.2 for the group $G = \mathscr{H}(M; \mu\text{-reg})$. Suppose X is a compact subset of M. Note that $G_X = \mathscr{H}_X(M; \mu\text{-reg})$.

Suppose C is a compact subset of M with $X \subset C$ and U is a neighborhood of C in M. Consider the pull-back diagram:

$$\begin{array}{ccc} p^*G_X & \xrightarrow{p'} & G_X \\ \pi' \downarrow & & \downarrow \pi \\ \mathscr{E}^G_X(U,M) & \xrightarrow[p]{} & \mathscr{E}^G_X(C,M), \end{array} \qquad \text{where} \quad \pi(h) = h|_C \quad \text{and} \quad p(f) = f|_C.$$

By Theorem 4.2 the pair (U,C) has WEP_G. Hence it has LSP_G and also LSP_{G_X}. Thus the next assertion follows from Lemma 2.4.

LEMMA 4.1. *The induced map $\pi' : p^*G_X \to \mathscr{E}^G_X(U,M)$ is a principal G_C-bundle.*

Suppose N is a compact n-submanifold of M such that $\mu(\partial_+ N) = 0$ and $X \subset \mathrm{Int}_M N$. Take any compact n-submanifold N_1 of M such that $\mu(\partial_+ N_1) = 0$ and N_1 is obtained from N by adding an outer collar of $\partial_+ N$. We obtain the pull-back diagram:

$$\begin{array}{ccc} p^*G_X & \xrightarrow{p'} & G_X \\ \pi' \downarrow & & \downarrow \pi \\ \mathscr{E}^G_X(N_1,M) & \xrightarrow[p]{} & \mathscr{E}^G_X(N,M), \end{array} \qquad \begin{array}{l} \text{where} \quad \pi(g) = g|_N, \quad p(f) = f|_N \quad \text{and} \\ \\ \qquad p^{-1}(i_N) = \mathscr{E}^G_N(N_1,M). \end{array}$$

LEMMA 4.2. *There exists a path $h : [0,1] \to G_X$ such that*

$$h_0 = \mathrm{id}_M, \quad h_1(N_1) = N \quad \textit{and} \quad h_t(N_1) \subset N_1, \quad h_t(N) \subset N \quad (t \in [0,1]).$$

PROOF.

(1) Let m denote the Lebesgue measure on $\boldsymbol{R}$. We can find a bicollar $E = \partial_+ N \times [a,b]$ $(a < 0,\ b > 1)$ of $\partial_+ N$ in $M - X$ and $\nu \in \mathscr{M}^\partial_g(\partial_+ N)$ such that

(i) $\partial_+ N = \partial_+ N \times \{0\}$, $\partial_+ N_1 = \partial_+ N \times \{1\}$ and (ii) $\mu|_E = \nu \times (m|_{[a,b]})$.

This follows from the following observation. First take any bicollar $E' = \partial_+ N \times [-1, 2]$ of $\partial_+ N$ in $M - X$ which satisfies (i) and the weaker condition (ii)′ $\mu(\partial_+ N \times \{-1\}) = \mu(\partial_+ N \times \{2\}) = 0$. Let $\mathscr{C}(\partial_+ N) = \{F_1, \ldots, F_m\}$ and set $E'_i = F_i \times [-1, 2]$ $(i = 1, \ldots, m)$. Choose any $\nu \in \mathscr{M}^\partial_g(\partial_+ N)$ such that $\nu(F_i) = \mu(F_i \times [0, 1])$ $(i = 1, \ldots, m)$. For each $i = 1, \ldots, m$, determine $a_i < 0$ and $b_i > 1$ by $|a_i|\nu(F_i) = \mu(F_i \times [-1, 0])$ and $(b_i - 1)\nu(F_i) = \mu(F_i \times [1, 2])$, and reparametrize $F_i \times [-1, 0]$ to $F_i \times [a_i, 0]$ and $F_i \times [1, 2]$ to $F_i \times [1, b_i]$. We can apply Theorem 3.1 on $F_i \times [a_i, 0]$, $F_i \times [0, 1]$ and $F_i \times [1, b_i]$ to obtain a new identification $E'_i = F_i \times [a_i, b_i]$ so that $\mu|_{E'_i} = \nu \times (m|_{[a_i, b_i]})$. Take a, b such that $\max_i a_i < a < 0$ and $1 < b < \min_i b_i$, and set $E = \bigcup_i (F_i \times [a, b])$.

(2) Choose $\lambda \in \mathscr{H}_\partial([a, b])$ such that λ is piecewise affine and $\lambda(0) = a/2$, $\lambda(1) = 0$. We obtain two isotopies

$$\lambda_t \in \mathscr{H}_\partial([a, b]) \quad (t \in [0, 1]) \text{ defined by } \lambda_t(s) = (1 - t)s + t\lambda(s) \text{ and}$$
$$g_t \in \mathscr{H}_{\partial_+ N \times \{a, b\}}(\partial_+ N \times [a, b]) \quad (t \in [0, 1]) \text{ defined by } g_t(y, s) = (y, \lambda_t(s)).$$

Note that $\lambda_0 = \mathrm{id}$, $\lambda_1([a, 1]) = [a, 0]$, $\lambda_t([a, 0]) \subset [a, 0]$ and $\lambda_t([a, 1]) \subset [a, 1]$. Since λ_t is also piecewise affine, it is seen that λ_t is $m|_{[a,b]}$-biregular. Then each g_t is $\nu \times (m|_{[a,b]})$-biregular by Proposition 3.2. Finally, the required isotopy $h_t \in \mathscr{H}_{E^c}(M; \mu\text{-reg}) \subset G_X$ $(t \in [0, 1])$ is defined by $h_t|_E = g_t$. □

By Lemmas 4.1, 4.2 and 2.5 we have the following conclusions.

Lemma 4.3.
(1) *The induced map* $\pi' : p^* G_X \to \mathscr{E}^G_X(N_1, M)$ *is a principal* G_N*-bundle.*
(2) *The map* $p : \mathscr{E}^G_X(N_1, M) \to \mathscr{E}^G_X(N, M)$ *is a homotopy equivalence.*
(3) *There exists a strong deformation retraction* χ_t $(t \in [0, 1])$ *of* $\mathscr{E}^G_N(N_1, M)$ *onto the singleton* $\{i_{N_1}\}$.
(4) *The map* $p' : p^* G_X \to G_X$ *is a homotopy equivalence.*

Corollary 4.1. *Suppose* X *is a compact subset of* M *and* N *is a compact* n*-submanifold of* M *such that* $\mu(\partial N) = 0$ *and* $X \subset \mathrm{Int}_M N$. *Then the restriction map*

$$\mathscr{H}_N(M; \mu\text{-reg}) \subset \mathscr{H}_X(M; \mu\text{-reg}) \xrightarrow{\pi} \mathscr{E}^{\mathscr{H}(M;\mu\text{-reg})}_X(N, M) \quad \textit{defined by } \pi(h) = h|_N$$

is a fibration up to homotopy equivalences and has the exact sequence for homotopy groups.

5. Weak extension theorem for measure-preserving homeomorphisms.

Throughout this section M is an n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$. In this section we combine the weak extension theorem for $G = \mathscr{H}(M; \mu\text{-reg})$ (Theorem 4.2) and the selection theorem for μ-biregular measures (Theorems 3.2 and 3.3) in order to obtain the weak extension theorems for the groups $H = \mathscr{H}(M; \mu)$ and $F = \ker c^\mu$. We also discuss a non-ambient weak deformation of measure-preserving embeddings (Theorem 5.3). Some application to the group $H_c = \mathscr{H}_c(M; \mu)$ endowed with the Whitney topology is provided in Section 6.

5.1. Weak extension theorem for $\mathscr{H}(M; \mu)$.

We obtain the weak extension theorem for $\mathscr{H}(M; \mu)$ in a general form (Theorem 5.1, cf. [**8**, Theorem 4.12]). This answers Problem 2.1 and also leads us to the weak extension theorem for $\ker c^\mu$ in Section 5.2. (Recall that M is an n-manifold, $\mu \in \mathscr{M}_g^\partial(M)$, $G = \mathscr{H}(M; \mu\text{-reg})$ and $H = \mathscr{H}(M; \mu)$.)

For $A, B \in \mathscr{B}(M)$, consider the subset $G^{A,B}$ of G defined by

$$G^{A,B} = \{h \in G \mid h|_A \in \mathscr{E}(A, M; \mu) \text{ and } \mu(h(L)) = \mu(L) \ (L \in \mathscr{C}(M - B))\}.$$

When $A = B$, we simply write G^A. For any $X \subset M$ we have the pair $(G_X^{A,B}, H_X)$ of subsets in G_X.

Lemma 5.1. *Suppose N is a compact n-submanifold of M with $\mu(\partial N) = 0$, $U \in \mathscr{B}(M)$ is a neighborhood of N in M and X is a closed subset of ∂M with $X \cap N = \emptyset$. Then the triple (M, U, N) has* $\mathrm{WEP}(G_X^N, H_X)$.

Proof.

Case 1: First we consider the case where M is connected.

Since $\mathscr{E}^{G_X^N}(U, M) \subset \mathscr{E}^*(U, M; \mu\text{-reg})$, by Theorem 4.2 applied to $(U, C) = (M - X, N)$, there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^{G_X^N}(U, M)$ and a map $\sigma : \mathscr{U} \times [0,1] \to (G_X)_1$ such that

(i) for each $f \in \mathscr{U}$ (a) $\sigma_0(f) = \mathrm{id}_M$, (b) $\sigma_1(f)|_N = f|_N$, (c) if $f = \mathrm{id}$ on $\partial_- U$, then $\sigma_t(f) = \mathrm{id}$ on ∂M,

(ii) $\sigma_t(i_U) = \mathrm{id}_M$ $(t \in [0,1])$.

(1) First we modify the map σ to achieve the following additional condition:
(i) (b′) $\sigma_1(f) \in H$.

Consider the induced map

$$\nu : \mathscr{U} \times [0,1] \longrightarrow \mathscr{M}_g^\partial(M; \mu)_{ew} \text{ defined by } \nu_t(f) = \sigma_t(f)^*\mu.$$

Since M is connected, each $L \in \mathscr{C}(N^c)$ meets $\partial_+ N$. Since $\partial_+ N$ is compact, it follows that $\mathscr{C}(N^c)$ is a finite set. We note that $\nu_1(f)|_L \in \mathscr{M}_g^\partial(L;\mu|_L)$ for any $f \in \mathscr{U}$ and $L \in \mathscr{C}(N^c)$. In fact, since $\nu_1(f) \in \mathscr{M}_g^\partial(M;\mu\text{-e-reg})$ and $\mu(\partial N) = 0$, we have $\nu_1(f)|_L \in \mathscr{M}_g^\partial(L;\mu|_L\text{-e-reg})$. It remains to show that $\nu_1(f)(L) = \mu(L)$. Since $f \in \mathscr{E}^{G_X^N}(U, M)$, there exists $h \in G_X^N$ such that $f = h|_U$. Then $k \equiv h^{-1}\sigma_1(f) \in \mathscr{H}_N(M)$. Since M is connected, we see that $N \cap L \neq \emptyset$, and since $k = \mathrm{id}$ on N, we have $k(L) = L$. Hence, $\sigma_1(f)(L) = h(L)$ and it follows that $\nu_1(f)(L) = \mu(\sigma_1(f)(L)) = \mu(h(L)) = \mu(L)$.

For each $L \in \mathscr{C}(N^c)$ we obtain the map $\mathscr{U} \to \mathscr{M}_g^\partial(L;\mu|_L)_{ew} \ :\ f \longmapsto \nu_1(f)|_L$. By the alternative description of the finite-ends weak topology and Lemma 3.1, this map is seen to be continuous (cf. [**11**, Lemma 3.2]). By Theorem 3.3 there exists a map

$$\eta_L : \mathscr{M}_g^\partial(L;\mu|_L)_{ew} \longrightarrow \mathscr{H}_\partial(L;\mu|_L\text{-reg})_1$$

such that $\eta_L(\nu)_*(\mu|_L) = \nu$ and $\eta_L(\mu|_L) = \mathrm{id}_L$.

Define the map $\tau_L : \mathscr{U} \times [0,1] \to \mathscr{H}_\partial(L;\mu|_L\text{-reg})_1$ by

$$\tau_L(f,t) = \eta_L((1-t)\mu|_L + t\nu_1(f)|_L).$$

Combining τ_L $(L \in \mathscr{C}(N^c))$, we obtain the map

$$\tau : \mathscr{U} \times [0,1] \longrightarrow \mathscr{H}_{N\cup\partial M}(M;\mu\text{-reg})_1$$
$$\text{defined by} \quad \tau(f,t) = \begin{cases} \tau_L(f,t) & \text{on } L \in \mathscr{C}(N^c) \\ \mathrm{id} & \text{on } N. \end{cases}$$

Note that $\tau_0(f) = \mathrm{id}_M$ and $\tau_1(f)_*\mu = \nu_1(f)$. Define a map

$$\sigma' : \mathscr{U} \times [0,1] \longrightarrow \mathscr{H}_X(M;\mu\text{-reg})_1 \quad \text{by} \quad \sigma'_t(f) = \begin{cases} \sigma_{2t}(f) & (t \in [0,1/2]) \\ \sigma_1(f)\tau_{2t-1}(f) & (t \in [1/2,1]). \end{cases}$$

Then the map σ' satisfies the conditions (i) (a), (b), (c) and (ii). The condition (i) (b′) is verified by

$$\sigma'_1(f)_*\mu = \sigma_1(f)_*\tau_1(f)_*\mu = \sigma_1(f)_*\nu_1(f) = \sigma_1(f)_*\sigma_1(f)^*\mu = \mu.$$

(2) To see that the triple (M, U, N) has $\mathrm{WEP}(G_X^N, H_X)$, we construct a map

$s : \mathscr{U} \times [0,1] \to H_X$ such that

(iii) for each $f \in \mathscr{U}$ (a) $s_0(f) = \mathrm{id}_M$, (b) $s_1(f)|_N = f|_N$, (c) if $f = \mathrm{id}$ on $\partial_- U$, then $s_t(f) = \mathrm{id}$ on ∂M,

(iv) $s_t(i_U) = \mathrm{id}_M$ $(t \in [0,1])$.

Consider the induced map

$$\nu' : \mathscr{U} \times [0,1] \longrightarrow \mathscr{M}_g^\partial(M;\mu)_{ew} \quad \text{defined by} \quad \nu'_t(f) = \sigma'_t(f)^*\mu.$$

It is seen that $\nu'_0(f) = \nu'_1(f) = \mu$. By Theorem 3.3 there exists a map

$$\eta : \mathscr{M}_g^\partial(M;\mu)_{ew} \longrightarrow (G_\partial)_1 \quad \text{such that} \quad \eta(\nu)_*\mu = \nu \quad \text{and} \quad \eta(\mu) = \mathrm{id}_M.$$

The required map s is defined by $s_t(f) = \sigma'_t(f)\eta(\nu'_t(f))$ $((f,t) \in \mathscr{U} \times [0,1])$. The conditions (iii) and (iv) are easily verified. For example, (iii) (b) is seen by

$$s_1(f) = \sigma'_1(f)\eta(\nu'_1(f)) = \sigma'_1(f)\eta(\mu) = \sigma'_1(f) \quad \text{and} \quad s_1(f)|_N = \sigma'_1(f)|_N = f|_N.$$

Case 2: Next we treat the general case where M may not be connected.

By Lemma 2.6 we may assume that U is compact. Let $M_1, \dots, M_m$ denote the connected components of M which meet U. For each $i = 1, \cdots, m$, we set $(U_i, N_i, X_i) = (U, N, X) \cap M_i$ and $\mu_i = \mu|_{M_i}$. By Case 1, the triple (M_i, U_i, N_i) in M_i has WEP for $(G_i, H_i) = (\mathscr{H}_{X_i}(M_i;\mu_i\text{-reg})^{N_i}, \mathscr{H}_{X_i}(M_i;\mu_i))$. Since the pair (G_i, H_i) can be canonically identified with the subpair $(G_X^N(M_i), H_X(M_i))$ of (G_X^N, H_X) and $\mathscr{E}^{G_i}(U_i, M_i) = \mathscr{E}^{G_X^N(M_i)}(U_i, M) = \mathscr{E}^{G_X^N}(U_i, M) \cap \mathscr{E}(U_i, M_i)$, which is an open subset of $\mathscr{E}^{G_X^N}(U_i, M)$, it is seen that the triple (M_i, U_i, N_i) in M has WEP(G_X^N, H_X). Hence, by Lemma 2.7 $(\bigcup_i M_i, U, N)$ has WEP(G_X^N, H_X) and by Lemma 2.6 so is (M, U, N). □

THEOREM 5.1. *Suppose C is a compact subset of M, $U \in \mathscr{B}(M)$ is a neighborhood of C in M and X is a closed subset of ∂M with $X \cap C = \emptyset$. Then the triple (M, U, C) has* WEP$(G_X^{U,C}, H_X)$.

PROOF. By Lemma 2.2 (2)(i) and Remark 3.1, there exists a compact n-submanifold N of M such that

$$C \subset \mathrm{Int}_M N, \quad N \subset \mathrm{Int}_M U - X,$$
$$O - N \text{ is connected for each } O \in \mathscr{C}(M - C) \quad \text{and} \quad \mu(\partial N) = 0.$$

We show that $G^{U,C} \subset G^N$. Take any $h \in G^{U,C}$. Since $h|_U \in \mathscr{E}(U, M;\mu)$, we have

$h|_N \in \mathscr{E}(N, M; \mu)$. By the choice of N, for each $L \in \mathscr{C}(M - N)$ there exists a unique $O \in \mathscr{C}(M - C)$ such that $L = O - N$. Since $h \in G^{U,C}$, we have $\mu(h(O)) = \mu(O)$. Since $h|_U \in \mathscr{E}(U, M; \mu)$, $O \cap N \subset N \subset U$ and N is compact, it follows that $\mu(h(O \cap N)) = \mu(O \cap N) \leq \mu(N) < \infty$. Hence, $\mu(h(L)) = \mu(L)$. This means that $h \in G^N$.

By Lemma 5.1 the triple (M, U, N) has $\mathrm{WEP}(G_X^N, H_X)$ and by Lemma 2.6 we conclude that the triple (M, U, C) has $\mathrm{WEP}(G_X^{U,C}, H_X)$. □

Since $H_X \subset G_X^{U,C}$, the next statement is an immediate consequence of Theorem 5.1 and Lemma 2.6.

Corollary 5.1. *Suppose C is a compact subset of M, $U \in \mathscr{B}(M)$ is a neighborhood of C in M and X is a closed subset of ∂M with $X \cap C = \emptyset$. Then the triple (M, U, C) has $\mathrm{WEP}(\mathscr{H}_X(M; \mu))$.*

5.2. The weak extension theorem for $\ker c^\mu$.

Suppose M is a *connected* n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$. In this section we deduce the weak extension theorem for the group $F = \ker c^\mu$ (Theorem 5.2). (Recall that $G = \mathscr{H}(M; \mu\text{-reg})$ and $H = \mathscr{H}(M; \mu)$. Note that $H_c = F_c$ and $H(C) = F(C)$ for any compact subset C of M.)

Theorem 5.2. *Suppose C is a compact subset of M, U and V are open neighborhoods of C in M such that $V \cap O$ is connected for each $O \in \mathscr{C}(M - C)$. Then, the triple (V, U, C) has $\mathrm{WEP}(\ker c^\mu, \mathscr{H}_c(M; \mu))$.*

Proof.

(1) By Lemma 2.2 (2)(ii) and Remark 3.1, there exists a compact n-submanifold N of M such that

$$C \subset \mathrm{Int}_M N, \quad N \subset V,$$
$$N \cap O \text{ is connected for each } O \in \mathscr{C}(M - C) \quad \text{and} \quad \mu(\partial N) = 0.$$

Note that $\mathscr{C}(N - C) = \{N \cap O \mid O \in \mathscr{C}(M - C)\}$. Take compact subsets D and W of M such that $C \subset \mathrm{Int}_M D$, $D \subset \mathrm{Int}_M W$ and $W \subset U \cap \mathrm{Int}_M N$. Since $N \subset V$ and $W \subset U$, by Lemma 2.6 it suffices to show that the triple (N, W, C) has $\mathrm{WEP}(\ker c^\mu, \mathscr{H}_c(M; \mu))$.

Since $\mathscr{E}^F(W, M) \subset \mathscr{E}^*(W, M; \mu\text{-reg})$, by Theorem 4.2 there exists a neighborhood $\mathscr{U}$ of i_W in $\mathscr{E}^F(W, M)$ and a map

$$s : \mathscr{U} \to G(N) \ \text{ such that } \ s(f)|_D = f|_D \ \text{ and } \ s(i_W) = \mathrm{id}_M.$$

Replacing $\mathscr{U}$ by a smaller one, we may assume that $f(W) \subset N$ ($f \in \mathscr{U}$).

(2) Consider the n-manifold N and $\mu|_N \in \mathscr{M}_g^\partial(N)$. By Theorem 5.1 the triple (N, D, C) has WEP for

$$(G', H') = (\mathscr{H}_{\partial_+ N}(N; \mu|_N\text{-reg})^{D,C}, \mathscr{H}_{\partial_+ N}(N; \mu|_N)).$$

Let $\mathscr{E}^{G'}(D, N) \supset \mathscr{U}' \xrightarrow{\sigma'_t} H'$ be the associated LWE map. Each $h' \in H'$ has a canonical extension $\psi(h') \in H(N)$ and this defines the canonical homeomorphism $\psi : H' \cong H(N)$.

(3) We show that $s(f)|_N \in G'$ for any $f \in \mathscr{U}$. Since $s(f) \in G(N)$, we have $s(f)|_N \in \mathscr{H}_{\partial_+ N}(N; \mu|_N\text{-reg})$. Since $f \in \mathscr{E}^F(W, M)$, there exists $h \in F$ such that $f = h|_W$. Since $s(f)|_D = f|_D = h|_D \in \mathscr{E}(D, M; \mu)$ and $s(f)(N) = N$, it follows that $s(f)|_D \in \mathscr{E}(D, N; \mu|_N)$. Take any $L \in \mathscr{C}(N - C)$. Then there exists a unique $O \in \mathscr{C}(M - C)$ with $L = N \cap O$. Let $K = O - L = O - N$. Consider $g \equiv h^{-1}s(f) \in \mathscr{H}_D(M)$. Since M is connected, we have $O \cap D \neq \emptyset$ and since $g = \mathrm{id}$ on D, we have $g(O) = O$ and so $s(f)(O) = h(O)$. Since $s(f) \in G(N)$, it follows that

$$s(f)(K) = K \quad \text{and} \quad s(f)(L) = s(f)(O - K) = s(f)(O) - K = h(O) - K.$$

Thus, we have $\mu(s(f)(L)) = \mu(h(O) - K)$. Since

$$\mathrm{Fr}_M O \subset C, \quad O - K = L \subset N \quad \text{and} \quad K = s(f)(K) \subset s(f)(O) = h(O),$$

it follows that $O \in \mathscr{B}_c(M)$, $K \subset O \cap h(O)$ and $O - K$ is relatively compact in M. Since $h \in F$, by Lemma 3.3 we have $\mu(h(O) - K) = \mu(O - K) = \mu(L)$. Therefore, we have $\mu(s(f)(L)) = \mu(L)$. This means that $s(f)|_N \in G'$.

(4) By (3), for any $f \in \mathscr{U}$, we have $s(f)|_N \in G'$ and $f|_D = s(f)|_D = (s(f)|_N)|_D \in \mathscr{E}^{G'}(D, N)$. Thus, we obtain the continuous map $\phi : \mathscr{U} \to \mathscr{E}^{G'}(D, N)$ defined by $\phi(f) = f|_D$. Replacing $\mathscr{U}$ by a smaller one, we may assume that $\phi(\mathscr{U}) \subset \mathscr{U}'$. Finally, the associated LWE map $S_t : \mathscr{U} \to H(N)$ for WEP(F, H_c) of the triple (N, W, C) is defined by

$$S_t(f) = \psi \sigma'_t \phi(f). \qquad \square$$

Since $H_c \subset F$, the next statement is an immediate consequence of Theorem 5.2 and Lemma 2.6.

Corollary 5.2. *Suppose C is a compact subset of M, U and V are open neighborhoods of C in M such that $V \cap O$ is connected for each $O \in \mathscr{C}(M - C)$. Then the triple (V, U, C) has* WEP$(\mathscr{H}_c(M; \mu))$.

5.3. Non-ambient weak deformation of measure-preserving embeddings.

Suppose M is an n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$. In this section we obtain a non-ambient weak deformation theorem for measure-preserving embeddings. For $X \in \mathscr{B}(M)$, let $\mathscr{E}^*(X, M; \mu) = \mathscr{E}(X, M; \mu) \cap \mathscr{E}^*(X, M)$ with the compact-open topology.

THEOREM 5.3. *Suppose C is a compact subset of M and $U \in \mathscr{B}(M)$ is a neighborhood of C in M. Then there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^*(U, M; \mu)$ and a map $s : \mathscr{U} \times [0,1] \to \mathscr{E}^*(C, M; \mu)$ such that $s_0(f) = i_C$, $s_1(f) = f|_C$ $(f \in \mathscr{U})$ and $s_t(i_U) = i_C$ $(t \in [0,1])$.*

We call the map s a *local weak deformation* map (an LWD map) for the pair (U, C) in M.

LEMMA 5.2. *Suppose N is a compact n-submanifold of M with $\mu(\partial_+ N) = 0$ and $U \in \mathscr{B}(M)$ is a neighborhood of N in M. Then the pair (U, N) admits an* LWD *map in M.*

PROOF.
Case 1: First we treat the case where N is connected.
(1) By Lemma 3.2 there exists a bicollar $E = \partial_+ N \times [a, b]$ $(a < 0 < b)$ of $\partial_+ N$ in M such that

$$\partial_+ N = \partial_+ N \times \{0\}, \quad N \cap E = \partial_+ N \times [a, 0] \quad \text{and} \quad \mu|_E = \nu \times (m|_{[a,b]}),$$

where $\nu \in \mathscr{M}_g^\partial(\partial_+ N)$ and m is the Lebesgue measure on $\boldsymbol{R}$. Let $\mathscr{C}(\partial_+ N) = \{F_1, \dots, F_m\}$ and $E_i = F_i \times [a, b]$ $(i = 1, \dots, m)$. For notational simplicity, we use the following notations:

$$E(I) = \partial_+ N \times I, \quad E_i(I) = F_i \times I \quad (I \subset [a, b]) \quad \text{and}$$
$$N_t = (N - E) \cup E[a, t] \quad (t \in [a, b]).$$

Take $\varepsilon > 0$ such that $a < -3\varepsilon$, $3\varepsilon < b$, and define $\alpha_t \in \mathscr{H}_\partial([a, b])$ $(t \in (-2\varepsilon, 2\varepsilon))$ by the conditions:

$$\alpha_t(s) = s + t \ (s \in [-\varepsilon, \varepsilon]) \quad \text{and} \quad \alpha_t \text{ is affine on the intervals } [a, -\varepsilon] \text{ and } [\varepsilon, b].$$

For each $i = 1, \dots, m$, we obtain the isotopy

$$\phi_t^i = \mathrm{id}_{F_i} \times \alpha_t \in \mathscr{H}_{\partial_+ E_i}(E_i; \mu|_{E_i}\text{-reg}) \quad (t \in (-2\varepsilon, 2\varepsilon)).$$

Note that $\alpha_0 = \mathrm{id}_{[a,b]}$ and $\phi_0^i = \mathrm{id}_{E_i}$.

Take a small neighborhood $\mathscr{W}$ of i_N in $\mathscr{E}^*(N, M; \mu\text{-reg})$ such that for any $g \in \mathscr{W}$ and $i = 1, \ldots, m$,

$$E_i[a, -\varepsilon] \subset g(N) \cap E_i \subset E_i[a, \varepsilon], \quad N_{-\varepsilon} \subset g(N) \subset N_\varepsilon \quad \text{and} \quad g(F_i) \subset E_i(-\varepsilon, \varepsilon).$$

Then, for each $g \in \mathscr{W}$ and $i = 1, \ldots, m$, we have

(i) $(-\varepsilon - a)\nu(F_i) < \mu(g(N) \cap E_i) < (\varepsilon - a)\nu(F_i)$,

(ii) $\mu(\phi_t^i(g(N) \cap E_i)) = \mu(g(N) \cap E_i) + t\nu(F_i)$, since ϕ_t^i is μ-preserving on $E_i[-\varepsilon, \varepsilon]$.

For each $i = 1, \ldots, m$, consider the map

$$c_i : \mathscr{W} \longrightarrow \boldsymbol{R} \ \text{ defined by } \ c_i(g) = \mu(g(N) \cap E_i).$$

Since $\mu(g(\partial_+ N)) = 0$, the map c_i is seen to be continuous. Note that $c_i(g) \in ((-\varepsilon - a)\nu(F_i), (\varepsilon - a)\nu(F_i))$.

(2) Next we construct a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^*(U, M; \mu)$ and a map $\eta : \mathscr{U} \times [0, 1] \to \mathscr{E}^*(N, M; \mu\text{-reg})$ such that for any $f \in \mathscr{U}$ and $t \in [0, 1]$,

$$\text{(iii) } \eta_0(f) = i_N, \ \eta_1(f) = f|_N, \ \eta_t(i_U) = i_N \quad \text{and} \quad \text{(iv) } \mu(\eta_t(f)(N)) = \mu(N).$$

By Theorem 4.2 there exists a neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^*(U, M; \mu)$ and a map

$$\sigma : \mathscr{U} \times [0, 1] \longrightarrow \mathscr{H}_c(M; \mu\text{-reg})$$

such that $\sigma_0(f) = \mathrm{id}_M$, $\sigma_1(f)|_N = f|_N$ $(f \in \mathscr{U})$ and $\sigma_t(i_U) = \mathrm{id}_M$ $(t \in [0, 1])$.

Replacing $\mathscr{U}$ by a smaller one, we may assume that $\sigma_t(f)|_N \in \mathscr{W}$ $(f \in \mathscr{U}, t \in [0, 1])$. Consider the map

$$\gamma : \mathscr{U} \times [0, 1] \longrightarrow \mathscr{W} \subset \mathscr{E}^*(N, M; \mu\text{-reg}) \quad \text{defined by} \quad \gamma_t(f) = \sigma_t(f)|_N.$$

The map γ satisfies the condition (iii). To achieve the condition (iv) we modify the map γ.

We define the maps $\lambda^i : \mathscr{U} \times [0, 1] \to \boldsymbol{R}$ and $\tau^i : \mathscr{U} \times [0, 1] \to (-2\varepsilon, 2\varepsilon)$ by

$$\lambda_t^i(f) = (1 - t)c_i(i_N) + tc_i(f|_N) \quad \text{and} \quad c_i(\gamma_t(f)) + \tau_t^i(f)\nu(F_i) = \lambda_t^i(f).$$

Since $\lambda_t^i(f), c_i(\gamma_t(f)) \in ((-\varepsilon - a)\nu(F_i), (\varepsilon - a)\nu(F_i))$, we have

$$|\tau_t^i(f)|\nu(F_i) = |\lambda_t^i(f) - c_i(\gamma_t(f))| < 2\varepsilon\nu(F_i).$$

The map τ^i has the following properties:

(v) $\tau^i_0(f) = \tau^i_1(f) = \tau^i_t(i_U) = 0$,

(vi) $\mu(\phi^i_{\tau^i_t(f)}(\gamma_t(f)(N) \cap E_i)) = \mu(\gamma_t(f)(N) \cap E_i) + \tau^i_t(f)\nu(F_i) = \lambda^i_t(f)$.

The assertion (vi) follows from the property (1)(ii), while the assertion (v) follows from

$$\begin{aligned}
\tau^i_0(f)\nu(F_i) &= \lambda^i_0(f) - c_i(\gamma_0(f)) = c_i(i_N) - c_i(i_N) = 0,\\
\tau^i_1(f)\nu(F_i) &= \lambda^i_1(f) - c_i(\gamma_1(f)) = c_i(f|_N) - c_i(f|_N) = 0,\\
\tau^i_t(i_U)\nu(F_i) &= \lambda^i_t(i_U) - c_i(\gamma_t(i_U)) = c_i(i_N) - c_i(i_N) = 0.
\end{aligned}$$

The maps $\phi^i_{\tau^i}$ $(i = 1, \dots, m)$ are combined to induce the map

$$\phi : \mathscr{U} \times [0,1] \longrightarrow \mathscr{H}_{E^c}(M; \mu\text{-reg}) \text{ defined by } \phi_t(f)|_{E_i} = \phi^i_{\tau^i_t(f)} \quad (i = 1, \dots, m).$$

The desired map $\eta : \mathscr{U} \times [0,1] \to \mathscr{E}^*(N, M; \mu\text{-reg})$ is defined by $\eta_t(f) = \phi_t(f)\gamma_t(f)$. From (v) it follows that $\phi_0(f) = \phi_1(f) = \phi_t(i_U) = \mathrm{id}_M$, since

$$\phi_0(f)|_{E_i} = \phi_1(f)|_{E_i} = \phi_t(i_U)|_{E_i} = \phi^i_0 = \mathrm{id}_{E_i}.$$

Thus, the map η satisfies the condition (iii). To see the condition (iv), first note that

$$\begin{aligned}
\eta_t(f)(N) = \phi_t(f)\gamma_t(f)(N) &= \phi_t(f)\left(N_a \cup \left(\bigcup_i (\gamma_t(f)(N) \cap E_i)\right)\right)\\
&= N_a \cup \left(\bigcup_i \phi^i_{\tau^i_t(f)}(\gamma_t(f)(N) \cap E_i)\right).
\end{aligned}$$

Since f is μ-preserving, we have $\mu(f(N)) = \mu(N)$. Hence, from (vi) it follows that

$$\begin{aligned}
\mu(\eta_t(f)(N)) &= \mu(N_a) + \sum_i \mu(\phi^i_{\tau^i_t(f)}(\gamma_t(f)(N) \cap E_i)) = \mu(N_a) + \sum_i \lambda^i_t(f)\\
&= \mu(N_a) + (1-t)\sum_i c_i(i_N) + t\sum_i c_i(f|_N)\\
&= (1-t)\left(\mu(N_a) + \sum_i c_i(i_N)\right) + t\left(\mu(N_a) + \sum_i c_i(f|_N)\right)\\
&= (1-t)\mu(N) + t\mu(f(N)) = \mu(N).
\end{aligned}$$

(3) The required LWD map s is obtained as follows.

Theorem 3.2 yields a map $\chi : \mathscr{M}_g^\partial(N;\mu|_N) \to \mathscr{H}_\partial(N;\mu|_N\text{-reg})_1$ such that

$$\chi(\omega)_*(\mu|_N) = \omega \ (\omega \in \mathscr{M}_g^\partial(N;\mu|_N)) \quad \text{and} \quad \chi(\mu|_N) = \mathrm{id}_N.$$

By the condition (2)(iv) we have the map

$$\rho : \mathscr{U} \times [0,1] \longrightarrow \mathscr{M}_g^\partial(N;\mu|_N) \ \text{ defined by } \ \rho_t(f) = \eta_t(f)^*\mu.$$

Since $\rho_t(f) = \eta_t(f)^*\mu = ((\phi_t(f)\sigma_t(f))^*\mu)|_N$, the map ρ is the composition of the following maps:

$$\mathscr{U} \times [0,1] \xrightarrow{\rho_1} \mathscr{H}(M;\mu\text{-reg}) \xrightarrow{\rho_2} \mathscr{M}_g^\partial(M;\mu\text{-reg}) \xrightarrow{\rho_3} \mathscr{M}_g^\partial(N;\mu|_N\text{-reg}),$$
$$\text{where} \quad \rho_1(f,t) = \phi_t(f)\sigma_t(f), \quad \rho_2(h) = h^*\mu \quad \text{and} \quad \rho_3(\omega) = \omega|_N.$$

Since $\mu(\partial_+ N) = 0$, by Lemma 3.1 the third map is continuous. Thus the continuity of the map ρ follows from the continuity of these maps. Finally, the map

$$s : \mathscr{U} \times [0,1] \longrightarrow \mathscr{E}^*(N,M;\mu) \quad \text{is defined by} \quad s_t(f) = \eta_t(f)\chi(\rho_t(f)).$$

Since $s_t(f)^*\mu = \chi(\rho_t(f))^*(\eta_t(f)^*\mu) = \chi(\rho_t(f))^*\rho_t(f) = \mu|_N$, it follows that $s_t(f)$ is μ-preserving. If $t = 0, 1$ or $f = i_U$, then by (2)(iii), $\eta_t(f)$ is μ-preserving, and so $\rho_t(f) = \mu|_N$ and $s_t(f) = \eta_t(f)$. Hence, by (2)(iii) the map s satisfies the required conditions: $s_0(f) = i_N$, $s_1(f) = f|_N$ and $s_t(i_U) = i_N$.

Case 2: Next we treat the general case where N may not be connected.

Let $\mathscr{C}(N) = \{N_1, \cdots, N_m\}$. By Case 1, each pair (U, N_i) $(i = 1, \cdots, m)$ admits an LWD map in M

$$\mathscr{E}^*(U,M;\mu) \supset \mathscr{U}_i \xrightarrow{s_t^i} \mathscr{E}^*(N_i,M;\mu) \quad (t \in [0,1]).$$

For each $i = 1, \ldots, m$, choose a neighborhood U_i of N_i in U such that $U_i \cap U_j = \emptyset$ $(i \neq j)$.

We can find a small neighborhood $\mathscr{U}$ of i_U in $\mathscr{E}^*(U,M;\mu)$ such that $\mathscr{U} \subset \mathscr{U}_i$ and $s_t^i(f)(N_i) \subset U_i$ $(f \in \mathscr{U})$ for each $i = 1, \cdots, m$. An LWD map

$$s : \mathscr{U} \times [0,1] \longrightarrow \mathscr{E}^*(N,M;\mu)$$

for (U, N) is defined by $s_t(f)|_{N_i} = s_t^i(f) \quad (i = 1, \ldots, m)$. □

PROOF OF THEOREM 5.3. By Lemma 2.1 and Remark 3.1 there exists a compact n-submanifold N of M such that $\mu(\partial_+ N) = 0$ and $C \subset N \subset \mathrm{Int}_M U$. By Lemma 5.2 the pair (U, N) admits an LWD map

$$\mathscr{E}^*(U, M; \mu) \supset \mathscr{U} \xrightarrow{\sigma_t} \mathscr{E}^*(N, M; \mu) \quad (t \in [0, 1]).$$

An LWD map $s_t : \mathscr{U} \to \mathscr{E}^*(C, M; \mu)$ for (U, C) is defined by $s_t(f) = \sigma_t(f)|_C$. □

6. Groups of measure-preserving homeomorphisms endowed with the Whitney topology.

Suppose M is a *connected noncompact* n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$. In [**3**, Proposition 5.3] we have shown that the group $\mathscr{H}_c(M)_w$, endowed with the Whitney topology, is locally contractible. In this section we shall apply the weak extension theorem for $\mathscr{H}_c(M; \mu)$ (Corollary 5.2) to verify the local contractibility of the group $\mathscr{H}_c(M; \mu)_w$ endowed with the Whitney topology (Theorem 6.1).

6.1. Homeomorphism groups with the Whitney topology.

First we recall basic properties of the Whitney topology on homeomorphism groups (cf. [**3**, Section 4.3]). Suppose Y is a paracompact space and $\mathrm{cov}(Y)$ is the family of all open covers of Y. For maps $f, g : X \to Y$ and $\mathscr{U} \in \mathrm{cov}(Y)$, we say that f, g are *$\mathscr{U}$-near* and write $(f, g) \prec \mathscr{U}$ if every point $x \in X$ admits $U \in \mathscr{U}$ with $f(x), g(x) \in U$. For each $h \in \mathscr{H}(Y)$ and $\mathscr{U} \in \mathrm{cov}(Y)$, let

$$\mathscr{U}(h) = \{f \in \mathscr{H}(Y) \mid (f, h) \prec \mathscr{U}\}.$$

The Whitney topology on $\mathscr{H}(Y)$ is generated by the base $\mathscr{U}(h)$ ($h \in \mathscr{H}(Y)$, $\mathscr{U} \in \mathrm{cov}(Y)$). The symbol $\mathscr{H}(Y)_w$ denotes the group $\mathscr{H}(Y)$ endowed with the Whitney topology (while the symbol $\mathscr{H}(Y)$ denotes the group $\mathscr{H}(Y)$ with the compact-open topology). It is known that $G = \mathscr{H}(Y)_w$ is a topological group. Recall the notations $G_0 = \mathscr{H}_0(Y)_w$ (the identity component of G) and $G_c = \mathscr{H}_c(Y)_w$ (the subgroup of G consisting of homeomorphisms with compact support). In [**3**, Sections 4.1, 4.3] it is shown that $\mathscr{H}_0(Y)_w \subset \mathscr{H}_c(Y)_w$.

6.2. The box topology on topological groups.

The Whitney topology is closely related to box products (cf. [**3**]). Next we recall basic properties of (small) box products (cf. [**3**, Sections 1, 2]). The *box product* $\square_{n\geq 1} X_n$ of a sequence of topological spaces $(X_n)_{n\geq 1}$ is the product $\prod_{n\geq 1} X_n$ endowed with the box topology generated by the base consisting of boxes $\prod_{n\geq 1} U_n$ (U_n is an open subset of X_n). The *small box product* $\boxdot_{n\geq 1} X_n$ of a sequence of pointed spaces $((X_n, *_n))_{n\geq 1}$ is the subspace of $\square_{n\geq 1} X_n$ defined by

$$\boxdot_{n\ge1}X_n = \{(x_n)_{n\ge1} \in \square_{n\ge1}X_n \mid \exists m \ge 1 \text{ such that } x_n = *_n \ (n \ge m)\}.$$

It has the canonical distinguished point $(*_n)_{n\ge1}$. For a sequence of subsets $A_n \subset X_n$ $(n \ge 1)$, we set

$$\boxdot_{n\ge1}A_n = \boxdot_{n\ge1}X_n \cap \square_{n\ge1}A_n.$$

We say that a space X is (*strongly*) *locally contractible* at $x \in X$ if every neighborhood V of x contains a neighborhood U of x which is contractible in V (rel. x) (i.e., there is a homotopy $h : U \times [0,1] \to V$ such that $h_0 = \mathrm{id}_U$, $h_1(U) = \{x\}$ (and $h_t(x) = x$ $(t \in [0,1])$). A pointed space (X, x_0) is said to be *locally contractible* if X is locally contractible at any point of X and strongly locally contractible at x_0. It is easily seen that if a topological group G is locally contractible at the identity element e, then the pointed space (G, e) is locally contractible ([**3**, Remark 1.9]). The next lemma follows from a straightforward argument.

LEMMA 6.1 ([**3**, Proposition 1.10]). *If pointed spaces $(X_i, *_i)$ $(i \ge 1)$ are locally contractible, then the small box product $\boxdot_{i\ge1}(X_i, *_i)$ is also locally contractible as a pointed space.*

Suppose G is a topological group with the identity element $e \in G$. A sequence of closed subgroups $(G_n)_{n\ge1}$ of G is called a *tower* in G if it satisfies the following conditions:

$$G_1 \subset G_2 \subset G_3 \subset \cdots \quad \text{and} \quad G = \bigcup_{n\ge1} G_n.$$

Any tower $(G_n)_{n\ge1}$ in G induces the small box product $\boxdot_{n\ge1}(G_n, e)$ and the multiplication map

$$p : \boxdot_{n\ge1}(G_n, e) \longrightarrow G \quad \text{defined by} \quad p(x_1, \ldots, x_m, e, e, \ldots) = x_1 \cdots x_m.$$

Note that $\boxdot_{n\ge1}G_n$ is a topological group with the coordinatewise multiplication and the identity element $\boldsymbol{e} = (e, e, \ldots)$ and that the map p is well-defined and continuous ([**3**, Lemma 2.1]).

DEFINITION 6.1. We say that G *carries the box topology* with respect to $(G_n)_{n\ge1}$ if the map $p : \boxdot_{n\ge1}G_n \to G$ is an open map.

Recall that G is *the direct limit* of $(G_n)_{n\ge1}$ in the category of topological groups if any group homomorphism $h : G \to H$ to an arbitrary topological group H is continuous provided the restriction $h|G_n$ is continuous for each $n \ge 1$. If G

carries the box topology with respect to $(G_n)_{n\geq 1}$, then G is the direct limit of $(G_n)_{n\geq 1}$ in the category of topological groups ([**3**, Proposition 2.7]). Note that the map p is an open map if it is open at $\boldsymbol{e}$ (i.e., for any neighborhood U of $\boldsymbol{e}$ in $\boxdot_{n\geq 1} G_n$ the image $p(U)$ is a neighborhood of e in G). We say that a map $f : X \to Y$ has a local section at $y \in Y$ if there exists a neighborhood U of y in Y and a map $s : U \to X$ such that $fs = i_U$. If the map p has a local section $s : U \to \boxdot_{n\geq 1} G_n$ at $e \in G$, then (i) we can adjust s so that $s(e) = \boldsymbol{e}$ and so (ii) the map p is open at $\boldsymbol{e}$. Thus, the next lemma follows from Definition 6.1 and Lemma 6.1.

LEMMA 6.2. *Suppose the map $p : \boxdot_{n\geq 1} G_n \to G$ has a local section at e. Then*
(1) *G carries the box topology with respect to the tower $(G_n)_{n\geq 1}$,*
(2) *if the subgroups G_n $(n \geq 1)$ are locally contractible, then G is also locally contractible.*

LEMMA 6.3. *The map $p : \boxdot_{n\geq 1} G_n \to G$ has a local section at e if and only if for any (or some) subsequence $(G_{n(i)})_{i\geq 1}$ the multiplication map $p' : \boxdot_{i\geq 1} G_{n(i)} \longrightarrow G$ has a local section at e.*

PROOF. Consider the maps $\pi : \boxdot_{n\geq 1} G_n \to \boxdot_{i\geq 1} G_{n(i)}$ and $\eta : \boxdot_{i\geq 1} G_{n(i)} \to \boxdot_{n\geq 1} G_n$ defined by

$$\pi(\ldots, x_{n(i-1)+1}, \ldots, x_{n(i)}, \ldots) = (\ldots, \overset{\overset{i}{\vee}}{(x_{n(i-1)+1} \ldots x_{n(i)})}, \ldots) \quad \text{and}$$
$$\eta(\ldots, x_{i-1}, x_i, \ldots) = (\ldots, e, \underset{\underset{n(i-1)}{\wedge}}{x_{i-1}}, e, \ldots, e, \underset{\underset{n(i)}{\wedge}}{x_i}, \ldots), \quad \text{where } n(0) = 0.$$

The maps p and p' have the factorizations $p' = p\eta$ and $p = p'\pi$, from which follows the assertion. $\square$

6.3. Local contractibility of $\mathscr{H}_c(M;\mu)_w$.

Suppose M is a connected noncompact n-manifold and $\mu \in \mathscr{M}_g^\partial(M)$. Let $H = \mathscr{H}(M;\mu)$ and $F = \ker c^\mu$. (Recall that the subscript w means the Whitney topology. For example, $H_{c,w} = \mathscr{H}_c(M;\mu)_w$.)

Consider any sequence $(K_i)_{i\geq 1}$ of compact subsets of M such that $K_i \subset \mathrm{Int}_M K_{i+1}$ $(i \geq 1)$ and $M = \bigcup_{i\geq 1} K_i$. It induces a tower $H(K_i) = \mathscr{H}_{M-K_i}(M;\mu)$ $(i \geq 1)$ of $H_{c,w}$ and the multiplication map

$$p : \boxdot_{i\geq 1} H(K_i) \longrightarrow H_{c,w}, \quad p(h_1, \ldots, h_m, \mathrm{id}_M, \mathrm{id}_M, \ldots) = h_1 \cdots h_m.$$

THEOREM 6.1.
(1) *The multiplication map $p : \boxdot_{i\geq 1} H(K_i) \to \mathscr{H}_c(M;\mu)_w$ has a local section at id_M.*

(2) *The group* $\mathscr{H}_c(M;\mu)_w$ *carries the box topology with respect to the tower* $(H(K_i))_{i\geq 1}$.
(3) *The group* $\mathscr{H}_c(M;\mu)_w$ *is locally contractible.*

We need some preliminary lemmas. Consider a sequence of compact connected n-submanifolds $(M_i)_{i\geq 1}$ of M such that $M_i \subset \mathrm{Int}_M M_{i+1}$ $(i \geq 1)$ and $M = \bigcup_{i\geq 1} M_i$. Let $M_0 = \emptyset$ and $L_i = M_i - \mathrm{Int}_M M_{i-1}$ $(i \geq 1)$. There exists a sequence of compact n-submanifolds $(N_i)_{i\geq 1}$ of M such that $L_i \subset \mathrm{Int}_M N_i$ and $N_i \cap N_j \neq \emptyset$ if and only if $|i-j| \leq 1$. We call the sequence $(M_i, L_i, N_i)_{i\geq 1}$ an *exhausting sequence* for M.

LEMMA 6.4. *For any sequence* $(K_i)_{i\geq 1}$ *of compact subsets of* M *there exists an exhausting sequence* $(M_i, L_i, N_i)_{i\geq 1}$ *for* M *such that for each* $i \geq 1$ (i) $K_i \subset M_i$, (ii) $\mu(\partial_+ M_i) = 0$ *and* (iii) *the pair* (N_i, L_i) *has* WEP(F, H_c).

PROOF. By the repeated application of Lemma 2.1, we can find a sequence of compact connected n-submanifolds $(M_i)_{i\geq 1}$ of M such that

(i) $K_i \subset M_i \subset \mathrm{Int}_M M_{i+1}$, $\mu(\partial_+ M_i) = 0$ $(i \geq 1)$ and $M = \bigcup_{i\geq 1} M_i$,
(ii) L is noncompact and $M_{i+1} \cap L$ is connected for each $i \geq 1$ and each $L \in \mathscr{C}(M_i^c)$.

Let $M_i = \emptyset$ $(i \leq 0)$ and $M_i^j = M_j - \mathrm{Int}_M M_i$ $(j > i)$.

(1) First we show that the pair $(N, K) = (M_{i-1}^{j+1}, M_i^j)$ has WEP(F, H_c) for each $j > i \geq 0$. Let $\mathscr{C}(M_{i-1}^c) = \{C_1, \dots, C_m\}$ and set $(N_k, K_k) = (N \cap C_k, K \cap C_k)$ $(k = 1, \dots, m)$. Since $(N_k)_k$ is a disjoint finite family, by Lemma 2.7 it suffices to show that each pair (N_k, K_k) has WEP(F, H_c).

Note that $\mathscr{C}(K_k^c) = \{E_0, E_1, \cdots, E_\ell\}$, where

$$E_0 = M_i \cup \bigcup_{s\neq k} C_s \quad \text{and} \quad \{E \in \mathscr{C}(M_j^c) \mid E \subset C_k\} = \{E_1, \dots, E_\ell\}.$$

(If $i = 0$, we ignore E_0.) By the above condition (ii) it is seen that the intersections

$$N_k \cap E_0 = M_i \cap C_k \quad \text{and} \quad N_k \cap E_t = M_{j+1} \cap E_t \quad (t = 1, \dots, \ell)$$

are connected. Hence, we can apply Theorem 5.2 to $(V, U, C) = (\mathrm{Int}_M N_k, \mathrm{Int}_M N_k, K_k)$ to conclude that this triple has WEP(F, H_c). Thus, by Lemma 2.6 the pair (N_k, K_k) also has WEP(F, H_c).

(2) Now consider the subsequence $(M_{3i})_{i\geq 1}$. Let $L_i = M_{3i-3}^{3i}$ and $N_i = M_{3i-4}^{3i+1}$ $(i \geq 1)$. Then, it is seen that $(M_{3i}, L_i, N_i)_{i\geq 1}$ is an exhausting sequence for M and by (1) each pair (N_i, L_i) has WEP(F, H_c). □

Suppose $(M_i, L_i, N_i)_{i\geq 1}$ is an exhausting sequence for M. It induces a tower $(H(M_i))_{i\geq 1}$ of $H_{c,w}$ and the multiplication map $p : \boxdot_{i\geq 1} H(M_i) \to H_{c,w}$.

LEMMA 6.5. *If each pair (N_{2i}, L_{2i}) $(i \geq 1)$ has* WEP(H_c)*, then the map $p : \boxdot_{i\geq 1} H(M_i) \to H_{c,w}$ has a local section $s : \mathscr{U} \to \boxdot_{i\geq 1} H(M_i)$ at* id_M *such that* $s(\mathrm{id}_M) = (\mathrm{id}_M)_{i\geq 1}$.

PROOF. We use the following notations: Let $L_e = \bigcup_i L_{2i}$, $L_o = \bigcup_i L_{2i-1}$ and $N_e = \bigcup_i N_{2i}$. Consider the continuous maps defined by

(a) $r_e : H_{c,w} \to \boxdot_i \mathscr{E}^{H_c}(L_{2i}, M)$, $r_e(h) = (h|_{L_{2i}})_i$ and
$r : H_{c,w} \to \boxdot_i \mathscr{E}^{H_c}(N_{2i}, M)$, $r(h) = (h|_{N_{2i}})_i$,

(b) $\lambda : \boxdot_i H(N_{2i}) \to H_c(N_e)_w$, $\lambda((g_i)_i)|_{N_{2i}} = g_i|_{N_{2i}}$ and
$\lambda_o : \boxdot_i H(L_{2i-1}) \to H_c(L_o)_w$, $\lambda_o((h_i)_i)|_{L_{2i-1}} = h_i|_{L_{2i-1}}$,

(c) $\rho : \boxdot_i H(N_{2i}) \times \boxdot_i H(L_{2i-1}) \to H_{c,w}$, $\rho(\boldsymbol{g}, \boldsymbol{h}) = \lambda(\boldsymbol{g})\lambda_o(\boldsymbol{h})$.

Note that the map λ_o is a homeomorphism, since for any $h \in H_c(L_o)$ we have $h = \mathrm{id}$ on $\partial_+ M_i$ and $h(M_i) = M_i$, so that $h(L_i) = L_i$ $(i \geq 1)$.

First we construct a local section of the map ρ at id_M. By the assumption, for each $i \geq 1$ there exists a neighborhood $\mathscr{V}_i$ of the inclusion map $i_{N_{2i}}$ in $\mathscr{E}^{H_c}(N_{2i}, M)$ and a map

$$\sigma_i : \mathscr{V}_i \longrightarrow H(N_{2i}) \quad \text{such that} \quad \sigma_i(f)|_{L_{2i}} = f|_{L_{2i}} \ (f \in \mathscr{V}_i) \quad \text{and} \quad \sigma_i(i_{N_{2i}}) = \mathrm{id}_M.$$

Since $\boxdot_i \mathscr{V}_i$ is a neighborhood of $(i_{N_{2i}})_i$ in $\boxdot_i \mathscr{E}^{H_c}(N_{2i}, M)$, the preimage $\mathscr{U} = r^{-1}(\boxdot_i \mathscr{V}_i)$ is a neighborhood of id_M in $H_{c,w}$. The maps $(\sigma_i)_i$ determine the continuous maps

$$\sigma : \boxdot_i \mathscr{V}_i \longrightarrow \boxdot_i H(N_{2i}) \text{ defined by } \quad \sigma((f_i)_i) = (\sigma_i(f_i))_i \quad \text{and}$$
$$\eta = \lambda\,\sigma\,r : \mathscr{U} \longrightarrow H_c(N_e)_w.$$

For each $g \in \mathscr{U}$ we have $\eta(g) = g$ on L_e and $\eta(g)^{-1} g \in H_{c,L_e} = H_c(L_o)$. Thus we obtain the map

$$\phi : \mathscr{U} \longrightarrow H_c(L_o)_w \quad \text{defined by} \quad \phi(g) = \eta(g)^{-1} g.$$

The required local section $\zeta : \mathscr{U} \to \boxdot_i H(N_{2i}) \times \boxdot_i H(L_{2i-1})$ of the map ρ is defined by

$$\zeta(g) = (\sigma\, r(g), \lambda_o^{-1}\phi(g)).$$

In fact, we have

$$\rho\zeta(g) = \rho(\sigma r(g), \lambda_o^{-1}\phi(g)) = \lambda(\sigma r(g))\phi(g) = \eta(g)(\eta(g)^{-1}g) = g.$$

Note that $\zeta(\mathrm{id}_M) = ((\mathrm{id}_M)_i, (\mathrm{id}_M)_i)$.

For each $h \in \mathscr{U}$ the image $\zeta(h) = ((f_i)_i, (g_i)_i)$ satisfies the following conditions:

(i) $h = \lambda((f_i)_i)\,\lambda_o((g_i)_i) = (f_1 f_2 \cdots)(g_1 g_2 \cdots) = f_1 g_1 f_2 g_2 f_3 g_3 \cdots$.

(ii) $f_i \in H(N_{2i}) \subset H(M_{2i+1})$, $g_i \in H(L_{2i-1}) \subset H(M_{2i-1}) \subset H(M_{2i+2})$ $(i \geq 1)$.

(iii) $(\mathrm{id}_M, \mathrm{id}_M, f_1, g_1, f_2, g_2, \ldots) \in \boxdot_{i\geq 1} H(M_i)$ and $h = p(\mathrm{id}_M, \mathrm{id}_M, f_1, g_1, f_2, g_2, \ldots)$.

Therefore, the required local section $s : \mathscr{U} \to \boxdot_i H(M_i)$ of the map $p : \boxdot_i H(M_i) \to H_{c,w}$ is defined by

$$s(h) = (\mathrm{id}_M, \mathrm{id}_M, f_1, g_1, f_2, g_2, \ldots).$$

This completes the proof. □

Lemma 6.6. *Suppose N is a compact n-manifold, L is a (locally flat) $(n-1)$-submanifold of ∂N and $\nu \in \mathscr{M}_g^\partial(N)$. Then the group $\mathscr{H}_L(N;\nu)$ is locally contractible.*

Proof. In [**8**, Theorem 4.4] the case where $L = \emptyset$ or ∂N is verified. For the sake of completeness we include a proof. We may assume that N is connected.

(1) First we see that the group $G_L = \mathscr{H}_L(N;\nu\text{-reg})$ is locally contractible. Since G_L is a topological group, it suffices to show that it is semi-locally contractible at id_N, that is, a neighborhood of id_N contracts in G_L. Using a collar $L \times [0,2]$ of L in N (cf. Lemma 3.2), we have a deformation of G_L to $G_{L\times[0,1]}$ which fixes id_N. Applying Theorem 4.1 to $(C,U,D,E) = (N,N,L,L\times[0,1])$, we can find a neighborhood of id_N in $G_{L\times[0,1]}$ which contracts in G_L. These deformations are combined to yield a desired contraction of a neighborhood of id_N in G_L.

(2) Next we show that the group $H_L = \mathscr{H}_L(N;\nu)$ is a strong deformation retract (SDR) of G_L. By Theorem 3.2 the map $\pi : G \to \mathscr{M}_g^\partial(N;\nu)$ admits a section $s : \mathscr{M}_g^\partial(N;\nu) \to G_\partial \subset G_L$. This yields a homeomorphism of pairs

$$H_L \times (\mathscr{M}_g^\partial(N;\nu), \{\nu\}) \approx (G_L, H_L) \;:\; (h,\omega) \longmapsto s(\omega)h.$$

Since $\mathscr{M}_g^\partial(N;\nu)$ admits the "straight line contraction" to $\{\nu\}$, we obtain an SDR of G_L onto H_L.

Finally, the conclusion follows from the observations (1) and (2). □

PROOF OF THEOREM 6.1. (1), (3) By Lemma 6.4 there exists an exhausting sequence $(M_i, L_i, N_i)_{i\geq 1}$ for M such that $\mu(\partial_+ M_i) = 0$ $(i \geq 1)$ and each pair (N_i, L_i) $(i \geq 1)$ has WEP(H_c). By Lemma 6.5 the multiplication map $p' : \boxdot_{i\geq 1} H(M_i) \to H_{c,w}$ has a local section at id_M. By Lemma 6.3 this implies the assertion (1) (consider a mixed sequence of $(K_i)_i$ and $(M_i)_i$). By Lemma 6.6 the group $H(M_i) \cong \mathscr{H}_{\partial_+ M_i}(M_i; \mu|_{M_i})$ is locally contractible for each $i \geq 1$. Thus, by Lemma 6.2 (2) the group $H_{c,w}$ is also locally contractible.

(2) The assertion follows from (1) and Lemma 6.2 (1). □

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Tatsuhiko YAGASAKI

Division of Mathematics
Graduate School of Science and Technology
Kyoto Institute of Technology
Matsugasaki, Sakyo-ku
Kyoto 606-8585, Japan
E-mail: yagasaki@kit.ac.jp