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On restriction algebras of tensor algebras

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§ 1. The main results.

Let X be a compact (non-empty, Hausdorff) space, and $C(X)$ (resp. $D(X)$) the Banach algebra of all continuous (resp. bounded) complex-valued functions on X. Let Y be another compact space, and consider the Banach algebras

$$V(X, Y)=C(X)\widehat{\otimes}C(Y), \quad \text{and} \quad V_D(X, Y)=D(X)\widehat{\otimes}D(Y),$$

both being endowed with the projective tensor product norm (see [**13**; Chap. 1 and 2]). Then we have the natural imbeddings

$$V(X, Y)\subset V_D(X, Y)\subset D(X\times Y),$$

where the first one is an isometric homomorphism and the second one is a norm-decreasing one-to-one homomorphism (see Theorems 4.1 and 4.3 in [**7**]). For an arbitrary closed subset E of the product space $X\times Y$, we define the Banach algebras $V(E)$ and $\tilde{V}(E)$ as in [**14**]. Similarly, we define the algebra $V_D(E)$ as follows. The space $V_D(E)$ is the subalgebra of $D(E)$ consisting of all functions $f\in D(E)$ that have an expansion of the form

$$f(x, y)=\sum_{n=1}^{\infty} g_n(x)h_n(y) \qquad ((x, y)\in E),$$

where $g_n\in D(X)$, $h_n\in D(Y)$ and

$$M=\sum_{n=1}^{\infty} \|g_n\|_{D(X)}\cdot\|h_n\|_{D(Y)}<\infty;$$

the norm $\|f\|_{V_D(E)}$ is defined to be the infimum of the M's taken over all expansions of f in the above form. Thus we have

$$V(E)\subset \tilde{V}(E)\subset C(E) \quad \text{and} \quad V(E)\subset V_D(E)\subset D(E).$$

It is easy to see that these four imbeddings are all norm-decreasing. We also write $V_C(E)=V_D(E)\cap C(E)$, which is clearly a closed subalgebra of $V_D(E)$.

Let now E be an arbitrary subset of the product space $X\times Y$. It is called *rectangular* if $E=\pi_X(E)\times\pi_Y(E)$, where π_X and π_Y denote the canonical projections of $X\times Y$ onto X and onto Y, respectively. In this case, we define

$$\operatorname{leng}(E)=\min\{\operatorname{Card}(\pi_X(E)),\ \operatorname{Card}(\pi_Y(E))\}\ .$$

For an arbitrary subset F of $X\times Y$, we define $\operatorname{leng}(F)$ to be the supremum of $\operatorname{leng}(E)$ taken over all rectangular subsets E of F. Finally, a closed subset E of $X\times Y$ is called a *Varopoulos set* if $V(E)$ is not closed in $\tilde{V}(E)$ with respect to the $\tilde{V}(E)$-norm.

We now state our main theorems.

THEOREM 1. *Let E be a non-empty clopen subset of $X\times Y$. Then the natural imbeddings $V(E)\subset\tilde{V}(E)$ and $V(E)\subset V_D(E)$ are both isometric, and $V(E)=V_C(E)$. If, in addition, E is not a Helson set for the algebra $V(X, Y)$, then $\tilde{V}(E)\neq V(E)$.*

THEOREM 2. *Let E be a non-empty, closed and metrizable subset of $X\times Y$ with $\operatorname{leng}(E)=\infty$. Then E contains a closed set F satisfying the following conditions.*

(i) $$V(F)\subset V_C(F)\subset\tilde{V}(F)\,.$$

(ii) *The imbeddings $V(F)\subset V_C(F)$ and $V(F)\subset\tilde{V}(F)$ are both isometric while $V_C(F)\subset\tilde{V}(F)$ is norm-decreasing.*

(iii) *The spaces $V_C(F)$ and $\tilde{V}(F)$ are both non-separable, and there is a function $f\in V_C(F)$ (resp. $g\in\tilde{V}(F)$) such that $\Phi(f)\notin V(F)$ (resp. $\Phi(g)\notin V_C(F)$) for all non-constant entire functions $\Phi(z)$.*

THEOREM 3. *Suppose that both X and Y are compact, metrizable, perfect, and totally disconnected spaces, and that E is a compact subset of the product space $X\times Y$ which disobeys spectral synthesis. Then there is a countable set F whose accumulation points are all in E, such that $E\cup F$ is a Varopoulos set.*

Let now G be a locally compact abelian group, and $A(G)$ the Fourier algebra on G. Then, for every compact subset E of G, we define the restriction algebra $A(E)$ of $A(G)$ and the associated algebra $B(E)=\tilde{A}(E)$ as in [**2**]. A Varopoulos set for $A(G)$ is similarly defined as before.

THEOREM 4. *Let E be a compact, totally disconnected subset of the N-dimensional euclidean space R^N, and suppose that E disobeys spectral synthesis. Then there is a countable set F whose accumulation points are all in E, such that $E\cup F$ is a Varopoulos set.*

Our theorems are all closely related with those works of Y. Katznelson and O. C. McGehee in [**3**]. Before proving these theorems, we would like to make some remarks. The only non-trivial statement in Theorem 1 is that the imbedding $V(E)\subset V_D(E)$ is isometric. In fact, the last part in Theorem 1 is due to N. Th. Varopoulos [**14**; Theorem 1]; that the imbedding $V(E)\subset\tilde{V}(E)$ is isometric is contained in the author's paper [**8**; Theorem 4.6]; and that $V(E)=V_C(E)$ immediately follows from Theorem 4.3 in [**7**]. Our Theorem 2 yields a stronger conclusion than Theorem III in [3] does. Theorem 3 is an elaboration of a result of Varopoulos [**15**; Theorem 3 and Proposition 9.5]. Finally, Theorem 4 is a generalization of a theorem of Katznelson and McGehee [**3**; Theorem VI].

§2 contains some auxiliary propositions which will be used later in the proofs of Theorems 2 and 3; we also give the proof of Theorem 1 in the last part of §2. §3 is devoted to the proof of Theorem 2, and §4 to those of Theorems 3 and 4. In the last §5, we consider certain restriction algebras of Fourier algebras.

§2. Preliminaries.

Throughout this section, we denote by X and Y arbitrary compact spaces, and by $V'=BM$ the conjugate space of the Banach space $V(X, Y)$, whose elements are called bimeasures on $X\times Y$. For an arbitrary closed subset E of $X\times Y$, we put

$$I(E)=\{f\in V(X, Y);\ f=0 \text{ on } E\}\ ;$$

$$J(E)=\text{the closure of } \{g\in V(X, Y);\ E\cap \operatorname{supp}(g)=\emptyset\}\ ,$$

which are both closed ideals in $V(X, Y)$. Let us also write

$$V'(E)=\{B\in BM;\ \langle f, B\rangle=0 \text{ for all } f\in I(E)\}\ ;$$

$$BM(E)=\{B\in BM;\ \langle g, B\rangle=0 \text{ for all } g\in J(E)\}\ .$$

Thus we have $V'(E)\subset BM(E)$, and the conjugate space of $V(E)=V(X, Y)|_E$ is naturally identified with $V'(E)$. By definition, E is a set of spectral synthesis if and only if $I(E)=J(E)$, or equivalently, if and only if $V'(E)=BM(E)$.

We now denote by $\bar{X}$ and $\bar{Y}$ the maximal ideal spaces of the Banach algebras $D(X)$ and $D(Y)$, respectively. In other words, they are the Stone-Čech compactifications of the spaces X and Y endowed with the discrete topology. Thus, for any non-empty subset E of the product space $X\times Y$, we may identify $V_D(E)$ with the restriction algebra $V(\bar{E})$ of $V(\bar{X}, \bar{Y})$ in a natural way, where $\bar{E}$ denotes the closure of E in $\bar{X}\times\bar{Y}$. It follows at once that the maximal ideal space of $V_D(E)$ may be identified with $\bar{E}$, and that the spectrum of a function f in $V_D(E)$ is the closure of the set $f(E)$. On the other hand, the maximal ideal space of $V_C(E)$ is E, provided that E is compact in $X\times Y$. In fact, let f be any function in $V_C(E)$. It is trivial that f is invertible in $V_C(E)$ if and only if it is invertible in $V_D(E)$. It follows that the spectrum of f is the set $f(E)$, because E is compact and f is continuous on E. Let now m be any non-trivial multiplicative linear functional on $V_C(E)$. The above observation shows that $m(f)\in f(E)$ and so $|m(f)|\leqq\|f\|_{C(E)}$ for any functions f in $V_C(E)$. Since $V_C(E)$ is uniformly dense in $C(E)$, this assures that there is a unique point x of E such that $m(f)=f(x)$ for all f in $V_C(E)$, which clearly establishes our assertion.

Two subsets E and F of the product space $X\times Y$ are called *bidisjoint* if $\pi_X(E)\cap\pi_X(F)=\emptyset$ and $\pi_Y(E)\cap\pi_Y(F)=\emptyset$.

PROPOSITION 2.1. *Let $(E_n)_{n=0}^{\infty}$ be a sequence of pairwise bidisjoint compact subsets of $X\times Y$. Suppose that $E=\bigcup_{n=0}^{\infty}E_n$ is closed and that every E_n $(n=1, 2, \cdots)$ is relatively open in E, then we have:*

(a) *Every bimeasure $B\in BM(E)$ has a unique decomposition of the form*

$$B=\sum_{n=0}^{\infty}B_n\,;\; B_n\in BM(E_n) \qquad \text{for } n=0, 1, 2, \cdots,$$

where the series absolutely converges to B in the norm of $BM(E)$. In this case, we have

$$\|B\|_{BM}=\sum_{n=0}^{\infty}\|B_n\|_{BM}\,.$$

(b) *For an arbitrary function $f\in\tilde{V}(E)$, we have*

$$\|f\|_{\tilde{V}(E)}=\sup\{\|f\|_{\tilde{V}(E_n)}\,;\; n=0, 1, 2, \cdots\}\,.$$

(c) *Let $g\in V(E)$ and $h\in V_D(E)$, and suppose that there is a complex number c such that*

$$g=c=h \text{ on } E_0, \text{ and } \lim_n\|g-c\|_{V(E_n)}=0=\lim_n\|h-c\|_{V_D(E_n)},$$

then we have

$$\|g\|_{V(E)}=\sup\{\|g\|_{V(E_n)}\,;\; n=0, 1, 2, \cdots\}\,,$$

$$\|h\|_{V_D(E)}=\sup\{\|h\|_{V_D(E_n)}\,;\; n=0, 1, 2, \cdots\}\,.$$

PROOF. Let B be any element in $BM(E)$. Since each E_n $(n=1, 2, 3, \cdots)$ is relatively clopen in E, we can define the restriction, B_n, of B to E_n $(n=1, 2, \cdots)$. But then the bimeasures

$$B-\sum_{n=1}^{N}B_n,\, B_1,\, B_2,\, \cdots,\, B_N$$

have pairwise bidisjoint supports for all $N=1, 2, \cdots$; it follows from Lemma 2.2 in [8] that

$$\|B\|_{BM}=\|B-\sum_{n=1}^{N}B_n\|_{BM}+\sum_{n=1}^{N}\|B_n\|_{BM} \qquad (N=1, 2, \cdots)\,.$$

Therefore the series $\sum_{n=1}^{\infty}B_n$ absolutely converges in the norm of $BM(E)$. Putting $B_0=B-\sum_{n=1}^{\infty}B_n$, we see $\operatorname{supp}(B_0)\subset E_0$ and

$$\|B\|_{BM}=\lim_N \|B_0+\sum_{n=1}^N B_n\|_{BM}=\sum_{n=0}^\infty \|B_n\|_{BM},$$

which clearly establishes part (a).

Part (b) is an easy consequence of part (a) combined with the Hahn-Banach theorem.

Let now g be any function in $V(E)$ such that

$$g=c \text{ on } E_0, \quad\text{and}\quad \lim_n \|g-c\|_{V(E_n)}=0$$

for some complex number c. We first note that if there is a natural number N such that $g=c$ on E_n for all $n\geqq N$, then we have

$$\|g\|_{V(E)}=\max\{\|g\|_{V(E_n)};\ n=1, 2, \cdots, N\}.$$

In fact, this follows immediately from part (a) and the fact that sets $E_1, E_2, \cdots, E_{N-1}$, and $E\backslash(\bigcup_{n=1}^{N-1} E_n)$ are pairwise bidisjoint. For general g, we put

$$g_N=g \text{ on } \bigcup_{n=1}^N E_n, \quad\text{and}\quad g_N=c \text{ on } E\backslash(\bigcup_{n=1}^N E_n)$$

for $N=1, 2, \cdots$. It is easy to see from the above remark and our hypothesis that $(g_N)_{N=1}^\infty$ is a Cauchy sequence in $V(E)$ and that its limit is g. Hence we have the required equality.

Finally, let $h\in V_D(E)$ be as in part (c). Since the sets $(E_n)_{n=0}^\infty$ are pairwise bidisjoint in $X\times Y$, we see that the sets $\tilde{E}_0=\bar{E}\backslash(\bigcup_{n=1}^\infty \bar{E}_n)$, $\bar{E}_1, \bar{E}_2, \cdots$ are pairwise bidisjoint in $\bar{X}\times\bar{Y}$, and that all $\bar{E}_n$ are clopen in $\bar{E}$ $(n=1, 2, \cdots)$. Let h' be the function in $V(\bar{E})$ naturally corresponding to h, and observe that

$$h'=c \text{ on } \tilde{E}_0, \quad\text{and}\quad \lim_n \|h'-c\|_{V(\bar{E}_n)}=0.$$

Thus, the required equality follows from the one obtained in the preceding paragraph.

This completes the proof.

PROPOSITION 2.2. *Suppose that* $E=\bigcup_{n=0}^\infty E_n$ *is as in Proposition* 2.1, *and that* E_0 *is a set of spectral synthesis for the algebra* $V(X, Y)$. *Let also* f *be any function in* $C(E)$ *such that* $f=c$ *on* E_0 *for some complex number* c. *Then* f *belongs to* $V(E)$ *if and only if* $f\in V(E_n)$ *for all* n *and* $\lim_n \|f-c\|_{V(E_n)}=0$.

PROOF. Suppose that f belongs to $V(E)$. Since E_0 is a set of spectral synthesis and $f=c$ on E_0, it follows that, for any $\varepsilon>0$, there is a function g in $V(E)$ with $g=c$ on some neighborhood of E_0, such that $\|f-g\|_{V(E)}<\varepsilon$. Let N be any natural number such that $g=c$ on E_n for all $n\geqq N$; then we have

$$\varepsilon>\|f-g\|_{V(E)}\geqq\sup\{\|f-c\|_{V(E_n)};\ n\geqq N\},$$

which proves $\lim_n \|f-c\|_{V(E_n)}=0$.

The converse part is true even in the case that E_0 is not a set of spectral synthesis, as is easily seen from part (c) of Proposition 2.1.

This establishes our proof.

PROPOSITION 2.3. *Let G and H be two compact metrizable spaces, and let G_0 and H_0 be any dense subsets of G and of H, respectively. Then, for any closed subset K of $G\times H$, there is a sequence $(L_n)_{n=1}^{\infty}$ of finite subsets of $G_0\times H_0$ such that:*

(a) *The accumulation points of the set $L=\bigcup_{n=1}^{\infty} L_n$ are all in K;*

(b) *The set $K\cup L$ is a set of spectral synthesis for the algebra $V(G, H)$.*

If, in addition, both G and H are perfect, then such a sequence $(L_n)_{n=1}^{\infty}$ can be taken so that

(c) $$\operatorname{leng}(L_n)\geqq n \qquad (n=1, 2, \cdots).$$

PROOF. We may assume that G and H are metric spaces with metrics d_G and d_H, respectively. We define a metric d on $G\times H$ by setting

$$d((x, y), (x', y'))=\max\{d_G(x, x'), d_H(y, y')\}.$$

For any subset E of an arbitrary metric space, let us denote by $\Delta(E)$ and $U(E)$ the diameter of E and an arbitrary neighborhood of E, respectively.

We shall inductively choose two increasing sequences $(G_n=\{x_{nj}\}_j\subset G_0)_{n=1}^{\infty}$ and $(H_n=\{y_{nk}\}_k\subset H_0)_{n=1}^{\infty}$ of finite sets; two sequences $(\{\varphi_{nj}\}_j\subset C(G))_{n=1}^{\infty}$ and $(\{\psi_{nk}\}_k\subset C(H))_{n=1}^{\infty}$ of (finite) partitions of unity; and a sequence $(a_n)_{n=1}^{\infty}$ of positive real numbers subject to the following conditions.

(P_n) $$G_n \text{ is } a_n\text{-dense in } G;$$

(P'_n) $$H_n \text{ is } a_n\text{-dense in } H;$$

(Q_n) $$\varphi_{nj}\geqq 0;\ \varphi_{nj}=1 \text{ on some } U(x_{nj});\ \Delta(\operatorname{supp}\varphi_{nj})<3a_n;$$

(Q'_n) $$\psi_{nk}\geqq 0;\ \psi_{nk}=1 \text{ on some } U(y_{nk});\ \Delta(\operatorname{supp}\psi_{nk})<3a_n.$$

We do this as follows. For $n=1$, our choices may be quite arbitrary, and this starts our inductive choices. Soppose that the choices have been done for some natural number n. We then put

(1) $$L_n=\{(x_{nj}, y_{nk});\ K\cap\operatorname{supp}(\varphi_{nj}\otimes\psi_{nk})\neq\emptyset\},$$

(2) $$M_n=(G_n\times H_n)\backslash L_n,$$

and

(3) $$b_n=(6n)^{-1}\inf\{d(K, \operatorname{supp}(\varphi_{nj}\otimes\psi_{nk}));\ (x_{nj}, y_{nk})\in M_n\}.$$

Let us fix any positive real number $a_{n+1} < b_n$, and take any G_{n+1} with $G_n \subset G_{n+1} \subset G_0$, H_{n+1} with $H_n \subset H_{n+1} \subset H_0$, $\{\varphi_{n+1,j}\}_j \subset C(G)$ and $\{\psi_{n+1,k}\}_k \subset C(H)$ so that they satisfy all the conditions (P_{n+1}), (P'_{n+1}), (Q_{n+1}), and (Q'_{n+1}). This completes our inductive choices. We now claim that the set $L = \bigcup_{n=1}^{\infty} L_n$ satisfies the required conclusions (a) and (b). Part (a) is trivial since the sequence $(a_n)_{n=1}^{\infty}$ clearly tends to zero. To prove (b), we define

$$T_n f = \sum_{j,k} f(x_{nj}, y_{nk})\varphi_{nj} \otimes \psi_{nk}. \tag{4}$$

Then every T_n is a norm-decreasing linear operator on $V(G, H)$, and the sequence $(T_n)_{n=1}^{\infty}$ strongly converges to the identity operator on $V(G, H)$. (See [7; Theorem 2.1] and [8; Lemma 4.4].) Let f be any function in $V(G, H)$ vanishing on the set $K \cup L$, and let n be any natural number. We prove that each term in the right-hand side of (4) vanishes on some neighborhood of $K \cup L$. Let (x_{nj}, y_{nk}) be any point of $G_n \times H_n$. If this point belongs to $K \cup L$, then $f(x_{nj}, y_{nk}) = 0$. Otherwise, it belongs to M_n, and so the set $\operatorname{supp}(\varphi_{nj} \otimes \psi_{nk})$ has a distance at least $6nb_n$ apart from K, by (3). On the other hand, the set $\bigcup_{m>n} L_m$ has a distance at most $6a_{n+1}$ from K. It follows that $\varphi_{nj} \otimes \psi_{nk}$ vanishes on some neighborhood of $K \cup (\bigcup_{m>n} L_m)$. Further, since $L_m \subset G_n \times H_n$ and $(x_{nj}, y_{nk}) \notin L_m$ for all $m = 1, 2, \cdots, n$, it follows from (Q_n) and (Q'_n) that $\varphi_{nj} \otimes \psi_{nk}$ vanishes on some neighborhood of $\bigcup_{m=1}^{n} L_m$. Thus every $T_n f$ has compact support disjoint from $K \cup L$, and since $(T_n f)_{n=1}^{\infty}$ converges to f in norm, the set $K \cup L$ is a set of spectral synthesis.

We now suppose that both G and H are perfect. Preserving all the notation used before, define

$$L'_n = \{(x_{nj}, y_{nk}); \ d(K, \operatorname{supp}(\varphi_{nj} \otimes \psi_{nk})) < b_{n-1}\}$$

for all $n = 1, 2, \cdots$, where $b_0 = 4\Delta(G \times H)$. After the choices in the n'th step have been done, we choose this time a_{n+1} with $0 < a_{n+1} < b_n/2$ so that the conditions (P_{n+1}), (P'_{n+1}), (Q_{n+1}), and (Q'_{n+1}) automatically imply $\operatorname{leng}(L'_{n+1}) \geqq n+1$. Such a choice of a_{n+1} is possible since G and H are perfect. We then construct G_{n+1}, H_{n+1}, $\{\varphi_{n+1,j}\}_j$ and $\{\psi_{n+1,k}\}_k$ as before. It is trivial that the sequence $(L'_n)_{n=1}^{\infty}$ has all the required properties (a), (b), and (c).

This completes the proof.

We finish up this section by proving Theorem 1. Suppose that E is a clopen subset of $X \times Y$. Denoting by $M_F(E)$ the space of all measures with finite support contained in E, we then have

$$\|f\|_{V(E)} = \sup\left\{\left|\int_E f d\mu\right|; \ \mu \in M_F(E), \ \|\mu\|_{BM} \leqq 1\right\} \tag{1}$$

for all f in $V(E)$ (see [8; Lemma 4.4 and Theorem 4.5]). Since E is clopen, it is a finite union of rectangular subsets, and so $\bar{E}$ is clopen in $\bar{X}\times\bar{Y}$. Further, the set $\pi_X(E)\times\pi_Y(E)$ is dense in $\pi_{\bar{X}}(\bar{E})\times\pi_{\bar{Y}}(\bar{E})$ and

$$(\pi_X(E)\times\pi_Y(E))\cap\bar{E}=E.$$

It follows from Lemma 4.4 in [8] that the formula (1), with $V(E)$ replaced by $V_D(E)=V(\bar{E})$, is valid for all f in $V_D(E)$. (Note that for any finite set F, we have $V(F)=V_D(F)$ isometrically.) This implies, in particular, that the imbedding $V(E)\subset V_D(E)$ is isometric. Finally, the other statements in Theorem 1 have already been verified in [7], [8], [14], as was remarked in § 1. This completes the proof.

§ 3. Proof of Theorem 2.

Let X and Y be two compact spaces, and E a compact and metrizable subset of $X\times Y$ with $\mathrm{leng}(E)=\infty$. Then it is easy to see that E contains a point e such that $\mathrm{leng}(E\cap U)=\infty$ for all neighborhoods U of e. Thus Theorem 2 is an immediate consequence of the following.

THEOREM 2′. *Let X, Y and $E=\bigcup_{n=0}^{\infty}E_n$ be as in Proposition* 2.1. *Suppose also that $\lim_n \mathrm{leng}(E_n)=\infty$ and that E_0 consists of a single point e. Then E contains a closed set F for which we have* (i), (ii), *and* (iii) *in Theorem* 2.

PROOF. Let G and H be two compact, infinite, metrizable, abelian groups. Then the Malliavin-Varopoulos theorem states that the algebra $V(G,H)$ contains a real-valued function φ such that the closed ideals in $V(G,H)$ generated by each φ^k $(k=1,2,\cdots)$ are all distinct (see [11] and [10; Example 3]). We fix once and for all such a function φ, and write $K=\varphi^{-1}(0)$ and

$$\varphi(x,y)=\sum_{j=1}^{\infty}g_j(x)h_j(y)\qquad(x\in G,\ y\in H),\tag{1}$$

where

$$g_j\in C(G),\quad h_j\in C(H)\quad\text{and}\quad\sum_{j=1}^{\infty}\|g_j\|_\infty\cdot\|h_j\|_\infty<\infty.$$

We preserve all the notations in the proof of Proposition 2.3, and claim that, for every non-zero entire function $\Phi(z)$, we have

$$\liminf_n\|\Phi(\varphi)\|_{V(L_n)}>0.\tag{2}$$

We may suppose that $\Phi(0)=0$, and so $\Phi(z)=\sum_{n=N}^{\infty}c_nz^n$ $(|z|<\infty)$, where $N\geqq 1$ and $c_N\neq 0$. Let B be any bimeasure in $BM(K)$ such that $\langle\varphi^N,B\rangle=1$ but $\langle\varphi^{N+1}f,B\rangle=0$ for all f in $V(G,H)$. Then we have

$$c_N = \langle \Phi(\varphi), B \rangle = \lim_n \langle T_n(\Phi(\varphi)), B \rangle$$
$$= \lim_n \langle \Phi(\varphi), T_n^* B \rangle ,$$

where T_n^* denotes the conjugate operator of T_n. From the definition of T_n, it is clear that $T_n^* B$ is a measure in $M(L_n)$, so that we have

$$\lim_n |\langle \Phi(\varphi), T_n^* B \rangle| \leqq \liminf_n \|\Phi(\varphi)\|_{V(L_n)} \cdot \|B\|_{BM} .$$

Therefore we have (2).

Let now $E = \bigcup_{n=0}^{\infty} E_n$ be as in our Theorem. Replacing $(E_n)_{n=1}^{\infty}$ by its suitable subsequence and each E_n by its suitable subset, we may assume that $E_n = X_n \times Y_n$, where $X_n = \pi_X(E_n)$ and $Y_n = \pi_Y(E_n)$ $(n = 0, 1, 2, \cdots)$, and that there are continuous onto mappings

$$p_n : X_n \longrightarrow G_n, \quad \text{and} \quad q_n : Y_n \longrightarrow H_n \quad (n = 1, 2, \cdots) .$$

We can also assume that $X = \bigcup_{n=0}^{\infty} X_n$ and $Y = \bigcup_{n=0}^{\infty} Y_n$. Put $F_0 = E_0$, and $F_n = (p_n \times q_n)^{-1}(L_n')$ for all $n = 1, 2, \cdots$. We then claim that the set $F = \bigcup_{n=0}^{\infty} F_n$ has the required properties. It is trivial that F is a closed subset of E, and that every F_n is clopen in $X \times Y$ $(n = 1, 2, \cdots)$.

We first prove (i) and (ii). Let f be any function in $V(F)$; we have

$$(3) \qquad \|f\|_{V(F_n)} = \|f\|_{V_D(F_n)} = \|f\|_{\tilde{V}(F_n)} \qquad (n = 0, 1, 2, \cdots)$$

by Theorem 1. Since F_0 consists of a single point e, it is a set of spectral synthesis. Thus, Proposition 2.2 applies, and we have

$$\lim_n \|f - f(e)\|_{V_D(F_n)} = \lim_n \|f - f(e)\|_{V(F_n)} = 0 ,$$

which, combined with (3) and Proposition 2.1, gives

$$\|f\|_{V(F)} = \|f\|_{V_D(F)} = \|f\|_{\tilde{V}(F)}$$
$$= \sup \{\|f\|_{V(F_n)} ;\ n = 0, 1, 2, \cdots\} .$$

The properties that $V_C(F) \subset \tilde{V}(F)$ and that this imbedding is norm-decreasing, follow from Proposition 2.1.

We now prove that $V_C(F)$ is non-separable and that $V_C(F)$ contains a real-valued function f with the property that $\Phi(f)$ does not belong to $V(F)$ for every non-constant entire function $\Phi(z)$. We define a norm-decreasing linear operator P from $D(G)$ into $D(X)$ by setting

$$Pg = g \circ p_n \text{ on } X_n \ (n = 1, 2, \cdots), \quad \text{and} \quad Pg = 0 \text{ on } X_0 ,$$

and similarly a norm-decreasing linear operator Q from $D(H)$ into $D(Y)$ by setting

$$Qh = h \circ q_n \text{ on } Y_n \ (n=1, 2, \cdots), \quad \text{and} \quad Qh = 0 \text{ on } Y_0 .$$

Note then that

$$(4) \qquad (P \widehat{\otimes} Q)\psi = \psi \circ (p_n \times q_n) \text{ on } X_n \times Y_n \qquad (\psi \in V_D(G, H))$$

for $n=1, 2, \cdots$, and that

$$(5) \qquad \|(P \widehat{\otimes} Q)\psi\|_{V(F_n)} = \|\psi\|_{V(L'_n)} \qquad (\psi \in V_D(G, H))$$

for $n=1, 2, \cdots$, because the mapping $p_n \times q_n : X_n \times Y_n \to G_n \times H_n$ is a continuous surjection and $G_n \times H_n$ is a finite set (cf. [7; Theorem 2.1]). Let us now put $f = ((P \widehat{\otimes} Q)\varphi)|_F$, and prove that f has the required property. It is trivial from (1) and (4) that f is real-valued and belongs to $V_D(F)$. Since $f=0$ on F_0 and $\|f\|_{D(F_n)} = \|\varphi\|_{D(L'_n)}$ for all $n=1, 2, \cdots$, by (4), it follows that f is continuous and so belongs to $V_C(F)$. Let $\Phi(z)$ be any non-constant entire function. In order to prove that $\Phi(f)$ does not belong to $V(F)$, we may assume that $\Phi(0)=0$. We have by (2), (4), and (5)

$$\liminf_n \|\Phi(f)\|_{V(F_n)} = \liminf_n \|\Phi(\varphi)\|_{V(L'_n)}$$
$$\geqq \liminf_n \|\Phi(\varphi)\|_{V(L_n)} > 0 .$$

But $f=0$ on F_0 and F_0 is a set of spectral synthesis; it follows from Proposition 2.2 that $\Phi(f) \notin V(F)$. We now prove that $V_C(F)$ is non-separable. Let N be any natural number such that

$$(6) \qquad \inf \{\|f\|_{V(F_n)} ;\ n=N,\ N+1,\ N+2, \cdots\} = d_N > 0 ,$$

and let $\mathcal{M}$ be the family of all subsets of the index set $\{N, N+1, N+2, \cdots\}$. For any set $A \in \mathcal{M}$, define $f_A \in D(F)$ by setting

$$f_A = f \text{ on } \cup \{F_n ;\ n \in A\}, \quad \text{and} \quad f_A = 0 \text{ on } \cup \{F_n ;\ n \notin A\} .$$

It is trivial that f_A belongs to $C(F)$. Further, since the sets $(F_n)_{n=0}^{\infty}$ are pairwise bidisjoint, it is easy to see that $f_A \in V_D(F)$, so that $f_A \in V_C(F)$. On the other hand, if A and B are distinct elements of $\mathcal{M}$, then

$$\|f_A - f_B\|_{V_D(F)} \geqq d_N$$

by (6), and $\mathcal{M}$ has the cardinal number of continuum. This implies that $V_C(F)$ is non-separable.

We now prove that $\tilde{V}(F)$ is non-separable and that $\tilde{V}(F)$ contains a real-valued function g with the property that $\Phi(g) \notin V_C(F)$ for all non-constant entire functions $\Phi(z)$. That $\tilde{V}(F)$ is non-separable is trivial from (3) and the proof of the fact that $V_C(F)$ is non-separable. Recall now that the sets $(F_n)_{n=0}^{\infty}$ are pairwise bidisjoint and that

$$\text{leng}\,(F_n) \geqq \text{leng}\,(L'_n) \geqq n \qquad (n=1, 2, \cdots),$$

which follows from our construction of the sets F_n. It follows from a theorem of Varopoulos [**14**; Theorem 1 and its proof] that there exists a real-valued function g in $\tilde{V}(F)$ such that

$$\|\sum_{k=0}^{N} c_k g^k\|_{V(F)} = \sum_{k=0}^{N} |c_k| \tag{7}$$

for all complex numbers $(c_k)_{k=0}^{N}$ and all $N=0, 1, 2, \cdots$. Let $\Phi(z)$ be any non-constant entire function, and, to get a contradiction, suppose that $\Phi(g)$ belongs to $V_C(F)$; since $V_C(F)$ is self-adjoint and g is real-valued, we may assume that $\Phi(z)$ is real-valued on the real line. Property (7) assures that the spectrum of g contains the set $\{z;\, |z|=1\}$ of complex numbers (see [**6**; 5.3.4 and 5.4.2]). Therefore there exists a non-real complex number c such that $\Phi(g)-c$ is not invertible in $\tilde{V}(F)$, since $\Phi(z)$ is a non-constant entire function. But $\Phi(g)-c$ is invertible in $V_C(F)$ because $\Phi(g)$ is real-valued on F and c is non-real (recall that the maximal ideal space of $V_C(F)$ is F). It follows that

$$(\Phi(g)-c)^{-1} \in V_C(F) \subset \tilde{V}(F),$$

a contradiction.

This completes the proof.

REMARKS. (a) Suppose in Theorem 2′ that each E_n is rectangular and that either $\pi_X(E_n)$ or $\pi_Y(E_n)$ is perfect for infinitely many n, then the set F with the required properties can be chosen to be perfect. This is easily seen from the proof of Theorem 2′.

(b) The set F constructed in the proof of Theorem 2′ has the following additional properties (iii) and (iv).

(iii) The quotient algebras $V_C(F)/V(F)$ and $\tilde{V}(F)/V(F)$ are both non-separable.

(iv) Let $\Phi(t)$ be any function defined on the interval $I=[0, 1]$ of the real line. If $\Phi(t)$ operates in either $V(F)$ or $V_C(F)$, then $\Phi(t)$ is the restriction of a function defined and analytic in some neighborhood of I in the complex plane. On the other hand, if $\Phi(t)$ operates in $\tilde{V}(F)$, then $\Phi(t)$ is the restriction of an entire function.

We omit the proofs of these statements.

§ 4. Proofs of Theorems 3 and 4.

The proofs of Theorems 3 and 4 are very like that of Theorem VI in [3]. We first prove Theorem 3.

Let X, Y and E be as in Theorem 3. Since E is not a set of spectral synthesis, $I(E)$ contains a function f such that

$$\inf\{\|f+g\|_{V(X\times Y)};\ g\in J(E)\}>1. \tag{1}$$

Fixing such a function f, we take an arbitrary $\varepsilon>0$. Observe first that every point of $X\times Y$ is a set of spectral synthesis. It follows that there exists a finite open covering $(U_n)_{n=1}^N$ of E such that

$$U_n\cap E\neq\emptyset,\quad\text{and}\quad \|f\|_{V(U_n)}<\varepsilon\qquad(n=1,2,\cdots,N). \tag{2}$$

Since both X and Y are totally disconnected by hypothesis, we may assume that the sets $(U_n)_{n=1}^N$ are clopen, rectangular, and pairwise disjoint. We now apply Proposition 2.3 to each $U_n\cap E\subset U_n$; there is a countable sets F_n in U_n whose accumulation points are all in $U_n\cap E$, such that $(U_n\cap E)\cup F_n$ is a set of spectral synthesis. Since X and Y are perfect, we may assume that the sets $(F_n)_{n=1}^N$ are pairwise bidisjoint. Putting $F=\bigcup_{n=1}^N F_n$, we claim that

$$\|f\|_{V(E\cup F)}\geqq 1,\quad\text{and}\quad \|f\|_{\tilde{V}(E\cup F)}<\varepsilon. \tag{3}$$

Indeed, $E\cup F$ is a set of spectral synthesis, because it is a finite disjoint union of sets of spectral synthesis. Thus the first inequality in (3) is an easy consequence of (1). Let now μ be any measure in $M(E\cup F)$, and μ_n its restriction to the set $G_n=(\pi_X(F_n)\times\pi_Y(F_n))\cap U_n$ for $n=1,2,\cdots,N$. Then the sets $(G_n)_{n=1}^N$ are all rectangular and pairwise bidisjoint; it follows that

$$\sum_{n=0}^N\|\mu_n\|_{BM}\leqq\|\mu\|_{BM}, \tag{4}$$

as is easily seen from Lemma 2.2 in [8] or from part (a) of Proposition 2.1. On the other hand, since f belongs to $I(E)$ and μ is concentrated in $E\cup F$, we have

$$\left|\int_{E\cup F}f\,d\mu\right|\leqq\sum_{n=1}^N\left|\int_{(U_n\cap E)\cup F_n}f\,d\mu\right|$$
$$=\sum_{n=1}^N\left|\int_{G_n}f\,d\mu\right|\leqq\sum_{n=1}^N\|f\|_{V(U_n)}\cdot\|\mu_n\|_{BM}.$$

This, combined with (2) and (4), yields the second inequality in (3).

We can now complete the proof of Theorem 3 as follows. Choose any sequence $(V_n)_{n=1}^\infty$ of pairwise disjoint, rectangular, and clopen subsets of $X\times Y$ so that: every $V_n\cap E$ disobeys spectral synthesis; and the sequence $(V_n)_{n=1}^\infty$ converges to a single point. It is easy to see that such a choice is always possible. For each $n=1,2,\cdots$, let us take a countable subset F_n of V_n and a function f_n of $V(X,Y)$ so that: the accumulation points of F_n are all in $E_n=V_n\cap E$; and

$$\|f_n\|_{V(E_n\cup F_n)}\geqq 1,\quad\text{and}\quad\|f_n\|_{\tilde{V}(E_n\cup F_n)}<n^{-1}. \tag{5}$$

We may assume that f_n vanishes outside V_n, since V_n is clopen and rectangular. Putting $F=\bigcup_{n=1}^{\infty} F_n$, we see that $E\cup F$ is closed, and that

$$\|f_n\|_{\tilde{V}(E\cup F)}=\|f_n\|_{\tilde{V}(E_n\cup F_n)} \qquad (n=1, 2, \cdots).$$

This, combined with (5), implies that $V(E\cup F)$ is not closed in $\tilde{V}(E\cup F)$. The proof of Theorem 3 is complete.

A topological space is called residual if it does not contain any perfect subset. The following is a generalization of [**9**; Theorem 1] and [**15**; Proposition 8.6].

PROPOSITION 4.1. *Let G be a locally compact abelian group, and K any residual compact subset of G. Let also E be a closed subset of G disjoint from K. Then each of the following four statements implies the others.*

(a) *For any pseudomeasure $P\in PM(K)$ and $Q\in PM(E)$, we have $\|P\|_{PM} \leqq \|P+Q\|_{PM}$.*

(b) *Given $\varepsilon>0$, there is a function $f\in A(G)$ such that $\|f\|_{A(G)}<1+\varepsilon$, $f=1$ on some neighborhood of K, and $f=0$ on some neighborhood of E.*

(c) *There is a constant $\eta>0$ with the following property; given any $\varepsilon>0$ and any finite subset K_0 of K, there is a function $f\in A(G)$ such that $\|f\|_{A(G)} <1+\varepsilon$, $|f|>1-\varepsilon$ on K_0, and $||f|-1|>\eta$ on E.*

(d) *E is disjoint from the coset algebraically generated by K.*

PROOF. Suppose that (a) holds, and fix any function g in $A(G)$ such that $g=1$ on some neighborhood of K, and $g=0$ on some neighborhood of E. Let $I_0(E\cup K)$ be the ideal in $A(G)$ consisting of those functions in $A(G)$ which vanish on some neighborhood of $E\cup K$. Then the statement (b) is equivalent to saying that

$$\|g+J\|=\inf\{\|g+h\|_{A(G)};\ h\in J\}\leqq 1,$$

where J denotes the closure of $I_0(E\cup K)$. But this inequality is an easy consequence of (a) combined with the Hahn-Banach theorem.

Property (b) trivially implies property (c). Suppose that (c) holds. Let $\beta(\hat{G})$ be the Bohr compactification of $\hat{G}$. Then Property (c) assures that there exists a measure μ in $M(\beta(\hat{G}))$ such that $\|\mu\|_M=1$, $|\hat{\mu}|=1$ on K, and $||\hat{\mu}|-1| \geqq\eta$ on E. Then the set $\{x\in G;\ |\hat{\mu}(x)|=1\}$ is a coset of G containing K, as is easily proved. Thus (d) holds.

Finally suppose that (d) holds, and let us take arbitrary $P\in PM(K)$ and $Q\in PM(E)$. To obtain the required inequality, we may assume that P has a finite support F, since the set of such P's is dense in $PM(K)$ by a theorem of L.H. Loomis [**4**; Theorem 4]. Then, for any given $\varepsilon>0$, there is a function f in $A(G)$ satisfying the conditions in (b) with K replaced by F (see [**9**; Theorem 1]). It follows that we have

$$\|P\|_{PM}=\|f(P+Q)\|_{PM}\leqq(1+\varepsilon)\|P+Q\|_{PM},$$

which establishes (a). This completes the proof.

We now prove Theorem 4. Let E be a totally disconnected closed subset of T^N that disobeys spectral synthesis. Let Q denote the subgroup of T^N consisting of elements of finite order. Since E has no interior point and Q is countable, Baire's category theorem assures that $(E+x)\cap Q=\emptyset$ for some point of T^N. Therefore, without loss of generality, we may assume that $E\cap Q=\emptyset$. Take and fix any function f in $I(E)$ and any pseudo-measure P in $PM(E)$ such that $\langle f, P\rangle \geqq 1$ and $\|P\|_{PM}\leqq 1$. Then, note that we have $\|f\|_{A(K)}\geqq 1$ if K is a set of spectral synthesis and $E\subset K$.

Let $\varepsilon>0$ be arbitrary. Since E is totally disconnected, and since every set consisting of a single point is a set of spectral synthesis, there is a finitely many, open, disjoint covering $(U_j)_{j=1}^L$ of E such that: (a) $U_j\cap E$ is non-empty and closed; and (b) $\|f\|_{A(\bar{U}_j)}<\varepsilon$ for $j=1, 2, \cdots, L$. Let $p_1, p_2, \cdots, p_L$ be any distinct primes. For each j, let Q_j be the subgroup of T^N consisting of all elements whose orders are powers of p_j. Using the procedure of Herz (cf. [**1**; IX. 8]), we can find a countable subset F_j of $U_j\cap Q_j$ whose accumulation points are all in $U_j\cap E$, such that the set $(U_j\cap E)\cup F_j$ is a set of spectral synthesis. Let $F=\bigcup_{j=1}^L F_j$; it is clear that $E\cup F$ is a set of spectral synthesis, and hence $\|f\|_{A(E\cup F)}\geqq 1$. Take now any measure μ in $M(E\cup F)$, and let ν be the restriction of μ to the countable group $Q_1+Q_2+\cdots+Q_N$. It follows from Proposition 4.1 that $\|\nu\|_{PM}\leqq\|\mu\|_{PM}$. It is easy to check that

$$(E\cup F)\cap(Q_1+Q_2+\cdots+Q_L)=\bigcup_{j=1}^L F_j,$$

and so $\nu=\nu_1+\nu_2+\cdots+\nu_L$, where ν_j denotes the restriction of ν to F_j. But the sum $Q_1+Q_2+\cdots+Q_L$ is the direct sum of $(Q_j)_{j=1}^L$ in the usual sense; hence we have $\|\nu\|_{PM}\geqq(1/4)\sum_{j=1}^L\|\nu_j\|_{PM}$. It follows that

$$\left|\int_{E\cup F} f\,d\mu\right|=\left|\int_F f\,d\nu\right|\leqq\sum_{j=1}^L\left|\int_{F_j} f\,d\nu_j\right|$$

$$\leqq\varepsilon\sum_{j=1}^L\|\nu_j\|_{PM}\leqq 4\varepsilon\|\nu\|_{PM}\leqq 4\varepsilon\|\mu\|_{PM};$$

in other words, $\|f\|_{\tilde{A}(E\cup F)}\leqq 4\varepsilon$.

The remainder part of the proof is now easy (see [**3**; §4]), and our theorem is established.

§5. Certain restriction algebras of Fourier algebras.

Let G be a locally compact abelian group, and $\hat{G}$ its dual. Let also $\hat{G}_d$ be the discrete group of all, not necessarily continuous, characters of G.

Thus the dual of $\hat{G}_d$ is the Bohr compactification $\bar{G}$ of G_d, where G_d denotes the group G endowed with the discrete topology. For any non-empty subset E of G, we define three Banach algebras $A_D(E)$, $A_C(E)$, and $\tilde{A}_C(E)$ in the following way. The space $A_D(E)$ is a subalgebra of $D(E)$ consisting of those functions f in $D(E)$ that have an expansion of the form $f(x)=\sum_{n=1}^{\infty} c_n\gamma_n(x)$ where $(c_n)_{n=1}^{\infty}$ is a sequence of complex numbers with $M=\sum_{n=1}^{\infty}|c_n|<\infty$ and $(\gamma_n)_{n=1}^{\infty}$ a sequence of characters in $\hat{G}_d$; the norm of f in $A_D(E)$ is defined to be the infimum of the M's taken over all such expansions of f in the above form. It is easy to see that $A_D(E)$ can be naturally identified with the restriction algebra $A(\bar{E})$ of the Fourier algebra $A(\bar{G})$, where $\bar{E}$ denotes the closure of E in $\bar{G}$. We define $A_C(E)$ to be $A_D(E)\cap C(E)$, which is clearly a closed subalgebra of $A_D(E)$. The definition of $\tilde{A}_C(E)$ is now self-evident.

An independent compact subset X of G is called a *Rudin set* if every non-empty, relatively open subset of X carries a non-zero measure with Fourier-Stieltjes transform vanishing at infinity. It is well-known and easy to see that we have $\tilde{A}(X)=A(X)$ isometrically for such a set X. For the existence of such sets, we refer to [5] and [12].

PROPOSITION 5.1 (cf. [3; Theorem III]). *Let X and Y be infinite compact disjoint subsets of G whose union is independent (over the integers), and put $K=X+Y$.*

(i) *If X is not a Helson set, then $A(X)\subsetneqq A_C(X)=C(X)$.*

(ii) *If X is a Rudin set, then $A(X)=\tilde{A}(X)\subsetneqq A_C(X)=C(X)$.*

(iii) *If $X\cup Y$ is either countable or a Kronecker set, then $A(K)=A_C(K)\subsetneqq \tilde{A}(K)$.*

(iv) *If either X or Y is a Rudin set, then*

$$A(K)=\tilde{A}(K)\subsetneqq A_C(K)\subsetneqq \tilde{A}_C(K)\subsetneqq C(K)\,.$$

PROOF. Let $E=X\cup Y$, and let $D^*(E)$ be the multiplicative group consisting of all complex-valued functions on E of absolute value 1. By hypothesis, every function in $D^*(E)$ can be extended to a character of G (cf. [6; 5.1.3]). This clearly implies

$$D^*(E)=\{\gamma|_E\,;\ \gamma\in\hat{G}_d\}\,.$$

Suppose that f is a function in $C(\bar{E})$ of absolute value 1. Then $f|_E$ is in $D^*(E)$, and so there is a character γ in $\hat{G}_d$ with $f=\gamma$ on E. Since both f and γ are continuous on $\bar{E}$, this implies $f=\gamma$ on $\bar{E}$. Hence we see that $\bar{E}$ is a Kronecker set in $\bar{G}$. Further, we have $D(E)=C(\bar{E})|_E$, so that the maximal ideal space of $D(E)$ may be identified with $\bar{E}$ in a trivial way. In fact, it suffices to note that every function f in $D(E)$ can be written in the form

$f=\sum_{n=1}^{\infty} c_n f_n$ on E, where $(c_n)_{n=1}^{\infty}$ is an absolutely summable sequence of complex numbers and $(f_n)_{n=1}^{\infty}$ a sequence of functions in $D^*(E)$. It follows, in particular, that

$$A_D(E)=A(\bar{E})|_E=C(\bar{E})|_E=D(E),$$

so that $A_C(E)=C(E)$.

Parts (i) and (ii) are now trivial.

To prove part (iii), note that we have $A(K)=V(X, Y)$ under our hypothesis. Varopoulos [**13**] proved this in the case that E is a Kronecker set. The proof in the case that E is countable, is not trivial, but can be easily done by applying a theorem of Loomis [**4**; Theorem 4]; we omit the details. Thus part (iii) follows from Theorem 1.

Finally suppose that X is a Rudin set. Since $\bar{X}$ and $\bar{Y}$ are disjoint and their union is a Kronecker set, we have by Varopoulos' theorem

$$A_D(K)=A(\bar{X}+\bar{Y})|_K=V(\bar{X}, \bar{Y})|_K=V_D(X, Y),$$

with trivial identifications, and so

$$A_C(K)=V_D(X, Y)\cap C(X\times Y)=V(X, Y)$$

by Theorem 1. These observations clearly establish part (iv), since a Rudin set is not a Helson set.

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