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The nilpotency of elements of the stable homotopy groups of spheres

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§ 0. Introduction.

In this paper we shall describe the theory of extended powers of CW-complexes. Our main application is to demonstrate the following conjecture of M. G. Barratt:

CONJECTURE. *Any element of positive stem of the stable homotopy groups of spheres is nilpotent.*

The extended n-th power $D_n(X)$ of a CW-complex X is defined by $WS_n \times_{S_n} X^{(n)}/WS_n \times_{S_n}(base\ point)^{(n)}$, where WS_n is an acyclic S_n-free complex as S_n being the n-th symmetric group (for details of the definition, see § 1). The study of constructions of this kind was initiated by N. E. Steenrod [**18**]. For $n =$ a prime, various applications of extended powers to homotopy theory have been done by J. F. Adams, M. G. Barratt, D. S. Kahn, M. Mahowald and H. Toda. Also R. J. Milgram treated the D_4 construction to apply it to the Arf invariant problem.

The basic idea of the proof of the conjecture is given by H. Toda in [**19**]. That is, roughly speaking, the study of the stable homotopy type of $D_n(X)$ for $X = S^k$ or $S^k \cup_p e^{k+1}$ may lead us to the conjecture. We shall describe two ways of attacking the conjecture, which are given in Part I and Part II, respectively. The second method gives the comprehensive result (Corollaries 8.2 and 8.4), but the estimate of exponent t in $\alpha^t = 0$ is very large. On the other hand, though it gives only a restricted result (Corollary 4.2), the first method gives much better estimate of exponent than that of the second one.

The paper is organized as follows.

§7. The adjoint of h_p and the theorem of Kahn-Priddy.

§8. The nilpotency of elements of $\pi_*^s(S^0)$.

The part I consists of §1–§4 and the part II consists of §5–§8. In §1 and §2, the definition and the basic properties of the extended power $D_n(X)$ and the Γ-spectrum are given. In §3, we shall show that if X is $S^k\cup_p e^{k+1}$ a Moore space of type $(\boldsymbol{Z}_p, k)$, then the associated Γ-spectrum $\boldsymbol{D}_X$ is mod p homotopy equivalent to a generalized Eilenberg-MacLane spectrum. Then in §4, we prove that if $\alpha\in\pi_i^s(S^0; p)$ is of order p, then α is nilpotent. In Part II, §5 and §6 give an analysis of non-additivity of D_p construction and more detailed study of $D_p(X)$. Then in §7 and §8, we give the proof of the conjecture by use of the theorem of Kahn-Priddy [8].

I would like to thank M. G. Barratt for many helpful conversations.

PART I.

§1. Extended powers of CW-complexes.

Throughout this paper we work in the category of based CW-complexes. All maps and homotopies will mean base point preserving maps and base point preserving homotopies unless otherwise stated. Let S_n be the symmetric group on n-letters. Let G be a subgroup of S_n. Then for any space X, G acts on the smash product $X^{(n)}$ of n-copies of X as the permutations. Let WG denote a G-free acyclic complex, G acting from the right and $WG^{(r)}$ denotes the r-skeleton of WG. Let $WG_+^{(r)}$ be the one point disjoint union of $WG^{(r)}$ with trivial G-action on the point. We regard the added point as a base point.

DEFINITION 1.1. The extended n-th power of X with respect to G is defined by

$$D_G^{(r)}(X)\equiv(WG_+^{(r)})\wedge_G X^{(n)}.$$

For a map $f;\ X\to Y$, the extended power of f is defined by

$$D_G^{(r)}(f)=\mathrm{id}\wedge_G f^{(n)};\ D_G^{(r)}(X)\longrightarrow D_G^{(r)}(Y)$$

where $f^{(n)}=f\wedge\cdots\wedge f;\ X^{(n)}\to Y^{(n)}$ denotes the smash product of f.

The following functorial properties are obvious. 1). $D_G^{(r)}(\mathrm{id})=\mathrm{id}$; 2). $D_G^{(r)}(f\circ g)=D_G^{(r)}(f)\circ D_G^{(r)}(g)$; 3). If $f\sim g$ homotopic, then $D_G^{(r)}(f)\sim D_G^{(r)}(g)$; 4). $D_G^{(0)}(X)=X^{(n)}$ and $D_G^{(0)}(f)=f^{(n)}$.

For our purpose, the case that $G=S_n$ or a p-Sylow subgroup of S_n for a prime p will be important. In those cases $D_G^{(r)}(X)$ is denoted by $D_n^{(r)}(X)$ and $D_{n,p}^{(r)}(X)$, respectively. If r is infinite, $D_G^{(r)}(X)$ is denoted simply by $D_G(X)$.

We consider the direct product $S_n\times S_m$ as a subgroup of S_{n+m} by letting S_n act on the former n-letters and S_m act on the latter m-letters. The wreath product $S_n \int S_m$ acts on the set $\{(i, j);\ 1 \leqq i \leqq n \text{ and } 1 \leqq j \leqq m\}$. So we may condider $S_n \int S_m$ as a subgroup of S_{nm}. These inclusions define equivariant maps

$$WS_n \times WS_m \longrightarrow WS_{n+m}\,;\ S_n \times S_m\text{ - equivariant}$$

$$WS_n \times (WS_m)^n \longrightarrow WS_{nm}\,;\ S_n \int S_m\text{ - equivariant}\,,$$

and we can construct the following natural maps in the obvious ways

$$\mu_{n,m}\,;\ D_n(X) \wedge D_m(X) \longrightarrow D_{n+m}(X)$$

$$\psi_{n,m}\,;\ D_n(D_m(X)) \longrightarrow D_{nm}(X)\,.$$

PROPOSITION 1.2. *$\mu_{n+m,l}(\mu_{n,m} \wedge 1_{D_l(X)})$ is homotopic to $\mu_{n,m+l}(1_{D_n(X)} \wedge \mu_{m,l})$. Let $T;\ D_n(X) \wedge D_m(X) \to D_m(X) \wedge D_n(X)$ be the switching function, then $\mu_{n,m}T$ is homotopic to $\mu_{m,n}$ if X is connected and simply connected.*

PROOF. The first statement is clear from definition. Let $\sigma \in S_{n+m}$ be the element defined by $\sigma(i) = i+m$ for $1 \leqq i \leqq n$ and $\sigma(i) = i-n$ for $n+1 \leqq i \leqq n+m$. Let $T'\,;\ S_n \times S_m \to S_m \times S_n$ be the switching function. Then we have the following commutative diagram

$$\begin{array}{ccc} S_n \times S_m & \xrightarrow{T'} & S_m \times S_n \\ \downarrow & & \downarrow \\ S_{n+m} & \xrightarrow{\sigma_*} & S_{n+m} \end{array}$$

where σ_* indicates the inner automorphism defined by σ. Let $B\sigma_*\colon BS_{n+m} \to BS_{n+m}$ be the map of the classifying space. Then using obstruction theory we can easily see that $B\sigma_*$ is freely homotopic to the identity (not necessarily preserving base points). Taking a suitable model of WS_{n+m}, S_{n+m} acts also on WS_{n+m} from the left. Let $l_\sigma\colon WS_{n+m} \to WS_{n+m}$ be the left action of σ. l_σ is equivariant, for we consider WS_{n+m} as the right S_{n+m}-space. Since $B\sigma_* \sim \mathrm{id}$, we have easily that l_σ is (freely) equivariant homotopic to the identity. Now $D_n(X) \wedge D_m(X) \xrightarrow{T} D_m(X) \wedge D_n(X) \xrightarrow{\mu_{n,m}} D_{n+m}(X)$ is induced from $(WS_n \times WS_m)_+ \wedge X^{(n+m)} \xrightarrow{WT' \wedge \sigma} (WS_m \times WS_n)_+ \wedge X^{(n+m)} \xrightarrow{Wj \wedge 1} (WS_{n+m})_+ \wedge X^{(n+m)}$, and this is (freely) equivariant homotopic to $(WS_n \times WS_m)_+ \wedge X^{(n+m)} \xrightarrow{Wj \wedge 1} (WS_{n+m})_+ \wedge X^{(n+m)}$. Hence we have that $\mu_{n,m}T$ is (freely) homotopic to $\mu_{m,n}$ and if X is connected and simply connected, the proposition follows. Q.E.D.

Now let us consider a pair (X, S^k), i.e., a space X and an inclusion $i;\ S^k \to X$, and consider the composition map $g_n\,;\ D_n(X) \wedge S^k \xrightarrow{1 \wedge i} D_n(X) \wedge X$

$\xrightarrow{\mu_{n,1}} D_{n+1}(X)$.

DEFINITION 1.3. For a pair (X, S^k) with $k>0$, we define a spectrum $\boldsymbol{D}_X = \{(D_X)_t, \varepsilon_t\}$ by $(D_X)_{nk+i} = D_n(X) \wedge S^i$ for $0 \leqq i < k$ and $\varepsilon_{nk+i} = \text{id}$ for $0 \leqq i < k-1$, $\varepsilon_{nk+k-1} = g_n$.

A spectrum $\boldsymbol{M} = \{M_n, \varepsilon_n\}$ is called a ring spectrum with unit if there exists maps $i_n ; S^n \to M_n$ and $\phi_{n,m} ; M_n \wedge M_m \to M_{n+m}$ such that the following diagrams are homotopy commutative.

$$\begin{array}{ccc} M_n \wedge S^m & \xrightarrow{1\wedge i_m} & M_n \wedge M_m \\ & {\scriptstyle\varepsilon}\searrow & \downarrow{\scriptstyle \phi_{n,m}} \\ & & M_{n+m} \end{array} \qquad \begin{array}{ccc} S^n \wedge M_m & \xrightarrow{i_n \wedge 1} & M_n \wedge M_m \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \phi_{n,m}} \\ M_m \wedge S^n & \xrightarrow[\varepsilon]{} & M_{n+m} \end{array}$$

$$\begin{array}{ccc} M_n \wedge M_m \wedge S^1 & \xrightarrow{\phi_{n,m}\wedge 1} & M_{n+m} \wedge S^1 \\ \downarrow{\scriptstyle 1\wedge\varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ M_n \wedge M_{m+1} & \xrightarrow[\phi_{n,m+1}]{} & M_{n+m+1} \end{array} \qquad \begin{array}{ccc} M_n \wedge S^1 \wedge M_m & \xrightarrow{1\wedge T} & M_n \wedge M_m \wedge S^1 \\ & & \downarrow{\scriptstyle \phi_{n,m}\wedge 1} \\ \downarrow{\scriptstyle \varepsilon\wedge 1} & & M_{n+m}\wedge S^1 \\ & & \downarrow{\scriptstyle \varepsilon} \\ M_{n+1} \wedge M_m & \xrightarrow[\phi_{n,m+1}]{} & M_{n+m+1} \end{array}$$

where T denotes the switching function and ε is the corresponding structure map. Also the homotopy associativity and homotopy commutativity are defined in the usual way.

THEOREM 1.4. *If X is 1-connected, then $\boldsymbol{D}_X$ is a homotopy associative and homotopy commutative ring spectrum with unit.*

PROOF. The ring structure is defined by $\mu_{n,m} ; D_n(X) \wedge D_m(X) \to D_{n+m}(X)$ and by the obvious extension to all $(D_X)_i$. The unit is defined by i_{kn+i}; $S^{kn+i} \xrightarrow{i^{(m)}\wedge 1} X^{(n)} \wedge S^i \xrightarrow{j\wedge 1} D_n(X) \wedge S^i$ where j is the natural inclusion. Let us check the homotopy commutativity of the last diagram. In order to do that, it suffices to check the homotopy commutativity of

$$\begin{array}{ccc} D_n(X) \wedge S^k \wedge D_m(X) & \xrightarrow{1\wedge T} & D_n(X) \wedge D_m(X) \wedge S^k \\ & & \downarrow{\scriptstyle \mu_{n,m}\wedge 1} \\ \downarrow{\scriptstyle g_n\wedge 1} & & D_{n+m}(X) \wedge S^k \\ & & \downarrow{\scriptstyle g_{n+m}} \\ D_{n+1}(X) \wedge D_m(X) & \xrightarrow[\mu_{n+1,m}]{} & D_{n+m+1}(X) . \end{array}$$

But this is clear by Proposition 1.2. The rest of the proof is similar. Q.E.D.

DEFINITION 1.5. A homotopy associative ring spectrum $\boldsymbol{M}$ is called a Γ-spectrum if for any n and r there exist an integer k and a map

$$\theta\,;\ D_n^{(r)}(M_k) \longrightarrow M_{nk}$$

such that $\theta\,|\,D_n^{(0)}(M_k)\,;\ D_n^{(0)}(M_k) \to M_{nk}$ is homotopic to $n-1$ fold iteration of the multiplication.

EXAMPLE. The spectrum $\boldsymbol{D}_X$ is of course a Γ-spectrum by means of $\phi_{n,m}\,;\ D_n(D_m(X)) \to D_{nm}(X)$. Also various Thom spectra such as $\boldsymbol{MO}$, $\boldsymbol{MSO}$, $\boldsymbol{MU}$ and $\boldsymbol{S}$, the sphere spectrum are Γ-spectra. For the sphere spectrum, we have

PROPOSITION 1.6. *For any integers n and r, there exists an integer k such that for any N, there exists a retraction $r\,;\ D_n^{(r)}(S^{kN}) \to S^{knN}$.*

PROOF. $D_n^{(r)}(S^k)$ is the one point compactification of $WS_n^{(r)} \times_{S_n} (\boldsymbol{R}^k)^n$. Therefore $D_n^{(r)}(S^k)$ is considered as the Thom complex of the vector bundle over $BS_n^{(r)}$ induced from $k \cdot \gamma$, where $\gamma\,;\ BS_n \to BO(n)$ denotes the regular representation. Since S_n is a finite group, γ has finite order k in $\widetilde{KO}(BS_n^{(r)})$. This proves the proposition.

§ 2. Cohomology group of $D_n(X)$.

Let X be a space with a base point $*$. Let $Q(X) = \varinjlim \Omega^n S^n(X)$ denotes the infinite loop space. Let Γ^+ be the free monoid functor of Barratt [4], i.e., for suitable model of WS_n

$$\Gamma^+(X) = \cup\, WS_n \times X^n/R$$

where the equivalence relation R is the following;

$$(w\,;\ x_1, \cdots, x_n) \sim (w\,;\ x_{\sigma(1)}, \cdots, x_{\sigma(n)}) \qquad \text{for } \sigma \in S_n\,,$$
$$(w\,;\ x_1, \cdots, x_{n-1}, *) \sim (Tw\,;\ x_1, \cdots, x_{n-1})$$

where $w \in WS_n$ and $T;\ WS_n \to WS_{n-1}$ indicates some S_{n-1} equivariant map. Let $\Gamma(X)$ be the universal group of $\Gamma^+(X)$. For the details of definitions, see [4] and [7].

THEOREM 2.1 (Barratt-Quillen). *$\Gamma(X) \cong Q(X)$, homotopy equivalent. If X is connected, then $\Gamma^+(X) \cong \Gamma(X) \cong Q(X)$.*

For the proof, see [7]. Note that for a connected space X the above homotopy equivalence may be given by

$$\Gamma^+(X) \xrightarrow{\Gamma^+(j)} \Gamma^+(Q(X)) \xrightarrow{D.L} Q(X)$$

where $j;\ X \to Q(X)$ is the natural inclusion and $D.L$ denotes the Dyer-Lashof map, see [11]. Now let p be a prime then we have

THEOREM 2.2 (Barratt). *If X is connected, then there exists a natural splitting*

$$\tilde{H}_*(\Gamma^+(X);\, \boldsymbol{Z}_p) \cong \sum_{n=1}^{\infty} \tilde{H}_*(D_n(X);\, \boldsymbol{Z}_p)\,.$$

PROOF. Let $X_+ = X \vee S^0$. We consider the point $+$ as the base point. Let $p;\, X_+ \to X$ be the obvious map. $\Gamma_n(X)$ denotes $\bigcup_{i=1}^{n} WS_i \times X^i/R$, so Γ_n filters Γ^+. Apparently $\Gamma_n(X_+) = \coprod_{i=1}^{n} WS_i \times_{S_i} X^i$, $\Gamma_n(X)/\Gamma_{n-1}(X) = D_n(X)$ and $\Gamma_n(X_+)/\Gamma_{n-1}(X_+) = D_n(X_+) = (WS_n \times_{S_n} X^n)_+$. Then we have the following commutative diagram

$$\begin{array}{ccccc}
\Gamma_{n-1}(X_+) & \longrightarrow & \Gamma_n(X_+) & \longrightarrow & D_n(X_+) \\
\big\downarrow \Gamma_{n-1}(p) & & \big\downarrow \Gamma_n(p) & & \big\downarrow D_n(p) \\
\Gamma_{n-1}(X) & \longrightarrow & \Gamma_n(X) & \longrightarrow & D_n(X)
\end{array}$$

where the horizontal sequences are cofiberations. If X is connected, then $\tilde{H}_*(X_+) \cong H_*(X)$ and $D_n(p)_*;\, \tilde{H}_*(D_n(X_+);\, \boldsymbol{Z}_p) \to \tilde{H}_*(D_n(X);\, \boldsymbol{Z}_p)$ is an epimorphism. Apparently we have a natural splitting

$$\tilde{H}_*(\Gamma_n(X_+);\, \boldsymbol{Z}_p) \cong \tilde{H}_*(\Gamma_{n-1}(X_+);\, \boldsymbol{Z}_p) + \tilde{H}_*(D_n(X_+);\, \boldsymbol{Z}_p)$$

and this gives the required splitting of $\tilde{H}_*(\Gamma_n(X);\, \boldsymbol{Z}_p)$ or $\tilde{H}_*(\Gamma^+(X);\, \boldsymbol{Z}_p) \cong \tilde{H}_*(Q(X);\, \boldsymbol{Z}_p)$. Q.E.D.

Now we recall the homology of $Q(X)$, [6]. Let Q^i be the mod p Dyer-Lashof operation. For a class x of dim q, $Q^i(x)$ is defined by $Q_{(2i-q)(p-1)}(x)$ if p odd and by $Q_{i-q}(x)$ if $p=2$, where Q_j denotes the operation in [6]. The degree of Q^i is i for $p=2$, and $2i(p-1)$ for odd p. Let β denote the homology Bockstein operation. We use the following convention.

a). $p=2$. Let $I=(s_1, \cdots, s_k)$ be a sequence of non-negative integers. We denote $Q^{s_1} \circ \cdots \circ Q^{s_k}$ by Q^I. We define the degree, length and excess of I by $d(I) = \sum_{j=1}^{k} s_j$; $l(I) = k$; $e(I) = s_1 - \sum_{j=2}^{k} s_j$. I is called admissible if $2s_j \geqq s_{j+1}$ for $2 \leqq j \leqq k$.

b). p is odd. Let $I=(\varepsilon_1, s_1, \cdots, \varepsilon_k, s_k)$ be a sequence of integers such that $s_i \geqq 0$ and $\varepsilon_i = 0$ or 1 for all i. We denote $\beta^{\varepsilon_1} Q^{s_1} \cdots \beta^{\varepsilon_k} Q^{s_k}$ by Q^I. We define $d(I) = \sum_{j=1}^{k} (2s_j(p-1) - \varepsilon_j)$; $l(I) = k$; $e(I) = 2s_1 - \sum_{j=2}^{k} (2s_j(p-1) - \varepsilon_j)$. I is called admissible if $ps_j - \varepsilon_j \geqq s_{j-1}$ for $2 \leqq j \leqq k$.

THEOREM 2.3 (Dyer-Lashof). *If X is connected, then $H_*(Q(X);\, \boldsymbol{Z}_p)$ is isomorphic to a free commutative graded algebra generated by all $Q^I x_j$, where $\{x_j\}$ is a $\boldsymbol{Z}_p$-basis of $\tilde{H}_*(X;\, \boldsymbol{Z}_p)$ and I is admissible with $e(I) > \deg x_j$.*

For the proof, see [6].

Now let $x=\prod_{i=1}^{t} Q^{I_i}(x_i)$ be any monomial of $H_*(Q(X);\boldsymbol{Z}_p)$. We define the height of x by $h(x)=\sum_{i=1}^{t} p^{l(I_i)}$ and $h(1)=0$. This is well defined. Let $A_n(X)$ denote the sub-module of $H_*(Q(X);\boldsymbol{Z}_p)$ spanned by all monomials of height n.

PROPOSITION 2.4. *If X is connected,* $H_*(D_n(X);\boldsymbol{Z}_p)\cong A_n(X)$.

PROOF. Let $M_n(X)$ be the image of $H_*(\Gamma_n(X);\boldsymbol{Z}_p)\xrightarrow{j_{n*}} H_*(\Gamma^+(X);\boldsymbol{Z}_p)\cong H_*(Q(X);\boldsymbol{Z}_p)$. Then since $\tilde{H}_*(Q(X);\boldsymbol{Z}_p)\cong\sum\tilde{H}_*(D_n(X);\boldsymbol{Z}_p)$ (Theorem 2.2), it is enough to prove that $M_n(X)\cong$ submodule of $H_*(Q(X);\boldsymbol{Z}_p)$ spanned by all monomials of height $\leqq n$. Let $S_n(p)$ be a p-Sylow subgroup of S_n. Then $S_{p^r}(p)\cong\boldsymbol{Z}_p\int\cdots\int\boldsymbol{Z}_p$, the r-fold wreath product of $\boldsymbol{Z}_p$ with itself. Let $n=a_kp^k+\cdots+a_0$ be the p-adic expansion of n. Then $S_n(p)\cong S_{p^k}(p)^{a_k}\times\cdots\times S_1(p)^{a_0}$. Let $B_{n,p}(X)$ denote $WS_n(p)\times_{S_n(p)}X^n$ and $B_n(X)$ denote $WS_n\times_{S_n}X^n$. Then we have an obviovs natural map $B_{n,p}(X)\to B_n(X)$. Clearly

$$B_{n,p}(X)\cong B_{p^k,p}(X)^{a_k}\times\cdots\times B_{1,p}(X)^{a_0}$$

and $B_{p^k,p}(X)\cong B_{p,p}(\cdots(B_{p,p}(X)\cdots)$.

Consider the map

$$\phi_n;\ \coprod_{i=0}^{n} B_{i,p}(X)\longrightarrow\coprod_{i=0}^{n} B_i(X)=\Gamma_n(X_+)\xrightarrow{\Gamma_n(p)}\Gamma_n(X).$$

Since $B_{i,p}(X)\to B_i(X)$ is a finite covering of degree prime to p, ϕ_{n*}; $H_*(\coprod B_{i,p}(X);\boldsymbol{Z}_p)\to H_*(\Gamma_n(X);\boldsymbol{Z}_p)$ is an epimorphism. Therefore we have $M_n(X)\cong$ Image of $(j_n\circ\phi_n)_*$. Note that $H_*(B_{i,p}(X);\boldsymbol{Z}_p)$ is written by iterated Dyer-Lashof operations, and we can define the height of elements in $H_*(B_{i,p}(X);\boldsymbol{Z}_p)$. Then by comparing the height, we can see that $M_n(X)$ is spanned by all monomials of height $\leqq n$. Q. E. D.

Now let us assume that X is $k-1$ connected and $H_k(X;\boldsymbol{Z}_p)\cong Z_p$. Let i; $S^k\to X$ be a map representing a generator $z\in H_k(X;\boldsymbol{Z}_p)$. z also denotes the corresponding element in $H_k(Q(X);\boldsymbol{Z}_p)$. Consider the map

$$g_{n-1};\ D_{n-1}(X)\wedge S^k\longrightarrow D_n(X)$$

and the commutative diagram

$$\begin{array}{ccccccc}
(WS_{n-1}\times_{S_{n-1}}X^{n-1})\times S^k & \xrightarrow{1\times i} & (WS_{n-1}\times_{S_{n-1}}X^{n-1})\times X & \to & WS_n\times_{S_n}X^n & \xrightarrow{D.L} & Q(X)\\
\downarrow & & \downarrow & & \downarrow & & \\
D_{n-1}(X)\wedge S^k & \xrightarrow{1\wedge i} & D_{n-1}(X)\wedge X & \xrightarrow{\mu_{n-1,1}} & D_n(X) & &
\end{array}$$

where vertical maps are obvious identification. Let σ_k; $H_i\to H_{i+k}$ be the suspension, and let $\alpha=\times z$; $H_i(Q(X);\boldsymbol{Z}_p)\to H_{i+k}(Q(X);\boldsymbol{Z}_p)$ be the homomorphism defined by product with z. Clearly $\times z$; $A_{n-1}\to A_n$ and we have

$$g_{n-1*}\sigma_k=\alpha\,;\ A_{n-1}\longrightarrow A_n\,.$$

THEOREM 2.5. *We assume that k is even if p is odd. Then g_{n-1*}; $H_i(D_{n-1}(X)\wedge S^k;\,\boldsymbol{Z}_p)\to H_i(D_n(X);\,\boldsymbol{Z}_p)$ is monomorphic for any i, and isomorphic for $i<kn+\frac{2p-3}{p}n$.*

PROOF. It is enough to consider α; $A_{n-1}\to A_n$. Since $H_*(Q(X);\,\boldsymbol{Z}_p)$ is a free graded algebra, under the assumption, α is clearly monomorphic. We prove that α is epimorphism up to appropriate dimension. First we consider the case of $p=2$. Let $x=\prod^t Q^{I_i}x_i$ be a monomial of height n. Since Q^{I_i} is admissible and $e(Q^{I_i})>\deg x_i$, we have easily

$$(*)\qquad \begin{aligned}\deg(x)&\geqq\sum_{i=1}^t 2^{l(I_i)}\deg(x_i)+\sum_{i=1}^t(2^{l(I_i)}-1)\\&=\sum 2^{l(I_i)}\deg(x_i)+n-t\\&\geqq nk+n-t\,.\end{aligned}$$

Now let s be the number of 1's in the sum $\sum 2^{l(I_i)}(=n)$. Then we may assume that $I_i=\emptyset$ (the void set) for $1\leqq i\leqq s$. Also we have easily

$$(**)\qquad 2(t-s)+s\leqq n\,.$$

Suppose that $\deg x_i>k$ for $1\leqq i\leqq s$, then by (*), we have

$$(***)\qquad \deg(x)\geqq nk+n-t+s$$

and together with (**), this implies $\deg(x)\geqq nk+\frac{n}{2}$. Therefore if $\deg(x)<nk+\frac{n}{2}$, at least one of x_i, $i=1, 2, \cdots, s$, has degree k which must be z. Thus α; $A_{n-1}\to A_n$ is epimorphism in dim. $<nk+\frac{n}{2}$.

Next we consider the case of p odd. Also consider an element $x=\prod^t Q^{I_i}(x_i)$ of height $n=\sum p^{l(I_i)}$. By a simple calculation we have the following; If $\deg x_i$ is even and $e(I_i)>\deg x_i$, then

$$\deg Q^{I_i}x_i\geqq p^{l(I_i)}\deg x_i+\frac{2p-3}{p-1}(p^{l(I_i)}-1)\,.$$

If $\deg x_i$ is odd and $e(I_i)>\deg x_i$, then

$$\deg Q^{I_i}x_i\geqq p^{l(I_i)}\deg x_i+\frac{p-2}{p-1}(p^{l(I_i)}-1)\,.$$

Let t' be the number of x_i's such that $\deg x_i$ is even. Then

$$\begin{aligned}\deg x &\geqq \sum_{\deg x_i \text{ even}} \left(p^{l(I_i)} \deg x_i + \frac{2p-3}{p-1}(p^{l(I_i)}-1)\right) \\ &\quad + \sum_{\deg x_i \text{ odd}} \left(p^{l(I_i)} \deg x_i + \frac{p-2}{p-1}(p^{l(I_i)}-1)\right) \\ &\geqq \sum^{t'} \left(p^{l(I_i)} k + \frac{2p-3}{p-1}(p^{l(I_i)}-1)\right) \\ &\quad + \sum^{t-t'} \left(p^{l(I_i)}(k+1) + \frac{p-2}{p-1}(p^{l(I_i)}-1)\right) \\ &= kn + \frac{p-2}{p-1}(n-t) + n - t'.\end{aligned}$$

Now let s be the number of 1's in $\sum p^{l(I_i)}$. Then we may assume $I_i = \emptyset$ for $1 \leqq i \leqq s$. Suppose that $\deg x_i > k$ for any i, $1 \leqq i \leqq s$. Let s_1 and s_2 be the number of x_i's of even degree and the number of those of odd degree for $1 \leqq i \leqq s$, respectively. Clearly $t' \geqq s_1$ and $p(t-s)+s \leqq n$ and we have

$$\begin{aligned}\deg x &\geqq kn + \frac{p-2}{p-1}(n-t) + n - t' + 2s_1 + s_2 \\ &\geqq kn + \frac{p-2}{p-1}(n-t) + 2s \geqq kn + \frac{2p-3}{p} n .\end{aligned}$$

Then the rest of the proof is the same as that of $p=2$.

§ 3. Homotopy type of Γ-spectra.

Let $M_k = S^k \cup_p e^{k+1}$ denotes the Moore space of type $(\boldsymbol{Z}_p, k)$. Then the purpose of this section is to prove

THEOREM 3.1. *We assume that k is even if p is odd. If $k \geqq 2$ then $\boldsymbol{D}_{M_k}$ has the same* mod p *homotopy type of a wedge of Eilenberg-MacLane spectra $\Sigma^i \boldsymbol{K}(\boldsymbol{Z}_p)$.*

Now we call a Γ-spectrum $\boldsymbol{M}$ mod p connected if the stable homology group $H_0(\boldsymbol{M}; \boldsymbol{Z}_p) \cong \boldsymbol{Z}_p$ for a prime p generated by the unit $\boldsymbol{S} \xrightarrow{i} \boldsymbol{M}$. A generator of $H^0(\boldsymbol{M}; \boldsymbol{Z}_p))$ is denoted by u. Then we have

COROLLARY 3.2. *Let $\boldsymbol{M}$ be a* mod p *connected Γ-spectrum. Assume that $\beta u \neq 0$, β being the cohomology Bockstein operation, then $\boldsymbol{M}$ has the same* mod p *homotopy type of a wedge of $\Sigma^i \boldsymbol{K}(\boldsymbol{Z}_p)$.*

The proof of Theorem 3.1 and Corollary 3.2. will be given in the end of this section.

Let π be the cyclic group of order p and let $D_\pi(X)$ be the cyclic extended power of X (see § 1). $H^n(B\pi; \boldsymbol{Z}_p) \cong \boldsymbol{Z}_p$ is generated by w_1^n for $p=2$ and by $(\beta w_1)^{n/2}$ or $w_1(\beta w_1)^{(n-1)/2}$ for odd p. Let $e_i \in H_i(B\pi; \boldsymbol{Z}_p)$ be the dual homology class.

Let $\{x_i\}$, $i=1, 2, \cdots$, be a $\boldsymbol{Z}_p$-basis of $\tilde{H}_*(X; \boldsymbol{Z}_p)$. Then a $\boldsymbol{Z}_p$-basis of $\tilde{H}_*(D_\pi(X); \boldsymbol{Z}_p)$ is given by

$$e_i \otimes x_j^p,\ i=0, 1, \cdots, \quad \text{and} \quad j=1, 2, \cdots,$$

$$e_0 \otimes (x_{j_1} \otimes \cdots \otimes x_{j_p}),\ j_s \neq j_t \qquad \text{for some } s \text{ and } t,$$

where we identify $e_0 \otimes (x_{j_1} \otimes \cdots \otimes x_{j_p})$ with $e_0 \otimes (x_{j_{\sigma(1)}} \otimes \cdots \otimes x_{j_{\sigma(p)}})$ for any $\sigma \in \pi$. Let Sq_*^n, $\mathcal{P}_*^n$ and Δ denote the dual of the Steenrod square, and the dual of the Steenrod reduced power and the homology Bockstein operation, respectively. Then we have

THEOREM 3.3. *Let x be an element of $H_q(X; \boldsymbol{Z}_p)$ then*

$$Sq_*^n(e_{n+c} \otimes x^2) = \sum_i \binom{q+c}{n-2i} e_{c+2i} \otimes (Sq_*^i x)^2,$$

$$\mathcal{P}_*^n(e_{2n(p-1)+c} \otimes x^p) = \sum \binom{\left[\frac{c}{2}\right]+qm}{n-pi} e_{c+2ip(p-1)} \otimes (\mathcal{P}_*^i x)^p$$

$$-\mu(q)\varepsilon(c+1) \sum \binom{\left[\frac{c+1}{2}\right]+qm-1}{n-pi-1} e_{c+p+2ip(p-1)} \otimes (\mathcal{P}_*^i \Delta x)^p$$

and

$$\Delta(e_c \otimes x^p) = \varepsilon(c) e_{c-1} \otimes x^p,$$

where c may be negative, $m=(p-1)/2$, $\mu(q) \not\equiv 0(p)$ and $\varepsilon(s)=1$ if s is even, $\varepsilon(s)=0$ if s is odd.

PROOF. The last formula is easy to see. The second one is proved in [**16**] and the first one is similar and easier and the proof is left to the reader.

Let Sq^{Δ_i} be the dual of the Milnor generator ζ_i in the dual Steenrod algebra $\mathcal{A}_{2*}$. Also $\mathcal{P}^{\Delta_i}$ and Q_i denote the dual of ξ_i and the dual of τ_i in $\mathcal{A}_{p*}$, respectively. Now let $D_{p^r,p}(X)$ be the extended power with respect to a p-Sylow subgroup of S_{p^r}. Then $D_{p^r,p}(X) \cong D_\pi \circ \cdots \circ D_\pi(X)$, r times iteration of the cyclic extended power. Let $u \in H^q(X; \boldsymbol{Z}_p)$. Then the external reduced power $P(u) \in H^{pq}(D_\pi(x); \boldsymbol{Z}_p)$ is defined in [**18**], and we can consider the iteration $P^{(r)}(u) \in H^{p^r q}(D_{p^r,p}(X); \boldsymbol{Z}_p)$. Then we have

PROPOSITION 3.4. *Let X be $k-1$ connected and let $u \in H^k(X; \boldsymbol{Z}_p)$ be a class such that $\beta u \neq 0$. Then*

$$Sq^{\Delta_i} P^{(r)}(u) \neq 0 \qquad \text{for } p=2,$$

$$\mathcal{P}^{\Delta_i} P^{(r)}(u) \neq 0 \quad \text{and} \quad Q_i P^{(r)}(u) \neq 0 \qquad \text{for odd } p,$$

for any i, $0 \leqq i \leqq r$.

PROOF. Under the assumption of the proposition, there exists a map $f;\ S^k\cup_p e^k=M_k\to X$ such that $f^*(u)\neq 0$. Therefore by the naturality of the operations, it suffices to prove for $X=S^k\cup_p e^{k+1}$ and $u\in H^k(S^k\cup_p e^{k+1};\ \boldsymbol{Z}_p)$, a generator. We shall prove it in the term of the dual operations.

Let $x\in H_k(S^k\cup_p e^{k+1};\ \boldsymbol{Z}_p)$ be the dual of u and let $y\in H_{k+1}(S^k\cup_p e^{k+1};\ \boldsymbol{Z}_p)$ be an element such that $\Delta y=x$. Note that $P^{(r)}(u)$ is the dual of $x^{p^r}=x\otimes\cdots\otimes x$. Let $z_i,\ i=1,\cdots,m$ be homology classes of $D_{p^r,p}(S^k\cup_p e^{k+1})$. We call the sequence $\{z_1,\cdots,z_m\}$ essential if $z_{i+1}=Sq_*^{2^{t_i}}z_i$ for any i, $1\leqq i\leqq m-1$ for $p=2$ and if $z_{i+1}=\mathcal{P}_*^{p^{t_i}}z_i$ or $z_{i+1}=\beta z_i$ for any i, $1\leqq i\leqq m-1$ for p odd. By definition [13], $Sq^{\Delta_i}=[Sq^{2^{i-1}},\ Sq^{\Delta_{i-1}}]$, $\mathcal{P}^{\Delta_i}=[\mathcal{P}^{p^{i-1}},\ \mathcal{P}^{\Delta_i}]$ and $Q_0=\beta$, $Q_i=[\mathcal{P}^{p^{i-1}},\ Q_{i-1}]$ for $i\geqq 1$. Therefore

$$Sq^{\Delta_i}=Sq^1Sq^2\cdots Sq^{2^{i-1}}+\text{other terms}$$

$$\mathcal{P}^{\Delta_i}=\pm\mathcal{P}^1\mathcal{P}^p\cdots\mathcal{P}^{p^{i-1}}+\text{other terms}$$

and

$$Q_i=\pm\beta\mathcal{P}^1\mathcal{P}^p\cdots\mathcal{P}^{p^{i-1}}+\text{other terms}.$$

First we assume $p=2$ and k is even. Consider the elements $e_1\otimes y^2$ and $e_0\otimes x^2=x^2$ in $H_*(D_\pi(S^k\cup_2 e^{k+1});\ \boldsymbol{Z}_2)$. Then by Theorem 3.3, the only essential sequence from $e_1\otimes y^2$ to x^2 is $\{e_1\otimes y^2\xrightarrow{Sq_*^1}e_0\otimes y^2\xrightarrow{Sq_*^2}x^2\}$. Then by induction using Theorem 3.3, we have the following. In $D_{2^r,2}(S^k\cup_2 e^{k+1})$, the only one essential sequence from $e_1\otimes(e_1\otimes\cdots\otimes(e_1\otimes y^2)^2\cdots)^2$ to x^{2^r} is $\{e_1\otimes(\cdots(e_1\otimes y^2)^2\cdots)^2\xrightarrow{Sq_*^1}e_0\otimes(e_1\otimes\cdots\otimes(e_1\otimes y^2)^2\cdots)^2\xrightarrow{Sq_*^2}\cdots\to e_0\otimes(e_0\otimes\cdots\otimes(e_0\otimes y^2)^2\cdots)^2\xrightarrow{Sq_*^{2^r}}x^{2^r}\}$. This proves that $Sq^{\Delta_i}P^{(r)}(u)\neq 0$ for $i\leqq r+1$. Now let k be odd. Note that there exists a map $g;\ S^{2k}\cup_2 e^{2k+1}\to D_\pi(S^k\cup_2 e^{k+1})$ which is degree 1 on the $2k$ cell. Then by the naturality we have $Sq^{\Delta_i}P^{(r)}(u)\neq 0$ for $i\leqq r$.

Next we shall consider the case $p=$ odd. By Theorem 3.3, we have that in $H_*(D_\pi(S^k\cup_p e^{k+1});\ \boldsymbol{Z}_p)$, $\Delta(e_{p-1}\otimes y^p)=e_{p-2}\otimes y^p$ and $\mathcal{P}_*^1(e_{p-2}\otimes y^p)=*e_0\otimes x^p$, where $*\not\equiv 0(p)$. Furthermore the only essential sequence from $e_{p-1}\otimes y^p$ to x^p is $\{e_{p-1}\otimes y^p\xrightarrow{\Delta}e_{p-2}\otimes y^p\xrightarrow{\mathcal{P}_*^1}x^p\}$. Then also by an induction, we can see that in $D_{p^r,p}(S^k\cup_p e^{k+1})$, $\{e_{p-1}\otimes(e_{p-1}\otimes\cdots\otimes(e_{p-1}\otimes y^p)^p\cdots)^p\xrightarrow{\Delta}e_{p-2}\otimes(e_{p-1}\otimes\cdots\otimes(e_{p-1}\otimes y^p)^p\cdots)^p\xrightarrow{\mathcal{P}_*^1}e_0\otimes(e_{p-2}\otimes(e_{p-1}\otimes\cdots\otimes(e_{p-1}\otimes y^p)^p\cdots)^p\to\cdots\to e_0\otimes(e_0\otimes\cdots\otimes e_0\otimes(e_{p-2}\otimes y^p)^p\cdots)^p\xrightarrow{\mathcal{P}_*^{p^{r-1}}}x^{p^r}\}$ is the only one essential sequence from the first class to x^{p^r}. Thus we have $\mathcal{P}^{\Delta_i}P^{(r)}(u)\neq 0$ and $Q_iP^{(r)}(u)\neq 0$ for $i\leqq r$. This completes the proof. Q. E. D.

PROOF OF THEOREM 3.1. Let $X=S^k\cup_p e^{k+1}$. Assume that $k\geqq 2$ and k is even if p is odd. In this case, the stable cohomology $H^*(\boldsymbol{D}_X;\ \boldsymbol{Z}_p)$ is a connected coalgebra over $\mathcal{A}_p$ with unit by Theorem 1.4. Let $u\in H^0(\boldsymbol{D}_X; \boldsymbol{Z}_p)$ be a unit. Let $\phi; \mathcal{A}_p\rightarrow H^*(\boldsymbol{D}_X; \boldsymbol{Z}_p)$ be the homomorphism of coalgebras defined by $\phi(a)=a(u)$ for $a\in\mathcal{A}_p$. Under the assumption on k, the natural map $D_{p^r,p}(X)\rightarrow D_{p^r}(X)$ has the degree $\not\equiv 0(p)$ on the $p^r k$ cell. Then by the naturality of cohomology operations and by Proposition 3.4 and by the fact that the dual of ζ_i for $p=2$ and ξ_i, τ_i for p odd generates the sub-module of primitive elements of $\mathcal{A}_{p*}$, it follows that ϕ is monomorphic. Then by Theorem of [**14**], we have $H_*(\boldsymbol{D}_X; \boldsymbol{Z}_p)$ is a free $\mathcal{A}_p$-module. Therefore $\boldsymbol{D}_X$ is a wedge of Eilenberg—MacLane spectra $\boldsymbol{K}(\boldsymbol{Z}_p)$. This completes the proof.

PROOF OF COROLLARY 3.2. Let $\boldsymbol{M}$ be a connected Γ-spectrum with unit. Then clearly $H^*(\boldsymbol{M}; \boldsymbol{Z}_p)$ is a connected coalgebra over $\mathcal{A}_p$ with unit. Therefore it is sufficient to show that $\phi; \mathcal{A}_p\rightarrow H^*(\boldsymbol{M}; \boldsymbol{Z}_p)$ is monomorphic. Since $\beta u\neq 0$ for the unit u, there is a map $f; S^k\cup_p e^{k+1}\rightarrow M_k$ for large k such that f^* is non trivial. $\boldsymbol{M}$ is a Γ-spectrum, so for any n and r, there exist an integer k and a map $\theta; D_n^{(r)}(M_k)\rightarrow M_{nk}$. Consider the map

$$D_{n,p}^{(r)}(S^k\cup_p e^{k+1}) \xrightarrow{D_{n,p}^{(r)}(f)} D_{n,p}^{(r)}(M_k) \longrightarrow D_n^{(r)}(M_k) \longrightarrow M_{nk}\,.$$

Then by the naturality and by Proposition 3.4, we can see that ϕ is monomorphic. This implies immediately the corollary.

REMARK. As a corollary, we have well-known result; $\boldsymbol{MO}\cong\vee S^i\boldsymbol{K}(\boldsymbol{Z}_2)$.

§ 4. Applications to the stable homotopy groups of spheres.

Let $\pi_*^s(S^0; p)$ denote the p primary component of the stable homotopy groups of spheres. Then we have

THEOREM 4.1. *Let $\gamma\in\pi_*^s(S^0; p)$ be of order p. Then for any integer n and for any $\alpha\in\pi_t^s(S^0; p)$ such that $0<t<\left[\frac{2p-3}{p}n\right]-1$, we have $\alpha\gamma^n=0$, where $[x]$ indicates the integer part of a rational x.*

As an immediate corollary, we have

COROLLARY 4.2. *Let $\gamma\in\pi_t^s(S^0; p)$ be of order p, and let n be the smallest integer such that $t=\text{stem}\,\gamma<\left[\frac{2p-3}{p}n\right]-1$. Then we have $\gamma^{n+1}=0$.*

PROOF OF THEOREM 4.1. Let $\gamma\in\pi_t^s(S^0; p)$. Note that if p and k are odd, then $\gamma^2=0$, and the theorem is apparently true. Therefore we may assume that k is even if p is odd. Let $f; S^{k+N}\rightarrow S^N$, N large enough, be a representative of γ. By assumption, $p\gamma=0$. Therefore f extends to a map

$$\tilde{f};\ S^{k+N}\underset{p}{\cup} e^{k+N+1}\longrightarrow S^N\,.$$

Now by Proposition 1.6, it follows that for any n and r, we can choose

N such that there exists a retraction r; $D_n^{(r)}(S^N) \to S^{Nn}$. Note that N must be even if $n \geqq 2$. Consider the following commutative diagram

$$\begin{array}{ccc}
D_n^{(r)}(S^{k+N} \cup_p e^{k+N+1}) & \xrightarrow{D_n^{(r)}(\tilde{f})} & D_n^{(r)}(S^N) \\
\uparrow{\scriptstyle D_n^{(r)}(i)} & & \| \\
D_n^{(r)}(S^{k+N}) & \xrightarrow{D_n^{(r)}(f)} & D_n^{(r)}(S^N) \\
\uparrow{\scriptstyle j} & & \downarrow{\scriptstyle r} \\
D_n^{(0)}(S^{k+N}) = S^{n(k+N)} & \xrightarrow{f^{(n)}} & S^{nN}
\end{array}$$

where i and j are the natural inclusions. Since $k+N$ is even if p is odd, by Theorem 2.5 and by Theorem 3.1 we have $D_n(S^{k+N} \cup_p e^{k+N+1})$ is $\bmod p$ stably homotopy equivalent to a wedge of $S^i \boldsymbol{K}(\boldsymbol{Z}_p)$ up to dimension $n(k+N)+\frac{2p-3}{p}n$. Therefore taking r and N large enough, we have $D_n^{(r)}(S^{k+N} \cup_p e^{k+N+1})$ is $\bmod p$ homotopy equivalent to a product of Eilenberg-MacLane complexes up to this dimension. Then from the commutativity of the above diagram, it follows that $f^{(n)}g \sim 0$ for any g; $S^{n(k+N)+i} \to S^{n(k+N)}$ for $0 < i < \left[\frac{2p-3}{p}n\right]-1$. This completes the proof.

Now let $\boldsymbol{M}$ be a Γ-spectrum. Then the stable homotopy group $\pi_*^s(\boldsymbol{M})$ turns out to be an associative commutative ring. Then by a similar argument, we can see

THEOREM 4.3. *Let $\gamma \in \pi_*^s(\boldsymbol{M})$ be an element of order p and satisfy that γ_*; $H_*(\boldsymbol{S}; \boldsymbol{Z}_p) \to H_*(\boldsymbol{M}; \boldsymbol{Z}_p)$ is trivial. Then γ is nilpotent.*

REMARK. As far as known element in $\pi_*^s(S^0)$ are concerned, the above result is not best possible. For most elements of the 2-primary components, it is known that $\alpha^4 = 0$. For the odd component, it is known [**19**] that $\beta_1^{p^2+1} = 0$ for $\beta_1 \in \pi_{2p(p-1)-2}^s(S^0, p)$.

PART II.

§ 5. Further properties of $D_n(X)$.

Let $\overset{m}{\vee} A_i = A_1 \vee \cdots \vee A_m$ be the wedge sum of A_i. We denote by i_k; $A_k \to \overset{m}{\vee} A_i$ and π_k; $\overset{m}{\vee} A_i \to A_k$ the natural inclusion and projection map, respectively. For an integer n, $w = (s_1, \cdots, s_m)$ is called a partition of n of length m if each s_i is non-negative and $\sum_{i=1}^{m} s_i = n$. Then we define

$$f_w=\mu_w(D_{s1}(i_1)\wedge\cdots\wedge D_{sm}(i_m))\,;\ D_{s1}(A_1)\wedge\cdots\wedge D_{sm}(A_m)$$
$$\longrightarrow D_{s1}(\overset{m}{\vee}A_i)\wedge\cdots\wedge D_{sm}(\overset{m}{\vee}A_i)\longrightarrow D_n(\overset{m}{\vee}A_i)$$

where μ_w denotes an obvious generalization of the pairing $\mu_{i,j}\,;\ D_i(X)\wedge D_j(X)\to D_{i+j}(X)$, and $D_0(X)$ stands for S^0 for any X. Consider the maps

$$f=\vee f_w\,;\ \vee D_{s1}(A_1)\wedge\cdots\wedge D_{sm}(A_m)\longrightarrow D_n(\overset{m}{\vee}A_i)$$

the wedge sum of f_w for all partitions of n of length m.

THEOREM 5.1. *For any n and m, f is a homotopy equivalent.*

PROOF. By definition, $D_n(\overset{m}{\vee}A_i)=(WS_{n+})\wedge_{S_n}(A_1\vee\cdots\vee A_m)^{(n)}$. $(A_1\vee\cdots\vee A_m)^{(n)}=\vee(A_{i_1}\wedge\cdots\wedge A_{i_n})$, $i_s=1,2,\cdots,m$. Let $w=(s_1,\cdots,s_m)$ be a partition of n of length m. Then I_w denotes the minimal S_n-invariant subspace of $(A_1\vee\cdots\vee A_m)^{(n)}$ including $A_1^{(s1)}\wedge\cdots\wedge A_m^{(sm)}$. Then we have an equivariant splitting $(A_1\vee\cdots\vee A_m)^{(n)}\cong\underset{\text{all partitions}}{I_w}$ and thus

$$D_n(\overset{m}{\vee}A_i)\cong\vee(WS_{n+}\wedge_{S_n}I_w)\,.$$

Now the subgroup $G=S_{s1}\times\cdots\times S_{sm}$ of S_n acts on $A_1^{(s1)}\wedge\cdots\wedge A_m^{(sm)}$ and the inclusion $A_1^{(sm)}\wedge\cdots\wedge A_m^{(sm)}\to I_w$ is equivariant with respect to the inclusion $G\to S_n$. Consider the following induced maps

$$WG_+\wedge_G(A_1^{(s1)}\wedge\cdots\wedge A_m^{(sm)})\longrightarrow WS_{n+}\wedge_G(A_1^{(s1)}\wedge\cdots\wedge A_m^{(sm)})$$
$$\longrightarrow WS_{n+}\wedge_{S_n}I_w\,.$$

The second map is easily seen to be a homeomorphism and the first one is a homotopy equivalence. But

$$WG_+\wedge_G(A_1^{(s1)}\wedge\cdots\wedge A_m^{(sm)})\cong D_{s1}(A_1)\wedge\cdots\wedge D_{sm}(A_m)$$

and it is easily seen that the resulting map is equal to f_w. This proves the theorem.

COROLLARY 5.2. *Let $\vee g_i\,;\ \vee A_i\to B$ be the wedge sum of g_i. Then $D_n(\vee g_i)f\,;\ \vee D_{s1}(A_i)\wedge\cdots\wedge D_{sm}(A_m)\to D_n(B)$ is homotopic to $\vee\mu_w(D_{s1}(g_1)\wedge\cdots\wedge D_{sm}(g_m))$, where the sum is taken all over the partitions of n.*

Now let σ be an element of S_m. Then σ defines a map $\sigma^*\,;\ X_1\times\cdots\times X_m\to X_{\sigma^{-1}(1)}\times\cdots\times X_{\sigma^{-1}(m)}$ by $\sigma(x_1,\cdots,x_m)=(x_{\sigma^{-1}(1)},\cdots,x_{\sigma^{-1}(m)})$. This induces maps $\sigma^*\,;\ X_1\wedge\cdots\wedge X_m\to X_{\sigma^{-1}(1)}\wedge\cdots\wedge X_{\sigma^{-1}(m)}$ and $\sigma^*\,;\ X_1\vee\cdots\vee X_m\to X_{\sigma^{-1}(1)}\vee\cdots\vee X_{\sigma^{-1}(m)}$. Let $w=(s_1,\cdots,s_m)$ be a partition of n. Then $\sigma^*w=(s_{\sigma^{-1}(1)},\cdots,s_{\sigma^{-1}(m)})$ defines a partition of n. Then we have

LEMMA 5.3. *Assume that A_i is 1-connected for all i. Then the following diagram is homotopy commutative*

$$\begin{array}{ccc} D_{s1}(A_1)\wedge\cdots\wedge D_{sm}(A_m) & \xrightarrow{f_w} & D_n(A_1\vee\cdots\vee A_m) \\ \downarrow \sigma^* & & \downarrow D_n(\sigma^*) \\ D_{s_{\sigma^{-1}(1)}}(A_{\sigma^{-1}(1)})\wedge\cdots\wedge D_{s_{\sigma^{-1}(m)}}(A_{\sigma^{-1}(m)}) & \xrightarrow[f_{\sigma^* w}]{} & D_n(A_{\sigma^{-1}(1)}\vee\cdots\vee A_{\sigma^{-1}(m)})\,. \end{array}$$

PROOF. Consider the diagram

$$\begin{array}{ccc} D_{s1}(A_1)\wedge\cdots\wedge D_{sm}(A_m) & \xrightarrow{\sigma^*} & D_{s_{\sigma^{-1}(1)}}(A_{\sigma^{-1}(1)})\wedge\cdots\wedge D_{s_{\sigma^{-1}(m)}}(A_{\sigma^{-1}(m)}) \\ \downarrow \alpha & & \downarrow \beta \\ D_{s1}(\vee A_i)\wedge\cdots\wedge D_{sm}(\vee A_i) & \xrightarrow{\sigma^*} & D_{s_{\sigma^{-1}(1)}}(\vee A_i)\wedge\cdots\wedge D_{s_{\sigma^{-1}(m)}}(\vee A_i) \\ & \mu\searrow \quad \swarrow\mu & \\ & D_n(\vee A_i) & \end{array}$$

where $\alpha=D_{s1}(i_1)\wedge\cdots\wedge D_{sm}(i_m)$ and $\beta=D_{s_{\sigma-1(1)}}(i_{\sigma^{-1}(1)})\wedge\cdots\wedge D_{s_{\sigma-1(m)}}(i_{\sigma^{-1}(m)})$. Then the commutativity of the rectangle diagram is obvious and the triangle diagram is homotopy commutative if all A_i are 1-connected by Theorem 1.4. Then the lemma immediately follows. Q. E. D.

Now we take $A_i=B=S^k$ for any i. Let ι_k; $S^k\to S^k$ denote the identity. Then $\pi=\iota_k\vee\cdots\vee\iota_k$; $\overset{m}{\vee}S^k\to S^k$ is the natural projection. Let ϕ; $S^k\to\overset{m}{\vee}S^k$ be the standard comultiplication map. Then $\pi\phi=m\iota_k$; $S^k\to S^k$. By Corollary 5.2, applying D_n we have

$$D_n(m\iota_k)\sim(\bigvee_w\mu_w(D_{s1}(\iota_k)\wedge\cdots\wedge D_{sm}(\iota_k)))f^{-1}D_n(\phi) \tag{5.4}$$
$$\sim(\bigvee_w\mu_w)f^{-1}D_n(\phi)$$

where f^{-1} indicates a homotopy inverse of f.

Let α_w; $D_n(\overset{m}{\vee}S^k)\to D_{s1}(S^k)\wedge\cdots\wedge D_{sm}(S^k)$ be the composition

$$D_n(\overset{m}{\vee}S^k)\xrightarrow{f^{-1}}\bigvee_w D_{s1}(S^k)\wedge\cdots\wedge D_{sm}(S^k)\xrightarrow{\text{Proj.}}D_{s1}(S^k)\wedge\cdots\wedge D_{sm}(S^k)\,.$$

Note that in the stable category, we have

$$D_n(m\iota_k)\sim\sum_w\mu_w\alpha_w D_n(\phi)\,. \tag{5.5}$$

Now let us consider the diagram

$$\begin{array}{ccccccc} D_n(S^k) & \xrightarrow{D_n(\phi)} & D_n(\overset{m}{\vee}S^k) & \xrightarrow{\alpha_w} & D_{s1}(S^k)\wedge\cdots\wedge D_{sm}(S^k) & \xrightarrow{\mu_w} & D_n(S^k) \\ \| & & \downarrow D_n(\sigma^*) & & \downarrow\sigma^* & & \| \\ D_n(S^k) & \xrightarrow{D_n(\phi)} & D_n(\overset{m}{\vee}S^k) & \xrightarrow{\alpha_{\sigma^* w}} & D_{s_{\sigma-1(1)}}(S^k)\wedge\cdots\wedge D_{s_{\sigma-1(m)}}(S^k) & \xrightarrow{\mu_{\sigma^* w}} & D_n(S^k)\,. \end{array}$$

We assume that $k \geqq 2$. Then by Lemma 5.2, the second and third rectangles are homotopy commutative. Also as is well known, $\phi \sim \sigma^*\phi$ for any $\sigma \in S_m$ if $k \geqq 2$. Hence the first rectangle is homotopy commutative.

Let Ω be the set of all partitions of n of length m. Then S_m acts on Ω by $w \to \sigma^* w$. Let t_i, d_i, $i=1, 2, \cdots, m$, be integers such that $t_1 = 0 < t_2 < \cdots < t_m$, $\sum_{i=1}^{m} d_i = m$ and $\sum_{i=1}^{m} t_i d_i = n$. Then (t_i, d_i) defines a partition of n by arranging t_i with multiplicity d_i (d_i may be 0). Then clearly all such (t_i, d_i) give representatives of the set Ω/S_m. Note that $\#$\{the orbit set through (t_i, d_i)\} $= m!/\prod(d_i!)$ with convention $0!=1$.

Now we put $n=p$, a prime, and $m \equiv 0(p)$. For (t_i, d_i) satisfying the above condition we have

$$p = t_1 d_1 + t_2 d_2 + \cdots + t_m d_m \geqq d_2 + \cdots + d_m$$

and $p = d_2 + \cdots + d_m$ if and only if $d_2 = p$ and $d_3 = \cdots = d_m = 0$ and $t_2 = 1$. Then by a simple calculation we can see that every (t_i, d_i) satisfies $(d_1 = m-p,\ d_2 = p,\ d_3 = \cdots = d_m = 0)$ or $(d_1 > m-p,\ d_2 < p, \cdots, d_m < p)$. For an integer t, we define $\nu_p(t)$ by $t = \prod_{p \in \{\text{all primes}\}} p^{\nu_p(t)}$. Then apparently we have

$$\nu_p(m!/(p!)(m-p)!) = \nu_p(m) - 1 \qquad \text{if} \quad m \equiv 0(p).$$

$$\nu_p(m!/(d_1!) \cdots (d_m!)) = \nu_p(m) \qquad \text{if} \quad d_1 > m-p \text{ and } d_i < p \text{ for } i > 1.$$

If $d_1 = m-p$ and $d_2 = p$ (hence $t_2 = 1$), the corresponding partition w is $(0, \cdots, 0, \underbrace{1, \cdots, 1}_{p})$. Then α_w is a map from $D_p(\overset{m}{\vee} S^k)$ to S^{pk}. Let π; $\overset{m}{\vee} S^k \to \overset{p}{\vee} S^k$ be the map shrinking the first $m-p$ spheres. Then we have the following commutative diagram

$$\begin{array}{ccccc} & & D_p(S^k) & & \\ & \swarrow^{D_p(\phi)} & & \searrow^{D_p(\phi')} & \\ D_p(\overset{m}{\vee} S^k) & & \xrightarrow{D_p(\pi)} & & D_p(\overset{p}{\vee} S^k) \\ & \searrow_{\alpha_w} & & \swarrow_{\alpha_{w'}} & \\ & & S^{pk} & & \end{array}$$

where ϕ, ϕ' are the respective comultiplication, $w = (0, \cdots, 0, 1, \cdots, 1)$ and $w' = (1, \cdots, 1)$. Then we denote by h_p the composition $\alpha_w D_p(\phi) \sim \alpha_{w'} D_p(\phi')$; $D_p(S^k) \to S^{pk}$. Let j; $S^{pk} \to D_p(S^k)$ be the inclusion. Then summing up the preceding argument, we have

THEOREM 5.6. *Let $m=p^r a$, $r\geqq 1$ and $a\not\equiv 0(p)$. Then in the stable category, there exists a map g; $D_p(S^k)\rightarrow D_p(S^k)$ such that* $D_p(m\iota_k)\sim p^r g+\binom{m}{p}jh_p$.

REMARK 5.7. If $p=2$, the theorem can be written precisely as $D(m\iota_k)\sim m\iota_{D_2(S^k)}+\binom{m}{2}jh_2$.

§ 6. Some properties of h_p.

Let f; $X\rightarrow Y$ be a map. $C_f=Y\cup_f CX$ denotes the mapping cone of f. Consider the map $D_p(f)$; $D_p(X)\rightarrow D_p(Y)$. Then we define a map

$$e;\quad D_p(Y)\cup_{D_p(f)}C(D_p(X))\longrightarrow D_p(Y\cup_f CX)$$

by

$$e(w;\ x_1\wedge\cdots\wedge x_p),\ t)=(w;\ (x_1,\ t)\wedge\cdots\wedge(x_p,\ t)).$$

Apparently $e\,|\,D_p(Y)$; $D_p(Y)\rightarrow D_p(Y\cup_f CX)$ is the natural inclusion. We assume that the chain map $f_\#$; $C_*(X;\boldsymbol{Z}_p)\rightarrow C_*(Y;\boldsymbol{Z}_p)$ is trivial, then so is $D_p(f)$. Then we have canonical splittings

$$H_{q+1}(C_f;\boldsymbol{Z}_p)\cong H_{q+1}(Y;\boldsymbol{Z}_p)+H_q(X;\boldsymbol{Z}_p)$$

$$H_{q+1}(C_{D_p(f)};\boldsymbol{Z}_p)\cong H_{q+1}(D_p(Y);\boldsymbol{Z}_p)+H_q(D_p(X);\boldsymbol{Z}_p).$$

Denote by $\hat{x}\in H_{q+1}(C_f;\boldsymbol{Z}_p)$ the corresponding element to $x\in H_q(X;\boldsymbol{Z}_p)$ and similarly for $C_{D_p(f)}$.

Note that the natural map $D_\pi(X)\rightarrow D_p(X)$ induces an epimorphism of $H_*(\ ;\boldsymbol{Z}_p)$ for the cyclic subgroup of order p. Therefore any element of $H_*(D_p(X);\boldsymbol{Z}_p)$ is written as $e_i\otimes x^p$ or $e_0\otimes(x_1\otimes\cdots\otimes x_p)$ for x, $x_i\in H_*(X;\boldsymbol{Z}_p)$. Then by Theorem 2 of [**19**] immediately we have

PROPOSITION 6.1. $e_*(\widehat{e_i\otimes x^p})=\lambda e_{i-p+1}\otimes\hat{x}^p$ *for* $\lambda\not\equiv 0(p)$ *if* $i-p+1\geqq 0$, *and* $=0$ *if* $i-p+1<0$.

Now from Theorem 5.6 it follows that the degree of the map $S^{pk}\subset D_p(S^k)\xrightarrow{h_p}S^{pk}$ is 0 mod p. Then we have

THEOREM 6.2. *Let $k=p^t a$, $a\not\equiv 0(p)$. Assume that a is even if p odd. Then the functional $\beta\mathcal{P}^i$-(Sq^i- if $p=2$) operation of h_p; $D_p(S^k)\rightarrow S^{pk}$ is non trivial for $1\leqq i<p^t$ (non trivial for $2\leqq i<2^t$ if $p=2$).*

PROOF. We may prove the theorem for a large suspension of h_p so that h_p is in the stable range. Then by Theorem 5.6 we have

$$D_p(p\iota_k)\sim pg+jh_p;\qquad D_p(S^k)\longrightarrow D_p(S^k).$$

Let $M_p^{k+1}=S^k\cup_p e^{k+1}$ be a Moore space. Let $x\in H_k(M_p^{k+1};\boldsymbol{Z}_p)$ and $y\in H_{k+1}(M_p^{k+1};\boldsymbol{Z}_p)$ denote the generators such that $\varDelta y=x$. Also we denote by

x the corresponding generator of $H_k(S^k; \boldsymbol{Z}_p)$. By Proposition 2.4, we can easily see that a $\boldsymbol{Z}_p$-basis of $H_*(D_p(M_p^{k+1}); \boldsymbol{Z}_p)$ is given by

$$\text{(i)}\quad e_0\otimes x^p,\ \ e_0\otimes(x^{p-1}y),\ \ e_{2i(p-1)-1}\otimes x^p,\ \ e_{2i(p-1)}\otimes x^p,$$
$$e_{(2i-1)(p-1)-1}\otimes y^p \ \text{ and } \ e_{(2i-1)(p-1)}\otimes y^p,\quad i=1,2,\cdots.$$

Then by Theorem 3.3 and by the naturality of the operation for the inclusion $D_\pi(M_p^{k+1})\to D_p(M_p^{k+1})$ and by $k=ap^t$, we have

$$\text{(ii)}\quad Sq_*^i(e_i\otimes x^2)=\binom{k}{i}e_0\otimes x^2=0 \qquad \text{for}\quad i<2^t$$

$$Sq_*^i(e_{i-2}\otimes y^2)=\binom{k-1}{i-2}e_0\otimes x^2\neq 0 \qquad \text{for}\quad i<2^t+2,$$

and

$$\text{(iii)}\quad (\beta\mathcal{P}^i)_*(e_{(2i-1)(p-1)}\otimes y^p)=\binom{\frac{(p-1)k}{2}}{i-1}e_0\otimes x^p\neq 0 \qquad \text{for}\quad i<p^t+1.$$

Furthermore by the structure of $H_*(D_p(S^k); \boldsymbol{Z}_p)$, we can easily see that $\beta\mathcal{P}^i=0$ on $H^{pk}(D_p(S^k); \boldsymbol{Z}_p)$. Thus if $i<p^t$, the functional $\beta\mathcal{P}^i$- (Sq^i- if $p=2$) operation of h_p is non trivial if and only if so is for jh_p; $D_p(S^k)\to D_p(S^k)$.

Next consider $C_{D_p(p\iota_k)}$ the mapping cone of $D_p(p\iota_k)$; $D_p(S^k)\to D_p(S^k)$. Apparently $(p\iota_k)_\#=0$, $C_*(S^k; \boldsymbol{Z}_p)\to C_*(S^k; \boldsymbol{Z}_p)$. Then by Proposition 6.1 and by (ii) and (iii) above, we have

$$\text{(iv)}\quad (Sq^i)_*(\widehat{e_{i-1}\otimes x^2})\neq 0 \qquad \text{for}\quad 2\leqq i<2^t+2$$

$$(\beta\mathcal{P}^i)_*(\widehat{e_{2i(p-1)}\otimes x^p})\neq 0 \qquad \text{for}\quad 1\leqq i<p^t+1$$

in $H_*(C_{D_p(p\iota_k)}; \boldsymbol{Z}_p)$. This implies that the functional $\beta\mathcal{P}^i$- (Sq^i- if $p=2$) operation of $D_p(p\iota_k)$ is non trivial for $1\leqq i<p^t$ ($2\leqq i<2^t$ if $p=2$). Finally consider the map $p\iota_{D_p(S^k)}$; $D_p(S^k)\to D_p(S^k)$. The cofibre of $p\iota_{D_p(S^k)}$ may be considered as $D_p(S^k)\wedge(S^N\cup_p e^{N+1})$ in the stable range. Then $\beta\mathcal{P}_*^i(Sq_*^i)$ is easily computed by (ii) and (iii) and by the Cartan formula. i.e., $\beta\mathcal{P}^iH^{pk+N}(D_p(S^k)\wedge(S^N\cup_p e^{N+1}); \boldsymbol{Z}_p)=0$ for $i>0$ and $Sq^iH^{2k+N}=0$ for $2\leqq i<2^t$. Thus the functional operations of $p\iota_{D_p(S^k)}$ are trivial.

Therefore if the functional operation of h_p for i in that range is trivial, then so is for $D_p(p\iota_k)$ by the additivity of functional operation, but this is a contradiction. This completes the proof.

REMARK. M. Barratt pointed out that the cofibre of the map h_2 is homotopy equivalent to $\Sigma^2D_2(S^{k-1})$. Furthermore we can see that the cofibration $D_2(S^k)\xrightarrow{h_2}S^{2k}\to Ch_2$ is equivalent to the usual cofibration of stunted projective spaces $\Sigma^kRP_k^\infty\to S^{2k}\to\Sigma^{k+1}RP_{k-1}^\infty$. Then if $p=2$, the theorem can be proved directly from this fact.

For odd prime p, the composition map $D_\pi(S^k)\to D_p(S^k)\xrightarrow{h_p} S^{pk}$ is also denoted by h_p. Then we have

COROLLARY 6.3. *Theorem 6.2 holds for the map* $h_p\,;\,D_\pi(S^k)\to S^{pk}$.

THEOREM 6.4. $D_2^{(r)}(S^k)$ *is stably homotopy equivalent to* RP^r_+, *the one point union of the real projective space, if and only if* $k\equiv 0(2^{\phi(r)})$, *where* $\phi(r)=\#\{i\,;\,0<i\leqq r,\ i\equiv 0,1,2$ *or* 4 *mod* $8\}$ *denotes the Adams number.*

PROOF. It is known in [1] that the J-group $\tilde{J}(RP^r)$ is isomorphic to $\boldsymbol{Z}_{2^{\phi(r)}}$ generated by $\xi-1$, ξ is the canonical line bundle over RP^r. Note that $D_2^{(r)}(S^k)$ is the Thom complex of the bundle $k\xi+k$. Then by the result of [3], the theorem is proved.

Now let $L_p^r=S^{2r+1}/\pi$ denote a mod p lens space of dimension $2r+1$. We may consider $L_p^r=B\pi^{(2r+1)}$.

THEOREM 6.5. *Let p be an odd prime and k even. Then* $D_\pi^{(2r+1)}(S^k)$ *is stably homotopy equivalent to* L^r_{p+} *if and only if* $k\equiv 0(p^{[\frac{r}{p-1}]})$.

PROOF. As is easily seen, if k is even $D_\pi(S^k)$ is the Thom complex of the complex vector boundle over $B\pi$ induced from $\frac{k}{2}\cdot\gamma$, where $\gamma\,;\,B\pi\to BU(p)$ denotes the complex regular representation. Let ξ be the canonical line bundle over CP^∞, and let $\pi\,;\,B\pi\to CP^\infty=BS^1$ be the canonical projection. $\pi^*\xi$ denotes the induced bundle and $(\pi^*\xi)^i$ the i-th tensor product. Then apparently we have that $D_\pi^{(2r+1)}(S^k)$ is the Thom complex of the vector bundle

$$(k/2)(1+\pi^*\xi+(\pi^*\xi)^2+\cdots+(\pi^*\xi)^{p-1})$$

over $B\pi^{(2r+1)}=L_p^r$.

Consider the J-homomorphism $\tilde{J}\,;\,\tilde{K}(L_p^r)\to\tilde{J}(L_p^r)$. Then it is shown in [9] that

$$\begin{aligned}\tilde{J}(L_p^r)&\cong \boldsymbol{Z}_p^{[\frac{r}{p-1}]} && \text{if}\quad r\not\equiv 0(4)\\ &\cong \boldsymbol{Z}_p^{[\frac{r}{p-1}]}+\boldsymbol{Z}_2 && \text{if}\quad r\equiv 0(4)\end{aligned}$$

and the cyclic group $\boldsymbol{Z}_p^{[\frac{r}{p-1}]}$ is generated by $\tilde{J}(r(\sigma))$. Here $\sigma=\pi^*\xi-1$ and $r\,;\,K(X)\to KO(X)$ denotes the realization homomorphism. It is also known that $\tilde{J}(r(\sigma))=-\tilde{J}(r(\sigma^2))=\cdots=-\tilde{J}(r(\sigma^{p-1}))$. Now

$$\begin{aligned}1+\pi^*\xi+\cdots+(\pi^*\xi)^{p-1}&=1+(\sigma+1)+\cdots+(\sigma+1)^{p-1}\\ &=((\sigma+1)^p-1)/\sigma\\ &=p+\binom{p}{2}\sigma+\cdots+\binom{p}{p-1}\sigma^{p-2}+\sigma^{p-1}.\end{aligned}$$

Therefore we have

$$\tilde{J}(r(1+\cdots+(\pi^*\xi)^{p-1}))=\left(\binom{p}{2}-\binom{p}{3}+\cdots+1\right)J(r(\sigma))$$
$$=(p-1)\tilde{J}(r(\sigma)).$$

Then by the result of [3], we have that $D_\pi^{(2r+1)}(S^k)$ is stably homotopy equivalent to L_{p+}^r if and only if $k/2\equiv 0(p^{\left[\frac{r}{p-1}\right]})$, i. e. $k\equiv 0(p^{\left[\frac{r}{p-1}\right]})$. This completes the proof.

§ 7. The adjoint map of h_p and the theorem of Kahn-Priddy.

Let $Q(S^0)=\varinjlim \Omega^n S^n$ be the infinite loop space of sphere and $Q_0(S^0)$ the component of the trivial map of $Q(S^0)$. In [6], Dyer and Lashof have shown that for any prime p, there exists a map $\alpha_\pi; B\pi\rightarrow Q_0(S^0)$ such that the adjoint map

$$\mathrm{Ad}(\alpha_\pi);\ \Sigma B\pi \longrightarrow Q(S^1)$$

satisfies $\mathrm{Ad}(\alpha_\pi)_*(\sigma(e_{i+p-1}))=Q_i(s_1)$. Here π is a cyclic group of order p, σ denotes the suspension isomorphism and $s_1\in H_1(S^1; \boldsymbol{Z}_p)$, $e_{i+p-1}\in H_{i+p-1}(B\pi; \boldsymbol{Z}_p)$ are generators. $\mathrm{Ad}(\alpha_\pi)$ can be factored through $\tilde{Q}(S^1)$, the universal covering space of $Q(S^1)$. Then by use of the spectral sequence of the path fibre space $Q_0(S^0)\rightarrow P\rightarrow\tilde{Q}(S^1)$, they have shown that

(7.1) $H_*(Q_0(S^0); \boldsymbol{Z}_p)$ is a free commutative algebra generated by $\alpha_{\pi*}(e_{2i(p-1)})$ and $\alpha_{\pi*}(e_{2i(p-1)-1})$ ($\alpha_{\pi*}(e_i)$ if $p=2$) for all i, as an algebra over the algebra of the Dyer-Lashof operations.

In [5], Barratt-Kahn-Priddy have constructed a map

$$w;\ BS_\infty \longrightarrow Q_0(S^0)$$

such that $w_*; H_*(BS_\infty; \boldsymbol{Z}_p)\rightarrow H_*(Q_0(S^0); \boldsymbol{Z}_p)$ is an isomorphism of algebras over the Dyer-Lashof algebra, where $BS_\infty=\lim BS_n$ denotes the classifying space of the infinite symmetric group and the Dyer-Lashof operation is defined by the wreath product $WS_n\underset{S_n}{\times}(BS_m)^n\rightarrow BS_{nm}$. Let $\gamma_p; B\pi\rightarrow Q_0(S^0)$ be the map obtained by restricting w on $B\pi\subset BS_p$. By the definition of w, $\gamma_{p*}(e_i)=Q_i(J)*\underbrace{(-J)*\cdots*(-J)}_{p\ \text{times}}$, where $J\in H_0(S^0; \boldsymbol{Z}_p)$ is the class represented by the point different from the base point and $*$ denotes the loop product (c. f. [12]). Therefore we may consider $\alpha_\pi=\gamma_p$.

Now we introduce a filtration in $H_*(Q_0(S^0); \boldsymbol{Z}_p)$ defined by Tsuchiya [21]. For a basis element $x=\prod_{k=1}^{m} Q^{I_k}(\gamma_{p*}(e_{i_k}))$, the height of x is defined by $h(x)=\sum_{k=1}^{m} p^{l(I_k)}$, where $l(I_k)$ denotes the length of I_k. Then the filtration $G_0=$

$H_*(Q_0(S^0); \boldsymbol{Z}_p) \supset G_1 \supset G_2 \supset \cdots$, is defined by putting $G_t =$ subspace spanned by all basis of height $\geqq t$. Apparently $G_1 = \tilde{H}_*(Q_0(S^0); \boldsymbol{Z}_p)$, $G_s \cdot G_t \subset G_{s+t}$ and G_1/G_2 is spanned by $\gamma_{p*}(e_{2i(p-1)-1})$ and $\gamma_{p*}(e_{2i(p-1)})$ (by $\gamma_{2*}(e_i)$ if $p=2$).

For a connected space X (in paticular $X = S^1$), $H_*(Q(X); \boldsymbol{Z}_p)$ has the similar filtration using the height defined in § 2. This filtration is denoted by F_t. Then we have,

LEMMA 7.2. *Let* $f; \sum B\pi^{(r)} \to Q(S^1)$ *be a map satisfying* $f_*(\sigma(e_{i+p-1})) \equiv Q_i(s_1)$ mod $F_{p+1}H_*(Q(S^1); \boldsymbol{Z}_p)$ *for* $i+p-1 \leqq r$. *Then for the adjoint map* $\mathrm{Ad}(f); B\pi^{(r)} \to Q_0(S^0)$, *we have* $\mathrm{Ad}(f)_*(e_{i+p-1}) \equiv \gamma_{p*}(e_{i+p-1})$ mod G_2.

PROOF. Let $\tilde{Q}(S^1)$ be the universal covering space of $Q(S^1)$. It is shown [6] that $H_*(\tilde{Q}(S^1); \boldsymbol{Z}_p)$ is isomorphic to the subalgebra of $H_*(Q(S^1); \boldsymbol{Z}_p)$ generated by $Q^I(s_1)$, where $s_1 \in H_1(S^1; \boldsymbol{Z}_p)$ is a generator and I admissible, $e(I) > 1$ and $I \neq \emptyset$ for p odd. For $p=2$, it is easy to see that $H_*(\tilde{Q}(S^1); \boldsymbol{Z}_2)$ is isomorphic to the subalgebra generated by all $Q^I(s_1)$ and s_1^2 where I admissible, $e(I) > 1$ and $I \neq \emptyset$. Let $\tilde{f}; \sum B\pi^{(r)} \to \tilde{Q}(S^1)$ be a lift of f. Then $f_*(\sigma(e_{i+p-1})) \equiv Q_i(s_1)$ mod F_{p+1} implies that in $H_*(Q(S^1); \boldsymbol{Z}_p)$, $f_*(\sigma(e_{i+p-1})) - Q_i(s_1)$ is decomposable of $Q_j(s_1)$, $j \geqq 0$, or of the form $Q_jQ_k(x)$ for some x. Therefore easily we have that $\tau(\tilde{f}_*(\sigma(e_{i+p-1}))) \equiv \tau(Q_i(s_1))$ mod G_2 where τ is the transgression. Hence $\mathrm{Ad}(f)_*(e_{i+p-1}) \equiv \gamma_{p*}(e_{i+p-1})$ mod G_2.

THEOREM 7.3. *Let* $h; \Sigma^\infty RP^r \to S^\infty$ *be a map such that the functional* Sq^n-*operation is non-trivial for* $r+1 \geqq n \geqq 2$. *Then the adjoint map* $\mathrm{Ad}(h); RP^r \to Q_0(S^0)$ *satisfies that* $\mathrm{Ad}(h)_*(e_{n-1}) \equiv \gamma_{2*}(e_{n-1})$ mod G_2.

PROOF. First we assume that $n=2$. Let $i; S^1 \to RP^r$ be the inclusion. Then if the functional Sq^2-operation of h is non-trivial, the map $\Sigma^\infty(S^1) \to \Sigma^\infty RP^r \to S^\infty$ is essential. Hence $\mathrm{Ad}(h); RP^r \to Q_0(S^0)$ satisfies that $\mathrm{Ad}(h)_*; \pi_1(RP) \to \pi_1(Q_0(S^0)) \cong \boldsymbol{Z}_2$ is an isomorphism and so is for $H_1(\ ; \boldsymbol{Z}_2)$.

Now we assume $n > 2$. We restrict $h; \Sigma^\infty RP^r \to S^\infty$ on $\Sigma^\infty RP^{n-1}$. Then as is well-known, h is the suspension of a map

$$h'; \quad \Sigma^n RP^{n-1} \longrightarrow S^n .$$

Let $\Sigma^n RP^{n-1} \to S^n \to C \to \Sigma^{n+1} RP^{n-1}$ be the induced cofibre sequence. Then taking the functor Q, we have a sequence of fibering

$$Q\Sigma^n RP^{n-1} \longrightarrow QS^n \longrightarrow QC \longrightarrow Q\Sigma^{n+1} RP^{n-1} .$$

Let $s_n \in H_n(S^n; \boldsymbol{Z}_2)$ and $\sigma^{n+1}(e_i) \in H_{n+1+i}(\Sigma^{n+1} RP^{n-1}; \boldsymbol{Z}_2)$ be generators. Note that $H_*(C; \boldsymbol{Z}_2) \cong H_*(S^n; \boldsymbol{Z}_2) + H_*(\Sigma^{n+1} RP^{n-1}; \boldsymbol{Z}_2)$. Therefore $H_*(QC; \boldsymbol{Z}_2)$ has the following $\boldsymbol{Z}_2$-basis in dim. $\leqq 2n$; $s_n, \sigma^{n+1}(e_1), \cdots, \sigma^{n+1}(e_{n-1}), s_n^2$.

Consider the cohomology spectral sequence of the path fibration $\Omega QC \to P \to QC$. Let $u \in H^n(QC; \boldsymbol{Z}_2)$ be a generator and $\sigma(u) \in H^{n-1}(\Omega QC; \boldsymbol{Z}_2)$ the suspension of u. Then by assumption, $Sq^n u = u^2 \neq 0$. Hence $d_n(u \otimes \sigma(u)) = u^2 \neq 0$. Then we can see easily that the $\boldsymbol{Z}_2$-basis of $H_*(\Omega QC; \boldsymbol{Z}_2)$ in dim. $\leqq$

$2n-1$ is given by $\{\tau(s_n), \tau(\sigma^{n+1}(e_1)), \cdots, \tau(\sigma^{n+1}(e_{n-2})), \tau(s_n^2)\}$, where τ denotes the transgression.

Next consider the homology spectral sequence of $\Omega Q\Sigma^n RP^{n-1} \to \Omega QS^n \to \Omega QC$. $\Omega Q\Sigma^n RP^{n-1} = Q\Sigma^{n-1}RP^{n-1}$, $\Omega QS^n = QS^{n-1}$ and ΩQC is simply connected, for $n \geqq 3$. Then the spectral sequence turns out to be an exact sequence

$$H_{2n-2}(Q\Sigma^{n-1}RP^{n-1}; \boldsymbol{Z}_2) \longrightarrow H_{2n-2}(QS^{n-1}; \boldsymbol{Z}_2) \longrightarrow H_{2n-2}(\Omega QC; \boldsymbol{Z}_2) \xrightarrow{\partial} H_{2n-3}(Q\Sigma^{n-1}RP^{n-1}; \boldsymbol{Z}_2).$$

Then by the structure of $H_*(\Omega QC; \boldsymbol{Z}_2)$ given above, it is easily seen that $\partial(\tau(\sigma^{n+1}(e_{n-2}))) \neq 0$ and the map $\mathrm{Ad}(h') = \Omega Q(h') \circ j; \Sigma^{n-1}RP^{n-1} \to Q(\Sigma^{n-1}RP^{n-1}) \to Q(S^{n-1})$ satisfies that $\mathrm{Ad}(h')_*(\sigma^{n-1}(e_{n-1})) = Q^{n-1}(s_{n-1})$. Then by taking further adjoint and by the result of [6] and by the fact that $H_*(Q(S^1); \boldsymbol{Z}_p)/F_{p+1}$ is spanned by s_1 and $Q_i(s_1)$, $i \geqq 0$, we can see that $\mathrm{Ad}(h'); \Sigma RP^{n-1} \to Q(S^1)$ satisfies $\mathrm{Ad}(h')_*(e_{n-1}) \equiv Q^{n-1}(s_i) \bmod F_{p+1}$. Hence so is $\mathrm{Ad}(h)$ and the theorem follows from Lemma 7.2. Q.E.D.

We shall prove the similar result for odd prime. In order to do that, we prepare the following lemma for odd prime p.

LEMMA 7.4. *The suspension homomorphism $\Sigma^\infty; [\Sigma^{2m-1}B\pi^{(2n(p-1))}, S^{2m-1}] \to [\Sigma^\infty B\pi^{(2n(p-1))}, S^\infty]$ of the homotopy sets is an epimorphism if $m > n$. Here $B\pi^{(2n(p-1))}$ denotes the $2n(p-1)$ skeleton of $B\pi$.*

PROOF. Consider ther natural inclusion $i; S^{2m-1} \to \Omega^2 S^{2m+1}$. We may assume that i is a fibre map and Q_2^{2m-1} denotes the fibre. In [**20**], it is shown that the p-primary component $\pi_i(Q_2^{2m-1}; p) = 0$ for $i < 2mp-3$. For any complex K, we have the suspension exact sequence

$$[K, Q_2^{2m-1}] \longrightarrow [K, S^{2m-1}] \xrightarrow{\Sigma^2} [K, \Omega^2 S^{2m+1}].$$

Then if $K = \Sigma^3 L$ for some L, we can see that $\Sigma^2; [K, S^{2m-1}] \otimes Q_p \to [\Sigma^2 K, S^{2m+1}] \otimes Q_p$ is an epimorphism if $\dim K \leqq 2mp-3$. Here Q_p denotes the ring of fractions whose denominator is prime to p. Note that $H_*(B\pi^{(2n(p-1))}$ consists of elements of order p. Therefore

$$\Sigma^2; [\Sigma^{2m-1}B\pi^{(2n(p-1))}, S^{2m-1}] \longrightarrow [\Sigma^{2m+1}B\pi^{(2n(p-1))}, S^{2n+1}]$$

is an epimorphism if $2m-1+2n(p-1) \leqq 2mp-3$, i.e., $m \geqq n+1$. This proves the lemma.

THEOREM 7.5. *Let $h; \Sigma^\infty B\pi^{(r)} \to S^\infty$ be a map such that the functional $\beta\mathcal{P}^n$-operation of h is non-trivial for $2n(p-1) \leqq r$. Then the adjoint map $\mathrm{Ad}(h); B\pi^{(r)} \to Q_0(S^0)$ satisfies that $\mathrm{Ad}(h)_*(e_{2n(p-1)}) \equiv \gamma_{p^*}(e_{2n(p-1)}) \bmod G_2$.*

PROOF. Consider the restriction of $h; \Sigma^\infty B\pi^{(2n(p-1))} \to S^\infty$. Then by Lemma 7.4, h is the suspension of a map $h'; \Sigma^{2n+1}B\pi^{(2n(p-1))} \to S^{2n+1}$. Consider the sequence of cofibrations

$$\Sigma^{2n+1}B\pi^{(2n(p-1))} \longrightarrow S^{2n+1} \longrightarrow C \longrightarrow \Sigma^{2n+2}B\pi^{(2n(p-1))}$$

where C denotes the mapping cone of h'. Then as in the proof of Theorem 7.3, we have a sequence of fibrations

$$Q\Sigma^{2n+1}B\pi^{(2n(p-1))} \longrightarrow QS^{2n+1} \longrightarrow QC \longrightarrow Q\Sigma^{2n+2}B\pi^{(2n(p-1))}.$$

Let $s_{2n+1} \in H_{2n+1}(S^{2n+1}; \boldsymbol{Z}_p)$, $\sigma^{2n+2}(e_i) \in H_{2n+2+i}(\Sigma^{2n+2}B\pi^{(2n(p-1))}; \boldsymbol{Z}_p)$ be generators of appropriate groups, and also denote the corresponding elements in $H_*(QS^{2n+1}; \boldsymbol{Z}_p)$ and $H_*(Q\Sigma^{2n+2}B\pi^{(2n(p-1))}; \boldsymbol{Z}_p)$. Since $\tilde{H}_*(C; \boldsymbol{Z}_p) \cong \tilde{H}_*(S^{2n+1}; \boldsymbol{Z}_p) + \tilde{H}_*(\Sigma^{2n+2}B\pi^{(2n(p-1))}; \boldsymbol{Z}_p)$, we can easily see that in dim.$\leqq 2np+2$, $H_*(QC; \boldsymbol{Z}_p)$ is a free commutative algebra generated by $H_*(C; \boldsymbol{Z}_p)$ and the degree of any monomial is less than p.

We shall consider the cohomology spectral sequence of the path fibre space $\Omega QC \to P \to QC$. Let $\bar{s}_{2n+1} \in H^{2n+1}(QC; \boldsymbol{Z}_p)$ be the dual of s_{2n+1} and $\sigma(\bar{s}_{2n+1}) \in H^{2n}(\Omega QC; \boldsymbol{Z}_p)$ denote the suspension of $\bar{s}_{2n+1}$. By assumption, $\beta\mathcal{P}^n\bar{s}_{2n+1} = \overline{\sigma^{2n+2}(e_{2n(p-1)})}$ in $H^*(C; \boldsymbol{Z}_p)$. Hence so is in $H^*(QC; \boldsymbol{Z}_p)$. Then by Kudo's transgression theorem [**10**], we have

$$d_{2n(p-1)+1}(\bar{s}_{2n+1} \otimes \sigma(\bar{s}_{2n+1})^{p-1}) = \beta\mathcal{P}^n\bar{s}_{2n+1} \neq 0.$$

Then by simple calculation, we can deduce that in dim.$\leqq 2np+1$, $H_*(\Omega QC; \boldsymbol{Z}_p)$ is free commutative algebra generated by

$$\tau(s_{2n+1}),\ \tau(\sigma^{2n+2}(e_1)), \cdots, \tau(\sigma^{2n+2}(e_{2n(p-1)-2})),$$

where τ denotes the transgression. In particular, we have that all elements of $H_{2np+1}(\Omega QC; \boldsymbol{Z}_p)$ are decomposable.

Next consider the fibration

$$\Omega Q(\Sigma^{2n+1}B\pi^{(2n(p-1))}) \longrightarrow \Omega QS^{2n+1} \longrightarrow \Omega QC$$

and the associated homology spectral sequence. Note that $\Omega Q(\Sigma^{2n+1}B\pi^{(2n(p-1))}) = Q(\Sigma^{2n}B\pi^{(2n(p-1))})$ and $\Omega QS^{2n+1} = QS^{2n}$. In dim.$\leqq 2np+1$, $H_*(Q\Sigma^{2n}B\pi^{(2n(p-1))}; \boldsymbol{Z}_p)$ is a free commutative algebra such that every monomial has degree $< p$. Consider the element $\sigma^{2n}(e_{2n(p-1)}) \in H_{2np}(Q(\Sigma^{2n}B\pi^{(2n(p-1))}); \boldsymbol{Z}_p)$, which is indecomposable. Then by the structure of $H_*(\Omega QC; \boldsymbol{Z}_p)$, we have $\sigma^{2n}(e_{2n(p-1)}) \neq 0$ in E^∞ of the spectral sequence. This implies immediately that the map

$$\mathrm{Ad}(h);\ \Sigma^{2n}B\pi^{(r)} \longrightarrow Q(\Sigma^{2n}B\pi^{(r)}) \longrightarrow Q(S^{2n})$$

satisfies $\mathrm{Ad}(h)_*(\sigma^{2n}(e_{2n(p-1)})) = Q^n(s_{2n}) \in H_{2np}(QS^{2n}; \boldsymbol{Z}_p)$. Then taking further adjoint, we can easily see as in Theorem 7.3, that

$$\mathrm{Ad}(h);\ \Sigma B\pi^{(r)} \longrightarrow Q(S^1)$$

satisfies $\mathrm{Ad}(h)_*(\sigma^1(e_{2n(p-1)}) \equiv Q^n(s_1) \bmod F_{p+1}$. Hence the theorem follows from Lemma 7.2.

We shall now state the theorem of Kahn-Priddy [8].

THEOREM 7.6 (Kahn-Priddy). *Let π be a cyclic group of order p (p may be 2). Let h; $\Sigma^\infty B\pi^{(r)}\to S^\infty$ be a map such that* $\mathrm{Ad}(h)_*(e_{2i(p-1)})\equiv\gamma_{p*}(e_{2i(p-1)})$ $\mathrm{mod}\, G_2$ *for* $2i(p-1)<r$ ($\mathrm{Ad}(h)_*(e_1)=\gamma_{2*}(e_1)$ *if* $p=2$). *Then the homomorphism of p-primary components*

$$h_*;\ \pi_i^s(B\pi^{(r)};p)\longrightarrow\pi_i^s(S^0;p)$$

is an epimorphism for $0<i<r$.

REMARK 7.7. In [8], the theorem is proved for a map h; $\Sigma^\infty B\pi^{(\infty)}\to S^\infty$. But the argument is easily seen valid when restricted on appropriate skeletons.

COROLLARY 7.8. *Let h; $\Sigma^\infty B\pi^{(r)}\to S^\infty$ be a map such that the functional $\beta\mathcal{P}^i$-operation is non-trivial for $2i(p-1)<r$ (the functional Sq^2-operation is non-trivial if $p=2$). Then h_*; $\pi_i^s(B\pi^{(r)};p)\to\pi_i^s(S^0;p)$ is an epimorphism for $0<i<r$.*

§ 8. The nilpotency of elements of $\pi_*^s(S^0)$.

In the sequel, we denote by α; $S^k\to S^0$ a stable map α; $S^{k+N}\to S^N$ for large N. By Proposition 1.6, for any r, s and p, there exists a number N and a retraction R; $D_p^{(r)}(S^{sN})\to S^{psN}$. Therefore we can use the notation R; $D_p^{(r)}(S^0)\to S^0$ for any r and p. Let $\phi(r)$ be the Adams number. Then we have

THEOREM 8.1. *Let α; $S^k\to S^0$ be an element of order 2^m and k even. Given an integer n, let r be the maximal integer such that $nk\equiv 0(2^{\phi(r)})$. Then for any $\beta\in\pi_i^s(S^0;2)$ for $0<i<r$, we have $2^{m-1}(\alpha^{2n}\beta+2\gamma)=0$ for some $\gamma\in\pi_*^s(S^0;2)$.*

PROOF. Let h_2; $D_2(S^{nk})\to S^{2nk}$ be the map defined in § 5. By assumption and by Theorem 6.4, $D_2^{(r)}(S^{nk})=\Sigma^{2nk}(RP_+^{(r)})\cong S^{2nk}\vee\Sigma^{2nk}RP^{(r)}$. Since k is even, the functional Sq^2 operation of h_2 is non-trivial by Theorem 6.2. Then by Corollary 7.6, we have

$$h_{2*};\ \pi_{i+2nk}(D_2^{(r)}(S^{nk});2)\longrightarrow\pi_{i+2nk}(S^{2nk};2)$$

is an epimorphism for $0<i<r$. So given $\beta\in\pi_i^s(S^0;2)$, we may choose $\tilde\beta\in\pi_{i+2nk}(D_2^{(r)}(S^{nk});2)$ so that $h_{2*}(\tilde\beta)=\beta$.

Consider the product α^n. Since α is of order 2^m, so is α^n. Therefore the following composition

$$D_2^{(r)}(S^{nk})\xrightarrow{D_2^{(r)}(2^m)}D_2^{(r)}(S^{nk})\xrightarrow{D_2^{(r)}(\alpha^n)}D_2^{(r)}(S^0)\xrightarrow{R}S^0$$

is null-homotopic. By Theorem 5.6 we have

$$2^mRD_2^{(r)}(\alpha^n)+\binom{2^m}{2}\alpha^nh_2\sim 0;\ D_2^{(r)}(S^{nk})\longrightarrow S^0.$$

Then by composing $\tilde{\beta}$, we have

$$2^m RD_2^{(r)}(\alpha^n)\tilde{\beta}+2^{m-1}(2^m-1)\alpha^{2n}\beta\sim 0\,.$$

Hence by putting $RD_2^{(r)}(\alpha^n)\tilde{\beta}=\gamma$, the proof is completed.

COROLLARY 8.2. *Any element of* $\pi_i^s(S^0;2)$, $i>0$, *is nilpotent.*

PROOF. It is sufficient to prove for elements of even stem. Let $\alpha;S^k\to S^0$ be of order 2^m and k even. We take n such that $nk\equiv 0(2^{\phi(k+1)})$. Then in Theorem 8.1, we can take α itself as β. Thus we have $2^{m-1}(\alpha^{2n+1}+2\gamma)=0$ for some γ. Since $2^m\alpha=0$, by composing α we have $2^{m-1}\alpha^{2n+2}=0$. Then iterating this argument, we have $\alpha^t=0$ for some t.

REMARK. Unfortunately the exponent t given in this corollary is much bigger than that given in Corollary 4.2. For an element of order 2, the estimation of t is approximately $2^{\left[\frac{k+3}{2}\right]}$, whereas that of Corollary 4.2 is $2k+3$.

Now we shall state the corresponding results for odd component. The arguments are similar and the proof will be omitted.

THEOREM 8.3. *Let k be even and let $\alpha;S^k\to S^0$ be of order p^m. Given an integer n, let r be the maximal integer such that $nk\equiv 0(p^{\left[\frac{r}{p-1}\right]})$. Then for any i, $0<i<2r$ and for any $\beta\in\pi_i^s(S^0;p)$, there exists $\gamma\in\pi_*^s(S^0;p)$ such that $p^{m-1}(\alpha^n\circ\beta+p\gamma)=0$.*

COROLLARY 8.4. *Any element of* $\pi_i^s(S^0;p)$, $i>0$, *is nilpotent.*

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