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Cohomologies over commutative Hopf algebras

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with **Appendix**

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In [2], Sweedler has investigated a cohomology theory for module algebras over a given cocommutative Hopf algebra.

The purpose of this paper is to discuss some dual theories of [2]. In section 2, we give the definitions of cohomology groups for comodules, comodule coalgebras, comodule Hopf algebras and comodule algebras over a given commutative Hopf algebra. A familiar example of commutative Hopf algebras is the coordinate ring of an affine algebraic group. Section 3 deals with relations between these cohomology groups. Sections 4 and 5 contain the extension theory of coalgebras and Hopf algebras, which is the precisely dual statements of [2]. In section 6, we compute the cohomology groups for a special comodule algebra. Finally, section 7 gives a result on the conjugacy of the coradical splittings of commutative Hopf algebras over a field of characteristic 0.

§ 1. Preliminaries.

All vector spaces are over the ground field k. Our notation and terminology are essentially those used in [3]. One difference; if C is a coalgebra and $\phi: V\to C\otimes V$ is the structure map of a (left) C-comodule V, we sometimes write $\phi(v)=\sum v_{(C)}\otimes v_{(V)}$ for all $v\in V$.

1.1. DEFINITIONS. Let H be a Hopf algebra. The unit map $u_H: k\to H \cong H\otimes k$ gives k the structure of a left H-comodule. An algebra D which is a left H-comodule is called a left *H-comodule algebra* if $M_D: D\otimes D\to D$ and $u_D: k\to D$ are H-comodule maps. ($D\otimes D$ has the natural H-comodule structure.)

A coalgebra B which is a left H-comodule is called a left *H-comodule coalgebra* if $\Delta_B: B\to B\otimes B$ and $\varepsilon_B: B\to k$ are H-comodule maps.

A Hopf algebra L which is a left H-comodule is called a left *H-comodule Hopf algebra* (or *H-Hopf action* on L) if M_H, u_H, Δ_H and ε_H are H-comodule maps.

(1.1.1) If D is an algebra and $\psi: D\rightarrow H\otimes D$ is the structure map of a (left) H-comodule, then the followings are equivalent:

a) M_D and u_D are H-comodule maps.

b) ψ is an algebra map.

(1.1.2) If B is a coalgebra and $\psi: B\rightarrow H\otimes B$ is the structure map of a (left) H-comodule, then the followings are equivalent:

a) Δ_B and ε_B are H-comodule maps.

b) $(1\otimes\Delta_B)\psi=(M_H\otimes 1\otimes 1)(1\otimes T\otimes 1)(\psi\otimes\psi)\Delta_B$,
$(1\otimes\varepsilon_B)\psi=(u_H\otimes 1)\varepsilon_B$. (Where $T(x\otimes y)=y\otimes x$.)

c) For $b\in B$,
$\sum b_{(H)}\otimes b_{(B)(1)}\otimes b_{(B)(2)}=\sum b_{(1)(H)}b_{(2)(H)}\otimes b_{(1)(B)}\otimes b_{(2)(B)}$,
$\sum\varepsilon(b_{(B)})b_{(H)}=\varepsilon(b)1_H$.

1.2. EXAMPLES.

(1.2.1) Let H^A be the underlying algebra of H and let H^A have the left H-comodule structure induced by comultiplication. Thus H^A is a left H-comodule algebra.

(1.2.2) For any positive integer n, let $\otimes^n H$ denote $H\otimes\cdots\otimes H$ n-times. $\otimes^n H$ has the algebra structure on the tensor product of algebras and has a left H-comodule structure where $\psi(h_1\otimes\cdots\otimes h_n)\in H\otimes(\otimes^n H)$ is defined to be $\Delta_H(h_1)\otimes\cdots\otimes h_n$. Thus $\otimes^n H$ is a left H-comodule algebra. If we let $\otimes^0 H$ denote k, then $\otimes^0 H$ is a left H-comodule Hopf algebra. (k has the trivial Hopf algebra structure.)

(1.2.3) Let G be an affine algebraic group defined over k and let X be an affine variety defined over k. Let $H=k[G]$ and $A=k[X]$ be the coordinate rings of G and X respectively. Then it is well known that H has the Hopf algebra structure induced by the group structure on G. To give an action of G on X as a variety is equivalent to giving a H-comodule algebra structure on A.

(1.2.4) Let V be a left H-comodule with the structure map $\psi: V\rightarrow H\otimes V$. Then the left H-comodule coalgebra attached to V, denoted by $B(V)$, is defined as follows:

$B(V)=k\oplus V$ as a vector space and a coalgebra structure is defined by

$$\Delta:\ B(V)\ni\lambda+v\longmapsto\lambda\otimes 1+v\otimes 1+1\otimes v\in B(V)\otimes B(V),$$
$$\varepsilon:\ B(V)\ni\lambda+v\longmapsto\lambda\in k.$$

The left H-comodule structure map $\tilde{\psi}: B(V)\rightarrow H\otimes B(V)$ is defined by

$$\tilde{\psi}(\lambda+v)=\lambda 1_H\otimes 1+\psi(v)\qquad(\lambda\in k,\ v\in V).$$

One easily checks that $B(V)$ is a left H-comodule coalgebra.

(1.2.5) Let H^C be the underlying coalgebra of H and let H^C have the left

H-comodule structure induced by $\phi(h)=\sum h_{(1)}S(h_{(3)})\otimes h_{(2)}\in H\otimes H^C$, where S is the antipode of H.

$$\begin{aligned}(M_H\otimes 1\otimes 1)(1\otimes T\otimes 1)(\phi\otimes\phi)\Delta_H(h)\\ &=\sum h_{(1)}S(h_{(3)})h_{(4)}S(h_{(6)})\otimes h_{(2)}\otimes h_{(5)}\\ &=\sum h_{(1)}(h_{(3)})S(h_{(5)})\otimes h_{(2)}\otimes h_{(4)}\\ &=\sum h_{(1)}S(h_{(4)})\otimes h_{(2)}\otimes h_{(3)}\\ &=(1\otimes\Delta_H)\phi(h)\,.\end{aligned}$$

Thus H^C is a left H-comodule coalgebra. If H is commutative as an algebra then the antipode S is an algebra map and hence so ϕ is. Therefore $H=H^A=H^C$ is also a left H-comodule Hopf algebra.

(1.2.6) Let us now be in the situation (1.2.3), where X is an affine algebraic group. Then to give an action of G on X as an algebraic group is equivalent to giving a H-comodule Hopf algebra structure on A.

1.3. Convolution algebras. Let D be a left H-comodule algebra and let B be a left H-comodule coalgebra. Hom(B, D) has the following algebra structure. For $f, g\in$ Hom(B, D) the product $f*g$ is $M_D(f\otimes g)\Delta_B$. The unit of Hom(B, D) is $u_D\varepsilon_B$. This product is called convolution. If D is a commutative algebra and B is a cocommutative coalgebra then it is clear that Hom(B, D) is a commutative algebra. Hom$_H(B, D)$ denotes the H-comodule maps from B to D. Reg(B, D) denotes the multiplicative group of invertible elements of Hom(B, D) and Reg$_H(B, D)$ denotes Hom$_H(B, D)\cap$ Reg(B, D).

(1.3.1) Hom$_H(B, D)$ is a subalgebra of Hom(B, D).
Reg$_H(B, D)$ is a subgroup of Reg(B, D).

Proof. For $f, g\in$ Hom$_H(B, D)$ we show that $f*g\in$ Hom$_H(B, D)$.

$$\begin{aligned}\phi_D(f*g)&=\phi_D M_D(f\otimes g)\Delta_B\\ &=(M_H\otimes M_D)(1\otimes T\otimes 1)(\phi_D\otimes\phi_D)(f\otimes g)\Delta_B\quad(\phi_D\text{ is an algebra map})\\ &=(M_H\otimes M_D)(1\otimes T\otimes 1)(1\otimes f\otimes 1\otimes g)(\phi_B\otimes\phi_B)\Delta_B\quad(f, g\in\mathrm{Hom}_H(B, D))\\ &=(M_H\otimes M_D)(1\otimes 1\otimes f\otimes g)(1\otimes T\otimes 1)(\phi_B\otimes\phi_B)\Delta_B\\ &=(1\otimes M_D)(1\otimes f\otimes g)(M_H\otimes 1\otimes 1)(1\otimes T\otimes 1)(\phi_B\otimes\phi_B)\Delta_B\\ &=(1\otimes M_D)(1\otimes f\otimes g)(1\otimes\Delta_B)\phi_B\quad(\text{since }(1.1.2))\\ &=(1\otimes f*g)\phi_B\,.\end{aligned}$$

Q. E. D.

Let L be a left H-comodule Hopf algebra. Alg(L, D) denotes the algebra maps from L to D and Alg$_H(L, D)$ denotes Alg$(L, D)\cap$ Hom$_H(L, D)$.

(1.3.2) If D is a commutative algebra, then

a) Alg(L, D) is a subgroup of Reg(L, D).

b) $\mathrm{Alg}_H(L, D)$ is a subgroup of $\mathrm{Alg}(L, D)$.

PROOF. a) For $f \in \mathrm{Alg}(L, D)$ the inverse of f with respect to the convolution product is fS_L. Since D is commutative fS_L is an algebra map. b) For $f \in \mathrm{Alg}_H(L, D)$ we show that fS is a H-comodule map.

$$\begin{aligned} \phi fS &= (1\otimes f)\phi S \qquad \text{(since } f \text{ is } H\text{-comodule map)} \\ &= (1\otimes f)(1\otimes S)\phi \qquad \text{(see [1], (4.4) Lemma)} \\ &= (1\otimes fS)\phi\,. \end{aligned}$$

Q. E. D.

1.4. Cotensor products. If W is a right H-comodule and V is a left H-comodule the *cotensor product* of W and V is the space $W\square_H V$ such that the sequence

$$0 \longrightarrow W\square_H V \longrightarrow W\otimes V \xrightarrow{\phi_W\otimes 1-1\otimes\phi_V} W\otimes H\otimes V$$

is an exact sequence of k-spaces. If A is a right H-comodule algebra and D is a left H-comodule algebra then $A\square_H D$ is a subalgebra of $A\otimes D$. Note that $V\mapsto W\square_H V$ is a covariant functor from left H-comodules to k-spaces. In fact if $f\colon V\to V'$ is a H-comodule map then the corresponding map $W\square_H V \to W\square_H V'$ is given by the following diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & W\square_H V & \longrightarrow & W\otimes V & \xrightarrow{\phi_W\otimes 1-1\otimes\phi_V} & W\otimes H\otimes V \\ & \downarrow & & \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle 1\otimes 1\otimes f} \\ 0 \longrightarrow & W\square_H V' & \longrightarrow & W\otimes V' & \xrightarrow{\phi_W\otimes 1-1\otimes\phi_{V'}} & W\otimes H\otimes V'\,. \end{array}$$

§ 2. Definition of cohomologies.

We assume from now on that our Hopf algebra H is commutative. Let C be the category whose objects are commutative left H-comodule algebras and morphisms are H-comodule algebra maps which are by definition H-comodule maps as well as algebra maps. Let $\mathcal{A}$ be the category of abelian groups.

2.1. EXAMPLES. We consider some examples of covariant functors from C to $\mathcal{A}$.

(2.1.1) Let V be any left H-comodule. We have the functor F from C to $\mathcal{A}$; if $D\in C$ then $F(D)=\mathrm{Hom}_H(V, D)$, and if $D\xrightarrow{\alpha} D'$ then $F(\alpha)\colon \mathrm{Hom}_H(V, D) \to \mathrm{Hom}_H(V, D')$ is given by the rule $F(\alpha)(x)=\alpha x$.

(2.1.2) Let B be any cocommutative left H-comodule Hopf algebra. We have the functor F; if $D\in C$ then $F(D)=\mathrm{Reg}_H(B, D)$, and $D\xrightarrow{\alpha} D'$ in C then $F(\alpha)(x)=\alpha x$.

(2.1.3) Let L be any cocommutative left H-comodule Hopf algebra. We have the functor F; if $D \in \mathcal{C}$ then $F(D) = \mathrm{Alg}_H(L, D)$.

(2.1.4) Let W be any right H-comodule. We have the functor F; if $D \in \mathcal{C}$ then $F(D) = W \square_H D$.

(2.1.5) Let A be any commutative right H-comodule algebra. We have the functor F; if $D \in \mathcal{C}$ then $F(D) = U(A \square_H D)$, the multiplicative group of the invertible elements of $A \square_H D$.

2.2. We form a semi-cosimplicial complex ([**2**]) in $\mathcal{C}$, whose objects are $\{\otimes^{n+1} H\}_{n \geqq 0}$ of Example (1.2.2). The object of n-degree is $\otimes^{n+1} H$ for $n = 0, 1, 2, \cdots$. The coface operators are given by $\partial_i : \otimes^n H \to \otimes^{n+1} H$, $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \otimes \cdots \otimes \Delta(x_{i+1}) \otimes \cdots \otimes x_n$ for $i = 0, 1, \cdots, n-1$ and $\partial_n : \otimes^n H \to \otimes^{n+1} H$, $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \otimes \cdots \otimes x_n \otimes 1$. The codegeneracy operators are given by $s_i : \otimes^{n+2} H \to \otimes^{n+1} H$, $x_0 \otimes x_1 \otimes \cdots \otimes x_{n+1} \mapsto x_0 \otimes \cdots \otimes x_i \otimes \varepsilon(x_{i+1}) x_{i+2} \otimes \cdots \otimes x_{n+1}$ for $i = 0, 1, \cdots, n$. One easily checks all the coface and codegeneracy operators identities.

2.3. Let $F : \mathcal{C} \to \mathcal{A}$ be any covariant functor. We apply this functor F to the above semi-cosimplicial complex to obtain a semi-cosimplicial complex $\{F(\otimes^{n+1} H)\}_{n \geqq 0}$ in $\mathcal{A}$. The homology of $\{F(\otimes^{n+1} H)\}_{n \geqq 0}$ is defined by means of the differential $d^{n-1} : F(\otimes^n H) \to F(\otimes^{n+1} H)$ where $d^{n-1} = \sum_{i=0}^{n} (-1)^i F(\partial_i)$. Thus we have

$$F(\otimes^1 H) \xrightarrow{d^0} F(\otimes^2 H) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} F(\otimes^{n+1} H) \xrightarrow{d^n} \cdots .$$

The cohomology of F over H is defined to be the homology of the above complex and the n-th group $H^n(F, H)$ is $\mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$ for $n > 0$ and $\mathrm{Ker}\, d^0$ for $n = 0$.

Com-$H^n(V, H)$, *Coalg-*$H^n(B, H)$, *Hopf-*$H^n(L, H)$, *Hoch-*$H^n(W, H)$ and *Alg-*$H^n(A, H)$ denote the n-th cohomology group of F as in Examples (2.1.1), (2.1.2), (2.1.3), (2.1.4) and (2.1.5) respectively.

2.4. There is a normal subcomplex of our complex $\{F(\otimes^{n+1} H), d^n\}_{n \geqq 0}$. For $n > 0$ let $N^{n+1} = \bigcap_{i=0}^{n} \mathrm{Ker}(F(s_i))$, where $F(s_i) : F(\otimes^{n+2} H) \to F(\otimes^{n+1} H)$. For $n = 0$ let $N^0 = F(\otimes^1 H)$. Then $\{N^n, d^n | N^n\}_{n \geqq 0}$ is a subcomplex of $\{F(\otimes^{n+1} H), d^n\}_{n \geqq 0}$. The injection map induces an isomorphism of homology. This is the dual result in [**5**, Theorem 6.1].

2.5.

(2.5.1) PROPOSITION. *Let V be a left H-comodule. Then the map $\Phi : \mathrm{Hom}_H(V, \otimes^n H) \to \mathrm{Hom}(V, \otimes^{n-1} H)$ defined by $\Phi(f) = (\varepsilon_H \otimes 1) f$ is a linear isomorphism. The inverse map $\Psi : \mathrm{Hom}(V, \otimes^{n-1} H) \to \mathrm{Hom}_H(V, \otimes^n H)$ is given by $\Psi(g) = (1 \otimes g)\phi$.*

PROOF. It is clear.

(2.5.2) COROLLARY. *Let B be a cocommutative left H-comodule coalgebra and let L be a cocommutative left H-comodule Hopf algebra. Then the above isomorphism induces the following group isomorphisms;*

$$\mathrm{Reg}_H(B, \otimes^n H) \cong \mathrm{Reg}(B, \otimes^{n-1} H)$$

$$\mathrm{Alg}_H(L, \otimes^n H) \cong \mathrm{Alg}(L, \otimes^{n-1} H).$$

(2.5.3) PROPOSITION. *Let W be a right H-comodule. Then the map $\Phi: W \square_H (\otimes^n H) \to W \otimes (\otimes^{n-1} H)$ defined by $\Phi(w \otimes h_1 \otimes \cdots \otimes h_n) = \varepsilon(h_1) w \otimes h_2 \otimes \cdots \otimes h_n$ is a linear isomorphism. The inverse map Ψ is given by $\Psi(w \otimes h_1 \otimes \cdots \otimes h_{n-1}) = \phi(w) \otimes h_1 \otimes \cdots \otimes h_{n-1}$.*

(2.5.4) COROLLARY. *Let A be a commutative right H-comodule algebra. Then the above isomorphism induces the following group isomorphism;*

$$U(A \square_H (\otimes^n H)) \cong U(A \otimes (\otimes^{n-1} H)).$$

2.6. *We present the standard complex to compute Com-$H^n(V, H)$, Coalg-$H^n(B, H)$, etc., by means of* (2.5).

(2.6.1) $\{\mathrm{Hom}(V, \otimes^n H), D^n\}_{n \geqq 0}$.

The differential $D^{n-1}: \mathrm{Hom}(V, \otimes^{n-1} H) \to \mathrm{Hom}(V, \otimes^n H)$ is defined by

$$D^{n-1}(f) = (1 \otimes f)\phi - (\Delta \otimes 1 \cdots) f + (1 \otimes \Delta \otimes 1 \cdots) f$$
$$- \cdots \pm (1 \otimes \cdots 1 \otimes \Delta) f \mp f \otimes u_H.$$

Then the complex $\{\mathrm{Hom}(V, \otimes^n H), D^n\}_{n \geqq 0}$ is isomorphic to the complex $\{\mathrm{Hom}_H(V, \otimes^{n+1} H), d^n\}_{n \geqq 0}$ which defines the cohomology *Com-$H^n(V, H)$*.

(2.6.2) $\{\mathrm{Reg}(B, \otimes^n H), D^n\}_{n \geqq 0}$.

The differential $D^{n-1}: \mathrm{Reg}(B, \otimes^{n-1} H) \to \mathrm{Reg}(B, \otimes^n H)$ is defined by

$$D^{n-1}(f) = [(1 \otimes f)\phi] * [(\Delta \otimes 1 \cdots) f^{-1}] * [(1 \otimes \Delta \otimes 1 \cdots) f]$$
$$* \cdots * [(1 \otimes \cdots 1 \otimes \Delta) f^{\pm 1}] * [f^{\mp 1} \otimes u_H],$$

where $\mathrm{Reg}(B, \otimes^{n-1} H) \ni f^{-1}$ is the $*$-inverse of f. Then the complex $\{\mathrm{Reg}(B, \otimes^n H), D^n\}_{n \geqq 0}$ is isomorphic to the complex $\{\mathrm{Reg}_H(B, \otimes^{n+1} H), d^n\}_{n \geqq 0}$ which defines the cohomology *Coalg-$H^n(B, H)$*.

(2.6.3) $\{\mathrm{Alg}(L, \otimes^n H), D^n\}_{n \geqq 0}$.

The differential $D^{n-1}: \mathrm{Alg}(L, \otimes^{n-1} H) \to \mathrm{Alg}(L, \otimes^n H)$ is defined by the restriction of (2.6.2). Then the complex $\{\mathrm{Alg}(L, \otimes^n H), D^n\}_{n \geqq 0}$ is isomorphic to the complex $\{\mathrm{Alg}_H(L, \otimes^{n+1} H), d^n\}_{n \geqq 0}$ which defines the cohomology *Hopf-$H^n(L, H)$*.

(2.6.4) $\{W \otimes (\otimes^n H), D^n\}_{n \geqq 0}$.

The differential $D^{n-1}: W \otimes (\otimes^{n-1} H) \to W \otimes (\otimes^n H)$ is defined by

$$D^{n-1}(w \otimes h_1 \otimes \cdots \otimes h_{n-1})$$
$$= \phi(w) \otimes h_1 \otimes \cdots \otimes h_{n-1} - w \otimes \Delta(h_1) \otimes h_2 \otimes \cdots \otimes h_{n-1}$$
$$+ w \otimes h_1 \otimes \Delta(h_2) \otimes h_3 \otimes \cdots \otimes h_{n-1}$$
$$- \cdots \pm w \otimes h_1 \otimes \cdots \otimes h_{n-2} \otimes \Delta(h_{n-1}) \mp w \otimes h_1 \otimes \cdots \otimes h_{n-1} \otimes 1.$$

Then the complex $\{W\otimes(\otimes^n H), D^n\}_{n\geqq 0}$ is isomorphic to the complex $\{W\square_H(\otimes^{n+1}H), d^n\}_{n\geqq 0}$ which defines the cohomology *Hoch*-$H^n(W, H)$. Note that this cohomology equals the Hochschild cohomology [6, p. 191].

(2.6.5) $\{U(A\otimes(\otimes^n H)), D^n\}_{n\geqq 0}$.

The differential $D^{n-1}: U(A\otimes(\otimes^{n-1}H))\rightarrow U(A\otimes(\otimes^n H))$ is defind by

$$\begin{aligned} D^{n-1}&(a\otimes h_1\otimes\cdots\otimes h_{n-1}) \\ &=(\psi(a)\otimes h_1\otimes\cdots\otimes h_{n-1})(a\otimes\Delta(h_1)\otimes h_2\otimes\cdots)^{-1}(a\otimes h_1\otimes\Delta(h_2)\otimes\cdots) \\ &\qquad\cdots(a\otimes h_1\otimes\cdots\otimes h_{n-2}\otimes\Delta(h_{n-1}))^{\pm 1}(a\otimes h_1\otimes\cdots\otimes h_{n-1}\otimes 1)^{\mp 1}. \end{aligned}$$

Then the complex $\{U(A\otimes(\otimes^n H)), D^n\}_{n\geqq 0}$ is isomorphic to the complex $\{U(A\square_H(\otimes^{n+1}H)), d^n\}_{n\geqq 0}$ which defines the cohomology *Alg*-$H^n(A, H)$.

2.7. We can find a normal subcomplex of our standard complex which is isomorphic to $\{N^n, d^n|N^n\}_{n\geqq 0}$. In fact we define (for $n>0$);

$$\begin{aligned} &\mathrm{Hom}_+(V, \otimes^n H) \\ &\quad=\{f\in\mathrm{Hom}(V, \otimes^n H)\mid(\varepsilon\otimes 1\otimes\cdots)f=(1\otimes\varepsilon\otimes\cdots)f=\cdots=0\} \\ &\mathrm{Reg}_+(B, \otimes^n H) \\ &\quad=\{f\in\mathrm{Reg}(B, \otimes^n H)\mid(\varepsilon\otimes 1\otimes\cdots)f=(1\otimes\varepsilon\otimes\cdots)f=\cdots=u\varepsilon\} \\ &\mathrm{Alg}_+(L, \otimes^n H) \\ &\quad=\{f\in\mathrm{Alg}(L, \otimes^n H)\mid(\varepsilon\otimes 1\otimes\cdots)f=(1\otimes\varepsilon\otimes\cdots)f=\cdots=u\varepsilon\} \\ &W\otimes_+(\otimes^n H)=\{\textstyle\sum w\otimes h_1\otimes\cdots\otimes h_n\in W\otimes(\otimes^n H)\mid \\ &\qquad\qquad\textstyle\sum w\otimes\varepsilon(h_1)h_2\otimes\cdots\otimes h_n=w\otimes h_1\otimes\varepsilon(h_2)h_3\otimes\cdots=\cdots=0\} \\ &U_+(A\otimes(\otimes^n H))=\{\textstyle\sum a\otimes h_1\otimes\cdots h_n\in U(A\otimes(\otimes^n H))\mid \\ &\qquad\qquad\textstyle\sum a\otimes\varepsilon(h_1)h_2\otimes\cdots h_n=\sum a\otimes h_1\otimes\varepsilon(h_2)h_3\otimes\cdots\otimes h_n \\ &\qquad\qquad\qquad=\cdots=1\} \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Hom}_+(V, \otimes^0 H)=\mathrm{Hom}(V, \otimes^0 H)=\mathrm{Hom}(V, k)=V^* \\ &\mathrm{Reg}_+(B, \otimes^0 H)=\mathrm{Reg}(B, \otimes^0 H)=\mathrm{Reg}(B, k)=U(B^*) \\ &\mathrm{Alg}_+(L, \otimes^0 H)=\mathrm{Alg}(L, \otimes^0 H)=\mathrm{Alg}(L, k)=G(L^0) \\ &W\otimes_+(\otimes^0 H)=W\otimes(\otimes^0 H)=W\otimes k\cong W \\ &U_+(A\otimes(\otimes^0 H))=U(A\otimes(\otimes^0 H))\cong U(A). \end{aligned}$$

Then $\{\mathrm{Hom}_+(V, \otimes^n H), D^n|\mathrm{Hom}_+(V, \otimes^n H)\}_{n\geqq 0}$ is a normal subcomplex and the inclusion map induces an isomorphism of homology. $\{\mathrm{Reg}_+(B, \otimes^n H)\}_{n\geqq 0}$,

$\{\mathrm{Alg}_+(L\otimes^n H)\}_{n\geqq 0}$, etc., similar.

2.8. $H^0(\ ,H)$ and $H^1(\ ,H)$.

(2.8.1) $Com\text{-}H^0(V, H)=\{\tau\in V^*\mid(1\otimes\tau)\phi(v)=\tau(v)1$ for all $v\in V\}$.

If $f\colon V\to H$ is a linear map then f is a 1-cocycle if and only if

$$\Delta f(v)=(1\otimes f)\phi(v)+f(v)\otimes 1 \qquad \text{for all} \quad v\in V.$$

We denote by V^H the set $\{v\in V\mid\phi(v)=1\otimes v\}$. In case $V=V^H$ this reduces to $\Delta f(v)=1\otimes f(v)+f(v)\otimes 1$ so that $f(v)\in H$ is a primitive element of H. 1-coboundary is one of the form $D^0(\tau)$ for $\tau\in V^*$. For $v\in V$, $D^0(\tau)(v)=(1\otimes\tau)\phi(v)-\tau(v)1$.

(2.8.2) $Coalg\text{-}H^0(B, H)=\{\tau\in U(B^*)\mid(1\otimes\tau)\phi(b)=\tau(b)1$ for all $b\in B\}$.

If $f\in\mathrm{Reg}(B, H)$ then f is a 1-cocycle if and only if

$$\sum f(b)_{(1)}\otimes f(b)_{(2)}=\sum b_{(1)(H)}f(b_{(2)})\otimes f(b_{(1)(B)}) \qquad \text{for all} \quad b\in B.$$

In case $B=B^H$ this reduces to $\sum f(b)_{(1)}\otimes f(b)_{(2)}=\sum f(b_{(2)})\otimes f(b_{(1)})$ so that f is a coalgebra map if $f\in\mathrm{Reg}_+(B, H)$, since B is cocommutative. 1-coboundary is one of the form $D^0(\tau)$ for $\tau\in U(B^*)$. For $b\in B$, $D^0(\tau)(b)=\sum b_{(1)(H)}\tau(b_{(1)(B)})\cdot\tau^{-1}(b_{(2)})$.

(2.8.3) $Hopf\text{-}H^0(L, H)=\{\tau\in\mathrm{Alg}(L, k)\mid(1\otimes\tau)\phi(l)=\tau(l)1$ for all $l\in L\}$.

If $f\in\mathrm{Alg}(L, H)$ then f is a 1-cocycle if and only if

$$\sum f(l)_{(1)}\otimes f(l)_{(2)}=\sum l_{(1)(H)}f(l_{(2)})\otimes f(l_{(1)(L)}) \qquad \text{for all} \quad l\in L.$$

In case $L=L^H$ this reduces that f is a Hopf algebra map if $f\in\mathrm{Alg}_+(L, H)$.

(2.8.4) $Hoch\text{-}H^0(W, H)=\{w\in W\mid\phi(w)=w\otimes 1\}=W^H$.

If $\sum w\otimes h\in W\otimes H$ is a 1-cocycle then $\sum w\otimes\Delta(h)=\sum\phi(w)\otimes h+\sum w\otimes h\otimes 1$. 1-coboundary is one of the form $\phi(w)-w\otimes 1$ for $w\in W$.

(2.8.5) $Alg\text{-}H^0(A, H)=\{a\in U(A)\mid\phi(a)=a\otimes 1\}=U(A)\cap A^H$.

If $\sum a\otimes h\in U(A\otimes H)$ is a 1-cocycle then $\sum a\otimes\Delta(h)=(\sum\phi(a)\otimes h)(\sum a\otimes h\otimes 1)$. 1-coboundary is one of the form $\phi(a)(a^{-1}\otimes 1)$ for $a\in U(A)$.

2.9. Let $F, F'\colon \mathcal{C}\to\mathcal{A}$ be covariant functors and let $\eta\colon F\to F'$ be a natural transformation from F to F'. Then η induces a morphism of complexes $\tilde\eta$ from $\{F(\otimes^{n+1}H)\}_{n\geqq 0}$ to $\{F'(\otimes^{n+1}H)\}_{n\geqq 0}$. Suppose η is a pointwise monomorphism, that is, $\eta_D\colon F(D)\to F'(D)$ is a monomorphism for all $D\in\mathcal{C}$. Then there is an exact sequence of complexes:

$$0\longrightarrow\{F(\otimes^{n+1}H)\}_{n\geqq 0}\xrightarrow{\tilde\eta}\{F'(\otimes^{n+1}H)\}_{n\geqq 0}\longrightarrow\mathrm{Coker}\,\tilde\eta\longrightarrow 0.$$

This gives rise to the long exact cohomology sequence:

$$0\longrightarrow H^0(F, H)\longrightarrow H^0(F', H)\longrightarrow H^0(\mathrm{Coker}\,\tilde\eta)\longrightarrow H^1(F, H)\longrightarrow\cdots$$

$$\cdots\longrightarrow H^n(F', H)\longrightarrow H^n(\mathrm{Coker}\,\tilde\eta)\longrightarrow H^{n+1}(F, H)\longrightarrow H^{n+1}(F', H)\longrightarrow\cdots.$$

We can also consider the situation where $\eta\colon F\to F'$ is a pointwise epimorphism.

§ 3. Comparisons.

3.1. Let V be a finite dimensional vector space. There is a natural linear isomorphism $\gamma: V^*\otimes H\to\mathrm{Hom}(V, H)$, γ is given by $\gamma(\xi\otimes h)(v)=\xi(v)h$. The inverse map $\sigma:\mathrm{Hom}(V, H)\to V^*\otimes H$ is given by $\sigma(f)=\sum_{i=1}^n\xi_i\otimes f(v_i)$, where $\{v_i\}$ is a base of V and $\{\xi_i\}$ its dual base.

Suppose $\phi: V\to H\otimes V$ gives a left H-comodule structure (that is, $(\varepsilon\otimes 1)\phi=id_V$, $(1\otimes\phi)\phi=(\Delta\otimes 1)\phi$). Define $\rho: V^*\to V^*\otimes H$ by $\rho(\xi)=\sigma((1\otimes\xi)\phi)$.

(3.1.1) LEMMA. *(V^*, ρ) is a right H-comodule.*

PROOF. For any j,

$$\begin{aligned}(1\otimes\varepsilon)\rho(\xi_j)&=(1\otimes\varepsilon)(\textstyle\sum_{i=1}^n\xi_i\otimes(1\otimes\xi_j)\phi(v_i))\\&=\textstyle\sum_{i=1}^n\xi_i\otimes\xi_j(\varepsilon\otimes 1)\phi(v_i)\\&=\textstyle\sum_{i=1}^n\xi_i\otimes\xi_j(v_i)\\&=\xi_j\,.\end{aligned}$$

Hence we have $(1\otimes\varepsilon)\rho=id_V$. Next we show that $(\rho\otimes 1)\rho=(1\otimes\Delta)\rho$. If we denote $\phi(v_i)$ by $\sum_{k=1}^n h_{ik}\otimes v_k$, then $\rho(\xi_i)=\sum_{k=1}^n\xi_k\otimes(1\otimes\xi_i)\phi(v_k)=\sum_{k=1}^n\xi_k\otimes h_{ki}$.

$$\begin{aligned}(1\otimes 1\otimes\xi_j)(1\otimes\phi)\phi(v_k)&=(1\otimes 1\otimes\xi_j)(1\otimes\phi)(\textstyle\sum_{i=1}^n h_{ki}\otimes v_i)\\&=(1\otimes 1\otimes\xi_j)(\textstyle\sum_{i,l=1}^n h_{ki}\otimes h_{il}\otimes v_l)\\&=\textstyle\sum_{i=1}^n h_{ki}\otimes h_{ij}\,.\end{aligned}$$

Now $(\rho\otimes 1)\rho(\xi_j)=\sum_{i=1}^n(\rho\otimes 1)(\xi_i\otimes h_{ij})=\sum_{i,k=1}^n\xi_k\otimes h_{ki}\otimes h_{ij}$.

$$\begin{aligned}(1\otimes\Delta)\rho(\xi_j)&=(1\otimes\Delta)(\textstyle\sum_{k=1}^n\xi_k\otimes(1\otimes\xi_j)\phi(v_k))\\&=\textstyle\sum_{k=1}^n\xi_k\otimes(1\otimes 1\otimes\xi_j)(\Delta\otimes 1)\phi(v_k)\\&=\textstyle\sum_{k=1}^n\xi_k\otimes(1\otimes 1\otimes\xi_j)(1\otimes\phi)\phi(v_k)\\&=\textstyle\sum_{i,k=1}^n\xi_k\otimes h_{ki}\otimes h_{ij}\,.\end{aligned}$$

Hence we have $(\rho\otimes 1)\rho=(1\otimes\Delta)\rho$. Q.E.D.

(3.1.2) PROPOSITION. *Let V be a finite dimensional left H-comodule. Then Com-$H^n(V, H)$ and Hoch-$H^n(V^*, H)$ are canonically isomorphic for all n. The isomorphism is induced by a canonical isomorphism between the standard complex to compute Com-$H^n(V, H)$ and the standard complex to compute Hoch-$H^n(V^*, H)$.*

PROOF. One easily checks that the natural linear isomorphisms $\mathrm{Hom}(V, \otimes^n H)\to V^*\otimes(\otimes^n H)$ form a morphism of complexes.

3.2. Let V be a left H-comodule and let $S(V)$ be the symmetric algebra of V. $S(V)$ has a canonical Hopf algebra structure [**3**, Proposition 3.2.3]. Define $\phi: S(V)\to H\otimes S(V)$ by the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi_V} & H\otimes V \\ \downarrow i & & \downarrow 1\otimes i \\ S(V) & \xrightarrow{\phi} & H\otimes S(V) \end{array}$$

where $i: V\rightarrow S(V)$ is the natural injection.

(3.2.1) LEMMA. *$(S(V), \phi)$ is a left H-comodule Hopf algebra.*

PROOF. It is clear that $S(V)$ is a left H-comodule. Since ϕ is an algebra map $S(V)$ is a left H-comodule algebra. And since $S(V)$ is generated by V as an algebra, to show b) in (1.1.2), it suffices to show the following two equalities:

($'$) $(1\otimes\Delta)\phi(v)=(M\otimes 1\otimes 1)(1\otimes T\otimes 1)(\phi\otimes\phi)\Delta(v)$ for all $v\in V$.

($''$) $(1\otimes\varepsilon)\phi(v)=(u_H\otimes 1)\varepsilon(v)$ for all $v\in V$.

$$\begin{aligned}(M&\otimes 1\otimes 1)(1\otimes T\otimes 1)(\phi\otimes\phi)\Delta(v)\\ &=(M\otimes 1\otimes 1)(1\otimes T\otimes 1)(\phi\otimes\phi)(v\otimes 1+1\otimes v)\\ &=(M\otimes 1\otimes 1)(1\otimes T\otimes 1)(\textstyle\sum v_{(H)}\otimes v_{(V)}\otimes 1\otimes 1+\sum 1\otimes 1\otimes v_{(H)}\otimes v_{(V)})\\ &=\textstyle\sum v_{(H)}\otimes v_{(V)}\otimes 1+\sum v_{(H)}\otimes 1\otimes v_{(V)}\\ &=(1\otimes\Delta)\phi(v)\,.\end{aligned}$$

Hence the equation ($'$) holds. Since $\varepsilon(V)=0$, ($''$) is clear. Q. E. D.

(3.2.2) PROPOSITION. *Com-$H^n(V, H)$ and $Hopf$-$H^n(S(V), H)$ are canonically isomorphic for all n.*

PROOF. By the universal mapping property of $S(V)$, $\mathrm{Hom}(V, \otimes^n H)$ is in 1-1 correspondence with $\mathrm{Alg}(S(V), \otimes^n H)$. This map induces the isomorphism between the standard complex to compute Com-$H^n(V, H)$ and the standard complex to compute $Hopf$-$H^n(S(V), H)$.

3.3. Let V be a left H-comodule and let $B(V)$ be the left H-comodule coalgebra attached to V (see (1.2.3)). We consider $Coalg$-$H^n(B(V), H)$. Let D be any commutative algebra. There is a natural linear isomorphism φ from $\mathrm{Hom}(B(V), D)$ to $D\oplus\mathrm{Hom}(V, D)$, since $B(V)=k\oplus D$ as a space. $D\oplus\mathrm{Hom}(V, D)$ has an algebra structure induced by φ. Thus $(\lambda, f)(\mu, g)=(\lambda\mu, \lambda g+\mu f)$, where $\lambda, \mu\in D$ and $f, g\in\mathrm{Hom}(V, D)$. The unit of $D\oplus\mathrm{Hom}(V, D)$ is $(1, 0)$. Let (λ, f) be in $D\oplus\mathrm{Hom}(V, D)$. If λ is invertible in D then $(\lambda, f)(\lambda^{-1}, -\lambda^{-2}f)=(1, 0)$ and hence (λ, f) is invertible in $D\oplus\mathrm{Hom}(V, D)$. Conversely if (λ, f) is invertible then λ is invertible in D. Thus we have the following Lemma.

(3.3.1) LEMMA. *The map* $\mathrm{Reg}(B(V), \otimes^n H)\rightarrow U(\otimes^n H)\otimes\mathrm{Hom}(V, \otimes^n H)$, *$(\lambda, f)\mapsto(\lambda, \lambda^{-1}f)$ is a group isomorphism.*

(3.3.2) PROPOSITION. *$Coalg$-$H^n(B(V), H)\cong Coalg$-$H^n(k, H)\oplus Com$-$H^n(V, H)$.*

Proof. The natural projection $B(V)\to k$ induces a group monomorphism $\mathrm{Reg}(k, \otimes^n H)\to \mathrm{Reg}(B(V), \otimes^n H)$. By (3.3.1), its cokernel is $\mathrm{Hom}(V, \otimes^n H)$. Thus we have the short exact sequence of complexes;

$$0 \longrightarrow \{\mathrm{Reg}(k, \otimes^n H)\} \longrightarrow \{\mathrm{Reg}(B(V), \otimes^n H)\} \longrightarrow \{\mathrm{Hom}(V, \otimes^n H)\} \longrightarrow 0.$$

Moreover the exact sequence splits, the splitting map is induced by the natural inclusion $k \hookrightarrow B(V)$. Q. E. D.

3.4. Let B be a cocommutative left H-comodule coalgebra. The linear dual B^* has an algebra structure. Suppose B is finite dimensional. Then the linear isomorphism γ from $B^*\otimes H$ to the convolution algebra $\mathrm{Hom}(B, H)$ is an algebra isomorphism.

(3.4.1) Lemma. *B^* is a commutative right H-comodule algebra.*

Proof. We show that the composite $B^* \xrightarrow{\rho} B^*\otimes H \xrightarrow{\gamma} \mathrm{Hom}(B, H)$ is an algebra map. For $\xi_1, \xi_2 \in B^*$,

$$\begin{aligned}[(1\otimes\xi_1)\phi]*[(1\otimes\xi_2)\phi] &= M(1\otimes\xi_1\otimes 1\otimes\xi_2)(\phi\otimes\phi)\Delta \\ &= (M\otimes\xi_1\otimes\xi_2)(1\otimes T\otimes 1)(\phi\otimes\phi)\Delta \\ &= (1\otimes\xi_1\otimes\xi_2)(M\otimes 1\otimes 1)(1\otimes T\otimes 1)(\phi\otimes\phi)\Delta \\ &= (1\otimes\xi_1\otimes\xi_2)(1\otimes\Delta)\phi .\end{aligned}$$

Q. E. D.

(3.4.2) Proposition. *Suppose B is finite dimensional. Then Coalg-$H^n(B,H)$ and Alg-$H^n(B^*, H)$ are canonically isomorphic for all n. The isomorphism is induced by a canonical isomorphism between the standard complex to compute Coalg-$H^n(B, H)$ and the standard complex to compute Alg-$H^n(B^*, H)$.*

3.5. For a commutative algebra A the Amitsur cohomology group of A is denoted $H^n(A)$. Note that the Amitsur complex of A is the complex $\{U(\otimes^{n+1}A), E^n\}_{n\geq 0}$ and the differential $E^{n-1}: U(\otimes^n A)\to U(\otimes^{n+1}A)$ is defined by $E^{n-1}(x)=e_0(x)e_1(x)^{-1}\cdots e_n(x)^{\pm 1}$, where $e_i: \otimes^n A\to\otimes^{n+1}A$, $a_1\otimes\cdots\otimes a_n \mapsto a_1\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_n$.

Suppose A is a left H-comodule algebra. We have an algebra map $\Omega: \otimes^{n+1}A\to A\otimes(\otimes^n H)$. This is given by

$$\begin{aligned}\Omega(a_1\otimes\cdots\otimes a_{n+1}) = \sum a_1 a_{2(0)}a_{3(0)}\cdots a_{n+1(0)}\otimes a_{2(1)}a_{3(1)}\cdots a_{n+1(1)} \\ \otimes\cdots\otimes a_{n(n-1)}a_{n+1(n-1)}\otimes a_{n+1(n)},\end{aligned}$$

where we use the Sweedler's notation, for $a\in A$, $\phi(a)=\sum a_{(0)}\otimes a_{(1)}\in A\otimes H$, and we inductively define:

$$\sum a_{(0)}\otimes a_{(1)}\otimes\cdots\otimes a_{(n)} = (\phi\otimes 1\otimes\cdots)(\sum a_{(0)}\otimes a_{(1)}\otimes\cdots\otimes a_{(n-1)}).$$

Proposition. *Ω induces a morphism of complexes*

$$\tilde{\Omega}: \{U(\otimes^{n+1}A), E^n\}_{n\geq 0} \longrightarrow \{U(A\otimes(\otimes^n H)), D^n\}_{n\geq 0}.$$

Therefore there is a morphism from $H^n(A)$ to Alg-$H^n(A, H)$.

§ 4. Extensions and crossed products.

Let B be a cocommutative left H-comodule coalgebra and let L be a cocommutative left H-comodule Hopf algebra.

4.1. We say that a triple (C, f, ω) is a *coalgebra extention* of B by H if:

(1) C is a coalgebra

(2) $f: C \to B$ is a coalgebra map and surjective

(3) $\omega: C\otimes H \to C$ is a coalgebra map (we denote $\omega(c\otimes h) = c \leftharpoonup h$)

such that the followings hold:

(a) $C\otimes H \overset{\omega}{\underset{1\otimes\varepsilon}{\rightrightarrows}} C \overset{f}{\longrightarrow} B$ is exact (i. e., $C/\mathrm{Im}(\omega - 1\otimes\varepsilon) \cong B$ as a space)

(b) (C, ω) is a right H-module

(c) The following diagram is commutative:

$$\begin{array}{ccccc} C & \xrightarrow{\ \Delta\ } & C\otimes C & \xrightarrow{\ 1\otimes f\ } & C\otimes B \\ \downarrow{\scriptstyle \Delta} & & & & \uparrow \\ C\otimes C & & & & \omega\otimes 1 \\ \downarrow{\scriptstyle T} & & & & \\ C\otimes C & \xrightarrow{\ 1\otimes f\ } & C\otimes B & \xrightarrow{\ 1\otimes \phi\ } & C\otimes H\otimes B \end{array}$$

i. e., $\sum c_{(1)}\otimes f(c_{(2)}) = \sum c_{(2)} \leftharpoonup f(c_{(1)})_{(H)} \otimes f(c_{(1)})_{(B)}$ for all $c\in C$.

We say that a triple (C, f, ω) is a *Hopf extension* of L by H if:

(1) C is a Hopf algebra

(2) $f: C \to L$ is a Hopf algebra map and surjective

(3) $\omega: C\otimes H \to C$ is a Hopf algebra map

such that the above conditions (a), (b) and (c) hold.

A morphism of coalgebra extensions (of B by H) from (C, f, ω) to (C', f', ω') is a coalgebra map $\gamma: C \to C'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} C\otimes H & \xrightarrow{\ \omega\ } & C & \overset{f}{\searrow} & \\ \downarrow{\scriptstyle \gamma\otimes 1} & & \downarrow{\scriptstyle \gamma} & & B \\ C'\otimes H & \xrightarrow{\ \omega'\ } & C' & \underset{f'}{\nearrow} & \end{array}$$

(i. e., γ is an H-module map and $f = f'\gamma$).

A morphism of Hopf extensions (of L by H) from (C, f, ω) to (C', f', ω') is a Hopf algebra map $\gamma: C \to C'$ such that the above diagram is commutative.

4.2. The co-smash product.

We define the coalgebra $B \flat H$ to be $B\otimes H$ as a space. (We write $b \flat h$ for $b\otimes h$ when thought of as an element of $B \flat H$, $b\in B$, $h\in H$.) The coalgebra structure is defined by

$$\Delta : B\otimes H \xrightarrow{\Delta\otimes\Delta} B\otimes B\otimes H\otimes H \xrightarrow{1\otimes\phi\otimes 1\otimes 1} B\otimes H\otimes B\otimes H\otimes H$$
$$\xrightarrow{1\otimes 1\otimes T\otimes 1} B\otimes H\otimes H\otimes B\otimes H \xrightarrow{1\otimes M\otimes 1\otimes 1} B\otimes H\otimes B\otimes H$$

$$\varepsilon : B\otimes H \xrightarrow{\varepsilon\otimes\varepsilon} k\otimes k \cong k$$

i. e.,
$$\Delta(b \flat h) = \sum b_{(1)} \flat b_{(2)(H)}h_{(1)} \otimes b_{(2)(B)} \flat h_{(2)}$$
$$\varepsilon(b \flat h) = \varepsilon_B(b)\varepsilon_H(h) .$$

$B \flat H$ is called the *co-smash product* of B with H.

We define the Hopf algebra $L \flat H$ to be $L\otimes H$ as an algebra and to be $L \flat H$ as a coalgebra. The antipode is defined by

$$S : L\otimes H \xrightarrow{S_L\otimes S_H} L\otimes H \xrightarrow{\phi\otimes 1} H\otimes L\otimes H$$
$$\xrightarrow{T\otimes 1} L\otimes H\otimes H \xrightarrow{1\otimes S\otimes 1} L\otimes H\otimes H \xrightarrow{1\otimes M} L\otimes H .$$

$L \flat H$ is called the *co-smash product* of L with H (see [**1**], (4.2)).

4.3. The crossed product.

We now introduce crossed products. Suppose $\sigma : B \to H\otimes H$ is a linear map. $B \flat_\sigma H$ is the space $B\otimes H$ with comultiplication defined by

$$\Delta(b \flat_\sigma h) = \sum b_{(1)} \flat_\sigma b_{(2)(H)}b_{(3)\sigma(1)}h_{(1)} \otimes b_{(2)(B)} \flat_\sigma b_{(3)\sigma(2)}h_{(2)} ,$$

where we use a new notation, for $b\in B$,

$$\sigma(b) = \sum b_{\sigma(1)} \otimes b_{\sigma(2)} \in H\otimes H .$$

Note that when $\sigma = (u_H\otimes u_H)\varepsilon_B$ then $B \flat_\sigma H$ is precisely $B \flat H$.

(4.3.1) Lemma.

(a) *The comultiplication in* $B \flat_\sigma H$ *is coassociative if and only if*

$$[(1\otimes\sigma)\phi] * [(1\otimes\Delta)\sigma] = [(\Delta\otimes 1)\sigma] * [\sigma\otimes u] .$$

(b) $\varepsilon_B\otimes\varepsilon_H$ *is the counit in* $B \flat_\sigma H$ *if and only if*

$$\sum \varepsilon(b_{\sigma(1)})b_{\sigma(2)} = \varepsilon(b)1_H = \sum \varepsilon(b_{\sigma(2)})b_{\sigma(1)} \quad \textit{for all} \quad b\in B .$$

Proof. (a) Suppose $B \flat_\sigma H$ is coassociative. Then $(\Delta\otimes 1)\Delta(b\otimes h) = (1\otimes\Delta)\Delta(b\otimes h)$ for all $b\in B$, $h\in H$. The left hand side equals,

$$(*) \qquad \sum b_{(1)} \otimes b_{(2)(H)}b_{(3)\sigma(1)}b_{(4)(H)(1)}b_{(5)\sigma(1)(1)}h_{(1)} \otimes b_{(2)(B)}$$
$$\otimes b_{(3)\sigma(2)}b_{(4)(H)(2)}b_{(5)\sigma(1)(2)}h_{(2)} \otimes b_{(4)(B)} \otimes b_{(5)\sigma(2)}h_{(3)} .$$

And the right hand side equals,

$$(**) \qquad \sum b_{(1)} \otimes b_{(2)(H)}b_{(3)\sigma(1)}h_{(1)} \otimes b_{(2)(B)(1)} \otimes b_{(2)(B)(2)(H)}b_{(2)(B)(3)\sigma(1)}b_{(3)\sigma(2)(1)}h_{(2)}$$
$$\otimes b_{(2)(B)(2)(B)} \otimes b_{(2)(B)(3)\sigma(2)}b_{(3)\sigma(2)(2)}h_{(3)} .$$

Applying $\varepsilon\otimes 1\otimes\varepsilon\otimes 1\otimes\varepsilon\otimes 1$ to (*) and (**) and equating shows σ satisfies the identity in (a). Conversely, suppose σ satisfies the identity in (a). Applying $1\otimes\Delta\otimes 1$ to the first identity in c), (1.1.1) yields

$$(***)\qquad \sum b_{(H)}\otimes b_{(B)(1)}\otimes b_{(B)(2)}\otimes b_{(B)(3)} = \sum b_{(1)(H)}b_{(2)(H)}\otimes b_{(1)(B)(1)}\otimes b_{(1)(B)(2)}\otimes b_{(2)(B)}\,.$$

$$\begin{aligned}(**) &= \sum b_{(1)}\otimes b_{(2)(H)}b_{(3)\sigma(1)}h_{(1)}\otimes b_{(2)(B)(1)}\otimes b_{(2)(B)(3)(H)}\\ &\quad\cdot b_{(2)(B)(2)\sigma(1)}b_{(3)\sigma(2)(1)}h_{(2)}\otimes b_{(2)(B)(3)(B)}\otimes b_{(2)(B)(2)\sigma(2)}b_{(3)\sigma(2)(2)}h_{(3)}\\ &\qquad\qquad\text{(since } b_{(2)(B)} \text{ is a cocommutative element)}\\ &= \sum b_{(1)}\otimes b_{(2)(H)}b_{(3)(H)}b_{(4)\sigma(1)}h_{(1)}\otimes b_{(2)(B)(1)}\otimes b_{(3)(B)(H)}\\ &\quad\cdot b_{(2)(B)(2)\sigma(1)}b_{(4)\sigma(2)(1)}h_{(2)}\otimes b_{(3)(B)(B)}\otimes b_{(2)(B)(2)\sigma(2)}b_{(4)\sigma(2)(2)}h_{(3)}\\ &\qquad\qquad\text{(since (***))}\\ &= \sum b_{(1)}\otimes b_{(2)(H)}b_{(3)(H)(1)}b_{(4)\sigma(1)}h_{(1)}\otimes b_{(2)(B)(1)}\otimes b_{(3)(H)(2)}\\ &\quad\cdot b_{(2)(B)(2)\sigma(1)}b_{(4)\sigma(2)(1)}h_{(2)}\otimes b_{(3)(B)}\otimes b_{(2)(B)(2)\sigma(2)}b_{(4)\sigma(2)(2)}h_{(3)}\\ &= \sum b_{(1)}\otimes b_{(2)(H)}b_{(3)(H)}b_{(4)(H)(1)}b_{(5)\sigma(1)}h_{(1)}\otimes b_{(2)(B)}\otimes b_{(4)(H)(2)}\\ &\quad\cdot b_{(3)(B)\sigma(1)}b_{(5)\sigma(2)(1)}h_{(2)}\otimes b_{(4)(B)}\otimes b_{(3)(B)\sigma(2)}b_{(5)\sigma(2)(2)}h_{(3)}\\ &= \sum b_{(1)}\otimes b_{(2)(H)}b_{(3)(H)(1)}b_{\underline{(4)(H)}}b_{\underline{(5)\sigma(1)}}h_{(1)}\otimes b_{(2)(B)}\otimes b_{(3)(H)(2)}\\ &\quad\cdot b_{\underline{(4)(B)\sigma(1)}}b_{\underline{(5)\sigma(2)(1)}}h_{(2)}\otimes b_{(3)(B)}\otimes b_{(4)(B)\sigma(2)}b_{(5)\sigma(2)(2)}h_{(3)}\\ &= \sum b_{(1)}\otimes b_{(2)(H)}b_{(3)(H)(1)}b_{\underline{(4)\sigma(1)(1)}}b_{\underline{(5)\sigma(1)}}h_{(1)}\otimes b_{(2)(B)}\otimes b_{(3)(H)(2)}\\ &\quad\cdot b_{\underline{(4)\sigma(1)(2)}}b_{\underline{(5)\sigma(2)}}h_{(2)}\otimes b_{(3)(B)}\otimes b_{(4)\sigma(2)}h_{(3)}\\ &\qquad\qquad\text{(since the identity in (a))}\\ &= (*)\quad \text{(by the index permutation (345))}\,.\end{aligned}$$

(b) is clear. Q. E. D.

(4.3.2) We define $f\colon B\flat_\sigma H\to B$, $b\flat_\sigma h\mapsto\varepsilon(h)b$ and $\omega\colon B\flat_\sigma H\otimes H\to B\flat_\sigma H$, $b\flat_\sigma h\otimes g\mapsto b\flat_\sigma hg$. When σ satisfies the conditions of (4.3.1) Lemma one easily verifies that $(B\flat_\sigma H, f, \omega)$ is a coalgebra extention of B by H. We call it a *crossed product* (extention).

(4.3.3) Let σ be in $\mathrm{Alg}_+(L, H\otimes H)$ and $D^2(\sigma)=(u\otimes u\otimes u)\varepsilon$; i.e., normal 2-cocycle. We define the Hopf algebra $L\flat_\sigma H$ to be $L\otimes H$ as an algebra and to be $L\flat_\sigma H$ as a coalgebra. The antipode is defined by

$$S\colon L\otimes H\xrightarrow{\ \Delta\otimes 1\ } L\otimes L\otimes H\xrightarrow{\ 1\otimes\sigma^{-1}\otimes 1\ } L\otimes H\otimes H\otimes H$$
$$\xrightarrow{\ 1\otimes 1\otimes S\otimes 1\ } L\otimes H\otimes H\otimes H\xrightarrow{\ 1\otimes M(M\otimes 1)\ } L\otimes H$$

$$\xrightarrow{\phi\otimes S} H\otimes L\otimes H \xrightarrow{1\otimes S\otimes 1} H\otimes L\otimes H$$

$$\xrightarrow{T\otimes 1} L\otimes H\otimes H \xrightarrow{1\otimes S\otimes 1} L\otimes H\otimes H$$

$$\xrightarrow{1\otimes M} L\otimes H,$$

where $\sigma^{-1}: L\to H\otimes H$ is the $*$-inverse of σ.

Thus $L\, \natural_\sigma H$ is a Hopf extension of L by H.

§5. Cleft extensions and H^2.

5.1. A coalgebra extension (C, f, ω) of B by H is called *cleft* if there is an H-module map in $\mathrm{Reg}(C, H)$. (Regard H as a right H-module via M_H.) A Hopf extension (M, f, ω) of L by H is called cleft if there is an H-module map in $\mathrm{Alg}(M, H)$.

Note that if $\gamma: (C, f, \omega)\to(C', f', \omega')$ is a morphism of extensions and (C', f', ω') is cleft then so is (C, f, ω).

5.2. Examples.

(5.2.1) H may be viewed as a coalgebra (or Hopf) extension of k by H if we put $f=\varepsilon$ and $\omega=M_H$. The identity map on H is an H-module map which is invertible (since H has the antipode). Thus H is a cleft coalgebra (or Hopf) extension of k by H.

(5.2.2) Let G_2 be an affine algebraic group over an algebraically closed field k and let M be its coordinate ring. Let G_1 be a closed normal subgroup of G_2 which is commutative and let L be its coordinate ring. L is a cocommutative Hopf algebra. The inclusion map $G_1\hookrightarrow G_2$ induces a surjective Hopf algebra map f from M to L. Let G_3 be the quotient algebraic group of G_2 by G_1 and let H be its coordinate ring. We can consider H as a sub Hopf algebra of M.

$$1\longrightarrow G_1 \xrightarrow{i} G_2 \xrightarrow{p} G_3 \longrightarrow 1$$

$$L \xleftarrow{f} M \hookleftarrow H.$$

G_1 has a G_3-module structure: $x^g=sxs^{-1}$ $(g=p(s))$, $g\in G_3$, $x\in G_1$. Hence L has a H-comodule Hopf algebra structure ((1.2.6)). Define $\omega: M\otimes H\to M$ by $m\otimes h\mapsto mh$. Then it is easily shown that (M, f, ω) is a Hopf extension of L by H. Suppose that there exists a morphism of varieties $\alpha: G_3\to G_2$ such that $p\circ\alpha=id_{G_3}$. The corresponding algebra map $\tau: M\to H$ is the identity on H. This means that τ is an H-module map so that (M, f, ω) is a cleft extension of L by H.

5.3.

(5.3.1) LEMMA. *Let (C, f, w) be a coalgebra extension of B by H.*

(a) *If $\tau \in \mathrm{Reg}(C, H)$ is an H-module map then $\tau^{-1}\omega = M_H(\tau^{-1} \otimes S)$.*

(b) *If (C, f, ω) is cleft then there is an H-module map τ in $\mathrm{Reg}(C, H)$ such that $\varepsilon_H \tau = \varepsilon_C$.*

PROOF. (a) Since τ is an H-module map it satisfies $\tau\omega = M_H(\tau \otimes 1)$ in $\mathrm{Reg}(C \otimes H, H)$. One easily verifies that the inverse of $\tau\omega$ in $\mathrm{Reg}(C \otimes H, H)$ is $\tau^{-1}\omega$ and the inverse of $M_H(\tau \otimes 1)$ is $M_H(\tau^{-1} \otimes S)$. By the uniqueness of inverses we are done.

(b) By the assumption there is an H-module map τ in $\mathrm{Reg}(C, H)$. We define $\tau' : C \to H$, $\tau'(c) = \sum \varepsilon(\tau^{-1}(c_{(2)}))\tau(c_{(1)})$. A calculation shows that $\tau' \in \mathrm{Reg}(C, H)$ and $\varepsilon_H \tau' = \varepsilon_C$. Next we show that τ' is an H-module map.

$$\begin{aligned}\tau'\omega(c \otimes h) &= \sum \varepsilon(\tau^{-1}(c_{(2)} \leftharpoonup h_{(2)}))\tau(c_{(1)} \leftharpoonup h_{(1)}) \\ &= \sum \varepsilon(\tau^{-1}(c_{(2)})S(h_{(2)}))\tau(c_{(1)})h_{(1)} \qquad \text{(by (a))} \\ &= \sum \varepsilon(\tau^{-1}(c_{(2)}))\tau(c_{(1)})h \\ &= M_H(\tau' \otimes 1)(c \otimes h)\,. \end{aligned}$$

Q. E. D.

(5.3.2) LEMMA. *If $B \flat_\sigma H$ is a crossed porduct extension then the H-module map $\tau : B \flat_\sigma H \to H$, $b \flat_\sigma h \mapsto \varepsilon(b)h$ is invertible if $\sigma \in \mathrm{Reg}(B, H \otimes H)$. The inverse is given by $b \flat_\sigma h \mapsto \sum S(b_{\sigma^{-1}(1)})b_{\sigma^{-1}(2)}S(h)$.*

PROOF. It is clear.

5.4.

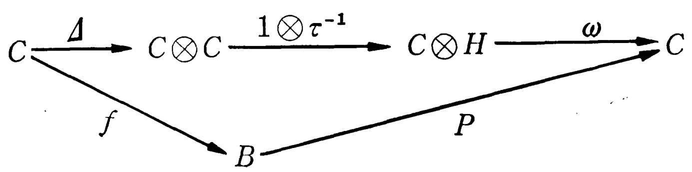
(5.4.1) LEMMA. *Let (C, f, ω) be a cleft coalgebra extension of B by H and $\tau \in \mathrm{Reg}(C, H)$ an H-module map.*

(a) *The composite*

$$C \xrightarrow{\ \Delta\ } C \otimes C \xrightarrow{\ f \otimes \tau\ } B \otimes H$$

is a linear isomorphism.

(b) *There is a map $P : B \to C$ such that the following diagram is commutative:*



And the composite

$$B \otimes H \xrightarrow{\ P \otimes 1\ } C \otimes H \xrightarrow{\ \omega\ } C$$

is the inverse isomorphism to the isomorphism given in (a).

PROOF. For $c\in C$ and $h\in H$,

$$\begin{aligned}
&\omega(1\otimes\tau^{-1})\Delta\omega(c\otimes h)\\
&\quad=\omega(1\otimes\tau^{-1})(\sum c_{(1)}\leftharpoonup h_{(1)}\otimes c_{(2)}\leftharpoonup h_{(2)}) \qquad (\omega \text{ is a coalgebra map})\\
&\quad=\omega(\sum c_{(1)}\leftharpoonup h_{(1)}\otimes\tau^{-1}(c_{(2)})S(h_{(2)})) \qquad (\text{by (a) in (5.3.1) Lemma})\\
&\quad=\sum\varepsilon(h)c_{(1)}\leftharpoonup\tau^{-1}(c_{(2)})\\
&\quad=\omega(1\otimes\tau^{-1})\Delta(1\otimes\varepsilon)(c\otimes h)\,.
\end{aligned}$$

Thus $\omega(1\otimes\tau^{-1})\Delta\omega=\omega(1\otimes\tau^{-1})\Delta(1\otimes\varepsilon)$. Since $C\otimes H\underset{1\otimes\varepsilon}{\overset{\omega}{\rightrightarrows}} C\overset{f}{\longrightarrow}B$ is exact the existence of the map $P\colon B\to C$ is guaranteed. Now

$$\begin{aligned}
\omega(P\otimes 1)(f\otimes\tau)\Delta(c)&=\omega(\sum Pf(c_{(1)})\otimes\tau(c_{(2)}))\\
&=\omega(\sum c_{(1)}\leftharpoonup\tau^{-1}(c_{(2)})\otimes\tau(c_{(3)}))\\
&=\sum c_{(1)}\leftharpoonup\tau^{-1}(c_{(2)})\tau(c_{(3)})\\
&=\sum c_{(1)}\varepsilon(c_{(2)})\\
&=c\,.
\end{aligned}$$

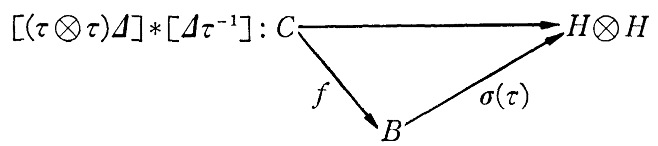
And

$$\begin{aligned}
&(f\otimes\tau)\Delta\omega(P\otimes 1)(f(c)\otimes h)\\
&\quad=(f\otimes\tau)\Delta(\sum c_{(1)}\leftharpoonup\tau^{-1}(c_{(2)})h)\\
&\quad=(f\otimes\tau)(\sum c_{(1)}\leftharpoonup\tau^{-1}(c_{(3)})_{(1)}h_{(1)}\otimes c_{(2)}\leftharpoonup\tau^{-1}(c_{(3)})_{(2)}h_{(2)})\\
&\quad=\sum f(c_{(1)})\varepsilon(\tau^{-1}(c_{(3)})_{(1)}h_{(1)})\otimes\tau(c_{(2)})\tau^{-1}(c_{(3)})_{(2)}h_{(2)}\\
&\qquad\qquad(\text{since } f\omega=f(1\otimes\varepsilon) \text{ and } \tau \text{ is an } H\text{-module map})\\
&\quad=\sum f(c_{(1)})\otimes\tau(c_{(2)})\tau^{-1}(c_{(3)})h\\
&\quad=f(c)\otimes h\,.
\end{aligned}$$

Thus C is isomorphic to $B\otimes H$. Q. E. D.

(5.4.2) LEMMA. *Let (C, f, ω) be a cleft coalgebra extension of B by H and $\tau\in\operatorname{Reg}(C, H)$ an H-module map such that $\varepsilon_H\tau=\varepsilon_C$.*

(a) *There is a map $\sigma(\tau)\colon B\to H\otimes H$ such that the following diagram is commutative:*



$\sigma(\tau)$ is a 2-cocycle in $\operatorname{Reg}_+(B, H\otimes H)$.

(b) *$\gamma_\tau\colon C\overset{\Delta}{\longrightarrow}C\otimes C\overset{f\otimes\tau}{\longrightarrow}B\otimes H=B\,\#_{\sigma(\tau)}H$ is an isomorphism of extensions.*

PROOF. (a) The map $\sigma(\tau)$ is given by

$$b \longmapsto \sum \tau(c_{(1)})\tau^{-1}(c_{(3)})_{(1)} \otimes \tau(c_{(2)})\tau^{-1}(c_{(3)})_{(2)} \qquad (f(c)=b).$$

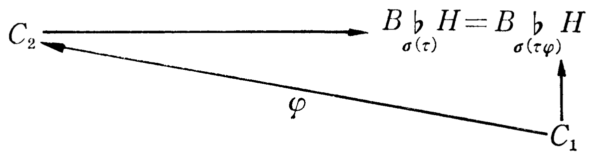
A calculation——involving the condition (c) in 4.1.——shows that $(\gamma_\tau \otimes \gamma_\tau)\Delta_C = \Delta_{B\flat_{\sigma(\tau)}H}\gamma_\tau$ and $(\varepsilon_B \otimes \varepsilon_H)\gamma_\tau = \varepsilon_C$. (5.4.1) Lemma implies γ_τ is bijective and thus $B\flat_{\sigma(\tau)}H$ is a (coassociative) coalgebra. Thus $\sigma(\tau)$ satisfies the condition of (4.3.1) Lemma. One easily verifies that $\sigma(\tau) \in \mathrm{Reg}(B, H \otimes H)$.

(b) is clear. Q. E. D.

(5.4.3) REMARK. If $C = B\flat_\sigma H$ for 2-cocycle $\sigma \in \mathrm{Reg}(B, H\otimes H)$ then $\tau \equiv \varepsilon \otimes 1 : B\flat_\sigma H \to H$, $b\flat_\sigma h \mapsto \varepsilon(b)h$ is an H-module map such that $\varepsilon\tau = \varepsilon$. One easily checks that $\sigma(\tau) = \sigma$ and γ_τ is the identity map on $B \otimes H$.

(5.4.4) LEMMA. *Let (C_i, f_i, ω_i) be coalgebra extensions of B by H for $i=1, 2$ and let $\varphi : C_1 \to C_2$ be a morphism of extensions. If (C_2, f_2, ω_2) is cleft then φ is an isomorphism.*

PROOF. Suppose $\tau \in \mathrm{Reg}(C_2, H)$ is an H-module map such that $\varepsilon\tau = \varepsilon$. Then $\tau\varphi \in \mathrm{Reg}(C_1, H)$ is an H-module map and $\sigma(\tau) = \sigma(\tau\varphi)$ (since we have $([(\tau\otimes\tau)\Delta] * [\Delta\tau^{-1}])\varphi = [(\tau\varphi \otimes \tau\varphi)\Delta] * [\Delta(\tau\varphi)^{-1}]$). Clearly the diagram,



is commutative. By (5.4.2) Lemma the horizontal and vertical maps are isomorphisms which implies φ is an isomorphism. Q. E. D.

(5.4.5) LEMMA. *Let σ and ρ be 2-cocycles in* $\mathrm{Reg}_+(B, H\otimes H)$. *Then the followings are equivalent:*

(a) *$B\flat_\sigma H \cong B\flat_\rho H$ as a coalgebra extension.*

(b) *σ and ρ are cohomologous: i.e., $\sigma * \rho^{-1} = D^1(e)$ for some $e \in \mathrm{Reg}_+(B, H)$.*

PROOF. (b)$\Rightarrow$(a). We define $\varphi : B\flat_\rho H \to B\flat_\sigma H$, $b\flat_\rho h \mapsto \sum b_{(1)}\flat_\sigma e(b_{(2)})h$. Then φ is a morphism of extensions.

(a)$\Rightarrow$(b). Suppose $\varphi : B\flat_\rho H \to B\flat_\sigma H$ is a morphism of extensions. Define $e : B \to H$, $b \mapsto (\varepsilon\otimes 1)\varphi(b\flat_\rho 1)$. We claim that $\varphi(b\flat_\rho 1) = \sum b_{(1)}\flat_\sigma e(b_{(2)})$ for all $b \in B$. The map $\tau : B\flat_\sigma H \to H$, $b\flat_\sigma h \mapsto \varepsilon(b)h$ is an H-module map in $\mathrm{Reg}(C, H)$. By (5.4.1) Lemma the composite $(f\otimes\tau)\Delta_{B\flat_\sigma C}$ is bijective so that it suffices to show the following equality,

$$(f\otimes\tau)\Delta\varphi(b\flat_\rho 1) = (f\otimes\tau)\Delta(\textstyle\sum b_{(1)}\flat_\sigma e(b_{(2)}))\,.$$

$$\begin{aligned}(f\otimes\tau)\Delta\varphi(b\flat_\rho 1) &= (f\otimes\tau)(\varphi\otimes\varphi)(\textstyle\sum b_{(1)}\flat_\rho b_{(2)(H)}b_{(3)\rho(1)} \otimes b_{(2)(B)}\flat_\rho b_{(3)\rho(2)}) \\ &= \textstyle\sum \varepsilon(b_{(2)(H)})\varepsilon(b_{(3)\rho(1)})b_{(1)} \otimes e(b_{(2)(B)})b_{(3)\rho(2)}\end{aligned}$$

(since $f\varphi = f$ and τ is an H-module map)

$$=\sum b_{(1)}\otimes e(b_{(2)})$$
$$=(f\otimes\tau)\Delta(\sum b_{(1)}\flat_{\sigma}e(b_{(2)}))\qquad\text{(by (5.4.3) Remark)}.$$

Thus

$$\Delta\varphi(b\flat_{\rho}1)=\Delta(\sum b_{(1)}\flat_{\sigma}e(b_{(2)}))$$
$$(*)\qquad =\sum b_{(1)}\flat_{\sigma}b_{(2)(H)}b_{(3)\sigma(1)}e(b_{(4)})_{(1)}\otimes b_{(2)(B)}\flat_{\sigma}b_{(3)\sigma(2)}e(b_{(4)})_{(2)}$$
$$(\varphi\otimes\varphi)\Delta(b\flat_{\rho}1)$$
$$(**)\qquad =\sum\varphi(b_{(1)}\flat_{\rho}1)\leftharpoonup b_{(2)(H)}b_{(3)\rho(1)}\otimes\varphi(b_{(2)(B)}\flat_{\rho}1)b_{(3)\rho(2)}.$$

Equating (*) and (**) and applying $\varepsilon\otimes 1\otimes\varepsilon\otimes 1$ implies

$$\sigma*[\Delta e]=[(1\otimes e)\phi]*[e\otimes u]*\rho.$$

Also $\varepsilon e=\varepsilon$. Thus if we show $e\in\mathrm{Reg}(B,H)$ it follows $e\in\mathrm{Reg}_{+}(B,H)$ and $\sigma*\rho^{-1}=D^{1}(e)$.

A calculation shows $ef=[\tau\varphi]*[\tau^{-1}]$. Thus $(ef)^{-1}=[\tau^{-1}\varphi]*[\tau]$ in $\mathrm{Reg}(B\flat_{\rho}H,H)$. $[\tau^{-1}\varphi]*[\tau]$ induces a map $e':B\to H$ such that $e'f=[\tau^{-1}\varphi]*[\tau]$. Now $(e*e')f=ef*e'f=u_{H}\varepsilon_{B\flat_{\rho}H}=u_{H}\varepsilon_{B}f$. Hence we have $e*e'=u_{H}\varepsilon_{B}$. Q.E.D.

5.5. THEOREM. *Let H be a commutative Hopf algebra and let B be a cocommutative left H-comodule coalgebra. Then there is a bijective correspondence between the isomorphism classes of cleft coalgebra extensions of B by H and Coalg-$H^{2}(B,H)$.*

PROOF. The correspondence is gotten by choosing a crossed product from the isomorphism class and passing to the cohomology class of the 2-cocycle determing the crossed product. Q.E.D.

Similar calculations show the next result about Hopf algebra extensions.

5.6. THEOREM. *Let L be a cocommutative left H-comodule Hopf algebra. Then there is a bijective correspondence between the isomorphism classes of cleft Hopf algebra extensions of L by H and Hopf-$H^{2}(L,H)$.*

§6. Cohomology of comodule algebras.

Let H be a commutative Hopf algebra and let A be a commutative right H-comodule algebra. Suppose that the ground field k is an algebraically closed field and k is algebraically closed in A and H.

The importance of this hypothesis resides in the following result ([4]), known as Ax-Lichtenbaum-Halperin's units theorem.

Suppose k is an algebraically closed field, X and Y commutative algebras over k and k is algebraically closed in X and Y. Then every invertible element in $X\otimes Y$ is of the form of $x\otimes y$ where x is an invertible element of X and y an invertible element of Y.

APPLICATION TO HOPF ALGEBRAS: *Under our condition of k, every invertible element of H is of the form of* λg *where* $\lambda \in k-\{0\}$ *and* $g \in G(H)=\{g \in H \mid g \neq 0,\ \Delta(g)=g \otimes g\}$.

We note that if A is finitely generated then our condition of k being algebraically closed in A is equivalent to zero being the only nilpotent element of A and Spec(A)——maximal or prime ideal spec——being Zariski connected. If H is finitely generated then H is the coordinate ring of an affine algebraic group. Since a Zariski connected affine algebraic group is actually irreducible we have that our condition of k guarantees that H is an integral domain.

6.1.

(6.1.1) Let a be an invertible element of A; i.e., $a \in U(A)$. Since the comodule structure map $\psi: A \rightarrow A \otimes H$ is an algebra map we have that $\psi(a)$ is an invertible element in $A \otimes H$. By the units theorem $\psi(a)=b \otimes g_a$ where b is an invertible element of A and g_a is a grouplike element of H; i.e., $g \in G(H)$. Since $(1 \otimes \varepsilon)\psi=id.$, we have that $b=a$ and $\psi(a)=a \otimes g_a$.

In case $A=k[X]$ and $H=k[G]$ as in Example (1.2.3) the above result implies that every invertible regular function a is a semi-invariant with weight g_a, that is, $a(x^t)=g_a(t)a(x)$ for all $x \in X$ and $t \in G$ where we denote the action of t on x by x^t. Note that grouplike elements of H are multiplicative characters of G.

In general the grouplike elements of H form a multiplicative subgroup of $U(H)$ $(S(g)=g^{-1})$. They are linearly independent. It is clear that the map $\xi: U(A) \rightarrow G(H)$, $a \mapsto g_a$ is a group homomorphism.

(6.1.2) PROPOSITION. *Alg-*$H^1(A, H) \cong G(H)/\mathrm{Im}\,\xi$. *In particular if* $A=A^H$ $(=\{a \in A \mid \psi(a)=a \otimes 1\})$ *then Alg-*$H^1(A, H) \cong G(H)$.

PROOF. The invertible elements in $A \otimes H$ are all of the form $a \otimes h$ where $a \in U(A)$ and $h \in G(H)$. Now

$$\begin{aligned} D^1(a \otimes h) &= (a \otimes \xi(a) \otimes h)(a^{-1} \otimes h^{-1} \otimes h^{-1})(a \otimes h \otimes 1) \\ &= a \otimes \xi(a) \otimes 1 . \end{aligned}$$

Thus if $c=a \otimes h$ is a 1-cocycle we can assume c is of the form $c=1 \otimes h$ where $h \in G(H)$. On the other hand for $a \in U(A)$,

$$D^0(a)=(a \otimes \xi(a))(a^{-1} \otimes 1)=1 \otimes \xi(a) .$$

Hence we have *Alg-*$H^1(A, H) \cong G(H)/\mathrm{Im}\,\xi$.

Finally if $A=A^H$ then $\mathrm{Im}\,\xi=\{1\}$. Q.E.D.

This gives rise to the exact sequence of groups:

$$1 \longrightarrow Alg\text{-}H^0(A, H) \longrightarrow U(A) \xrightarrow{\ \xi\ } G(H) \longrightarrow Alg\text{-}H^1(A, H) \longrightarrow 1 .$$

6.2.

(6.2.1) THEOREM. *Alg-*$H^n(A, H)=\{1\}$ *for* $n\geqq 2$.

PROOF. By [4], 3.0 Lemma the invertible elements in $A\otimes(\otimes^n H)$ are all of the form $a\otimes h_1\otimes\cdots\otimes h_n$ where $a\in U(A)$ and $\{h_i\}\subset G(H)$. A calculation shows

$$D^n(a\otimes h_1\otimes\cdots\otimes h_n)$$
$$=\begin{cases} a\otimes\xi(a)\otimes h_2\otimes h_2\otimes h_4\otimes h_4\otimes\cdots\otimes h_{n-1}\otimes h_{n-1}\otimes 1 & \text{if } n \text{ odd } (n\geqq 3),\\ 1\otimes\xi(a)h_1^{-1}\otimes 1\otimes h_2h_3^{-1}\otimes 1\otimes h_4h_5^{-1}\otimes\cdots\otimes h_{n-2}h_{n-1}^{-1}\otimes 1\otimes h_n & \text{if } n \text{ even } (n\geqq 2).\end{cases}$$

Thus if n is odd ($n\geqq 3$) and $c=a\otimes h_1\otimes h_2\otimes\cdots\otimes h_n$ is a cocycle we must have that $a=1$ and $h_2=h_4=\cdots=h_{n-1}=1$. Thus we can assume c is of the form

$$c=1\otimes h_1\otimes 1\otimes h_3\otimes\cdots\otimes 1\otimes h_n\,.$$

Then we have that

$$c=D^{n-1}(1\otimes h_1^{-1}\otimes 1\otimes h_3^{-1}\otimes\cdots\otimes 1\otimes h_{n-2}^{-1}\otimes h_n)$$

so that c is a coboundary.

If n is even ($n\geqq 2$) and c is a cocycle we have that $\xi(a)=h_1$, $h_2=h_3$, $h_4=h_5$, $\cdots h_{n-2}=h_{n-1}$ and $h_n=1$. This implies that c can be written

$$c=a\otimes\xi(a)\otimes h_2\otimes h_2\otimes h_4\otimes h_4\otimes\cdots\otimes h_{n-2}\otimes h_{n-2}\otimes 1\,.$$

Then we have that

$$c=D^{n-1}(a\otimes 1\otimes h_2\otimes 1\otimes h_4\otimes\cdots\otimes h_{n-2}\otimes 1)$$

so that c is a coboundary. Q. E. D.

(6.2.2) COROLLARY. *Coalg-*$H^n(k, H)=\{1\}$ *for* $n\geqq 2$.

PROOF. It is clear from (3.4.2) Proposition and (6.2.1) Theorem.

(6.2.3) COROLLARY. *Let* C *be a coalgebra which is a right* H*-module (with action* $\omega: C\otimes H\to C$*). Suppose* $C\otimes H \underset{1\otimes\varepsilon}{\overset{\omega}{\rightrightarrows}} C \xrightarrow{\varepsilon} k$ *is exact and there is an* H*-module map in* Reg(C, H)*. Then* $C\cong H$ *as a coalgebra.*

PROOF. It is very easy to see that (C, ε, ω) is a cleft coalgebra extension of k by H. Since *Coalg-*$H^2(k, H)=\{1\}$, it follows from 5.5 Theorem that $C\cong k\,\flat\, H$ as a coalgebra. Clearly, $k\,\flat\, H\cong H$ as a coalgebra so that (6.2.3) is proved.

§ 7. Application to coradical splittings.

7.1. Inner automorphisms.

We introduce inner automorphisms of Hopf algebras. Let M be a commutative Hopf algebra over a field k. For $\tau \in \mathrm{Alg}(M, k)$ we define the map $I(\tau): M \to M$ by $m \mapsto \sum \tau S(m_{(1)}) m_{(2)} \tau(m_{(3)})$. It is easy to see that $I(\tau)$ is a Hopf algebra endomorphism. And we have $I(\tau) I(\tau S) = id.$, which implies $I(\tau)$ is a Hopf algebra automorphism. We say that a Hopf algebra automorphism is *inner* if it is one of the form $I(\tau)$. Inner automorphisms form a group.

7.2. Let M be a commutative Hopf algebra over k of characteristic 0. Let H be the coradical of M, that is, H is the sum of all simple subcoalgebras of M. We know that H is a sub Hopf algebra of M ([**1**], (3.1)) and $L = M/M \cdot H^+$ is an irreducible Hopf algebra where $H^+ = \mathrm{Ker}\, \varepsilon_H$.

The purpose of this section is to prove the following Theorem.

THEOREM. *Suppose L is cocommutative. If $q, q': M \to H$ are Hopf algebra maps such that $q =$ identity on $H = q'$, then there exists an inner automorphism $I(\tau)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} M & \xrightarrow{I(\tau)} & M \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle q'} & \\ & H & \end{array}$$

REMARKS. (1) By [**7**, Theorem 1] there exists a Hopf algebra map $q: M \to H$ such that $q =$ identity on H, where M and H are as in the above Theorem. (2) The above Theorem with Remark (1) is similar in spirit to [**8**, Theorem 14.2].

7.3. We assume that k is of characteristic 0 and $L = M/M \cdot H^+$ is cocommutative. Let f denote the canonical projection $M \to L$.

(7.3.1) L is a left H-comodule Hopf algebra under the left H-comodule structure

$$\phi: L \longrightarrow H \otimes L, \qquad f(m) \longmapsto m_{(1)} S(m_{(3)}) \otimes f(m_{(2)}).$$

Indeed since $L = M/M \cdot H^+$ we have that L is a quotient M-comodule of M under the left M-comodule structure of Example (1.2.5). Hence it suffices to show that $\phi(L) \subset H \otimes L$. But this follows from [**7**, Lemma 5].

(7.3.2) *Hopf*-$H^1(L, H) = \{1\}$.

PROOF. Since L is irreducible cocommutative it follows from [**3**, Theorem 13.0.1] that L is isomorphic as a Hopf algebra to $U(V)$, the universal enveloping algebra of V, where $V = P(L) = \{v \in L \mid \Delta(v) = v \otimes 1 + 1 \otimes v\}$. Since L is commutative as an algebra we have $U(V) = S(V)$, the symmetric algebra of V. Thus we are done when we show that *Com*-$H^1(V, H) = \{1\}$ (by (3.2.2)).

This follows from the next Proposition.

7.4. PROPOSITION. *Let H be a co-semi-simple Hopf algebra over a field k (k is not necessarily of characteristic 0). For every left H-comodule V, we have $Com\text{-}H^1(V, H)=\{1\}$.*

PROOF. Since H is co-semi-simple there exists a linear map $x: H\to k$ such that $wx=\langle w, 1\rangle x$ for all $w\in H^*$ and $x(1)=1$ (see [3], Theorem 14.0.3), where we write $\langle w, h\rangle$ for $w(h)$ ($w\in H^*$, $h\in H$). Now

$$\begin{aligned}
& wx=\langle w, 1\rangle x && \text{for all } w\in H^* \\
& \Leftrightarrow \langle w\otimes x, \Delta(h)\rangle=\langle w, 1\rangle\langle x, h\rangle && \text{for all } w\in H^* \text{ and } h\in H \\
& \Leftrightarrow \sum\langle w, \langle x, h_{(2)}\rangle h_{(1)}\rangle=\langle w, \langle x, h\rangle 1\rangle \\
& \Leftrightarrow \langle w, (1\otimes x)\Delta(h)\rangle=\langle w, u_H x(h)\rangle \\
(*)\quad & \Leftrightarrow (1\otimes x)\Delta=u_H x\,.
\end{aligned}$$

Now let $f: V\to H$ be a 1-cocycle. Then we have (for $v\in V$)

$$(**)\qquad \Delta f(v)=(1\otimes f)\phi(v)+f(v)\otimes 1\,.$$

We denote by α the composite: $V\xrightarrow{f} H\xrightarrow{x} k$. For $v\in V$,

$$\begin{aligned}
D^0(\alpha)(v) &= (1\otimes\alpha)\phi(v)-\alpha(v)1 \\
&= (1\otimes x)(1\otimes f)\phi(v)-xf(v)1 \\
&= (1\otimes x)(\Delta f(v)-f(v)\otimes 1)-xf(v)1 && \text{by } (**) \\
&= xf(v)1-f(v)-xf(v)1 && \text{by } (*) \text{ and } \langle x, 1\rangle=1 \\
&= -f(v)\,.
\end{aligned}$$

Thus we have $D^0(-\alpha)=f$, whence f is a 1-coboundary. Q. E. D.

REMARK. The followings are equivalent: (a) H is co-semi-simple; (b) for every left H-comodule V, $Com\text{-}H^1(V, H)=\{1\}$; (c) for every left H-comodule V, $Com\text{-}H^n(V, H)=\{1\}$ ($n\geqq 1$). This is similar in spirit to [6, II, § 3, 3.7].

7.5. THE PROOF OF THE THEOREM.

We define the algebra map $\bar{F}: M\to H$ by

$$\bar{F}(m)=(q*q'^{-1})(m)=\sum q(m_{(1)})q'S(m_{(2)})\,.$$

Since $M\cdot H^+$ is evidently contained in the kernel of $\bar{F}$, we have the induced algebra map $F: L=M/M\cdot H^+\to H$. Now let f denote the canonical projection $M\to L$. For $l=f(m)\in L$,

$$\begin{aligned}
\sum F(l)_{(1)}\otimes F(l)_{(2)} &= \sum q(m_{(1)})_{(1)}q'S(m_{(2)})_{(1)}\otimes q(m_{(1)})_{(2)}q'S(m_{(2)})_{(2)} \\
&= \sum q(m_{(1)})q'S(m_{(4)})\otimes q(m_{(2)})q'S(m_{(3)})\,.
\end{aligned}$$

On the other hand,

$$\sum l_{(1)(H)}F(l_{(2)})\otimes F(l_{(1)(L)})$$
$$=\sum m_{(1)}S(m_{(3)})Ff(m_{(4)})\otimes Ff(m_{(2)})$$
$$=\sum q(m_{(1)})qS(m_{(4)})q(m_{(5)})q'S(m_{(6)})\otimes q(m_{(2)})q'S(m_{(3)}) \quad (\text{by } m_{(1)}S(m_{(3)})\in H)$$
$$=\sum q(m_{(1)})q'S(m_{(4)})\otimes q(m_{(2)})q'S(m_{(3)}).$$

This shows that $F: L\to H$ is a 1-cocycle. By (7.3.2) there exists a $\alpha\in \mathrm{Alg}(L, k)$ such that $F=D^0(\alpha)=[(1\otimes\alpha)\psi]*[\alpha^{-1}\otimes u]$. The equality $Ff=D^0(\alpha)f$ reduces

$$q=([\alpha^{-1}\otimes u]*[(1\otimes\alpha)\psi])f*q'.$$

Thus we have that for $m\in M$,

$$q(m)=\sum \alpha Sf(m_{(1)})m_{(2)}S(m_{(4)})\alpha f(m_{(3)})q'(m_{(5)})$$
$$=\sum \alpha fS(m_{(1)})q'(m_{(2)})q'S(m_{(4)})\alpha f(m_{(3)})q'(m_{(5)}) \qquad (\text{by } m_{(2)}S(m_{(4)})\in H)$$
$$=\sum \alpha fS(m_{(1)})q'(m_{(2)})\alpha f(m_{(3)})$$
$$=q'I(\alpha f)(m).$$

Hence we have $q=q'I(\tau)$ where $\tau=\alpha f$, and the Theorem is proved.

Appendix

By Mitsuhiro TAKEUCHI

1. Comparison with the Hochschild cohomology.

Let $\mathfrak{G}$ be a k-group-functor and $\mathfrak{M}$ a $\mathfrak{G}$-module-functor. In [**6**, II, § 3, 1.1] the Hochschild cohomology $H_0^n(\mathfrak{G}, \mathfrak{M})$ of $\mathfrak{G}$ with coefficients in $\mathfrak{M}$ is defined. Let H, V, B, L, W and A be just as in § 2.1. Let $\mathfrak{G}=\mathfrak{Sp}(H)$ be the affine group scheme of H, hence $\mathfrak{G}(R)=\mathrm{Alg}_k(H, R)$ for any k-model R. We define five $\mathfrak{G}$-module-functors $\mathfrak{M}_i$, $i=1, 2, \cdots, 5$, as follows:

$\mathfrak{M}_1(R)=\mathrm{Hom}_k(V, R)$ on which $\mathfrak{G}(R)$ acts as

$$(g\rightharpoonup x)(v)=\sum g(v_{(H)})x(v_{(V)}),\ g\in\mathfrak{G}(R),\ x\in\mathfrak{M}_1(R),\ v\in V$$

$\mathfrak{M}_2(R)=\mathrm{Reg}_k(B, R)$

$\mathfrak{M}_3(R)=\mathrm{Alg}_k(L, R)$.

The action of $\mathfrak{G}(R)$ on $\mathfrak{M}_2(R)$ (resp. $\mathfrak{M}_3(R)$) is induced from the action on $\mathfrak{M}_1(R)$ with V replaced by B (resp. by L).

$\mathfrak{M}_4(R)=R\otimes W$ with the $\mathfrak{G}(R)$-action

$$g\rightharpoonup(r\otimes w)=\sum rg(w_{(H)})\otimes w_{(W)},\ g\in\mathfrak{G}(R),\ r\in R,\ w\in W$$

$\mathfrak{M}_5(R)=U(R\otimes A)$.

The action of $\mathfrak{G}(R)$ on $\mathfrak{M}_5(R)$ is induced from the action on $\mathfrak{M}_4(R)$ with W replaced by A.

PROPOSITION. *$H_0^n(\mathfrak{G}, \mathfrak{M}_1) = Com\text{-}H^n(V, H)$, $H_0^n(\mathfrak{G}, \mathfrak{M}_2) = Coalg\text{-}H^n(B, H)$, $H_0^n(\mathfrak{G}, \mathfrak{M}_3) = Hopf\text{-}H^n(L, H)$, $H_0^n(\mathfrak{G}, \mathfrak{M}_4) = Hoch\text{-}H^n(W, H)$ and $H_0^n(\mathfrak{G}, \mathfrak{M}_5) = Alg\text{-}H^n(A, H)$.*

The proof may be omitted, since it is easy and standard.

In view of the above identifications, Theorems 5.5 and 5.6 are contained, in a sense, in [**6**, II, § 3, 2.3].

2. Non-abelian cohomology.

Let $\mathfrak{G}$ be a k-group-functor. By a *$\mathfrak{G}$-group-functor* we mean a (not necessarily commutative) k-group-functor $\mathfrak{M}$ on which $\mathfrak{G}$ acts as group-automorphisms. We define in the following $H_0^0(\mathfrak{G}, \mathfrak{M})$ and $H_0^1(\mathfrak{G}, \mathfrak{M})$ for any $\mathfrak{G}$-group-functor $\mathfrak{M}$.

First we define $H_0^0(\mathfrak{G}, \mathfrak{M}) = \mathfrak{M}^{\mathfrak{G}}(k)$. Next a morphism $\mathfrak{f}: \mathfrak{G} \to \mathfrak{M}$ is called a 1-cocycle if

$$\mathfrak{f}(gh) = \mathfrak{f}(g)[g \rightharpoonup \mathfrak{f}(h)], \qquad g, h \in \mathfrak{G}(R), \quad R \in \boldsymbol{M}_k .$$

Two 1-cocycles $\mathfrak{f}$ and $\mathfrak{f}'$ are said to be cohomologous if there is an $x \in \mathfrak{M}(k)$ such that

$$\mathfrak{f}'(g) = x_R^{-1}\mathfrak{f}(g)(g \rightharpoonup x_R), \qquad g \in \mathfrak{G}(R) .$$

This is an equivalence relation and the quotient space is denoted by $H_0^1(\mathfrak{G}, \mathfrak{M})$. This is a pointed set having the class of the identity cocycle as its base point.

Let $1 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 1$ be a k-model-wise exact sequence of $\mathfrak{G}$-group-functors. This means that

$$1 \longrightarrow \mathfrak{M}'(R) \longrightarrow \mathfrak{M}(R) \longrightarrow \mathfrak{M}''(R) \longrightarrow 1$$

is exact in the usual sense for any k-model R. Then just as in [J.-P. Serre, Corps locaux, p. 133], we have an exact sequence of pointed sets:

$$\begin{aligned} 1 \longrightarrow H_0^0(\mathfrak{G}, \mathfrak{M}') &\longrightarrow H_0^0(\mathfrak{G}, \mathfrak{M}) \longrightarrow H_0^0(\mathfrak{G}, \mathfrak{M}'') \\ \overset{\partial}{\longrightarrow} H_0^1(\mathfrak{G}, \mathfrak{M}') &\longrightarrow H_0^1(\mathfrak{G}, \mathfrak{M}) \longrightarrow H_0^1(\mathfrak{G}, \mathfrak{M}'') . \end{aligned}$$

Now let $\mathfrak{M}$ be a $\mathfrak{G}$-group-functor and form the semi-direct product $\bar{\mathfrak{G}} = \mathfrak{M} \cdot \mathfrak{G}$. Thus $\bar{\mathfrak{G}}(R) = \mathfrak{M}(R) \times \mathfrak{G}(R)$ and

$$(x, g)(y, h) = (x(g \rightharpoonup y), gh) \quad \text{in} \quad \bar{\mathfrak{G}}(R) .$$

Let $\pi: \bar{\mathfrak{G}} \to \mathfrak{G}$ be the canonical projection. Let $\sigma: \mathfrak{G} \to \bar{\mathfrak{G}}$ be a morphism such that $\pi \circ \sigma = 1$. Write $\sigma(g) = (\mathfrak{f}(g), g)$. Then σ is a homomorphism of k-group-functors if and only if $\mathfrak{f}$ is a 1-cocyle. Let $\sigma': \mathfrak{G} \to \bar{\mathfrak{G}}$ be another homomorphism such that $\pi\circ\sigma'=1$ and write $\sigma'(g)=(\mathfrak{f}'(g), g)$. Then $\mathfrak{f}$ and $\mathfrak{f}'$ are cohomologous if and only if there is an $x\in\mathfrak{M}(k)$ such that

$$\sigma'=\mathfrak{J}(x)\circ\sigma$$

where $\mathfrak{J}(x)$ denotes the inner-automorphism of $\mathfrak{G}$ determined by $(x, 1)\in\bar{\mathfrak{G}}(k)$. This is clear since

$$(x_R, 1)^{-1}(\mathfrak{f}(g), g)(x_R, 1)=(x_R^{-1}\mathfrak{f}(g)(g\rightharpoonup x), g)\,.$$

In particular $H_0^1(\mathfrak{G}, \mathfrak{M})=\{1\}$ means that any homomorphism of k-group-functors $\sigma:\mathfrak{G}\to\bar{\mathfrak{G}}$ such that $\pi\circ\sigma=1$ can be written as

$$\sigma(g)=(x_R, 1)^{-1}(1, g)(x_R, 1)\,,\qquad g\in\mathfrak{G}(R)$$

for some $x\in\mathfrak{M}(k)$.

LEMMA. *Suppose that k is of characteristic* 0. *Let $\mathfrak{G}=\mathfrak{Sp}(H)$ be an affine algebraic k-group with H co-semi-simple. Let $\mathfrak{M}$ be an affine algebraic unipotent k-group on which $\mathfrak{G}$ acts as group-automorphisms. Then* $H_0^1(\mathfrak{G}, \mathfrak{M})=1$.

PROOF. The case where $\mathfrak{M}$ is commutative. Then we have a canonical isomorphism of groups

$$\exp:\ \mathrm{Lie}(\mathfrak{M})_a \xrightarrow{\simeq} \mathfrak{M}$$

[**6**, IV, § 2, 4.1]. Notice that the action of $\mathfrak{G}$ on $\mathfrak{M}$ induces a natural linear representation: $\mathfrak{G}\to\mathfrak{GL}(\mathrm{Lie}(\mathfrak{M}))$. The above isomorphism can be easily seen to be $\mathfrak{G}$-equivalent. Since $H=\mathcal{O}(\mathfrak{G})$ is co-semi-simple, we have

$$H_0^1(\mathfrak{G}, \mathfrak{M})=H^1(\mathfrak{G}, \mathrm{Lie}(\mathfrak{M}))=0$$

by [**6**, II, § 3, 3.7].

General case. Let $\mathfrak{Z}$ be the center of $\mathfrak{M}$. Since $\mathfrak{Z}$ is characteristic, it is $\mathfrak{G}$-stable. The exact sequence of $\mathfrak{G}$-group-functors

$$1\longrightarrow\mathfrak{Z}\longrightarrow\mathfrak{M}\longrightarrow\mathfrak{M}\tilde{/}\mathfrak{Z}\longrightarrow 1$$

is k-model-wise exact, since $\mathfrak{Z}\simeq V_a$ for some vector space V and since [**6**, III, § 4, 6.6] holds with $\boldsymbol{\alpha}_k$ replaced by V_a. Hence we have an exact sequence

$$0=H_0^1(\mathfrak{G}, \mathfrak{Z})\longrightarrow H_0^1(\mathfrak{G}, \mathfrak{M})\longrightarrow H_0^1(\mathfrak{G}, \mathfrak{M}\tilde{/}\mathfrak{Z})=1$$

($H_0^1(\mathfrak{G}, \mathfrak{M}\tilde{/}\mathfrak{Z})=1$ by the induction hypothesis). Therefore $H_0^1(\mathfrak{G}, \mathfrak{M})=1$.

COROLLARY. *Let M be a commutative Hopf algebra over a field of characteristic* 0. *If M is finitely generated as an algebra then the statement of Theorem* 7.2 *holds, whether L is cocommutative or not.*

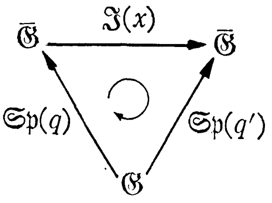
PROOF. Put $\mathfrak{G}=\mathfrak{Sp}(H)$, $\bar{\mathfrak{G}}=\mathfrak{Sp}(m)$, $\mathfrak{U}=\mathfrak{Sp}(L)$. Then the Hopf algebra maps

$$L\xleftarrow{\ f\ } M \underset{q}{\overset{\longleftarrow}{\longrightarrow}} H$$

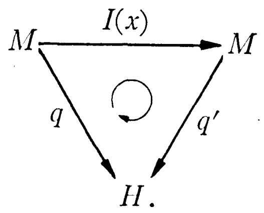
induce a split exact sequence of k-group-schemes

$$1 \longrightarrow \mathfrak{U} \longrightarrow \bar{\mathfrak{G}} \rightleftarrows \mathfrak{G} \longrightarrow 1 .$$

This permits us to identify $\bar{\mathfrak{G}}$ with the semi-direct product $\mathfrak{U}\cdot\mathfrak{G}$, where the action of $\mathfrak{G}$ on $\mathfrak{U}$ is determined through $\mathfrak{Sp}(q)$. Since $\mathfrak{U}$ is unipotent and H is co-semi-simple, we have $H_0^1(\mathfrak{G}, \mathfrak{U})=1$. Hence if $q' : M \rightarrow H$ is another Hopf algebra projection, then there is an $x \in \mathfrak{U}(k)$ such that



or equivalently



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