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On the global existence of real analytic solutions of linear differential equations (II)

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This paper is a continuation of our previous paper (Kawai [2]), where the global existence of real analytic solutions of a single linear differential equation with constant coefficients $P(D)u=f$ is discussed on a relatively compact open set $\Omega \subset \boldsymbol{R}^n$. Recently Hörmander [1] gave a necessary and sufficient condition for the solvability of $P(D)u=f$ assuming that Ω is *convex*. Once one restricts oneself to the consideration of convex sets, one can easily prove the global existence of real analytic solutions of the overdetermined system of linear differential equations with constant coefficients which is of "de Rham type" in the sense of Sato-Kawai-Kashiwara [6] by the aid of "micro-local" analysis (Cf. Kawai [3]). The purpose of this paper is to give a proof of this theorem. Note that the restriction to the consideration of "de Rham type" is a natural one in view of the above quoted results of Hörmander [1].

The system $\mathcal{M}$ of linear differential equations which we consider in this paper is always assumed to satisfy the following condition.

(1) It is with constant coefficients, i.e. it has the form $P(D)u=0$, where $P(D)$ is an $r\times l$ matrix of linear differential operators with constant coefficients of finite order.

(2) Its characteristic variety V is real and non-singular.

(3) It is purely d-dimensional for some d, i.e. $\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{M}, \mathcal{P})=0$ for $j\neq d$. (Here $\mathcal{P}$ denotes the sheaf of pseudo-differential operators.)

We consider the solvability of such equations on a relatively compact convex open set Ω with C^∞ boundary $\partial\Omega$ in $\boldsymbol{R}^n$. We denote by $\mathcal{A}(\Omega)$ and $\mathcal{A}(\bar{\Omega})$ the space of real analytic functions on Ω and $\bar{\Omega}$ respectively.

Our theorem is as follows:

THEOREM. *Let $\mathcal{M}$ and Ω satisfy the above conditions. Assume that the r-vector $f(x)$ of real analytic functions on Ω satisfies the compatibility conditions. Then the equation $P(D)u(x)=f(x)$ has a real analytic solution $u(x)$ in $\mathcal{A}(\Omega)^l$.*

PROOF. We first prove that

$$\rho: \operatorname{Ext}^0_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}/\mathcal{A}) \longrightarrow \operatorname{Ext}^0_{\mathcal{D}}(\Omega; \mathcal{M}, \mathcal{B}/\mathcal{A})$$

is surjective. Hereafter, when we use the same notations as in our previous paper (Kawai [2]) and Sato-Kawai-Kashiwara [6], we do not repeat their definitions.

Let $V_0 = \{(x, \sqrt{-1}\,\eta) \in \sqrt{-1}S^*\boldsymbol{R}^n \mid x \in \partial\Omega,\ \eta \in V$ and $\pi(b_{(x,\sqrt{-1}\eta)})$, the projection of the bicharacteristic manifold passing through $(x, \sqrt{-1}\eta)$ to $\boldsymbol{R}^n$, is tangent to $\partial\Omega\}$. If $\mu(x)$ is in $\operatorname{Ext}^0_{\mathcal{D}}(\Omega; \mathcal{M}, \mathcal{B}/\mathcal{A})$, then assumptions (2) and (3) imply that $\mu(x)$ can be extended uniquely to $\bar{\Omega} - \pi(V_0)$. (Sato-Kawai-Kashiwara [6] Chapter III Theorem 2.1.8.) Moreover for any $(x, \sqrt{-1}\eta) \in V_0$ we can find its open neighborhood ω so that $b_{(x,\sqrt{-1}\eta)}$ intersects $\sqrt{-1}S^*N_\omega \underset{N_\omega}{\times} V$ transversally there for a suitable hypersurface $N_\omega \subset \boldsymbol{R}^n$ passing through x. Then the above quoted theorem of Sato-Kawai-Kashiwara [6] shows $\mu(x)$ can be extended to ω as a microfunction solution of the equation $P(D)\mu = 0$ if we take ω sufficiently small. Since V_0 is compact, we can find $\{\omega_j\}_{j=1}^N$ so that $\bigcup_{j=1}^N \omega_j \supset V_0$ and $\mu(x)$ can be extended to ω_j. Taking a neighborhood U of $\bar{\Omega}$ sufficiently small, we can assume that $b_{(x,\sqrt{-1}\eta)}$ and $b_{(x',\sqrt{-1}\eta')}$ do not intersect mutually in $\sqrt{-1}S^*U - \sqrt{-1}S^*\Omega$ if $(x, \sqrt{-1}\eta) \in \omega_j \cap (\sqrt{-1}S^*U - \sqrt{-1}S^*\Omega)$ and $(x', \sqrt{-1}\eta') \in \omega_k \cap (\sqrt{-1}S^*U - \sqrt{-1}S^*\Omega)$ $(j \neq k)$. Since the support of the above extention of $\mu(x)$ is invariant under Hamiltonian flow associated with V, $\mu(x)$ is seen to be extended consistently to $\bar{\Omega}$. Thus we have proved the restriction map ρ is surjective.

On the other hand $\operatorname{Ext}^1_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}) = 0$ by a theorem of Komatsu [5] since $\bar{\Omega}$ is convex. Moreover $\operatorname{Ext}^2_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{A}) = 0$ by a theorem of Ehrenpreis, Malgrange and Komatsu, since $\bar{\Omega}$ is compact and convex. (See e.g. Komatsu [4].) Therefore the following exact sequence (4) implies $\operatorname{Ext}^1_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}/\mathcal{A}) = 0$.

$$\begin{aligned} (4) \qquad \cdots \longrightarrow \operatorname{Ext}^1_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}) &\longrightarrow \operatorname{Ext}^1_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}/\mathcal{A}) \\ &\longrightarrow \operatorname{Ext}^2_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{A}) \longrightarrow \cdots. \end{aligned}$$

Therefore the surjectivity of ρ implies the vanishing of $\operatorname{Ext}^1_{\mathcal{D},\partial\Omega}(\bar{\Omega}; \mathcal{M}, \mathcal{B}/\mathcal{A})$ since the following exact sequence holds:

$$\begin{aligned} (5) \qquad \cdots \longrightarrow \operatorname{Ext}^0_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}/\mathcal{A}) &\xrightarrow{\rho} \operatorname{Ext}^0_{\mathcal{D}}(\Omega; \mathcal{M}, \mathcal{B}/\mathcal{A}) \\ \longrightarrow \operatorname{Ext}^1_{\mathcal{D},\partial\Omega}(\bar{\Omega}; \mathcal{M}, \mathcal{B}/\mathcal{A}) &\longrightarrow \operatorname{Ext}^1_{\mathcal{D}}(\bar{\Omega}; \mathcal{M}, \mathcal{B}/\mathcal{A}) \longrightarrow \cdots. \end{aligned}$$

Now we use the existence theorem of Komatsu [5] and find easily that

$$(6) \qquad \operatorname{Ext}^2_{\mathcal{D},\partial\Omega}(\bar{\Omega}; \mathcal{M}, \mathcal{B}) = 0.$$

In fact, taking a fundamental system of open neighborhoods $\{U_j\}_{j=1}^\infty$ of $\bar{\Omega}$ so that they are convex, Komatsu's theorem implies that $\mathrm{Ext}^1_{\mathscr{D}}(\Omega\,;\mathscr{M},\mathscr{B})=\mathrm{Ext}^2_{\mathscr{D}}(U_j\,;\mathscr{M},\mathscr{B})=0$. Therefore the exact sequence

$$\begin{aligned}(7)\qquad &\cdots\longrightarrow \mathrm{Ext}^1_{\mathscr{D}}(\Omega\,;\mathscr{M},\mathscr{B})\longrightarrow \mathrm{Ext}^2_{\mathscr{D},U_j\cap(R^n-\Omega)}(U_j\,;\mathscr{M},\mathscr{B})\\ &\longrightarrow \mathrm{Ext}^2_{\mathscr{D}}(U_j\,;\mathscr{M},\mathscr{B})\longrightarrow\cdots\end{aligned}$$

proves the vanishing of $\mathrm{Ext}^2_{\mathscr{D},U_j\cap(R^n-\Omega)}(U_j\,;\mathscr{M},\mathscr{B})$, whence the vanishing of $\mathrm{Ext}^2_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{B})$ by the definition.

In passing, we consider the following exact sequence:

$$\begin{aligned}(8)\qquad &\cdots\longrightarrow \mathrm{Ext}^1_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{B}/\mathscr{A})\longrightarrow \mathrm{Ext}^2_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{A})\\ &\longrightarrow \mathrm{Ext}^2_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{B})\longrightarrow\cdots.\end{aligned}$$

Then the vanishing of $\mathrm{Ext}^1_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{B}/\mathscr{A})$ and $\mathrm{Ext}^2_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{B})$ proves that of $\mathrm{Ext}^2_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{A})$. Therefore we conclude that $\mathrm{Ext}^1_{\mathscr{D}}(\Omega\,;\mathscr{M},\mathscr{A})$ vanishes since the following exact sequence (9) exists:

$$\begin{aligned}(9)\qquad &\cdots\longrightarrow \mathrm{Ext}^1_{\mathscr{D}}(\bar{\Omega}\,;\mathscr{M},\mathscr{A})\longrightarrow \mathrm{Ext}^1_{\mathscr{D}}(\Omega\,;\mathscr{M},\mathscr{A})\\ &\longrightarrow \mathrm{Ext}^2_{\mathscr{D},\partial\Omega}(\bar{\Omega}\,;\mathscr{M},\mathscr{A})\longrightarrow\cdots.\end{aligned}$$

Note that the above quoted result of Ehrenpreis, Malgrange and Komatsu concerning the existence of real analytic solutions on a compact convex set means that $\mathrm{Ext}^1_{\mathscr{D}}(\bar{\Omega}\,;\mathscr{M},\mathscr{A})=0$.

The vanishing of $\mathrm{Ext}^1_{\mathscr{D}}(\Omega\,;\mathscr{M},\mathscr{A})$ is nothing but the global existence of a real analytic solution $u(x)$ of the equation $P(D)u(x)=f(x)$ when $f(x)$ satisfies the compatibility conditions. This completes the proof of the theorem.

REMARK. It is also possible to show the vanishing of $\mathrm{Ext}^j_{\mathscr{D}}(\Omega\,;\mathscr{M},\mathscr{A})$ $(j>1)$ by the aid of "micro-local" analysis under the same assumptions as in the theorem. This problem will be discussed somewhere else.

References

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Added in proof. Mr. T. Miwa has recently given a necessary and sufficient condition for the vanishing of $\mathrm{Ext}^1(\Omega\,;\mathcal{M},\mathcal{A})$ under the assumption that Ω is convex and that $\mathcal{M}$ is a system of linear differential equations with constant coefficients. His way of proof relies on the excellent and ingenious idea of Hörmander [1]. See T. Miwa: on the global existence of real analytic solutions of linear partial differential equations with constant coefficients, to appear in Proc. Japan Acad., **49**, No. 7.