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Explicit formula of the traces of Hecke operators for $\Gamma_0(N)$

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0.0. Let $\Gamma=\Gamma_0(N)$ be the congruence subgroup of level N, i. e. the group consisting of all two by two integral unimodular matrices $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c\equiv 0 \bmod N$. χ being a multiplicative character modulo N, let $S_0(\Gamma, \{k\}, \chi)$ be the space of cusp forms of weight k, and let $T(n)$ be the Hecke operator acting on $S_0(\Gamma, \{k\}, \chi)$ (see 1.1 and 5.5 for the definition).

When N is square free, M. Eichler ([**3**], [**4**]) gave an explicit formula for the trace $\operatorname{tr} T(n)$ of $T(n)$ on $S_0(\Gamma, \{k\}, \chi)$ with several interesting applications, and he suggested ([**3**] p. 168–169) that it might be interesting to refine the arithmetic of quaternion algebras so that one can handle the square level cases or principal congruence subgroups. Since then several authors took up the problem. For example, H. Shimizu [**9**] generalized Γ to be the higher dimensional (Hilbert modular type) ones (still with square free level), M. Yamauchi [**11**] generalized Γ to be the ones with level $4N'$ (where N' is odd and square free).

In this paper, we have again taken up the problem suggested by Eichler. We can give an explicit formula of $\operatorname{tr} T(n)$ for $\Gamma_0(N)$ (and its normal subgroups of Fricke type) for arbitrary N and for their analogues obtained from indefinite quaternions. In the following, we shall write down a 'ready to compute' formula of $\operatorname{tr} T(n)$ for $\Gamma_0(N)$ and its normal subgroups $\Gamma(\mathfrak{h})$ defined as follows. Let M be a divisor of N, let $\mathfrak{h}$ be a subgroup of $(\boldsymbol{Z}/M\boldsymbol{Z})^\times$, and let $\Gamma=\Gamma(\mathfrak{h})$ denote the subgroup of $\Gamma_0(N)$ consisting of the elements $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that a lies in $\mathfrak{h}$ modulo M. (For example, if we take $N=M^2$ and $\mathfrak{h}=\{1\}$, then $\Gamma(\mathfrak{h})$ is conjugate to the principal congruence subgroup $\Gamma(M)$.) Let χ be a character mod M, also considered as a linear character of $\Gamma(\mathfrak{h})$ via $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mapsto\chi(a)$. Assume n is prime to the level N and $k\geqq 2$, then, for any (M and) $\mathfrak{h}$, the trace of $T(n)$ acting on $S_0(\Gamma(\mathfrak{h}), k, \chi)$ is given by the following theorem.

0.1. THEOREM.

$$\operatorname{tr} T(n) = -\varphi(M)|\mathfrak{h}|^{-1} \sum_{s} a(s) \sum_{f} b(s,f)c(s,f) + \delta(\chi) \prod_{p|n} \frac{1-p^{\tau+1}}{1-p}$$
$$+\varphi(M)|\mathfrak{h}|^{-1}\delta(\sqrt{n})\frac{k-1}{12} \prod_{p|N} p^{\nu}\left(1+\frac{1}{p}\right) \prod_{p|M} \chi_p(\sqrt{n}).$$

Notation is: $N=\prod_{p|N} p^{\nu}$, $M=\prod_{p|M} p^{\mu}$, $n=\prod_{p|n} p^{\tau}$; $|\mathfrak{h}|=$ *the cardinality of* $\mathfrak{h}$; φ *is the Euler function,*

i. e. $\varphi(M)=|(\boldsymbol{Z}/M\boldsymbol{Z})^{\times}|$;

$$\delta(\chi)=\begin{cases} 1 & \text{if } k=2 \text{ and } \chi \text{ is a trivial character} \\ 0 & \text{otherwise} \end{cases};$$

$$\delta(\sqrt{n})=\begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise}. \end{cases}$$

The meaning of s, $a(s)$, f, $b(s,f)$ *and* $c(s,f)$ *are given in the following.*

Let s run through all the integers such that s^2-4n is not a positive non-square, hence by some positive integer t, s^2-4n has one of the following forms which we will call the case (p), (h), $(e1)$ or $(e23)$ respectively.

$$s^2-4n=\begin{cases} 0 & \cdots\cdots\ (p) \\ t^2 & \cdots\cdots\ (h) \\ t^2m \quad 0>m\equiv 1 \pmod 4 & \cdots\cdots\ (e1) \\ t^2 4m \quad 0>m\equiv 2,\,3 \pmod 4 & \cdots\cdots\ (e23) \end{cases} \Bigg\} (e).$$

Let $\Phi(X)=\Phi_s(X)=X^2-sX+n$, and let x and y be the solutions of $\Phi(X)=0$. Corresponding to the type of s, put:

$$a(s)=\begin{cases} (4M)^{-1}|x|(\operatorname{sgn} x)^k & \cdots\cdots\ (p) \\ (\operatorname{Min}\{|x|,\ |y|\})^{k-1}|x-y|^{-1}(\operatorname{sgn} x)^k n^{1-k/2} & \cdots\cdots\ (h) \\ \frac{1}{2}(x^{k-1}-y^{k-1})(x-y)^{-1}n^{1-k/2} & \cdots\cdots\ (e). \end{cases}$$

For each s fixed, corresponding to its type, let f run as follows:

$$f=\begin{cases} 1,2,\cdots,M & \cdots\cdots\ (p) \\ \text{all the positive divisors of } t & \cdots\cdots\ (h) \text{ and } (e) \end{cases}$$

and

$$b(s,f)=\begin{cases} 1 & \cdots\cdots\ (p) \\ \frac{1}{2}\varphi((s^2-4n)^{1/2}/f) & \cdots\cdots\ (h) \\ h((s^2-4n)/f^2)/w((s^2-4n)/f^2) & \cdots\cdots\ (e) \end{cases}$$

where φ is the Euler function, $h(d)$ (resp. $w(d)$) denotes the class number of primitive ideals (resp. 1/2 of the cardinality of the unit group) of the order of $\boldsymbol{Q}(\sqrt{d})$ with the discriminant d.

For a pair (s, f) fixed and a prime divisor p of N, let $\rho=\mathrm{ord}_p(f)$ and put

$$\tilde{A}=\{x\in\boldsymbol{Z}\mid \Phi(x)\equiv 0 \pmod{p^{\nu+2\rho}},\ 2x\equiv s \pmod{p^{\rho}}\}\ ,$$

$$\tilde{B}=\{x\in\tilde{A}\mid \Phi(x)\equiv 0 \pmod{p^{\nu+2\rho+1}}\}\ .$$

Let $A=A(s, f, p)$ (resp. $B=B(s, f, p)$) be a complete system of representatives of $\tilde{A}$ (resp. $\tilde{B}$) modulo $p^{\nu+\rho}$, and for each $\eta\in(\boldsymbol{Z}/M\boldsymbol{Z})^{\times}$, put:

$$A_{\eta}(s, f, p)=\{x\mid x\in A,\ x\equiv\eta \bmod p^{\mu}\}$$

$$B'_{\eta}(s, f, p)=\{s-x\mid x\in B,\ s-x\equiv\eta \bmod p^{\mu}\}$$

and

$$c_{\eta}(s, f, p)=\begin{cases}\text{the cardinality } |A_{\eta}(s, f, p)| \text{ of } A_{\eta}(s, f, p) \text{ if } (s^2-4n)/f^2\not\equiv 0 \bmod p\\ |A_{\eta}(s, f, p)|+|B'_{\eta}(s, f, p)| \text{ if } (s^2-4n)/f^2\equiv 0 \bmod p\,.\end{cases}$$

Then

$$c(s, f)=\sum_{\eta\in\mathfrak{h}}\chi(\eta)\prod_{p\mid N}c_{\eta}(s, f, p)\,.$$

If we assume that $\mathfrak{h}$ is a direct product $\prod_{p\mid M}\mathfrak{h}_p$ of $\mathfrak{h}_p\subset(\boldsymbol{Z}/p^{\mu}\boldsymbol{Z})^{\times}$ and χ_p denotes the restriction of χ to $(\boldsymbol{Z}/p^{\mu}\boldsymbol{Z})^{\times}$, $c(s, f)$ can be given by:

$$c(s, f)=\prod_{p\mid N}c(s, f, p)$$

with

$$c(s, f, p)=\begin{cases}(\sum_x \chi_p(x)) & \text{if } (s^2-4n)/f^2\not\equiv 0 \bmod p\\ (\sum_x \chi_p(x)+\sum_y \chi_p(y)) & \text{if } (s^2-4n)/f^2\equiv 0 \bmod p\end{cases}$$

where x runs over all elements of $\bigcup_{\eta\in\mathfrak{h}_p}A_{\eta}(s, f, p)$ and y runs over all elements of $\bigcup_{\eta\in\mathfrak{h}_p}B'(s, f, p)$.

0.2. A proof of 0.1 will be given in the final section. We shall explain the content of this paper as a way to it. In doing so, although our results cover the groups obtained from quaternions, for simplicity sake, we pretend as if we are considering as Γ only arithmetic subgroups of $GL_2(\boldsymbol{Q})$. In the first section, a result of Shimizu [8] is recalled (1.2. formula (1)). Let T be the Hecke operator corresponding to a union $\Xi=\cup\Gamma\alpha_j\Gamma$ of Γ-double cosets. Neglecting a limit process $\lim_{s\to 0}$, the main part of $\mathrm{tr}\,T$ is given as a sum $\sum_g \kappa(g)\lambda(g)$ where g is running through the quotient $\Omega/\tilde{\Gamma}$ of the set Ω of all the elliptic elements, hyperbolic elements fixing a cusp of Γ or parabolic elements in Ξ modulo Γ-conjugacy. κ and λ are functions on Ω, and κ depends only on the eigenvalues of g. By a simple Remark 1.4, we can replace the sum by a similar one extended over $\Omega/\tilde{\Gamma}^*$, i. e. $\sum_{g\in\Omega/\tilde{\Gamma}^*}\kappa(g)\lambda^*(g)$, where Γ^* is a group normalizing Γ in some nice way. For example, putting $R=\begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z}\end{pmatrix}$, we can take the unit group $R^\times$ as Γ^* for any $\Gamma(\mathfrak{h})\subset\Gamma_0(N)$ in 0.0. Then we observe (3.4) that $\lambda^*(g)=\lambda^*(g')$ if g is locally equivalent to g', i. e. for each p, there exists $x_p\in(R\otimes\boldsymbol{Z}_p)^\times$ such that $g'=x_pgx_p^{-1}$ (if $\Gamma^*=R^\times$). Hence it suffices to know the quotient of Ω by the local equivalence and the number $\sharp(g)$ of Γ^*-conjugacy classes contained in the local equivalence class of g. Define an open subgroup $\mathfrak{U}^*$ of the adele group $GL_2(\boldsymbol{A})$ by $\mathfrak{U}^*=\prod_p(R\otimes\boldsymbol{Z}_p)^\times$, then we have a natural injection

$$\theta:\ \Omega/\text{local equivalence}\longrightarrow GL_2(\boldsymbol{A})/\tilde{\mathfrak{U}}^*\,.$$

Using the strong approximation theorem and class field theory (3.5~3.8), the number $\sharp(g)$ can be written in terms of the class number of the order $R\cap(\boldsymbol{Q}+\boldsymbol{Q}g)$ in the $\boldsymbol{Q}$-algebra $\boldsymbol{Q}+\boldsymbol{Q}g$ generated by g (which gives the term $b(s,f)$ in 0.1).

If Ω contains no parabolic elements, or the class number of R is one, determination of Image θ can be completely localized, and (essentially) it amounts to determine (up to $R_p^\times$-equivalence) the optimal embeddings of the orders of $\boldsymbol{Q}_p+\boldsymbol{Q}_pg$ into R_p for each p. Then, by the theorem of Chevalley-Hasse-Noether [10], it is enough to consider a finite number of p's. If $R=\begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z}\end{pmatrix}$, it suffices to check the p's dividing N, and the optimal embeddings into the orders of the form $\begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ p^\nu\boldsymbol{Z}_p & \boldsymbol{Z}_p\end{pmatrix}$ are completely determined in § 2, and the computation for it corresponds to the term $c(s,f)$ in 0.1.

0.3. The following notation will be used. (Some of them were already used in 0.0~0.2.)

$\emptyset$ = the vacant set, $\boldsymbol{Z}$ = the ring of integers, $\boldsymbol{Q}$ (resp. $\boldsymbol{R}$; resp. $\boldsymbol{C}$) = the field of rational (resp. real; resp. complex) numbers. For a non-zero real number x, $\operatorname{sgn} x=1$ (resp. -1) if $x>0$ (resp. $x<0$).

Let S be a set, then $|S|$ = the cardinality of S. (However if $x\in\boldsymbol{C}$, $|x|$ = the ordinary absolute value.) If S is a subset of a group G, $\langle S\rangle$ = the subgroup of G generated by S.

Let S be a ring with the unity 1. $S^\times$ = the multiplicative group of the invertible elements of S. For any positive integer r, $M_r(S)$ = the ring of r by r matrices over S, and $GL_r(S)=M_r(S)^\times$. If S is commutative, $\det s$ (resp. $\operatorname{tr} s$) = the determinant (resp. trace) of s for $s\in M_r(S)$. For subsets S_{ij} $(1\leqq i, j\leqq r)$ of S, $(S_{ij})\subset M_r(S)$ denotes the set $\{s=(s_{ij})\in M_r(S)\,|\,s_{ij}\in S_{ij}\}$.

Let G be a group, H be a subgroup of G and Z be a subgroup of the center of G. We call an element g of G is *(H, Z)-equivalent* to another element g' of G, if there exist $h \in H$ and $z \in Z$ such that $g = zhgh^{-1}$. If $Z = \{1\}$, we simply call *H-equivalent* and write $g' \underset{H}{\sim} g$, i.e. $g' \underset{H}{\sim} g \Leftrightarrow {}^{\exists}h \in H$, $g' = hgh^{-1}$. For a subset X of G, and an equivalence relation $\sim$ in G, let $X/\sim$ denote the quotient of X by $\sim$, and sometimes let it denote also a complete system of representatives of X modulo $\sim$. If $\sim$ is the relation $\underset{H}{\sim}$ defined above, we write $X/\tilde{H}$ instead of $X/\sim$.

1.0. Let n be a positive integer. Let $\mathfrak{G}$ be a direct product of n copies of $GL_2(\boldsymbol{R})$. Let $\mathfrak{H}$ denote the complex upper half plane, $\mathfrak{H}_{\pm}$ the union of the upper and lower half plane, and $\mathfrak{H}^n$ (resp. $\mathfrak{H}_{\pm}^n$) be the product of n copies of $\mathfrak{H}$ (resp. $\mathfrak{H}_{\pm}$). For an element g (resp. z) of $\mathfrak{G}$ (resp. $\mathfrak{H}_{\pm}^n$), let $g^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$ $\in GL_2(\boldsymbol{R})$ (resp. $z^{(i)} \in \mathfrak{H}_{\pm}$) be its i-th coordinate for $i = 1, \cdots, n$. $\mathfrak{G}$ acts on $\mathfrak{H}^n$ by $g(z)^{(i)} = \dfrac{a^{(i)}z^{(i)} + b^{(i)}}{c^{(i)}z^{(i)} + d^{(i)}}$, as an analytic transformation group. The connected component of the identity $\mathfrak{G}^+$ of $\mathfrak{G}$ consists of elements $g = (g^{(i)})$ such that $\det g^{(i)} > 0$ for all i, and it is acting on $\mathfrak{H}^n$. The center $\mathfrak{Z}$ of $\mathfrak{G}$ consists of elements $g = (g^{(i)})$ such that all $g^{(i)}$ are scalars, it acts on $\mathfrak{H}_{\pm}^n$ trivially, and it can be identified with the direct product of n-copies of $\boldsymbol{R}^{\times}$. Let $\iota: \mathfrak{G} \rightarrow \mathfrak{G}/\mathfrak{Z}$ denote the canonical projection.

Following the formulation of Shimizu [8], we consider a triple $\{\Gamma, \Xi, \chi\}$ satisfying the following three conditions.

(Γ1) Γ is a subgroup of $\mathfrak{G}$. Put $\Gamma^+ = \mathfrak{G}^+ \cap \Gamma$, then $\iota(\Gamma^+)$ is irreducible and acting discontinuously on $\mathfrak{H}^n$ in the sense of [7], and it has a fundamental domain of finite volume satisfying the condition (F) in [7] p. 48.

(Γ2) Ξ is a finite union of distinct double cosets $\Gamma\alpha_j\Gamma$, $\alpha_j \in \mathfrak{G}$ $(1 \leqq j \leqq u)$. $\alpha_j\Gamma\alpha_j^{-1}$ is commensurable with Γ for any j.

(Γ3) $\chi: \langle\Xi\rangle \rightarrow GL_r(\boldsymbol{C})$ is a unitary representation of the group $\langle\Xi\rangle$ generated by Ξ, such that the kernel Γ_χ of χ in Γ is of finite index in Γ.

1.1. Let $\{k_i\} = \{k_1, \cdots, k_n\}$ be a n-tupple of positive integers. For a triple $(\Gamma, \{k_i\}, \chi)$, the space of cusp forms $S = S(\Gamma, \{k_i\}, \chi)$ is the $\boldsymbol{C}$-vector space of functions $f: \mathfrak{H}_{\pm}^n \rightarrow \boldsymbol{C}^r$ satisfying the following three conditions.

(S1) $f(z)$ is holomorphic on each connected component of $\mathfrak{H}_{\pm}^n$.

(S2) $f(\gamma z) = j(\gamma, z)^{-1}\chi(\gamma)f(z)$ for any $\gamma \in \Gamma$, where

$$j(g, z) = \prod_{i=1}^{n} (c^{(i)}z^{(i)} + d^{(i)})^{-k_i} |\det g^{(i)}|^{k_i/2} \qquad \text{for } g \in \mathfrak{G}.$$

(S3) In case $\mathfrak{H}^n/\Gamma^+$ is not compact, $f(z)$ is regular at every parabolic point z of Γ_χ, and the constant term in the Fourier expansion of f at z vanishes. As is easily seen, $S(\Gamma, \{k_i\}, \chi) \neq \{0\}$ only if the following condition (χ-k) is satisfied.

$$(\chi\text{-}k) \qquad \chi(\varepsilon)=\prod_{i=1}^{n}\operatorname{sgn}(\varepsilon^{(i)})^{k_i} \qquad \text{for any } \varepsilon\in\mathfrak{Z}\cap\Gamma.$$

Hence without any important loss of generality, we may and shall assume that $(\chi\text{-}k)$ is satisfied by $(\Gamma, \{k_i\}, \chi)$.

Let $\alpha\in\Xi$, $\Gamma\alpha\Gamma=\bigcup_{\nu=1}^{d}\beta_\nu\Gamma$ be the disjoint union, where d is called the degree of $\Gamma\alpha\Gamma$. The Hecke operator $T(\Gamma\alpha\Gamma)$ is a linear map on S defined by

$$(1) \qquad (T(\Gamma\alpha\Gamma)f)(z)=\sum_{\nu=1}^{d} j(\beta_\nu^{-1}, z)\chi(\beta_\nu)f(\beta_\nu^{-1}z).$$

For a formal sum $\eta=\sum_{j=1}^{t} c_j(\Gamma\alpha_j\Gamma)$, $c_j\in\boldsymbol{C}$, $\alpha_j\in\Xi$, define $T(\eta)=\sum c_jT(\Gamma\alpha_j\Gamma)$. η is called simple if each c_j is 1. The purpose of this paper is to compute the trace $\operatorname{tr}T(\eta)$. It is enough to do that when η is simple. Hence in the following, we assume

$$(2) \qquad \xi=\sum_{j=1}^{u}\Gamma\alpha_j\Gamma \text{ is simple, and } \Xi=\bigcup_{j=1}^{u}\Gamma\alpha_j\Gamma.$$

By the general method of Selberg or Eichler, to compute the trace $\operatorname{tr}T(\xi)$ is to compute a certain integral, and under certain circumstances the integral is expressible by a sum extended over some conjugacy classes induced by Γ. We have such a kind of formula in the following cases; by Eichler [4] where $n=1$ $k_i\geqq 2$ and χ is real; by Shimizu [8] where $k_i>2$ (in case $n\geqq 2$, under the mild assumption (F) on the shape of the fundamental domain of Γ); and by Saito [6] where $n=1$, $k_1=2$, $r=\deg\chi=1$.

1.2. It reads:

$$(1) \qquad \operatorname{tr}T(\xi)=t+t'+t'',$$

$$t'=\sum_{j=1}^{u}t'_j,$$

$$t'_j=\begin{cases}\operatorname{vol}(\mathfrak{H}_\pm^n/\Gamma)\operatorname{tr}\chi(\varepsilon)\prod_{i=1}^{n}(\operatorname{sgn}\varepsilon^{(i)})^{k_i}\left(\dfrac{k_i-1}{4\pi}\right) & \text{if } \varepsilon\in\mathfrak{Z}\cap\Gamma\alpha_j\Gamma\neq\emptyset\\ 0 \quad \text{if } \mathfrak{Z}\cap\Gamma\alpha_j\Gamma=\emptyset,\end{cases}$$

$$t''=2\deg\xi=\begin{cases}2\sum_{j=1}^{u}[\Gamma:\alpha_j^{-1}\Gamma\alpha_j\cap\Gamma] & \text{if } k=2,\ n=1 \text{ and } \chi \text{ trivial}\\ 0 & \text{otherwise},\end{cases}$$

$$t=-\lim_{s\to 0}s\sum_{g\in\Omega/\mathcal{L}}k(g)\lambda'(g),$$

where $\operatorname{vol}(\mathfrak{H}_\pm^n/\Gamma)$ denots the volume of the fundamental domain $\mathfrak{H}_\pm^n/\Gamma$ relative to the measure induced by dz, s in the formula of t is a complex variable, the meaning of $\Omega/\mathcal{L}$, $k(g)$ and $\lambda'(g)$ shall be given later on.

Let Ω_e (resp. Ω_h; resp. Ω_p) denote the set of all elliptic elements (resp. hyperbolic elements fixing the cusps of Γ; resp. parabolic elements) in Ξ. Put $\Omega=\Omega_e$ if $n>1$, and $\Omega=\Omega_e\cup\Omega_h\cup\Omega_p$ if $n=1$. Then in the terminology of 0.3, $\stackrel{\prime}{\sim}$ means the $(\Gamma, \mathfrak{Z}\cap\Gamma)$-equivalence, i. e.

$$g\stackrel{\prime}{\sim}g' \Leftrightarrow {}^{\exists}\varepsilon\in\mathfrak{Z}\cap\Gamma,\ {}^{\exists}\gamma\in\Gamma \quad \text{s. t.} \quad g'=\varepsilon\gamma g\gamma^{-1},$$

and $\Omega/\stackrel{\prime}{\sim}$ denote a complete system of representatives of Ω modulo $\stackrel{\prime}{\sim}$. Put

$$\Gamma'(g)=\{\gamma\in\Gamma \mid {}^{\exists}\varepsilon\in\mathfrak{Z}\cap\Gamma \quad \text{s. t.} \quad \gamma g\gamma^{-1}=\varepsilon g\}.$$

Let ζ_i and η_i be eigen values of $g^{(i)}$. If $n=1$, $g\in\Omega_p$ and x is a fixed point of g, we can find $t\in\mathfrak{G}$ such that $t(x)=\infty$. Then any parabolic element u of $\mathfrak{G}$ fixing x acts on $\mathfrak{H}_\pm$ as $t^{-1}ut(z)=z\pm m(u)$ by some positive real $m(u)$. Finally let $d(g)$ be the least positive value of $m(u)$ when u runs through $\Gamma'(g)$. Then we put:

$$(2)\qquad k(g)=\begin{cases} \prod_{i=1}^{n}(\zeta_i^{k_i-1}-\eta_i^{k_i-1})(\zeta_i-\eta_i)^{-1}(\det g^{(i)})^{1-k_i/2} & \text{if } g\in\Omega_e \\ 2(\mathrm{Min}\,\{|\zeta_1|, |\eta_1|\}^{k_1-1}|\zeta_1-\eta_1|^{-1}(\det g)^{1-k_1/2}(\operatorname{sgn}\zeta_1)^{k_1} & \text{if } g\in\Omega_h \\ \frac{1}{2}(\operatorname{sgn}\zeta_1)^{k_1} & \text{if } g\in\Omega_p; \end{cases}$$

$$\lambda'(g)=\begin{cases} s^{-1}\operatorname{tr}\chi(g)[\Gamma'(g):\mathfrak{Z}\cap\Gamma]^{-1} & \text{if } g\in\Omega,\ g\notin\Omega_p \\ (d(g)/m(g))^{1+s}\operatorname{tr}\chi(g) & \text{if } g\in\Omega_p. \end{cases}$$

We have tried to unify the expression of t for the convenience of later use. The following way is easier to understand:

$$(3)\qquad t=\begin{cases} t_e & \text{if } n>1 \\ t_e+t_h+t_p & \text{if } n=1, \end{cases}$$

$$t_e=-\sum_{g\in\Omega_e/\stackrel{\prime}{\sim}}\kappa(g)[\Gamma'(g):\mathfrak{Z}\cap\Gamma]^{-1}\operatorname{tr}\chi(g),$$

$$t_h=-\sum_{g\in\Omega_h/\stackrel{\prime}{\sim}}\kappa(g)[\Gamma'(g):\mathfrak{Z}\cap\Gamma]^{-1}\operatorname{tr}\chi(g),$$

$$t_p=-\lim_{s\to 0}s\sum_{g\in\Omega_p/\stackrel{\prime}{\sim}}\kappa(g)\Big(\frac{d(g)}{m(g)}\Big)^{1+s}\operatorname{tr}\chi(g).$$

To replace $\stackrel{\prime}{\sim}$ by something easier to deal with, let us introduce the following subgroup $\mathfrak{Z}_1$ of $\mathfrak{Z}$.

$$(4)\qquad \mathfrak{Z}_1=\{\varepsilon=(\varepsilon^{(i)})\in\mathfrak{Z} \mid \varepsilon^{(1)}>0\}.$$

Then $\mathfrak{Z}$ is a direct product of $\{\pm 1\}$ and $\mathfrak{Z}_1$, consequently $[\mathfrak{Z}\cap\Gamma:\mathfrak{Z}_1\cap\Gamma]$ is either 2 or 1 according as $\mathfrak{Z}_1$ contains $\mathfrak{Z}\cap\Gamma$ or not.

1.3. LEMMA. *Suppose Ξ satisfies the following condition* (Γ4).

(Γ4) (i) *If $n=1$, $\iota(\Gamma'(g))$ is commensurable with the group $\langle\iota(g)\rangle$ generated* *by* $\iota(g)$ *for any* $g\in\Omega_p$. (ii) *If* $\varepsilon\in\mathfrak{Z}\cap\Gamma$ *and* $\varepsilon^2=1$, *then* $\varepsilon=1$ *or* -1.

Put $\Gamma(g)=\{\gamma\in\Gamma\mid\gamma g\gamma^{-1}=g\}$ *and*

$$(1)\qquad [\mathfrak{Z}\cap\Gamma:\mathfrak{Z}_1\cap\Gamma]\lambda(g)=\begin{cases} s^{-1}\operatorname{tr}\chi(g)[\Gamma(g):\mathfrak{Z}\cap\Gamma]^{-1} & \text{if } g\in\Omega,\ g\notin\Omega_p \\ [\iota(\Gamma(g)):\langle\iota(g)\rangle]^{-(1+s)}\operatorname{tr}\chi(g) & \text{if } g\in\Omega_p, \end{cases}$$

where $[\iota(\Gamma(g)):\langle\iota(g)\rangle]=[\iota(\Gamma(g)):H][\langle\iota(g)\rangle:H]^{-1}$ *with* $H=\iota(\Gamma(g))\cap\langle\iota(g)\rangle$. *Then*

$$(2)\qquad t=-\lim_{s\to 0} s\sum_{g\in\Omega/\sim}\kappa(g)\lambda(g),$$

where $\sim$ *denote* $(\Gamma,\mathfrak{Z}_1\cap\Gamma)$*-equivalence.*

PROOF. By $(\Gamma 4)$ (i), we have $d(g)/m(g)=[\iota(\Gamma'(g)):\langle\iota(g)\rangle]^{-1}$. Since $xgx^{-1}=\varepsilon g$, $g\in\mathfrak{G}$, $\varepsilon\in\mathfrak{Z}$ implies $\det\varepsilon^{(i)}=(\varepsilon^{(i)})^2=1$, the assumption $(\Gamma 4)$ (ii) implies $\varepsilon=\pm 1$. We separate the case (I) where there exists $\gamma\in\Gamma$ s.t. $\gamma g\gamma^{-1}=-g$, and the case (II) where there is no $\gamma\in\Gamma$ s.t. $\gamma g\gamma^{-1}=-g$. Then we have

$$[\Gamma'(g):\Gamma(g)]=\begin{cases} [\mathfrak{Z}\cap\Gamma:\mathfrak{Z}_1\cap\Gamma] & \cdots\text{ Case (I)} \\ 1 & \cdots\text{ Case (II)}. \end{cases}$$

If the trace $\operatorname{tr} g$ of g is not zero, we can only have the case (II). In particular, if $g\in\Omega_p$, then $\Gamma(g)=\Gamma'(g)$. Consider the natural projection $p:\Omega/\sim\to\Omega/\overset{\prime}{\sim}$ and let $c(g)$ denote the cardinality of $p^{-1}(g)$ for $g\in\Omega/\overset{\prime}{\sim}$, then

$$c(g)=\begin{cases} 1 & \cdots\text{ Case (I)} \\ [\mathfrak{Z}\cap\Gamma:\mathfrak{Z}_1\cap\Gamma] & \cdots\text{ Case (II)}. \end{cases}$$

Since $\lambda'(g)=[\Gamma'(g):\Gamma(g)]^{-1}[\mathfrak{Z}\cap\Gamma:\mathfrak{Z}_1\cap\Gamma]\lambda(g)$ and $\sum_{g\in\Omega/\overset{\prime}{\sim}}\kappa(g)\lambda'(g)=\sum_{g\in\Omega/\sim}c(g)^{-1}\kappa(g)\lambda'(g)$, we get the desired formula (2).

1.4. REMARK. Both of the assumptions (i) and (ii) of $(\Gamma 4)$ are, as is readily seen, automatically satisfied by all arithmetic groups Γ in which we are mainly interested. The first of them might be satisfied by any Γ, but we do not have a proof.

Now we briefly discuss the situation where Γ is contained in the connected component $\mathfrak{G}^+$ of $\mathfrak{G}$.

Let D_ν $(0\leqq\nu\leqq 2^n-1)$ denote the connected compoment of $\mathfrak{H}^n_\pm$, and $D_0=\mathfrak{H}^n$. Let $S_\nu=S_\nu(\Gamma,\{k_i\},\chi)$ denote the subspace of $S(\Gamma,\{k_i\},\chi)$ consisting of the functions $f(z)$ having its support only on D_ν. If Ξ are contained in $\mathfrak{G}^+$, S is a direct sum of S_ν's, and each S_ν is invariant by $T(\xi)$. Let $T_\nu(\xi)$ denote the restriction of $T(\xi)$ on S_ν. In general, there is no reason to expect that $\operatorname{tr} T_\nu(\xi)$ is equal to $\operatorname{tr} T_\mu(\xi)$. However if there exists $g\in\mathfrak{G}$ such that $g\Gamma g^{-1}=\Gamma$, $g\Xi g^{-1}=\Xi$, $\chi(gxg^{-1})=\chi(x)$ for any $x\in\langle\Xi\rangle$ and $gD_\nu=D_\mu$, then as a representation of ξ, $T_\nu(\xi)$ is equivalent to $T_\mu(\xi)$, and they have the same trace. Hence suppose $\{\Gamma, \Xi, \chi\}$ satisfies the following.

(Γ5) There exists a subgroup Δ of $\mathfrak{G}$, which normalizes Γ, Ξ and χ, and permutes D_ν $(0 \leqq \nu \leqq 2^n-1)$ transitively.

Then $\operatorname{tr} T_\nu(\xi)$ is equal to $\operatorname{tr} T_\mu(\xi)$, and any of them, say $\operatorname{tr} T_0(\xi)$ is given by the following

(2) $\quad \operatorname{tr} T_0(\xi) = 2^{-n} \operatorname{tr} T(\xi)$.

1.5. LEMMA. *Let* $\{\Gamma, \Xi, \chi, \{k_i\}, \Gamma^*\}$ *satisfy the conditions* (Γ1)$\sim$(Γ3) 1.0, (Γ4) 1.3, (χ-k) 1.1 *and the following* (Γ^*) *and* (χ-k^*).

(Γ^*): *Γ^* is a subgroup of $\mathfrak{G}$, which normalizes Γ, Ξ and* $\operatorname{tr}\chi(gxg^{-1}) = \operatorname{tr}\chi(x)$ *for any $x \in \langle \Xi \rangle$ and $g \in \Gamma^*$.*

(χ-k^*) $$\chi(\varepsilon) = \prod_{i=1}^{n} (\operatorname{sgn} \varepsilon^{(i)})^{k_i} \qquad \textit{for any } \varepsilon \in \mathfrak{Z} \cap \Gamma^*.$$

If $[\Gamma^* : \Gamma]$ is finite, then t in 1.2 *is given by the following:*

$$t = -\lim_{s\to 0} s[\Gamma^* : \Gamma][\mathfrak{Z} \cap \Gamma^* : \mathfrak{Z}_1 \cap \Gamma^*]^{-1} \sum_{g \in \Omega/\overset{*}{\sim}} \kappa(g)\lambda^*(g),$$

where $\overset{*}{\sim}$ denotes $(\Gamma^*, \mathfrak{Z}_1 \cap \Gamma^*)$-equivalence, i. e. $g \overset{*}{\sim} g' \Leftrightarrow {}^\exists\varepsilon \in \mathfrak{Z}_1 \cap \Gamma^*$, ${}^\exists\gamma \in \Gamma^*$ such that $g' = \varepsilon\gamma^{-1}g\gamma$, and $\lambda^*(g)$ is defined by:

$$\lambda^*(g) = \begin{cases} 1/s \operatorname{tr}\chi(g)[\Gamma^*(g) : \mathfrak{Z} \cap \Gamma^*]^{-1} & \textit{if } g \in \Omega,\ g \notin \Omega_p \\ [\iota(\Gamma^*(g)) : \langle \iota(g) \rangle]^{-(1+s)} \operatorname{tr}\chi(g) & \textit{if } g \in \Omega_p. \end{cases}$$

PROOF. From the assumptions, it is easy to see that $g \overset{*}{\sim} g'$ implies $\kappa(g)\lambda(g) = \kappa(g')\lambda(g')$. Hence, let $c(g)$ denote the number of $\sim$ classes contained in the $\overset{*}{\sim}$ class of g, then $\sum_{g \in \Omega/\sim} \kappa(g)\lambda(g) = \sum_{g \in \Omega/\overset{*}{\sim}} c(g)\kappa(g)\lambda(g)$.

To compute $c(g)$, pick up two elements $\varepsilon_i\delta_i g\delta_i^{-1}$ ($\varepsilon_i \in \mathfrak{Z}_1 \cap \Gamma^*$, $\delta_i \in \Gamma^*$) from $\overset{*}{\sim}$ class of g and observe: $\varepsilon_1\delta_1 g\delta_1^{-1} \sim \varepsilon_2\delta_2 g\delta_2^{-1} \Leftrightarrow {}^\exists\varepsilon \in \mathfrak{Z}_1 \cap \Gamma$, ${}^\exists\gamma \in \Gamma$ such that $\varepsilon\varepsilon_1^{-1}\varepsilon_2\delta_1^{-1}\gamma\delta_2 g\delta_2^{-1}\gamma^{-1}\delta_1 = g \Leftrightarrow \varepsilon_2 \in \varepsilon_1(\mathfrak{Z}_1 \cap \Gamma)$, $\delta_2 \in \Gamma\delta_1\Gamma^*(g)$. Indeed, the first $\Leftrightarrow$ is obvious from the definitions. The second one holds because $\varepsilon\varepsilon_1^{-1}\varepsilon_2 \in \{\pm 1\} \cap (\mathfrak{Z}_1 \cap \Gamma^*) = \{1\}$.

Now $c(g) = [\mathfrak{Z}_1 \cap \Gamma^* : \mathfrak{Z}_1 \cap \Gamma]\,|\Gamma \backslash \Gamma^*/\Gamma^*(g)|$, and $|\Gamma\backslash\Gamma^*/\Gamma^*(g)| = [\Gamma^* : \Gamma]\cdot[\Gamma\Gamma^*(g) : \Gamma]^{-1} = [\Gamma^* : \Gamma][\Gamma^*(g) : \Gamma(g)]^{-1}$. If g is parabolic, $[\Gamma^*(g) : \Gamma(g)] = [\iota(\Gamma^*(g)) : \iota(\Gamma(g))][\mathfrak{Z} \cap \Gamma^* : \mathfrak{Z} \cap \Gamma]$. Hence $c(g)\lambda(g) = [\Gamma^* : \Gamma][\mathfrak{Z} \cap \Gamma^* : \mathfrak{Z}_1 \cap \Gamma^*]^{-1}\lambda^*(g)$, as wanted.

2.0. Let r be a Dedekind domain, and k be its quotient field. Let B be a finite dimensional algebra over k. A subset R of B is an r-order if firstly it is a finitely generated r-module such that $R \otimes k = B$, and secondly it is a subring of B containing the unity. Let K be another k-algebra and Λ be its r-order. An embedding ($=$injective k-homomorphism) $\varphi : K \to B$ is called *optimal* with respect to R/Λ if $\varphi(K)\cap R=\varphi(\Lambda)$. Since the restriction of φ to Λ determines φ, it may be called, as Eichler's original usage, *optimal embedding* of Λ into R. We are particularly interested in the case where K is a subalgebra of B generated by a single element, so let g be an element of B which generates K as a k-algebra. Let $C(g)$ denote the B conjugacy class of g, i. e. $C(g)=\{xgx^{-1}|x\in B^{\times}\}$ and for any r-order Λ of K, put $C(g,\Lambda)=\{xgx^{-1}\mid x\in B^{\times}, xKx^{-1}\cap R=x\Lambda x^{-1}\}$. Then $C(g)$ is the disjoint union $\bigcup_{\Lambda} C(g,\Lambda)$ where Λ runs through all the orders of K. Let Γ be a subgroup of $B^{\times}$ normalizing R i. e. $\gamma R\gamma^{-1}=R$ for any $\gamma\in\Gamma$. Two elements $g_i\in B$ $(i=1,2)$ are called Γ-equivalent if there exists $\gamma\in\Gamma$ such that $\gamma g_2\gamma^{-1}=g_1$. Two embeddings $\varphi_i: K\to B$ $(i=1,2)$ are called Γ-equivalent if there exists $\gamma\in\Gamma$ such that $\varphi_2(\gamma g\gamma^{-1})=\varphi_1(g)$.

If furthermore, B is a quaternion algebra over k, i. e. B is central simple over k and $[B:k]=4$, and g is not in the center k, then $K=k+kg$ is exactly the centralizer of g in B, and K is isomorphic to $K'=k+kg'$ if and only if g is conjugate to g' in B.

Let Ξ be a subset of R, which is normalized by Γ. Then assuming B to be a quaternion over k, as is easily seen from the definitions, the following assertions hold.

2.1. (i) Let $\mathrm{Emb}(g,\Xi)$ be the set of all the embeddings $\varphi: K\to B$ such that $\varphi(g)\in\Xi$. For $g'\in C(g)\cap\Xi$, define an embedding $\varphi_{g'}: K\to B$ by $\varphi_{g'}(g)=g'$, then $g'\mapsto\varphi_{g'}$ gives a bijective correspondence: $C(g)\cap\Xi\simeq\mathrm{Emb}(g,\Xi)$.

(ii) Let $\mathrm{Emb}(g,\Xi,R/\Lambda)$ denote the set of optimal embeddings $\varphi:\Lambda\to R$ such that $\varphi(g)\in\Xi$. Then the above correspondence $g'\mapsto\varphi_{g'}$ induces a bijection $C(g,\Lambda)\cap\Xi\simeq\mathrm{Emb}(g,\Xi,R/\Lambda)$.

(iii) Each $\mathrm{Emb}(g,\Xi,R/\Lambda)$ is normalized by Γ, and the correspondence $g'\mapsto\varphi_{g'}$ induces a bijection: $C(g,\Lambda)\cap\Xi/\tilde{\Gamma}\simeq\mathrm{Emb}(g,\Xi,R/\Lambda)/\tilde{\Gamma}$.

2.2. In the rest of this section 2.2~2.7, we assume, besides the assumptions and notation in 2.0, that r is a discrete valuation ring, $p=\pi r$ is its maximal ideal, and B is a 2 by 2 total matrix algebra $M_2(k)$ over k. (i) For an r-order R of B, the following four conditions are equivalent.

(1) R contains a subset which is $B^{\times}$-conjugate to $\begin{pmatrix} r & 0 \\ 0 & r\end{pmatrix}=\left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\in R \,\middle|\, a,d\in r\right\}$.

(2) R is $B^{\times}$-conjugate to $\begin{pmatrix} r & r \\ p^{\nu} & r\end{pmatrix}$ for some non-negative integer ν.

(3) R is the intersection of at most two maximal orders of B.

(4) R is either maximal or there exists a uniquely determined pair $\{R_1, R_2\}$ of distinct orders such that $R=R_1\cap R_2$.

If R satisfies one of the above conditions, it will be called a *split* order.
(ii) Let $N(R)$ denote the normalizer of R in $B^{\times}$, $N(R)=\{x\in B^{\times}\mid xRx^{-1}=R\}$.

If R is split and $\nu>0$, then $[N(R): k^{\times}R^{\times}]=2$. If R has the form $R=\begin{pmatrix} r & r \\ p^{\nu} & r\end{pmatrix}$, $N(R)$ is generated by $\begin{pmatrix} 0 & 1 \\ \pi^{\nu} & 0\end{pmatrix}$ and $k^{\times}R^{\times}$.

PROOF. (i) $(2)\Rightarrow(1)$ and $(4)\Rightarrow(3)$ are obvious from definitions. To see $(3)\Rightarrow(2)$, let $R_i=x_iM_2(r)x_i^{-1}$ $(i=1, 2)$ be maximal orders, and $R=R_1\cap R_2$. Then $x_1^{-1}Rx_1=M_2(r)\cap x_1^{-1}x_2M_2(r)x_2^{-1}x_1$. By the elementary divisors theorem, there exist u, $v\in M_2(r)^{\times}$ such that $x_1^{-1}x_2=u\begin{pmatrix} \pi^{\mu} & 0 \\ 0 & \pi^{\mu+\nu}\end{pmatrix}v$ with some integer μ and a non-negative integer ν. Hence $u^{-1}x_1^{-1}Rx_1u=\begin{pmatrix} r & r \\ p^{\nu} & r\end{pmatrix}$. To see $(1)\Rightarrow(4)$, suppose $R\supset\begin{pmatrix} r & 0 \\ 0 & r\end{pmatrix}$. Then, as is easily seen, R has the form $R=\begin{pmatrix} r & p^{a} \\ p^{b} & r\end{pmatrix}$ with integers a, b such that $\nu=a+b\geqq 0$. Let R' be a maximal order containing R, then since it contains $\begin{pmatrix} r & 0 \\ 0 & r\end{pmatrix}$, it has the form $R'=\begin{pmatrix} r & p^{c} \\ p^{-c} & r\end{pmatrix}$ with some integer c such that $a-\nu\leqq c\leqq a$. Set $R_1=\begin{pmatrix} r & p^{a} \\ p^{-a} & r\end{pmatrix}$ and $R_2=\begin{pmatrix} r & p^{a-\nu} \\ p^{\nu-a} & r\end{pmatrix}$. Then $R_1\cap R_2=R$. If we replace any of R_i, say R_2, by some R', then $R_1\cap R'$ is strictly bigger than R, that means the pair $\{R_1, R_2\}$ is uniquely determined by R.

(ii) Let $R=R_1\cap R_2$ and $x\in B^{\times}$ be in the normalizer of R. Then x permutes $\{R_1, R_2\}$, hence we have a homomorphism from $N(R)$ to the cyclic group of order 2. Since $N(R_i)=k^{\times}R_i^{\times}$ for maximal orders R_i, the kernel is obviously $k^{\times}R^{\times}$.

Since $\begin{pmatrix} 0 & 1 \\ \pi^{\nu} & 0\end{pmatrix}$ permutes $\begin{pmatrix} r & p^{-\nu} \\ p^{\nu} & r\end{pmatrix}$ and $M_2(r)$, the above map is surjective.

2.3. THEOREM. *Let r, k and B be as in 2.2, and $R=\begin{pmatrix} r & r \\ p^{\nu} & r\end{pmatrix}$. Let g be an integral element of B not in the center k, $K=k+kg$ and Λ be an r-order of K containing g. Then there is a uniquely determined non-negative integer ρ such that $[\Lambda: r+rg]=[r:p]^{\rho}$. Let $f(X)=X^2-sX+n\in r[X]$ be the minimal polynomial of g over k. Put $\tilde{F}=\{\xi\in r\mid f(\xi)\equiv 0 \bmod p^{\nu+2\sigma}$ and $2\xi\equiv s \bmod p^{\rho}\}$. Let F be a complete system of representatives of $\tilde{F}$ modulo $p^{\nu+2\rho}$, and put $F'=\{\eta\in F\mid f(\eta)\equiv 0 \bmod p^{\nu+2\rho+1}\}$. Define $\varphi_{\xi}: K\to B$, by $\varphi_{\xi}(g)=\begin{pmatrix} \xi & \pi^{\rho} \\ -\pi^{-\rho}f(\xi) & s-\xi\end{pmatrix}$ for $\xi\in F$ and $\varphi'_{\eta}: K\to B$ by $\varphi'_{\eta}(g)=\begin{pmatrix} s-\eta & -\pi^{-(\nu+\rho)}f(\eta) \\ \pi^{\nu+\rho} & \eta\end{pmatrix}$ for $\eta\in F'$.*

If $\pi^{-2\rho}(s^2-4n)$ is a unit of r or $\nu=0$ (resp. not a unit of r and $\nu>0$) the set $\{\varphi_{\xi}\mid \xi\in F\}$ (resp. $\{\varphi_{\xi}\mid\xi\in F\}\cup\{\varphi'_{\eta}\mid\eta\in F'\}$) gives a complete system of representatives of $\mathrm{Emb}(g, R, R/\Lambda)/\tilde{R}^{\times}$ *defined in* 2.1.

PROOF. It is a consequence of the following two Lemmas 2.4 and 2.5.

2.4. LEMMA. *Under the same assumptions and notation of* 2.3, *let $\varphi: K\to B$ be an injective k-algebra homomorphism. Let $\varphi(g)=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in R$. Then* $\mathrm{tr}\,\varphi(g)$

$=a+d=s$ *and* $\det\varphi(g)=ad-bc=n$, φ *induces an embedding of* $r+rg$ *into* R. *For such* φ, *the following four conditions are equivalent.*

(1) φ *is optimal with respect to* R/Λ.

(2) $(r+\varphi(g))\cap\pi^{\rho}R\neq\emptyset$ *and* $(r+\varphi(g))\cap\pi^{\rho+1}R=\emptyset$.

(3) b, $a-d\in p^{\rho}$, $c\in p^{\rho+\nu}$ *and furthermore, one of the following three conditions holds:* (I) $b\in\pi^{\rho}r^{\times}$, (II) $c\in\pi^{\nu+\rho}r^{\times}$, (III) *neither* (I) *nor* (II) *holds, i.e.* $b\in p^{\rho+1}$, $c\in p^{\nu+\rho+1}$, *and* $a-d\in\pi^{\rho}r^{\times}$.

(4) *There exists* $u\in N(R)$, *such that* $u\varphi(g)u^{-1}=\begin{pmatrix}\xi & \pi^{\rho}\\ -\pi^{-\rho}f(\xi) & s-\xi\end{pmatrix}$ *by some solution* $\xi\in r$ *of* $f(\xi)\equiv 0 \bmod p^{\nu+2\rho}$ *such that* $s-2\xi\equiv 0 \bmod p^{\rho}$.

PROOF. The first statement and the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are obvious. Since (4) implies (3), it suffices to show (3) $\Rightarrow$ (4).

Assume (I), i.e. $b\in\pi^{\rho}r^{\times}$, put $u=\begin{pmatrix}1 & 0\\ 0 & \pi^{-\rho}b\end{pmatrix}$ and $u\varphi(g)u^{-1}=(x_{ij})\in M_2(k)$. Then $u\in R^{\times}$ and $x_{12}=\pi^{\rho}$. Put $\xi=x_{11}$, then $\mathrm{tr}\,(x_{ij})=s$ implies $x_{22}=s-\xi$, and $\det(x_{ij})=n$ implies $x_{21}=-\pi^{-\rho}f(\xi)$, as desired. Assume (II), i.e. $c\in\pi^{\nu+\rho}r^{\times}$, put $\sigma=\begin{pmatrix}0 & 1\\ \pi^{\nu} & 0\end{pmatrix}$ and $\sigma\varphi(g)\sigma^{-1}=(x_{ij})$. Then $\sigma\in N(R)$, $x_{12}=\pi^{-\nu}c\in\pi^{\rho}r^{\times}$, hence (x_{ij}) satisfies the condition (I). Finally assume (III), and put $u=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $u\varphi(g)u^{-1}$ $=(x_{ij})$. Then $u\in R^{\times}$, and $x_{12}=-(a+c)+b+d\in\pi^{\rho}r^{\times}$ by the assumptions, again reducing the problem to the case (I).

2.5. LEMMA. *Under the same assumptions and notation as* 2.3, *let* φ, $\varphi'\in$ Emb$(g, R, R/\Lambda)$ *and* $\sigma=\begin{pmatrix}0 & 1\\ \pi^{\nu} & 0\end{pmatrix}$. *Define* ${}^{\sigma}\varphi$ *by* ${}^{\sigma}\varphi(g)=\sigma\varphi(g)\sigma^{-1}$. *Then we have the following.*

(i) φ *is* $N(R)$*-equivalent to* φ' *if and only if* φ *is* $R^{\times}$*-equivalent to either* φ' *or* ${}^{\sigma}\varphi'$. *If* $\nu=0$, $N(R)$*-equivalence is identical with* $R^{\times}$*-equivalence.*

(ii) *Let* ξ, $\xi'\in\tilde{F}$ *and* φ_{ξ}, $\varphi_{\xi'}$ *be as in* 2.3. *Then* φ_{ξ} *is* $R^{\times}$*-equivalent to* $\varphi_{\xi'}$ *if and only if* $\xi\equiv\xi' \bmod p^{\nu+\rho}$.

(iii) *Suppose* $\pi^{-2\rho}(s^2-4n)$ *is a unit in* r *(resp. not a unit in* r*), then* φ_{ξ} *is* $R^{\times}$*-equivalent to* ${}^{\sigma}\varphi_{\xi'}$ *if and only if* $\xi\equiv s-\xi' \bmod p^{\nu+\rho}$ *(resp.* $\xi\equiv s-\xi' \bmod p^{\nu+\rho}$ *and* $f(\xi')\not\equiv 0 \bmod p^{\nu+2\rho+1}$*).*

PROOF. (i) is obvious from (ii) 2.2. (ii): Suppose $\xi\equiv\xi' \bmod p^{\nu+\rho}$. Put $t=\pi^{-\rho}(\xi-\xi')\in p^{\nu}$, $u=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$ and $u\varphi_{\xi}(g)u^{-1}=(x_{ij})$. Then $u\in R^{\times}$, and direct computation shows that $x_{11}=\xi'$, $x_{12}=\pi^{\rho}$ hence $(x_{ij})=\varphi_{\xi'}(g)$ showing the if part of (ii). To see the only if part, suppose φ_{ξ} is $R^{\times}$-equivalent to $\varphi_{\xi'}$. Since every element of R is upper triangular mod p^{ν}, if $u\in R^{\times}$, $\pi^{-\rho}(u\varphi_{\xi}(g)u^{-1}-\xi)$ has the same diagonal entries as $\pi^{-\rho}(\varphi_{\xi}(g)-\xi)$ modulo p^{ν}, in particular $\xi\equiv\xi'$ mod $p^{\nu+\rho}$. (iii): Suppose $\pi^{-(\nu+2\rho)}f(\xi')$ is a unit, then ${}^{\sigma}\varphi_{\xi'}(g)$ satisfies the condition (I) of (3) 2.4, and it is $R^{\times}$-equivalent to $\begin{pmatrix}s-\xi' & \pi^{\rho}\\ -\pi^{-\rho}f(\xi') & \xi'\end{pmatrix}$. Hence by the above (ii), φ_ξ is $R^\times$-equivalent to ${}^\sigma\varphi_{\xi'}$ if and only if $\xi\equiv s-\xi' \bmod p^{\nu+\rho}$. Suppose $\pi^{-(\nu+2\rho)}f(\xi')$ is not a unit. Let $t\in r$ and put $u=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$, $u{}^\sigma\varphi_{\xi'}(g)u^{-1}=(x_{ij})$. Then $u\in R^\times$, $x_{11}\equiv s-\xi' \bmod p^{\nu+\rho}$, $x_{12}\equiv -\pi^{-(\nu+\rho)}f(\xi')+(2\xi'-s)t \bmod p^{\nu+\rho}$. Hence if $\pi^{-2\rho}(s^2-4n)$ is a unit or equivalently $\pi^{-\rho}(2\xi'-s)$ is a unit, by a proper choice of t, $\pi^{-\rho}x_{12}$ can be a unit, then (x_{ij}) is again $R^\times$-equivalent to $\begin{pmatrix}s-\xi' & \pi^\rho\\ -\pi^{-\rho}f(\xi') & \xi'\end{pmatrix}$. Thus φ_ξ is $R^\times$-equivalent to ${}^\sigma\varphi_{\xi'}$ if and only if $\xi\equiv s-\xi' \bmod p^{\nu+\rho}$ as above.

Finally suppose neither $\pi^{-(\nu+2\rho)}f(\xi')$ nor $\pi^{-2\rho}(s^2-4n)$ is a unit. Then again looking everything modulo $M_2(p^\nu)$, $R^\times$ is generated by the elements of the form $u=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$ and diagonal ones. Hence if $v\in R^\times$, $v{}^\sigma\varphi_{\xi'}(g)v^{-1}=(y_{ij})$, the above $x_{12}\equiv -\pi^{-(\nu+\rho)}f(\xi')+(2\xi'-s)t \bmod p^{\nu+\rho}$ implies that $\pi^{-\rho}y_{12}$ can not be a unit, thus ${}^\sigma\varphi_{\xi'}$ can not be $R^\times$-equivalent to φ_ξ.

2.6. COROLLARY. *Under the same assumptions and notation as in* 2.3, *let* φ *denote a regular representation of* K *by some base of* Λ.

(i) *Suppose* R *is maximal, then* $\varphi\in\mathrm{Emb}(g, R, R/\Lambda)$, *and* $\mathrm{Emb}(g, R, R/\Lambda)/\tilde{R}^\times$ *consists of a single element represented by* φ.

(ii) *Suppose* $\nu\leqq 1$, *then* $\mathrm{Emb}(g, R, R/\Lambda)/\widetilde{N(R)}$ *is either vacant or consists of a single element.*

The analogue of the first statement is valid for more general (not necessarily quaternion) algebras if we assume K to be a maximal field and Λ to be its maximal order, as shown by Chevalley, Hasse [**10**] and Noether. (ii) is proved in Eichler [**2**], and which makes his ideal theoretic approach to optimal embeddings possible.

PROOF. (i) If $\Lambda=r+rg_1$, then as mappings $K\to B$, $\mathrm{Emb}(g, R, R/\Lambda)$ can be identified with $\mathrm{Emb}(g_1, R, R/\Lambda)$. Hence without any important loss of generality, we may assume that $\rho=0$, and that φ is a regular representation by the base $\{1, g\}$ of Λ. If R is maximal, i.e. $\nu=0$, then any $\xi\in r$ gives a solution of $f(\xi)\equiv 0\ p^\nu$, and any ξ' is equal to ξ modulo p^ν. Taking $\xi=0$, we get $\varphi_\xi=\varphi$. (ii) If R is maximal $N(R)=R^\times$. Suppose $\nu=1$, and $f(X)\equiv 0 \bmod p^\nu$ has a solution. If $s^2-4n\equiv 0 \bmod p$, then $f(X)\equiv 0 \bmod p$ has only one solution. If $s^2-4n\not\equiv 0 \bmod p$, then $f(X)\equiv 0 \bmod p$ has two solutions. But in this case, φ_ξ is $N(R)$-equivalent to $\varphi_{\xi'}$ by (iii) 2.5.

2.7. COROLLARY. *Let* H *be a subset of* r *defined modulo* p^ν, *namely* H *is a full inverse of some subset* $\bar{H}$ *of* r/p^ν. *Put* $\Xi_H=\{(x_{ij})\in R \mid x_{11}\in H\}$. *Let* g, *and* $F=F(g,\nu)$ *be as in* 2.3. *If the discriminant* s^2-4n *of* f *is a unit of* r, *or* $\nu=0$ (*resp. not a unit of* r *and* $\nu>0$), *the set* $\{\varphi_\xi \mid \xi\in F\cap H\}$ (*resp.* $\{\varphi_\xi \mid \xi\in F\cap H\}\cup\{\varphi'_\eta \mid \eta\in F',\ s-\eta\in H\}$) *gives a complete system of representatives of* $\mathrm{Emb}(g, \Xi_H, R/\Lambda)/\tilde{R}^\times$.

PROOF. As we have already observed in the proof of 2.5, if $(x_{ij})\in R$,

$u \in R^{\times}$ and $u(x_{ij})u^{-1}=(y_{ij})$, then $x_{11} \equiv y_{11} \bmod p^{\nu}$. Hence $(x_{ij}) \in \Xi_H$ if and only if $(y_{ij}) \in \Xi_H$, thus this corollary is an immediate consequence of 2.3.

2.8. Let k be a p-field in the sense of [12] and B be the division quaternion over k. Let r (resp. R) be the maximal compact subring of k (resp. B). Let $K=k+kg$ be a k-algebra of rank 2 generated by g, and Λ be a maximal r-order of K. Obviously g is embeddable in R if and only if K is a field and g is integral over r, if that is so, since $B^{\times}$ normalizes R, $\mathrm{Emb}(g, R, R/\Lambda)/N(R)$ consists of a single point (identical representation). Hence, as is well known and as is easily seen, $\mathrm{Emb}(g, R, R/\Lambda)/\tilde{R}^{\times}$ consists of a single point corresponding to the identical representation (resp. two points corresponding to the identical representation and its transform by the prime element of B) if K is ramified (resp. unramified).

3.0. In the rest of this paper, let k be a totally real algebraic number field, and $r(k)$ be the ring of integers of k. For any place v of k, let $k(v)$ denote the completion of k with respect to v. If v is non-archimedian, let $r(v)$ and $p(v)$ denote the ring of integers of $k(v)$ and its maximal ideal. If v is archimedian, let us use the convention that $r(v)=k(v)$. Let m be the absolute degree of k, i.e. $[k:\boldsymbol{Q}]=m$. Let B be a quaternion algebra containing k as its center, and n be the number of archimedian places of k where $B \otimes k(v)$ is split. Assume B to be indefinite, i.e. $n>0$, and we can arrange the set of all archimedian places $\{v(1), \cdots, v(m)\}$ in such a way that there is an isomorphism

$$\tau_i : B \otimes k(v(i)) \longrightarrow M_2(\boldsymbol{R}) \qquad \text{for each } i \ (1 \leqq i \leqq n),$$

and $B \otimes k(v(i))$ is isomorphic to the Hamilton quaternion for each i $(n+1 \leqq i \leqq m)$. Let G denote a k-algebraic group such that the group of L-rational points G_L is canonically identified with $(B \otimes L)^{\times}$ for any field L containing k. In particular $G_{k(v)}$ is identified with $(B \otimes k(v))^{\times}$ for any v, and $G_{r(v)}$ is isomorphic to $M_2(r(v))^{\times}$ for almost all v. Let $\boldsymbol{A}=k_{\boldsymbol{A}}$ denote the adele ring of k, and $G_{\boldsymbol{A}}$ (resp. $B_{\boldsymbol{A}}$) denote the adelization of G (resp. B). The center of $B_{\boldsymbol{A}}$ will be identified with $k_{\boldsymbol{A}}$, and $G_{\boldsymbol{A}}$ will be identified with $B_{\boldsymbol{A}}^{\times}$. Let $G_{\infty}=\prod_{i=1}^{m} G_{k(v(i))}$ and G_{∞}^{+} be its topological connected component of the identity. As usual, we consider G_k as a subgroup of $G_{\boldsymbol{A}}$ (or of a partial product like G_{∞}) by the diagonal embedding.

By the isomorphism τ_i, $\prod_{i=1}^{n} G_{k(v(i))}$ is isomorphic to the direct product of n copies of $GL_2(\boldsymbol{R})$ which we write $\mathfrak{G}$ in §1. We shall identify $\prod_{i=1}^{n} G_{k(v(i))}$ with $\mathfrak{G}$ by τ_i, and consider G_k as a subgroup of $\mathfrak{G}$. Then, for $g \in G_k$, $g^{(i)}=$

$\tau_i(g)$ for $i=1, \cdots, n$. For example, let $\mathfrak{Z}_1$ be as in 1.2, and put $k_1^\times=\{x\in k^\times \mid \tau_1(x)>0\}$ then $\mathfrak{Z}_1\cap G_k=k_1^\times$.

3.1. Let's consider a quadruplet $\{\mathfrak{U}, \hat{\Xi}, \hat{\chi}, \mathfrak{U}^*\}$ satisfying the following four conditions (U1)$\sim$(U3) and (U*).

(U1) $\mathfrak{U}$ is an open subgroup of G_A (hence it contains G_∞^+), and $\mathfrak{U}/(k^\times\cap\mathfrak{U})G_\infty^+$ is compact.

(U2) $\hat{\Xi}$ is a union of a finite number of distinct $\mathfrak{U}$-double cosets $\mathfrak{U}\alpha\mathfrak{U}$ $(\alpha\in G_A)$.

(U3) $\hat{\chi}:\langle\hat{\Xi}\rangle\to GL_r(\boldsymbol{C})$ is a unitary representation of the group $\langle\Xi\rangle$ generated by Ξ such that $[\mathfrak{U}:\mathrm{Ker}\,\hat{\chi}\cap\mathfrak{U}]$ is finite.

(U*) $\mathfrak{U}^*$ is an open subgroup of G_A containing $\mathfrak{U}G_\infty$ with the following properties. $[\mathfrak{U}^*:\mathfrak{U}]<\infty$. If $g\in\mathfrak{U}^*$, then $g\mathfrak{U}g^{-1}=\mathfrak{U}$, $g\hat{\Xi}g^{-1}=\hat{\Xi}$ and $\mathrm{tr}\,\hat{\chi}(gxg^{-1})=\mathrm{tr}\,\hat{\chi}(x)$ for any $x\in\langle\hat{\Xi}\rangle$.

Set $\Gamma=\mathfrak{U}\cap G_k$, $\Gamma^*=\mathfrak{U}^*\cap G_k$, $\Xi=\hat{\Xi}\cap G_k$ and χ to be the restriction of $\hat{\chi}$ to $\langle\Xi\rangle$. Then, if Ξ is not vacant, $\{\Gamma, \Xi, \chi, \Gamma^*\}$ satisfies (Γ1), (Γ2), (Γ3) in 1.1, (Γ4) in 1.3 and (Γ^*) in 1.5.

PROOF. Γ is commensurable with $G_{T(k)}$. If B is a division quaternion, $\mathfrak{H}^n/\Gamma^+$ is compact. If not, Γ^+ is commensurable with the Hilbert modular group for which the condition (F) is well-known. (Γ2), (Γ3) and (Γ^*) follow directly from (U2), (U3) and (U*) respectively. As for (Γ4), we may assume $n=1$ and $\mathfrak{H}^n/\Gamma^+$ non-compact, hence $B=M_2(\boldsymbol{Q})$. Then Γ is commensurable with $SL_2(\boldsymbol{Z})$ where (Γ4) is certainly true.

3.2. Let k_i be n-tuple of positive integers, and we again assume the condition (χ-k) in 1.1 and (χ-k^*) in 1.5 for the above $\{\Gamma, \{k_i\}, \chi, \Gamma^*\}$. Now Ξ being the union $\Xi=\cup\Gamma\alpha_j\Gamma$ and $\xi=\sum\Gamma\alpha_j\Gamma$, $T(\xi)$ is operating on $S(\Gamma, \{k_i\}, \chi)$. Note however that the number of Γ-double cosets in Ξ is not nessecarily equal to that of $\mathfrak{U}$-double cosets in $\hat{\Xi}$. The condition for $\mathfrak{U}\alpha\mathfrak{U}\cap G_k=\Gamma\alpha\Gamma$ will be discussed in 3.7. By 1.4, to compute $\mathrm{tr}\,T(\xi)$ is to compute $\sum_g\kappa(g)\lambda^*(g)$ with g running through $\Omega/\overset{*}{\sim}$.

To reduce the sum $\sum_g\kappa(g)\lambda^*(g)$ to some easier form, we introduce a k-equivalence relation $\underset{k}{\sim}$ in G_A by: $g\underset{k}{\sim}g'\Leftrightarrow{}^\exists x\in G_k$, ${}^\exists\varepsilon\in\mathfrak{Z}_1\cap\Gamma^*$ such that $g'=\varepsilon xgx^{-1}$. For any subgroup $\mathfrak{V}$ of G_A, we call g is $\mathfrak{V}$-equivalent (or $\mathfrak{V}$-conjugate) to g', if there exists $x\in\mathfrak{V}$ such that $g'=xgx^{-1}$. For any subset X of G_A, let $X/\tilde{\mathfrak{V}}$ denote the quotient set. Let $\mathrm{Cl}_{\mathfrak{V}}(g)$ denote the $\mathfrak{V}$-conjugacy class of g i.e. $\mathrm{Cl}_{\mathfrak{V}}(g)=\{xgx^{-1}\mid x\in\mathfrak{V}\}$. For simplicity, we write $\mathrm{Cl}_{G_k}(g)=C(g)$ and $\mathrm{Cl}_{G_A}(g)=\hat{C}(g)$ in the following.

3.3. LEMMA. *$\Omega/\overset{*}{\sim}$ is a disjoint union of $\Xi\cap C(g)/\tilde{\Gamma}^*$ with g running through $\Omega/\underset{k}{\sim}$. Consequently, the sum $\sum_g\kappa(g)\lambda^*(g)$ appeared in* 1.5 *is given by:*

$$\sum \kappa(g)\lambda^*(g) = \sum_g \kappa(g) \sum_{g'} \lambda^*(g')$$

where g is running through $\Omega/\underset{k}{\sim}$ and for each g, g' is running through $\Xi \cap C(g)/\tilde{\Gamma}^*$.

PROOF. The equality $\Xi \cap C(g) = \Omega \cap C(g)$ is obvious. Since g_1 is Γ^*-eqivalent to g_2 only if $g_1 \overset{*}{\sim} g_2$, and $g_1 \overset{*}{\sim} g_2$ only if g_1 is k-equivalent to g_2, it is obvious that $\cup \Xi \cap C(g)/\tilde{\Gamma}^*$ covers $\Omega/\overset{*}{\sim}$ and $C(g_1)/\tilde{\Gamma}^*$ is disjoint to $C(g_2)/\tilde{\Gamma}^*$ if $g_1, g_2 \in \Omega/\tilde{k}$ and $g_1 \neq g_2$. It suffices to show that two distinct elements of $\cup C(g)/\tilde{\Gamma}^*$ is not $\overset{*}{\sim}$ equivalent. Let $x_1^{-1}g_1x_1 \overset{*}{\sim} x_2^{-1}g_2x_2$, then they are k-equivalent and $g_1 = g_2$. Thus there exists $\varepsilon \in \mathfrak{Z}_1 \cap \Gamma^*$ and $\gamma \in \Gamma^*$ such that $x_2^{-1}g_1x_2 = \varepsilon\gamma^{-1}x_1^{-1}g_1x_1\gamma$. Hence $\mathfrak{E} \in \{\pm 1\} \cap \mathfrak{Z}_1 = \{1\}$, they are Γ^*-equivalent.

3.4. LEMMA. *Let $g_0 \in \Omega$, and T be the centralizer of g_0 in G. Let $\theta: C(g_0)/\tilde{\Gamma}^* \to \hat{C}(g_0)/\tilde{\mathfrak{U}}^*$ be a map defined by: $\theta: \mathrm{Cl}_{\Gamma^*}(g) \mapsto \mathrm{Cl}_{\mathfrak{U}^*}(g)$. Let $\hat{g} = xg_0x^{-1} \in \hat{C}(g_0)$, $x \in G_A$.*

(i) $\mathrm{Cl}_{\mathfrak{U}^*}(\hat{g})$ *is in the image of θ if and only if $G_k \cap \mathfrak{U}^*xT_A \neq \emptyset$. Let $\sharp(\hat{g})$ denote the cardinality of $\theta^{-1}(\mathrm{Cl}_{\mathfrak{U}^*}(\hat{g}))$, then*

$$\sharp(\hat{g}) = |\Gamma^* \backslash \mathfrak{U}^*xT_A \cap G_k/T_k|.$$

(ii) *If g_0 is either elliptic or hyperbolic (resp. parabolic) put $\hat{\lambda}(\mathrm{Cl}_{\mathfrak{U}^*}(\hat{g})) = \hat{\lambda}(\hat{g}) = s^{-1}[T_k \cap x^{-1}\mathfrak{U}^*x : k^\times \cap x^{-1}\mathfrak{U}^*x]^{-1}\, \mathrm{tr}\, \hat{\chi}(\hat{g})$ (resp. $[\iota(T_k \cap x^{-1}\mathfrak{U}^*x) : \langle \iota(g_0) \rangle]^{-(1+s)}\mathrm{tr}\hat{\chi}(\hat{g})$). It is well defined. Looking λ^* as a function on $C(g_0)/\tilde{\Gamma}^*$, we have $\lambda^* = \hat{\lambda} \circ \theta$.*

(iii) *Consequently, $\sum_{g'} \lambda^*(g')$ in 3.2 is given by: $\sum_{g'} \lambda^*(g') = \sum_{\hat{g}} \sharp(\hat{g})\hat{\lambda}(\hat{g})$ where $\hat{g}$ runs through $\hat{\Xi} \cap \hat{C}(g)/\tilde{\mathfrak{U}}^*$.*

PROOF. Let $g_1 = x_1g_0x_1^{-1}$, $g_2 = x_2g_0x_2^{-1} \in \hat{C}(g_0)$. $\mathrm{Cl}_{\mathfrak{U}^*}(g_1) = \mathrm{Cl}_{\mathfrak{U}^*}(g_2)$ if and only if $x_1 \in \mathfrak{U}^*x_2T_A$. If $x_1, x_2 \in G_k$, $\mathrm{Cl}_{\Gamma^*}(g_1) = \mathrm{Cl}_{\Gamma^*}(g_2)$ if and only if $x_1 \in \Gamma x_2 T_k$. Hence (i) is obvious. Let $g_0 \in \Omega_e \cup \Omega_h$. If we replace x in $[T_k \cap x^{-1}\mathfrak{U}^*x : k \cap x^{-1}\mathfrak{U}^*x]$ by some element x' in $\mathfrak{U}^*xT_A$, then the value does not change. Hence $\hat{\lambda}$ is well defined as a function on $\hat{C}(g_0)/\tilde{\mathfrak{U}}^*$. If $\mathrm{Cl}_{\mathfrak{U}^*}(\hat{g}) = \theta(\mathrm{Cl}_{\Gamma^*}(g))$ with $g = xg_0x^{-1}$, $x \in G_k$, then $[T_k \cap x^{-1}\mathfrak{U}^*x : k^\times \cap x^{-1}\mathfrak{U}^*x] = [xT_kx^{-1} \cap \mathfrak{U}^* : k^\times \cap \mathfrak{U}^*] = [xT_Ax^{-1} \cap G_k \cap \mathfrak{U}^* : \mathfrak{Z} \cap G_k \cap \mathfrak{U}^*] = [\Gamma^*(g) : \mathfrak{Z} \cap \Gamma^*]$. Hence $\hat{\lambda}(\hat{g}) = \lambda^*(g)$ i.e. $\lambda^* = \hat{\lambda} \circ \theta$. If $g_0 \in \Omega_h$, the proof is similar. (iii) is immediate from (i) and (ii).

3.5. Let N denote the reduced norm map $N: G \to GL_1$, and also the induced map $N: G_A \to (GL_1)_A = k_A^\times$ of adele groups. Let $G^{(1)}$ be the kernel of N, then the strong approximation theorem holds in $G^{(1)}$ [1], namely $G_k^{(1)}G_\infty^{(1)}$ is dense in $G_A^{(1)}$. If $\mathfrak{B}$ is an open subgroup of G_A containing G_∞^+, then $G_k\mathfrak{B}$ contains $G_A^{(1)}$, hence $G_k\mathfrak{B}$ is a normal subgroup of G_A, and $G_A/G_k\mathfrak{B} = N(G_A)/N(G_k)N(\mathfrak{B})$. $N(G_A)$ consists of all the ideals $a = (a_v) \in k_A^\times$ such that $a_{v(i)} > 0$ for all $i > n$, and $N(G_k) = k^\times \cap N(G_A)$ [1].

For any algebraic group G and an open subgroup $\mathfrak{B}$ of G_A, the number of $(G_k, \mathfrak{B})$-double cosets $|G_k \backslash G_A/\mathfrak{B}|$ is finite (Finiteness of class number!).

We put $h_G(\mathfrak{B})=|G_k\backslash G_A/\mathfrak{B}|$, and abbreviate G if it is fixed. Thus $h(\mathfrak{B})=h_G(\mathfrak{B})$ $=[N(G_A):N(G_k)N(\mathfrak{B})]$.

3.6. LEMMA. *Let $\mathfrak{B}$ be an open subgroup of G_A containing G_∞. Let $\hat{H}$ be a subgroup of G_A and H be a normal subgroup of $\hat{H}$ contained in G_k. Suppose $|\mathfrak{B}\backslash\mathfrak{B}\hat{H}/H|$ is finite, then*

$$|G_k\cap\mathfrak{B}\backslash\mathfrak{B}\hat{H}\cap G_k/H|=h_G(\mathfrak{B})^{-1}[k_A^\times:k^\times N(\mathfrak{B}\hat{H})]|\mathfrak{B}\backslash\mathfrak{B}\hat{H}/H|.$$

PROOF. Since $\mathfrak{B}G_k$ is a normal subgroup of G_A, $\mathfrak{B}G_k\hat{H}$ is a subgroup normalizing $\mathfrak{B}G_k$. The index $r=[G_k\mathfrak{B}\hat{H}:G_k\mathfrak{B}]$ is given by; $r=[G_A:G_k\mathfrak{B}][G_A:G_k\mathfrak{B}\hat{H}]^{-1}=h(\mathfrak{B})[k_A^\times:k^\times N(\mathfrak{B}\hat{H})]^{-1}$. Let $\{g_1,\cdots,g_r\}$ be elements of $\hat{H}$ which generate $\mathfrak{B}\hat{H}G_k$ over $\mathfrak{B}G_k$. Since $\mathfrak{B}H$, and any of $G_k\mathfrak{B}g_i$, hence $\mathfrak{B}\hat{H}\cap G_k\mathfrak{B}g_i$, is a union of $(\mathfrak{B},H)$-double cosets, $\mathfrak{B}\backslash\mathfrak{B}\hat{H}/H$ is bijective with the disjoint union $\bigcup_{i=1}^{r}\mathfrak{B}\backslash\mathfrak{B}\hat{H}\cap G_k\mathfrak{B}g_i/H$. Now, the following two bijective correspondences imply our formula.

$$\mathfrak{B}\backslash\mathfrak{B}\hat{H}\cap G_k\mathfrak{B}g_i/H\simeq\mathfrak{B}\backslash\mathfrak{B}\hat{H}\cap G_k\mathfrak{B}/H\simeq G_k\cap\mathfrak{B}\backslash\mathfrak{B}\hat{H}\cap G_k/H.$$

3.7. COROLLARY. (i) *In the notation of* 3.4, *if $\sharp(\hat{g})\neq 0$, then*:

$$\sharp(\hat{g})=h_T(T_A\cap x^{-1}\mathfrak{U}^*x)h_G(\mathfrak{U}^*)^{-1}[k_A^\times:k^\times N(\mathfrak{U}^*T_A)].$$

(ii) *If $\mathfrak{B}'$ is an open subgroup of G_A, and $\mathfrak{B}$ is a subgroup of $\mathfrak{B}'$ of finite index, then:*

$$[\mathfrak{B}'\cap G_k:\mathfrak{B}\cap G_k]=h(\mathfrak{B})^{-1}h(\mathfrak{B}')[\mathfrak{B}':\mathfrak{B}].$$

(iii) *Let $\alpha\in G_k$. Then we have* (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (2).

(1) $\mathfrak{U}\alpha\mathfrak{U}=\mathfrak{U}\alpha\Gamma$, (2) $G_k\cap\mathfrak{U}\alpha\mathfrak{U}=\Gamma\alpha\Gamma$,

(3) $\deg\Gamma\alpha\Gamma=\deg\mathfrak{U}\alpha\mathfrak{U}$, (4) $h_G(\mathfrak{U})=h_G(\mathfrak{U}\cap\alpha^{-1}\mathfrak{U}\alpha)$,

(5) $k^\times N(\mathfrak{U})=k^\times N(\mathfrak{U}\cap\alpha^{-1}\mathfrak{U}\alpha)$.

PROOF. (i) If $\sharp(\hat{g})\neq 0$ i.e. $\mathfrak{U}^*xT_A\cap G_k\neq\emptyset$, we may assume $x\in G_k$. Then $\sharp(\hat{g})=|\Gamma^*\backslash\mathfrak{U}^*xT_A\cap G_k/T_k|=|\Gamma^*\backslash\mathfrak{U}^*xT_Ax^{-1}\cap G_k/xT_kx^{-1}|$. Apply 3.6, by taking $\mathfrak{B}=\mathfrak{U}^*$, $\hat{H}=xT_Ax^{-1}$ and $H=xT_kx^{-1}$.

(ii) In 3.6, take $\hat{H}=\mathfrak{B}'$ and $H=\{1\}$. (iii) By the definition, $\deg\Gamma\alpha\Gamma=[\Gamma:\Gamma\cap\alpha^{-1}\Gamma\alpha]$, $\deg\mathfrak{U}\alpha\mathfrak{U}=[\mathfrak{U}:\mathfrak{U}\cap\alpha^{-1}\mathfrak{U}\alpha]$, hence '(1) $\Leftrightarrow$ (3)' is obvious. '(3) $\Leftrightarrow$ (4)' is a special case of (ii) above. '(4) $\Leftrightarrow$ (5)' is by the definition. To see '(1) $\Rightarrow$ (2)', we have: (1) $\Leftrightarrow$ $|\mathfrak{U}\backslash\mathfrak{U}\alpha\mathfrak{U}/\Gamma|=1$, (2) $\Leftrightarrow$ $|\Gamma\backslash\mathfrak{U}\alpha\mathfrak{U}\cap G_k/\Gamma|$ in one hand, and: $1\leqq|\Gamma\backslash\mathfrak{U}\alpha\mathfrak{U}\cap G_k/\Gamma|=|\mathfrak{U}\backslash\mathfrak{U}\alpha\mathfrak{U}\cap\mathfrak{U}G_k/\Gamma^{-1}|\leqq|\mathfrak{U}\backslash\mathfrak{U}\alpha\mathfrak{U}/\Gamma|$, hence (1) $\Rightarrow$ (2).

3.8. LEMMA. *In the same notation as* 3.4, *put $K=k+kg_0$.* (i) *Then K is the centralizer of g_0 in B, hence $T_k=K^\times$ and $T_A=K_A^\times$. Put $d(g_0)=[k_A^\times:k^\times N(K_A^\times)N(\mathfrak{U}^*)]$.* (ii) *If g_0 is either elliptic or hyperbolic, then $d(g_0)=1$. Hence*

$\sharp(\hat{g})=h_G(\mathfrak{U}^*)^{-1}h_T(T_A\cap x^{-1}\mathfrak{U}^*x)$, *and* $\theta: C(g_0)/\tilde{\Gamma}^*\to\hat{C}(g_0)/\tilde{\mathfrak{U}}^*$ *is surjective.* (iii) *Assume* g_0 *is parabolic, and* $N(\mathfrak{U}^*)$ *contains* $r(v)^\times$ *for any* v, *then* $d(g_0)\leqq[k_A^\times:(k_A^\times)^2k^\times\prod r(v)^\times]$. *If the class number* $h(k)$ *of* k *is odd,* $d(g_0)=1$, *and* θ *is surjective. In general,* θ *is not surjective.*

PROOF. (i) g_0 is elliptic if and only if K is a totally imaginary quadratic extension of k. Suppose that is so, then by class field theory, the Galois group $\mathrm{Gal}(K/k)$ is isomorphic to $k_A^\times/k^\times N(K_A^\times)$, hence $d(g_0)=1$ or 2. If $d(g_0)=2$, $k^\times N(K_A^\times)\supset N(\mathfrak{U}^*)\supset N(G_\infty)\supset\prod_{i=1}^n k_{v(i)}^\times$. Hence $v(i)$ is unramified in K for $i=1,\cdots,n$ (cf. [**12**] Proposition 14, p. 277). It is a contradiction to our assumption that K is totally imaginary. g_0 is hyperbolic and fixing a cusp of Γ if and only if B is isomorphic to $M_2(k)$ and K is isomorphic to $k\oplus k$. Suppose that is so, then T is a split torus over k, hence the restriction of N to T_A is already surjective to $k_A^\times$. Finally g_0 is parabolic if and only if B is isomorphic to $M_2(k)$ and g_0 is $B^\times$-conjugate to $\begin{pmatrix} a & b\\ 0 & a\end{pmatrix}$, hence $K=k+kg_0$ is conjugate to $k+k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$. Then $N(K_A^\times)=N(k_A^\times)=k_A^{\times 2}$. Thus we have $d(g_0)=[k_A^\times:(k_A^\times)^2k^\times N(\mathfrak{U}^*)]$ as wanted.

4.0. Let k, $r(k)$, B be as in § 3, and R be an $r(k)$-order of B. Let $\mathfrak{a}$ be an integral two-sided ideal of R, i.e. $\mathfrak{a}$ is a finitely generated $r(k)$-module in R such that $\mathfrak{a}\otimes k=B$ and $R\mathfrak{a}R\subset\mathfrak{a}$. Let $S(\mathfrak{a})$ denote the set of all the non-archimedian places where $\mathfrak{a}_v=\mathfrak{a}\otimes r(v)$ is not equal to R_v. Then the quotient ring $R/\mathfrak{a}$ is canonically isomorphic to $\prod_{v\in S(\mathfrak{a})}R_v/\mathfrak{a}_v$, and we shall identify the one with the other. Let $\sigma: R\to R/\mathfrak{a}$ (resp. $\sigma_v: R_v\to R_v/\mathfrak{a}_v$ for $v\in S(\mathfrak{a})$; resp. $\sigma_{S(\mathfrak{a})}: \prod_{v\in S(\mathfrak{a})}R_v\to\prod_{v\in S(\mathfrak{a})}R_v/\mathfrak{a}_v=R/\mathfrak{a}$) denote the canonical homomorphism. Let $\chi^0:(R/\mathfrak{a})^\times\to GL_r(\boldsymbol{C})$ be a unitary representation of the unit group $(R/\mathfrak{a})^\times$ and χ_v^0 be its restriction to $(R_v/\mathfrak{a}_v)^\times$. For $g=(g_v)\in G_A$ with $\sigma_v(g_v)\in(R_v/\mathfrak{a}_v)^\times$, we put $\chi^*(g)=\prod_{v\in S(\mathfrak{a})}\chi_v^0\circ\sigma_v(g_v)$. We put $\mathfrak{U}^*=\prod R_v^\times$ and let Ξ^* be a finite union of $\mathfrak{U}^*$-double cosets.

Suppose there are given a normal subgroup $\mathfrak{h}$ of $(R/\mathfrak{a})^\times$ and a subgroup $\mathfrak{h}_\infty$ of G_∞/G_∞^+. Let $\mathfrak{S}=\mathfrak{S}(\mathfrak{h},\mathfrak{h}_\infty)=\mathfrak{S}_{S(\mathfrak{a})}\times\prod_{v\notin S(\mathfrak{a})}\mathfrak{S}_v\times\mathfrak{S}_\infty$ be a multiplicative semigroup in B_A defined by: $\mathfrak{S}_{S(\mathfrak{a})}=\sigma_{S(\mathfrak{a})}^{-1}(\mathfrak{h})$, $\mathfrak{S}_v=R_v$ for non-archimedian $v\notin S(\mathfrak{a})$ and $\mathfrak{S}_\infty=\sigma_\infty^{-1}(\mathfrak{h}_\infty)$ where $\sigma_\infty: G_\infty\to G_\infty/G_\infty^+$ is the canonical homomorphism.

Now we put $\mathfrak{U}=\mathfrak{U}^*\cap\mathfrak{S}$, $\hat{\Xi}=\Xi^*\cap\mathfrak{S}$ and $\hat{\chi}$ to be the restriction of χ^* to $\langle\hat{\Xi}\rangle$. Then, as is easily seen, the quadruplet $\{\mathfrak{U},\hat{\Xi},\hat{\chi},\mathfrak{U}^*\}$ satisfy the condition (U.1)$\sim$(U.3) and (U*) in 3.1. Finally we put $\Gamma=\mathfrak{U}\cap G_k$, $\Xi=\hat{\Xi}\cap G_k$, $\Gamma^*=\mathfrak{U}^*\cap G_k$ and χ to be the restriction of $\hat{\chi}$ to $\langle\Xi\rangle$. Then $\{\Gamma,\Xi,\chi,\Gamma^*\}$ satisfy

$(\Gamma 1)\sim(\Gamma 4)$ and (Γ^*) in § 1.

4.1. For $x=(x_v)\in G_A$, let $x^{-1}Rx$ denote the $r(k)$-order of B determined by $(x^{-1}Rx)_v=x_v^{-1}R_vx_v$ for any v. Let $g\in\Omega$, $K=k+kg$ and $\hat{C}(g)=\{xgx^{-1}\mid x\in G_A\}$. Noticing the fact $(x^{-1}Rx\cap K)_v=x_v^{-1}R_vx_v\cap K_v$ for $x\in G_A$, it is easy to establish the following.

(i) For $g'=xgx^{-1}\in\hat{C}(g)\cap\hat{\Xi}$, the $r(k)$-order $x^{-1}Rx\cap K$ of K is independent of the choice of x, so we write $\Lambda(g')=x^{-1}Rx\cap K$. $\Lambda(g')$ is an $r(k)$-order of K containing g. For an $r(k)$-order Λ of K, put $\hat{C}(g,\Lambda)=\{g'\in\hat{C}(g)\mid\Lambda(g')=\Lambda\}$. It is normalized by $\mathfrak{U}^*$ and it is vacant if $g\notin\Lambda$.

(ii) $\hat{C}(g,\Lambda)\cap C(g)=C(g,\Lambda)=\{xgx^{-1}\mid x\in G_k,\ x^{-1}Rx\cap K=\Lambda\}$. $\hat{\Xi}\cap\hat{C}(g)/\tilde{\mathfrak{U}}^*$ (resp. $\Xi\cap C(g)/\tilde{\Gamma}^*$) is a disjoint union $\bigcup_{\Lambda}\hat{\Xi}\cap\hat{C}(g,\Lambda)/\hat{\mathfrak{U}}^*$ (resp. $\Xi\cap C(g,\Lambda)/\tilde{\Gamma}^*$) where Λ runs through all the $r(k)$-orders of K containing g.

(iii) Since $\hat{\Xi}\cap\hat{C}(g,\Lambda)$ is normalized by $\mathfrak{U}^*$, we can consider the restriction of θ in 3.4,

$$\theta:\ \Xi\cap C(g,\Lambda)/\tilde{\Gamma}^*\longrightarrow\hat{\Xi}\cap\hat{C}(g,\Lambda)/\tilde{\mathfrak{U}}^*.$$

Let $g'\in\hat{\Xi}\cap\hat{C}(g,\Lambda)$. If g is either elliptic or hyperbolic, $\hat{\lambda}(g')$ is given by:

$$\hat{\lambda}(g')=s^{-1}[\Lambda^\times:r(k)^\times]^{-1}\operatorname{tr}\hat{\chi}(g').$$

By 3.7 (i) and 3.8, $\sharp(g')$ is given by:

$$\sharp(g')=\begin{cases} h(\Lambda)/h(R) & \text{if } g \text{ is either elliptic or hyperbolic},\\ h(\Lambda)/h(k) & \text{if } g \text{ is parabolic, } N(R_v^\times)\supset r(v)^\times \text{ and } h(k) \text{ is odd}.\end{cases}$$

4.2. (i) There exists a finite set S of non-archimedian places with the following property. If $v\notin S$, then $\mathfrak{a}_v=R_v=M_2(r(v))$, (hence $\mathfrak{S}_v=R_v$, χ_v is trivial) and $\Xi_v^*\cap\hat{C}_v(g)=R_v\cap\hat{C}_v(g)$ where Ξ_v^* (resp. $\hat{C}_v(g)$) is the projection of Ξ^* (resp. $\hat{C}(g)$) to $G_{k(v)}$.

(ii) Let $\hat{\Xi}_S$ (resp. $\hat{C}_S(g,\Lambda)$; resp. $\mathfrak{U}_S^*$) denote the projection of $\hat{\Xi}$ (resp. $\hat{C}(g,\Lambda)$; resp. $\mathfrak{U}^*$) to $G_S=\prod_{v\in S}G_{k(v)}$. Then, by (i) 2.6, the projection $g'\mapsto g'_S=\prod_{v\in S}g_v$ induces the following bijection:

$$(1)\qquad \hat{\Xi}\cap\hat{C}(g,\Lambda)/\tilde{\mathfrak{U}}^*\simeq\hat{\Xi}_S\cap\hat{C}_S(g,\Lambda)/\tilde{\mathfrak{U}}_S^*.$$

(iii) Suppose $\mathfrak{S}_S$ is a direct product $\mathfrak{S}_S=\prod_{v\in S}\mathfrak{S}_v$, we have a natural bijection (2) and, by 2.1, another bijection (3):

$$(2)\qquad \hat{\Xi}_S\cap\hat{C}_S(g,\Lambda)/\tilde{\mathfrak{U}}_S^*\simeq\prod_{v\in S}(\Xi_v^*\cap\mathfrak{S}_v\cap\hat{C}_v(g,\Lambda))/\tilde{R}_v^\times,$$

$$(3)\qquad \Xi_v^*\cap\mathfrak{S}_v\cap\hat{C}_v(g,\Lambda)/\tilde{R}_v^\times\simeq\operatorname{Emb}(g,\Xi_v^*\cap\mathfrak{S}_v,R_v/\Lambda_v)/\tilde{R}_v^\times.$$

Combining, the above (1), (2) and (3), if $g'\mapsto g'_S\mapsto(g'_v)_{v\in S}\mapsto(\varphi_v)_{v\in S}$, then

$\hat{\chi}(g')$ is given by:

(4) $$\hat{\chi}(g') = \prod_{v \in S(\mathfrak{a})} \chi^0 \circ \sigma_v \circ \varphi_v(g).$$

(iv) For $\eta \in \mathfrak{h}$, put $\mathfrak{S}^\eta = \mathfrak{S}^\eta_{S(\mathfrak{a})} \times \prod_{v \notin S(\mathfrak{a})} R_v \times \mathfrak{S}_\infty$, $\mathfrak{S}^\eta_{S(\mathfrak{a})} = \sigma^{-1}(\eta)$. Then $\mathfrak{S}^\eta_S = \prod_{v \in S} \mathfrak{S}^\eta_v$ and $\mathfrak{S} = \bigcup_{\eta \in \mathfrak{h}} \mathfrak{S}^\eta$ is a disjoint union. Suppose $\mathfrak{h}$ is in the center of $(R/\mathfrak{a})^\times$, then each $\mathfrak{S}^\eta$ is normalized by $\mathfrak{U}^*$.

Hence we have:

(5) $$\Xi_S \cap \hat{C}_S(g, \Lambda)/\tilde{\mathfrak{U}}^*_S \simeq \bigcup_{\eta \in \mathfrak{h}} \prod_{v \in S} \mathrm{Emb}(g, \Xi^*_v \cap \mathfrak{S}^\eta_v, R_v/\Lambda_v)/\tilde{R}^\times_v.$$

Let $c(\eta, v)$ denote the cardinality of $\mathrm{Emb}(g, \Xi^*_v \cap \mathfrak{S}^\eta_v, R_v/\Lambda_v)/\tilde{R}^\times_v$, then

(6) $$\sum_{g'} \mathrm{tr}\, \hat{\chi}(g') = \sum_{\eta \in \mathfrak{h}} \mathrm{tr}\, \chi^0(\eta) \cdot \prod_{v \in S} c(\eta, v)$$

where g' runs through $\hat{\Xi} \cap \hat{C}(g, \Lambda)/\tilde{\mathfrak{U}}^*$.

4.3. If g is either elliptic or hyperbolic, then there are only a finite number or $r(k)$-orders containing g in K. If g is parabolic, there are infinitely many of them. We shall see that it is enough to consider only a finite number of them. We may assume that $k = \boldsymbol{Q}$, $B = M_2(\boldsymbol{Q})$ and $\Xi \subset R \subset M_2(\boldsymbol{Z})$. Since $\underset{k}{\sim}$ is simply the $GL_2(\boldsymbol{Q})$-conjugacy, we have $g \underset{k}{\sim} g'$ if and only if $\det g = \det g'$ for any $g, g' \in \Omega_p$, and the deternimation of $\Omega_p/\underset{k}{\sim}$ is trivial. Let $g \in \Omega_p$, then its determinant is a square of its eigen value ζ, $\det g = \zeta^2$. Put $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $z = z(\zeta) = \zeta + y$, then g is $GL_2(\boldsymbol{Q})$-conjugate to z. A $\boldsymbol{Z}$-order Λ of $K = \boldsymbol{Q} + \boldsymbol{Q}y$ contains z if and only if it has the form $\Lambda = \Lambda_t = \boldsymbol{Z} + \boldsymbol{Z}(y/t)$ by some natural number t. Hence $\Xi \cap C(g)/\tilde{\Gamma}^*$ is a disjoint union of $\Xi \cap C(z(\zeta), \Lambda_t)/\tilde{R}^\times$ for $t = 1, 2, \cdots$. Put

(1) $$E = \mathrm{Emb}(y, R, R/\Lambda_1)/\tilde{R}^\times, \qquad E_t = E_t(\zeta) = \mathrm{Emb}(z(\zeta), \Xi, R/\Lambda_t)/\tilde{R}^\times$$

and

$$E'_t = E'_t(\zeta) = \mathrm{Emb}(z(\zeta), R, R/\Lambda_t)/\tilde{R}^\times \qquad \text{for } t = 1, 2, \cdots.$$

Our purpose here is to determine $E_t(\zeta)$ for any $\zeta \in (\det \Omega_p)^{1/2}$ and any t, then $\Xi \cap C(g, \Lambda_t)/\tilde{R}^\times = \{\varphi(z);\ \varphi \in E_t\}$. Let's take any ζ and fix it for a moment. For any $\psi \in E$, define the embedding $\psi'_t: K \to B$ as

(2) $$\psi'_t(z) = \zeta + t\psi(y).$$

Then $\psi'_t \in E'_t$ and, as is easily seen from definitions, the map $\psi \mapsto \psi'_t$ induces a bijection $E \simeq E'_t$, i.e. $E'_t = \{\psi'_t;\ \psi \in E\}$. Hence, as a principle, if we can determine E, then we can determine E'_t, and $E_t = \{\varphi \in E'_t;\ \varphi(z) \in \Xi\}$ as a subset of E'_t. In particular, if $g \in \Xi \cap C(g, \Lambda_t)$, then $g = \zeta + t\psi(y)$ by some $\psi \in E$, hence $\Gamma^*(g) = (\boldsymbol{Q} + \boldsymbol{Q}g) \cap R^\times = (\boldsymbol{Q} + \boldsymbol{Q}\psi(y)) \cap R^\times = \psi(\Lambda_1^\times) = \langle \pm(1 + \psi(y)) \rangle$, $\langle \iota(g) \rangle = \langle \iota(1 + t\zeta^{-1}\psi(y)) \rangle$. i.e.

(3) $[\iota(\Gamma^*(g)):\langle\iota(g)\rangle]=t|\zeta|^{-1}$.

Hence substituting $[\mathfrak{Z}\cap\Gamma^*:\mathfrak{Z}_1\cap\Gamma]=2$ and the above (3) into the formula (3) in 1.2 (modified by 1.5) we get

$$(4)\qquad t_p=-\lim_{s\to 0} s[\Gamma^*:\Gamma][\mathfrak{Z}\cap\Gamma^*:\mathfrak{Z}_1\cap\Gamma]^{-1}\sum_{g\in\Omega_p/\sim}\kappa(g)\lambda^*(g)$$
$$=-[R^\times:\Gamma]\sum_{\zeta}|\zeta|((\operatorname{sgn}\zeta)^k/4)\lim_{s\to 0} s\sum_{t=1} t^{-1+s}\sum_{\varphi\in E_t(\zeta)}\operatorname{tr}\chi\circ\varphi(\zeta+y)$$

where ζ runs over $(\det\Omega_p)^{1/2}$.

To make the sum $\sum_t$ finite, we choose an integer m with the following properties.

(5) If $g\equiv g' \bmod m$, then "$\chi(g)=\chi(g')$" and "$g\in\Xi$ if and only if $g'\in\Xi$".

Since $\mathfrak{U}^*$ is open, it is certainly possible to find such m, (for example, we have chosen M in 1.0 as m here) and we fix it for once for all. Then if $t\equiv u \bmod m$, and $\psi\in E$, then $\psi'_t\in E_t$ if and only if $\psi'_u\in E_u$, in particular $\sum_{\varphi\in E_t(\zeta)}=\sum_{\varphi\in E_u(\zeta)}$, and

$$(6)\qquad t_p=-[R^\times:\Gamma](\sum_{\zeta}|\zeta|(\operatorname{sgn}\zeta)^k/4)\frac{1}{m}\sum_{t=1}^{m}\sum_{\varphi\in E_t(\zeta)}\operatorname{tr}\chi\circ\varphi(\zeta+y).$$

If $h(R)=1$, $N(R_v^\times)\supset r(v)^\times$, $\theta: C(g)/\tilde{\Gamma}^*\to\hat{C}(g)/\tilde{\mathfrak{U}}^*$ is surjective by (iii) 3.8, or if χ is a trivial representation then by an obvious reason, we can replace the global sum $\sum_\varphi$ by the following local sum. If $\mathfrak{S}_S=\prod_{v\in S}\mathfrak{S}_v$,

$$(7)\qquad \sum_{\varphi}\operatorname{tr}\chi(\varphi(g))=h(R)^{-1}\prod_{v\in S}\sum_{\varphi_v}\operatorname{tr}\chi_v^0\circ\sigma_v\circ\varphi_v(g)$$

where φ_v is running through $\operatorname{Emb}(g,\Xi_v^*\cap\mathfrak{S}_v, R_v/\Lambda_v)/\tilde{R}_v^\times$.

If $\mathfrak{h}$ is in the center of $(R/\mathfrak{a})^\times$, then

$$(8)\qquad \sum_{\varphi}\operatorname{tr}\chi(\varphi(g))=\sum_{\eta\in\mathfrak{h}}\operatorname{tr}\chi^0(\eta)\prod_{v\in S}|\operatorname{Emb}(g,\Xi_v^*\cap\sigma_v^{-1}(\eta), R/\Lambda_v)/\tilde{R}_v^\times|.$$

4.4. LEMMA. *For a subgroup $\mathfrak{V}$ of G_A, and an element $\alpha\in G_A$, let $\mathfrak{V}(\alpha)$ denote the subgroup $\mathfrak{V}\cap\alpha^{-1}\mathfrak{V}\alpha$.*

(i) *If $\alpha\in\mathfrak{S}$, then the following four conditions are equivalent for $\mathfrak{U}=\mathfrak{U}^*\cap\mathfrak{S}$.*

(1) $\mathfrak{U}^*=\mathfrak{U}^*(\alpha)\mathfrak{U}$, (2) $\mathfrak{U}^*\alpha\mathfrak{U}^*\cap\mathfrak{S}=\mathfrak{U}\alpha\mathfrak{U}$,

(3) $\mathfrak{U}^*\alpha\mathfrak{U}^*=\mathfrak{U}^*\alpha\mathfrak{U}$, (4) $[\mathfrak{U}^*:\mathfrak{U}^*(\alpha)]=[\mathfrak{U}:\mathfrak{U}(\alpha)]$.

(ii) *If $\alpha\in\mathfrak{S}\cap G_k$ satisfies one of the above (1)$\sim$(4), and if $h(\mathfrak{U})=h(\mathfrak{U}(\alpha))$, then any of the following holds.*

(5) $(\mathfrak{U}^*\alpha\mathfrak{U}\cap\mathfrak{S})\cap G_k=\Gamma\alpha\Gamma$, (6) $h(\mathfrak{U}^*)=h(\mathfrak{U}^*(\alpha))$,

(7) $\deg\Gamma\alpha\Gamma=[\mathfrak{U}^*:\mathfrak{U}^*(\alpha)]$.

Consequently putting $\Xi^*=\mathfrak{U}^*\Gamma\alpha\Gamma\mathfrak{U}^*$, *we can recover* $\Gamma\alpha\Gamma$ *as* Ξ, *i.e.* $\Xi^*\cap\mathfrak{S}\cap G_k=\Gamma\alpha\Gamma$.

PROOF. (i) is elementary. (ii) (1) implies $\mathfrak{U}^*\alpha\mathfrak{U}^*\cap\mathfrak{S}=\mathfrak{U}\alpha\mathfrak{U}$ by (i). By 3.7 (iii), '$h(\mathfrak{U})=h(\mathfrak{U}(\alpha))$' implies $\mathfrak{U}\alpha\mathfrak{U}\cap G_k=\Gamma\alpha\Gamma$, and $\deg\Gamma\alpha\Gamma=[\mathfrak{U}:\mathfrak{U}(\alpha)]$ which is equal to $[\mathfrak{U}^*:\mathfrak{U}^*(\alpha)]$ by (4).

Since $\deg\Gamma\alpha\Gamma\leqq\deg R^\times\alpha R^\times\leqq[\mathfrak{U}^*:\mathfrak{U}^*(\alpha)]$, (7) implies (6).

4.5. LEMMA. *Let* $h(r(k)$ (*resp.* $D(k/\boldsymbol{Q})$; *resp.* ζ_k) *denote the class number* (*resp. discriminant over* $\boldsymbol{Q}$; *resp. zeta function*) *of* k. *Let* $S(B)$ *denote the set of all non-archimedian places where* $B\otimes k(v)$ *is a division algebra, and* $n(v)$ *denote the cardinality of* $r(v)/p(v)$. *Finally let* R_0 *be any maximal order containing* R, *then the volume of the fundamental domain of* $R^\times$ *is given by:*

$$(1)\qquad \mathrm{vol}(\mathfrak{H}_\pm^n/R^\times)=(\pi^{2m-n}h(R))^{-1}2^{2(n-m)+1}D(k/\boldsymbol{Q})^{3/2}h(r(k))\zeta_k(2)\times\prod_{v\in S(B)}(n(v)-1)\prod[(R_0)_v^\times:R_v^\times].$$

If $h(\mathfrak{U})=h(\mathfrak{U}^*)(=h(R))$, *then* $\mathrm{vol}(\mathfrak{H}_\pm^n/\Gamma)=[R^\times:\Gamma]\,\mathrm{vol}(\mathfrak{H}_\pm^n/R^\times)$ *is given by:*

$$(2)\qquad \mathrm{vol}(\mathfrak{H}_\pm^n/\Gamma)=[\mathfrak{U}^*:\mathfrak{U}]\,\mathrm{vol}(\mathfrak{H}_\pm^n/R^\times).$$

PROOF. By [**9**] Appendix,

$$\mathrm{vol}(\mathfrak{H}_\pm^n/R_0^\times)=(\pi^{2m-n}h(R_0))^{-1}2^{2(n-m)+1}D(k/\boldsymbol{Q})^{3/2}h(r(k))\zeta_k(2)\prod_{v\in S(B)}(n(v)-1).$$

We simply multiply it by $[R_0^\times:R]$ which is equal to $h(R)^{-1}h(R_0)\prod[(R_0)_v^\times:R]$ by 3.7 (ii). (2) is immediate by 3.7 (ii).

4.6. THEOREM. *Let everything be as in* 4.0. *Let* $\{\Gamma\alpha_j\Gamma\mid j=1,\cdots,u\}$ *be the set of distinct double cesets contained in* Ξ *and let* $\xi=\sum_{j=1}^{u}\Gamma\alpha_j\Gamma$. *Suppose* B *is not isomorphic to* $M_2(\boldsymbol{Q})$. *Then the trace* $\mathrm{tr}\,T(\xi)$ *of the Hecke operator* $T(\xi)$ *on the space of cusp forms* $S(\Gamma,\{k_i\},\chi)$ *is a sum* $\mathrm{tr}\,T(\xi)=t+t'$ *as in* 1.2, *and the each term is given as follows.*

$$(1)\qquad t'=\sum_{j=1}^{u}t'_j,$$

$$t'_j=\begin{cases}[R^\times:\Gamma]\,\mathrm{vol}(\mathfrak{H}_\pm^n/R^\times)\,\mathrm{tr}\,\chi(\varepsilon)\displaystyle\prod_{i=1}^{n}(\mathrm{sgn}\,\varepsilon^{(i)})^{k_i}\left(\frac{k_i-1}{4\pi}\right) & \textit{if } k^\times\cap\Gamma\alpha_j\Gamma\neq\emptyset\\ 0 & \textit{if } k^\times\cap\Gamma\alpha_j\Gamma=\emptyset\end{cases}$$

and $\mathrm{vol}(\mathfrak{H}_\pm^n/R^\times)$ *has been given in* 4.5.

$$(2)\qquad t=[R^\times:\Gamma]\sum_{g\in\Omega_e/\widetilde{k}}\frac{1}{2}\prod\frac{\zeta_i^{k_i-1}-\eta_i^{k_i-1}}{\zeta_i-\eta_i}(\det g^{(i)})^{1-k_i/2}\sum_{\Lambda}\frac{h(\Lambda)}{h(R)}\frac{c(g,\Lambda)}{[\Lambda^\times:r(k)]}$$

and $c(g,\Lambda)=\sum_{g'}\mathrm{tr}\,\chi(g')$ *where* g' *is running through* $\hat{\Xi}\cap\hat{C}(g,\Lambda)/\tilde{\mathfrak{U}}^*$.

S being a finite set of non-archimedian places defined in (i) 4.2, *if* $\mathfrak{S}_S = \prod_{v\in S}\mathfrak{S}_v$ *as in* (iii) 4.2, (*in particular if* $\mathfrak{h} = (R/\mathfrak{a})^\times$ *and* $\Gamma = R^\times$!), *then:*

$$(3) \qquad c(g, \Lambda) = \mathrm{tr} \prod_{v\in S} \sum_{\varphi_v} \chi_v^0 \circ \sigma_v \circ \varphi_v(g),$$

where φ_v *is running through the set* $\mathrm{Emb}(g, \Xi_v^* \cap \mathfrak{S}_v, R_v/\Lambda_v)/\tilde{R}_v^\times$. *If* $\mathfrak{h}$ *is in the center of* $(R/\mathfrak{a})^\times$ *as in* (iv) 4.2, *then*

$$(4) \qquad c(g, \Lambda) = \sum_{\eta\in\mathfrak{h}} \mathrm{tr}\, \chi^0(\eta) \prod_{v\in S} |\mathrm{Emb}(g, \Xi_v^* \cap \sigma_v^{-1}(\eta), R_v/\Lambda_v)/\tilde{R}_v^\times|.$$

All the notations have been given, we will briefly repeat the ones not appearing in 4.0. $r_1(k) = \{\varepsilon \in r(k)^\times \mid \varepsilon^{(1)} > 0\}$; Ω_e = the set of elliptic elements in Ξ ; $g \underset{k}{\sim} g' \Leftrightarrow {}^\exists\varepsilon \in r_1(k)^\times$, ${}^\exists x \in G_k$, $g' = \varepsilon x g x^{-1}$; $\{\zeta_i, \eta_i\}$ is the set of eigen values of $g^{(i)}$. Λ is running through the $r(k)$-orders of $k+kg$ containing g. $h(\Lambda)$ (resp. $h(R)$) denotes the class number of Λ (resp. R). $\mathrm{Emb}(g, \Xi_v^* \cap \mathfrak{S}_v, R_v/\Lambda_v)/\tilde{R}_v^\times$ denotes as in 2.1 (ii), a complete system of representatives of $R_v^\times$-equivalence of the optimal embeddings φ; $\Lambda_v \to R_v$ such that $\varphi(g) \in \Xi_v^* \cap \mathfrak{S}_v$.

PROOF. (1) is a repetition of (1) in 1.2. t in 1.2 can be converted to the right hand side of (2), by 1.4, 3.3, 3.4 (iii), 4.1 and 4.2 (i) applied in this order. (3) (resp. (4)) is a consequence of (ii) (resp. (iii)) 4.2.

4.7. THEOREM. *Let everything be as in* 4.6, *except that here we assume* B *is isomorphic to* $M_2(\boldsymbol{Q})$, *hence the argument of* 4.3 *is applicable. Then* $\mathrm{tr}\, T(\xi) = t + t' + t''$, t' *is given by the same formula* (1) *in* 4.6, $t'' = 2\deg\xi = \sum_{j=1}^{u} [\Gamma : \Gamma \cap \alpha_j^{-1}\Gamma\alpha_j]$, *and* t *is given as follows.* $t = t_e + t_h + t_p$, t_e *is given by the formula* (2) *in* 4.6, t_h *is given by the same formula as* t_e *if we replace* Ω_e *by the set of hyperbolic element* Ω_h *in* Ω, *and* t_p *is given by* (4) *or* (6) *in* 4.3. *If either* $h(R) = 1$ *or* χ *is a trivial representation, we can put them together in the following form.*

$$(1) \qquad t = [R^\times : \Gamma] \sum_{g \in \Omega/\underset{k}{\sim}} a(g) \sum_{\Lambda} b(\Lambda) c(g, \Lambda),$$

$$(2) \qquad a(g) = \begin{cases} \dfrac{1}{2}(\zeta^{k-1} - \eta^{k-1})(\zeta-\eta)^{-1}(\det g)^{1-k/2} & \text{if } g \text{ is elliptic} \\ (\mathrm{Min}\,\{|\zeta|, |\eta|\})^{k-1}|\zeta-\eta|^{-1}(\mathrm{sgn}\,\zeta)^k(\det g)^{1-k/2} & \text{if } g \text{ is hyperbolic} \\ \dfrac{1}{4M}|\zeta|(\mathrm{sgn}\,\zeta)^k & \text{if } g \text{ parabolic,} \end{cases}$$

$$(3) \qquad b(\Lambda) = \begin{cases} h(R)^{-1}h(\Lambda)[\Lambda^\times : r(k)^\times]^{-1} & \text{if } g \text{ is elliptic or hyperbolic} \\ h(R)^{-1}h(\Lambda) = h(R)^{-1} & \text{if } g \text{ is parabolic,} \end{cases}$$

where, for each g, $c(g, \Lambda)$ *has the same meaning as in* 4.6, *and it can be written as* (3) (*resp.* (4)) 4.6, *if* $\mathfrak{S} = \prod\mathfrak{S}_v$ (*resp.* $\mathfrak{h}$ *is in the center of* $(R/\mathfrak{a})^\times$).

Most of the notations are the same as in 4.6, we only explain the new ones. Ω is the set of elliptic or hyperbolic fixing a cusp of Γ or parabolic elements in Ξ; Λ is running through all the orders (resp. the orders such that $[\Lambda : \boldsymbol{Z}+\boldsymbol{Z}g] \leqq M$) of $\boldsymbol{Q}+\boldsymbol{Q}g$ containing g, if g is either elliptic or hyperbolic (resp. parabolic); $\{\zeta, \eta\}$ is the set of eigen values of g; M is a natural number such that $g' \equiv g'' \bmod M$ implies $g-g' \in \mathfrak{a}$.

PROOF. To get (1), we only need 4.3 besides the results which are needed for 4.6.

4.8. REMARK. (i) In the setting of 4.6 or 4.7, suppose $\mathfrak{h}_\infty = \{1\}$. Then Γ is acting on $\mathfrak{H}^n$. Furthermore assume $[R^\times : R^\times \cap \mathfrak{G}^+] = 2^n$, then the condition ($\Gamma$5) 1.3 is satisfied with $\Delta = R^\times$. Hence we have:

$$\operatorname{tr} T_0(\xi) = 2^{-n} \operatorname{tr} T(\xi).$$

(ii) In the formula of t in 4.6 or 4.7, it is not necessary to precisely determine $\Omega/\underset{k}{\sim}$ by the following reason. If g is either elliptic or hyperbolic, then by 3.8, $\operatorname{Emb}(g, \Xi, R/\Lambda)$ is not vacant if and only if $\operatorname{Emb}(g, \Xi_v, R_v/\Lambda_v)$ is not vacant for all v. Hence in $\sum_g$ we can let g run through some bigger set than $\Omega/\underset{k}{\sim}$ (for example the set of all elliptic elements g with $N(g) \in N(\Xi)$!), provided that we understand the term $\sum_{\varphi_v}$ represents zero if $\operatorname{Emb}(g, \Xi_v, R_v/\Lambda_v)$ is vacant. For parabolic points, as is seen in 4.3, we can take $\left\{\begin{pmatrix} \zeta & 1 \\ 0 & \zeta \end{pmatrix} \middle| \zeta^2 \in N(\Xi)\right\}$ instead of $\Omega_p/\underset{k}{\sim}$.

5.0. Let k, $r(k)$ and B be as in § 3 and § 4. Let $S(B)$ denote the set of all non-archimedian places v of k where $B \otimes k(v)$ is a division quaternion. An $r(k)$-order R of B will be called *split* if R_v is split in the sense of 2.2 whenever $v \notin S(B)$ and R_v is maximal whenever $v \in S(B)$. Let $S(R)$ denote the set of all the non-archimedian places where R_v is not maximal. Then, by the definition, there is a natural number $\nu(v)$ and an isomorphism $R_v \xrightarrow{\sim} \begin{pmatrix} r(v) & r(v) \\ p(v)^{\nu(v)} & r(v) \end{pmatrix}$ for each $v \in S(R)$. Let $\mu(v)$ be a non-negative integer not greater than $\nu(v)$. Identifying R_v with $\begin{pmatrix} r(v) & r(v) \\ p(v)^{\nu(v)} & r(v) \end{pmatrix}$ by the above isomorphism, let $\mathfrak{a}_v = \begin{pmatrix} p(v)^{\mu(v)} & r(v) \\ p(v)^{\nu(v)} & r(v) \end{pmatrix}$ for $v \in S(R)$, $\mathfrak{a}_v = R_v$ for $v \notin S(R)$ and $\mathfrak{a} = \bigcap_v (\mathfrak{a}_v \cap B)$. Then, as is easily seen, $\mathfrak{a}$ is a two sided ideal of R, $S(\mathfrak{a}) \subset S(R)$ and $R_v/\mathfrak{a}_v$ is isomorphic to $r(v)/p(v)^{\mu(v)}$ by the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a \bmod p(v)^{\mu(v)}$. Hence if $\chi^0 : (R/\mathfrak{a})^\times \to GL_r(\boldsymbol{C})$ is irreducible, then $r=1$, and it can be identified with a linear character of $\prod_{v \in S(\mathfrak{a})} (r(v)/p(v)^{\mu(v)})^\times$. For this $\{R, \mathfrak{a}, \chi^0\}$, we consider $\mathfrak{S} = \mathfrak{S}(\mathfrak{h}, \mathfrak{h}_\infty)$, $\mathfrak{U}^* = \prod R_v^\times$, $\mathfrak{U} = \mathfrak{S} \cap \mathfrak{U}^*$ etc. as in § 4. If $\alpha \in \mathfrak{S}$, a double coset

$\mathfrak{U}\alpha\mathfrak{U}$ meets with G_k if and only if $N(\alpha)r$ is a principal ideal ar of r such that $a^{(i)}>0$ for any archimedian places $v(i)$, $n<i\leqq m$ where $B\otimes k(v(i))$ is division.

Indeed, by 3.5, $\mathfrak{U}\alpha\mathfrak{U}$ meets with G_k if and only if $N(\alpha)$ is contained in $N(G_k\mathfrak{U})$. Since $\mathfrak{S}_\infty=\mathfrak{U}_\infty$, the archimedian part α_∞ of α is contained in $\mathfrak{U}_\infty$, hence $N(\alpha)$ is contained in $N(G_k\mathfrak{U})$ if and only if the non-archimedian part $N(\alpha)_f$ of $N(\alpha)$ is contained in $N(G_k\mathfrak{U})_f$. Since $N(\mathfrak{U})_f=N(\mathfrak{U}^*)_f=\prod r(v)^\times$, $N(\alpha)$ is contained in $N(G_k\mathfrak{U})$ if and only if $N(\alpha)_f$ is contained in $k^\times\prod r(v)^\times$, i.e. $N(\alpha)r$ is a principal ideal ar of the required property. Let us call a double coset $\mathfrak{U}^*\alpha\mathfrak{U}^*$ is *diagonalizable* if it satisfies the following condition (D).

(D) $\mathfrak{U}_v^*\alpha_v\mathfrak{U}_v^*$ contains a diagonal element for any $v\in S(R)$.

5.1. LEMMA. *Let $\mathfrak{U}^*\hat{\alpha}\mathfrak{U}^*$ be diagonalizable, and $\alpha\in\mathfrak{U}^*\hat{\alpha}\mathfrak{U}^*$.*

(i) *Then $\mathfrak{U}^*=\mathfrak{U}^*(\alpha)\mathfrak{U}$ and $h(\mathfrak{U}(\alpha))=h(\mathfrak{U})$ for any $\mathfrak{U}=\mathfrak{U}^*\cap\mathfrak{S}$.*

(ii) *If furthermore either $\mathfrak{h}_\infty=G_\infty/G_\infty^+$, or $r(k)^\times$ contains an element of any preassigned signature distribution, i.e. for any $I\subset\{1,\cdots,n\}$, there exists $\varepsilon\in r(k)^\times$ such that $\varepsilon^{(i)}>0$ for $i\in I$ and $\varepsilon^{(i)}<0$ for $i\notin I$. Then $h(\mathfrak{U}(\alpha))=h(R)$.*

PROOF. (i) If we prove the assertions for one α, then it obviously implies the assertions for any $\alpha\in\mathfrak{U}^*\hat{\alpha}\mathfrak{U}^*$. Hence we may assume α_v is diagonal for any $v\notin S(B)$. If $v\notin S(B)$, $\mathfrak{U}_v^*(\alpha_v)=R_v^\times\cap\alpha_v^{-1}R_v^\times\alpha_v\supset\begin{pmatrix}r(v)^\times & 0\\ 0 & r(v)^\times\end{pmatrix}$. Hence $\mathfrak{U}_v^*(\alpha_v)\mathfrak{U}_v\supset\begin{pmatrix}r(v)^\times & 0\\ 0 & r(v)^\times\end{pmatrix}\mathfrak{U}_v\supset R_v^\times$.

Since $\mathfrak{U}_v(\alpha_v)$ contains $\begin{pmatrix}1 & 0\\ 0 & r(v)^\times\end{pmatrix}$ for any $\mathfrak{S}$, $N(\mathfrak{U}_v(\alpha_v))=N(R_v^\times)$. If $v\in S(B)$, then α_v normalizes $R_v^\times$ and we have trivially $\mathfrak{U}_v^*(\alpha_v)=\mathfrak{U}_v^*$. (ii) $h(\mathfrak{U}(\alpha))=h(R)$ if and only if $k^\times N(\mathfrak{U}(\alpha))=k^\times N(\mathfrak{U}^*)$. Let $k_{\infty+}^\times$ be the connected component of $k_\infty^\times$. Since $k^\times N(\mathfrak{U}(\alpha))\supset k^\times\prod r(v)^\times k_{\infty+}^\times$, if $\mathfrak{h}_\infty=G_\infty/G_\infty^+$, $k^\times N(\mathfrak{U}(\alpha))$ obviously contains $N(\mathfrak{U}^*)$. Suppose $r(k)$ satisfies the assumption, then $k^\times\prod r(v)^\times k_{\infty+}^\times\supset k_\infty^\times k^\times\prod r(v)^\times\supset k^\times N(\mathfrak{U}^*)$.

5.2. Let $\mathfrak{b}$ be an integral ideal of k, and assume $\mathfrak{b}\otimes r(v)=r(v)$ for any $v\in S(R)$. Put $\Xi^*=\{\alpha\in\prod R_v\mid N(\alpha)r(k)=\mathfrak{b}\}$.

(i) Ξ^* is a union of diagonalizable $\mathfrak{U}^*$-double cosets. If $\mathfrak{b}$ is principal, each of the double cosets in Ξ^* meets with $\mathfrak{S}$, hence meets with $\mathfrak{S}\cap G_k$ by the last remark in 5.0. Thus $\Xi^*=\bigcup_{j=1}^{u}\mathfrak{U}^*\alpha_j\mathfrak{U}^*$ with $\alpha_i\in\Xi=\Xi^*\cap\mathfrak{S}\cap G_k$. Ξ is the set of all $\alpha\in R\cap\mathfrak{S}$ such that $N(\alpha)r=\mathfrak{b}$. By 5.1, $\Xi=\bigcup_{j=1}^{u}\Gamma\alpha_j\Gamma$. Let $\xi=\sum_{j=1}^{u}\Gamma\alpha_j\Gamma$, then again by 5.1, $\deg\xi=\sum_{j=1}^{u}[\mathfrak{U}^*:\mathfrak{U}^*(\alpha_j)]=|\Xi^*/\mathfrak{U}^*|=\prod_v|\Xi_v^*/\mathfrak{U}_v^*|$. Then, as is easily seen, $\deg\xi=\prod(n(v)^{\tau(v)}-1)(n(v)-1)^{-1}$, where $\mathfrak{b}\otimes r(v)=p(v)^{\tau(v)}$, $n(v)=[r(v):p(v)]$ and v runs through all the v's such that $\tau(v)>0$ and $v\notin S(B)$.

(ii) If $g\in G_k$, $\mathrm{ord}_v(N(g))=\tau(v)$, then $\mathrm{Emb}(g,\Xi_v^*,R_v/\Lambda_v)=\mathrm{Emb}(g,R_v,R_v/\Lambda_v)$, consequently $\mathrm{Emb}(g,\Xi_v^*\cap\sigma_v^{-1}(\eta),R_v/\Lambda_v)=\mathrm{Emb}(g,\sigma_v^{-1}(\eta),R_v/\Lambda_v)$ for any v, any $\eta\in(R/\mathfrak{a})^\times$ and any order Λ of $k+kg$. Put $S=S(B)\cup S(R)$, then

S satisfies the condition of 4.2 (i).

5.3. Let v_0 be a non-archimedian place outside of $S(B)$ corresponding a principal prime ideal (π_0) of $r(k)$. Let $\alpha \in G_A$ be defined by $\alpha_{v_0} = \begin{pmatrix} 1 & 0 \\ 0 & \pi_0 \end{pmatrix}$, $\alpha_v = 1$ for $v \neq v_0$, and put $\Xi^* = \mathfrak{U}^* \alpha \mathfrak{U}^*$. $\mathfrak{U}^* \alpha \mathfrak{U}^*$ is certainly diagonalizable and meets with $\mathfrak{S} \cap G_k$, hence by 5.1, there exists $\alpha_0 \in G_k \cap \mathfrak{S}$ such that $\Xi = \Gamma \alpha_0 \Gamma$. Let $\xi = \Gamma \alpha_0 \Gamma$, then $\deg \xi = |\Xi^*/\mathfrak{U}^*| = |\Xi^*_{v0}/\mathfrak{U}^*_{v0}| = n(v_0) = [r(v_0) : p(v_0)]$. (ii) If $v_0 \notin S(R)$, then this Ξ^* is a special case of 5.2, hence we assume $v_0 \in S(R)$. If $g \in G_k$, $\mathrm{ord}_v(N(g)) = \mathrm{ord}_v(\pi_0)$ for any $v \neq v_0$, then $\mathrm{Emb}(g, \Xi^*_v, R_v/\Lambda_v) = \mathrm{Emb}(g, R_v, R_v/\Lambda_v)$. For $v = v_0$, $\Xi^*_{v0} = \{x = (x_{ij}) \in R_v \mid \mathrm{ord}_{v0}(N(x)) = 1 \text{ and } x_{11} \not\equiv 0 \bmod p(v_0)\}$. $S = S(B) \cup S(R)$ satisfies the condition 4.2 (i).

5.4. THEOREM. *Let ξ be as in* 5.2 *or* 5.3. *Suppose B is not isomorphic to* $M_2(\boldsymbol{Q})$. *To compute* $\mathrm{tr}\, T(\xi)$, *it is sufficient to know the right-hand side of* (3) *or* (4) *in* 4.6. *It is certainly possible by* 2.3 (*and* 2.7 *for the latter* ξ) *for* $v \in S(R)$, *and by* 2.8 *for* $v \in S(B)$. *Indeed, it is enough to solve a finite number of quadratic congruence equations.*

If B is isomorphic to $M_2(\boldsymbol{Q})$, we have $h(R) = 1$. Hence we can compute $\mathrm{tr}\, T(\xi)$ in the similar way.

5.5. Let $k = \boldsymbol{Q}$, $r(k) = \boldsymbol{Z}$ and $B = M_2(\boldsymbol{Q})$. Then any split order of B is conjugate to $\begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z} \end{pmatrix}$ for certain natural number N. So let $R = \begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z} \end{pmatrix}$ and $\mathfrak{a} = \begin{pmatrix} M\boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z} \end{pmatrix}$ with some divisor M of N, and $\chi : (\boldsymbol{Z}/M\boldsymbol{Z})^\times \to \boldsymbol{C}^\times$ be a character mod M, such that $\chi(-1) = (-1)^k$. If $\mathfrak{S} = \mathfrak{S}(\mathfrak{h}, \mathfrak{h}_\infty)$ with $\mathfrak{h} = (R/\mathfrak{a})^\times$ and $\mathfrak{h}_\infty = \{1\}$, then as Γ we have a group customarily written as $\Gamma_0(N)$. Then, in this case $\chi(g) = \chi^0(a)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{S} \cap B^\times$, and we identified χ with χ^0 in § 0. In the following, we assume $\mathfrak{h}_\infty = \{1\}$, then Γ is operating on $\mathfrak{H}$, and since $\boldsymbol{Z}^\times$ has an element of any preassigned signature distribution, $\mathrm{tr}\,(T_0(\xi)) = \frac{1}{2} \mathrm{tr}\, T(\xi)$ by 4.8. Let ξ be the one defined in 5.2, and $\mathfrak{b}$ is generated by a natural number n. Explicitly, $\xi = \sum \Gamma \alpha \Gamma$ where $\Gamma \alpha \Gamma$ runs through all the distinct double cosets such that $\alpha = (\alpha_{ij}) \in R$, $\det \alpha = n$ and $\alpha_{11} \in \mathfrak{h} \bmod M$. We put $T(n) = T_0(\xi)$.

Let ξ be the one defined in 5.3, and $\pi_0 = n$ be a prime divisor of N, i.e. $\xi = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma$, we put $T(1, n) = T_0(\xi)$. Note that $T(n)$ or $T(1, n)$ is defined for each $\mathfrak{h} \subset (R/\mathfrak{a})^\times = (\boldsymbol{Z}/M\boldsymbol{Z})^\times$.

5.6. THEOREM. (i) *Let n be relatively prime to the level N, then* $\mathrm{tr}\, T(n)$ *is given by the formula in* 0.1. (ii) *Let n be a prime divisor of the level N. Then a formula for* $\mathrm{tr}\, T(1, n)$ *can be obtained from the formula for* $\mathrm{tr}\, T(n)$ *in* 0.1 *by the following two modifications. Firstly in the second term, replace*

$\prod_{p|n}\frac{1-p^{\tau+1}}{1-p}$ *by* n. *Secondly in the definition of* $\tilde{A}$ (*with* $p=n$), *add one more condition that* $x\not\equiv 0 \bmod n$.

5.7. Proof of 0.1 (resp. 5.6) is immediate from 4.7 and 5.2 (resp. 5.3), we can take S to be the set of all prime divisors of N, and M (in 4.7) $=M$ (in 5.5).

The first term represents $\frac{1}{2}t'$ and the third term represents $t''/2=\deg \xi$. To see that the first term represents $t/2$, we will give the correspondence of notations in 4.7 and that of 0.1.

$s\leftrightarrow g$ by 'the minimal polynomial of $g=X^2-sX+n$' and $a(g)=a(s)$; $f\leftrightarrow \Lambda$ by $[\Lambda : \boldsymbol{Z}+\boldsymbol{Z}g]=f$ and $b(\Lambda)=b(s,f)$. Finally define the embedding $\varphi_x : \Lambda_p\to R_p$, by $\varphi_x(g)=\begin{pmatrix} x & \pi^{\rho} \\ -\pi^{-\rho}f(x) & s-x\end{pmatrix}$ if $x\in A$ and $\varphi_x(g)=\begin{pmatrix} x & -\pi^{-\rho-\nu}f(x) \\ \pi^{\rho+\nu} & s-x\end{pmatrix}$ if $x\in s-B$. Then, $\mathrm{Emb}(g, R_p, R_p/\Lambda_p)/\tilde{R}_p^{\times}$ is given by $\{\varphi_x \mid x\in A\}$ if $(s^2-4n)/f^2\not\equiv 0 \bmod p$ and by the disjoint union $\{\varphi_x \mid x\in A\}\cup\{\varphi_x \mid x\in s-B\}$ if $(s^2-4n)/f^2\equiv 0 \bmod p$.

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