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A note on complex K-theory of infinite CW-complexes

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In the present paper we study certain properties of the complex cohomology and homology K-theories K^* and K_* defined on the category of based CW-complexes (or of the same homotopy type).

There exists a universal coefficient sequence

$$0 \longrightarrow \mathrm{Ext}(\tilde{K}_n(X), Z) \longrightarrow \tilde{K}^{n+1}(X) \longrightarrow \mathrm{Hom}(\tilde{K}_{n+1}(X), Z) \longrightarrow 0$$

between $\tilde{K}^*$ and $\tilde{K}_*$ [9]. So we can define a duality homomorphism $D: \chi(\tilde{K}_n(X)) \to \tilde{K}^{n+1}(X)$ by the composition

$$\chi(\tilde{K}_n(X)) \longrightarrow \mathrm{Ext}(\tilde{K}_n(X), Z) \longrightarrow \tilde{K}^{n+1}(X)$$

where $\chi(\tilde{K}_n(X))$ is the character group of the discrete abelian group $\tilde{K}_n(X)$. We give a necessary and sufficient condition that the duality homomorphism D is an isomorphism (Theorem 1). This theorem contains Vick's result [7] as a corollary.

Anderson-Hodgkin [1] computed the K^*-groups of the Eilenberg-MacLane spaces $K(\pi, n)$ for certain countable abelian groups π. The purpose of the present paper is to remove the countability restriction. First we dualize Anderson-Hodgkin's Theorem for a countable abelian group using the above universal coefficient sequence, and extend the dualized result to an arbitrary abelian group (Theorem 2). Then, dualizing it again we show the result of Anderson-Hodgkin without the assumption on the cardinality of an abelian group (Theorem 3).

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§ 1. A duality homomorphism.

1.1. Let $\tilde{K}^*$ and $\tilde{K}_*$ be the Z_2-graded reduced complex cohomology and homology K-theories represented by the unitary spectrum, which are defined on the category of based CW-complexes (or of the same homotopy type). We notice that $\tilde{K}^*$ and $\tilde{K}_*$ are additive and of finite type. $\tilde{K}^*$ and $\tilde{K}_*$ are related by the following universal coefficient sequence [9]: There exists a natural exact sequence

$$(1.1)\qquad 0 \longrightarrow \mathrm{Ext}\,(\tilde{K}_{n-1}(X), Z) \longrightarrow \tilde{K}^n(X) \longrightarrow \mathrm{Hom}\,(\tilde{K}_n(X), Z) \longrightarrow 0$$

for any based CW-complex X where n is regarded as an element of Z_2.

Let M_q, $\check{M}$ and S_l be the co-Moore spaces of type $(Z_q, 2)$, $(\hat{Z}, 2)$ and $(\hat{Z}/Z, 2)$ respectively given in [8]. We note that M_q, $\check{M}$ and S_l are Moore spaces of type $(Z_q, 1)$, $(Q/Z, 1)$ and $(Q, 1)$ respectively. We use these based CW-complexes to define complex cohomology K-theories with coefficients Z_q, $\hat{Z}$ and $\hat{Z}/Z$ and complex homology K-theories with coefficients Z_q, Q/Z and Q as follows:

$$(1.2)\qquad \begin{aligned} &\tilde{K}^n(\ ;Z_q)=\tilde{K}^{n+2}(\ \wedge M_q), &&\tilde{K}_n(\ ;Z_q)=\tilde{K}_{n+1}(\ \wedge M_q),\\ &\tilde{K}^n(\ ;\hat{Z})=\tilde{K}^{n+2}(\ \wedge \check{M}), &&\tilde{K}_n(\ ;Q/Z)=\tilde{K}_{n+1}(\ \wedge \check{M}),\\ &\tilde{K}^n(\ ;\hat{Z}/Z)=\tilde{K}^{n+2}(\ \wedge S_l) \text{ and } &&\tilde{K}_n(\ ;Q)=\tilde{K}_{n+1}(\ \wedge S_l) \end{aligned}$$

for each degree $n \in Z_2$.

New cohomology and homology theories are additive, too. By [8, Proposition 8] and [2, Theorem 3] we have natural isomorphisms

$$(1.3)\qquad \begin{aligned} &\tilde{K}^n(X;\hat{Z}) \cong \varprojlim_q \tilde{K}^n(X;Z_q), \qquad \tilde{K}_n(X;Q/Z) \cong \varinjlim_q \tilde{K}_n(X;Z_q)\\ &\text{and}\qquad \tilde{K}_n(X;Q) \cong \tilde{K}_n(X)\otimes Q \end{aligned}$$

for a based CW-complex X. The cofibration sequence $S^1 \to S_l \to \check{M} \to S^2$ ([8, Lemma 1]) induces the following exact sequences

$$(1.4)\qquad \begin{aligned} &\longrightarrow \tilde{K}^n(X) \xrightarrow{\iota} \tilde{K}^n(X;\hat{Z}) \xrightarrow{\kappa} \tilde{K}^n(X;\hat{Z}/Z) \xrightarrow{\delta} \tilde{K}^{n+1}(X) \longrightarrow\\ &\text{and}\\ &\longrightarrow \tilde{K}_n(X) \xrightarrow{i} \tilde{K}_n(X;Q) \xrightarrow{k} \tilde{K}_n(X;Q/Z) \xrightarrow{\partial} \tilde{K}_{n-1}(X) \longrightarrow \end{aligned}$$

corresponding to the coefficient sequences

$$0 \longrightarrow Z \longrightarrow \hat{Z} \longrightarrow \hat{Z}/Z \longrightarrow 0 \quad\text{and}\quad 0 \longrightarrow Z \longrightarrow Q \longrightarrow Q/Z \longrightarrow 0.$$

From (1.1) and the definition (1.2) we obtain

LEMMA 1. *There are natural isomorphisms*

$$\tilde{K}^*(X;Z_q) \cong \mathrm{Ext}\,(\tilde{K}_*(X;Z_q), Z), \quad \tilde{K}^*(X;\hat{Z}) \cong \mathrm{Ext}\,(\tilde{K}_*(X;Q/Z), Z)$$

and

$$\tilde{K}^*(X;\hat{Z}/Z) \cong \mathrm{Ext}\,(\tilde{K}_*(X;Q), Z)$$

for any based CW-complex X and $q>1$.

Since $\tilde{K}_*(X;Z_q)$ is a torsion abelian group, we can show by a parallel discussion to [8, (5.4)] that

$$(1.5)\qquad \tilde{K}_*(X;Z_q)=0 \quad \textit{if and only if} \quad \mathrm{Ext}\,(\tilde{K}_*(X;Z_q), Z)=0.$$

By use of Lemma 1 and (1.5), we get

LEMMA 2. *Let X be a based CW-complex and $q>1$. $\tilde{K}_*(X;Z_q)=0$ if and only if $\tilde{K}^*(X;Z_q)=0$.*

1.2. Let R be the field of real numbers. The injective resolution of Z given by $0\to Z\to R\to R/Z\to 0$ induces an exact sequence

$$\begin{aligned}0 &\longrightarrow \operatorname{Hom}(\tilde{K}_n(X), Z) \longrightarrow \operatorname{Hom}(\tilde{K}_n(X), R)\\ &\longrightarrow \operatorname{Hom}(\tilde{K}_n(X), R/Z) \longrightarrow \operatorname{Ext}(\tilde{K}_n(X), Z) \longrightarrow 0\end{aligned}$$

for any based CW-complex X. We recall that $\operatorname{Hom}(\tilde{K}_n(X), R/Z)$ is the character group of the discrete abelian group $\tilde{K}_n(X)$. So $\operatorname{Hom}(\tilde{K}_n(X), R/Z)$ is denoted by $\chi(\tilde{K}_n(X))$.

On the other hand, by (1.1) there exists a natural exact sequence

$$0 \longrightarrow \operatorname{Ext}(\tilde{K}_{n-1}(X), Z) \longrightarrow \tilde{K}^n(X) \longrightarrow \operatorname{Hom}(\tilde{K}_n(X), Z) \longrightarrow 0\,.$$

Hence we define a duality homomorphism D between $\chi(\tilde{K}_n(X))$ and $\tilde{K}^{n+1}(X)$ to be the composition

$$D:\chi(\tilde{K}_n(X)) \longrightarrow \operatorname{Ext}(\tilde{K}_n(X), Z) \longrightarrow \tilde{K}^{n+1}(X)\,. \tag{1.6}$$

The duality D is natural. Splicing the above two exact sequences together we obtain a long exact sequence

$$\longrightarrow \tilde{K}^n(X) \longrightarrow \operatorname{Hom}(\tilde{K}_n(X), R) \longrightarrow \chi(\tilde{K}_n(X)) \xrightarrow{D} \tilde{K}^{n+1}(X) \longrightarrow \tag{1.7}$$

involving the duality D.

Let X be a based CW-complex. A parallel discussion to [8, (5.4)] shows that

$$\operatorname{Hom}(\tilde{K}_n(X), R)=0 \qquad \text{if and only if} \qquad \tilde{K}_n(X)\otimes Q=0\,.$$

$\tilde{K}_*(\ ;Q)$ becomes an additive (Q)-homology theory by (1.3). So we apply Dold's theorem [3] (see also [8]) to get a natural isomorphism

$$\tilde{K}_*(X;Q)\cong \tilde{H}_{**}(X;Q) \tag{1.8}$$

where $H_{**}=H_{\mathrm{ev}}\oplus H_{\mathrm{od}}$ and H_{ev} (or H_{od}) denotes the direct sum of the even (or odd) dimensional ordinary homology groups. Hence we see

$$\operatorname{Hom}(\tilde{K}_*(X), R)=0 \qquad \textit{if and only if} \qquad \tilde{H}_*(X)\otimes Q=0\,. \tag{1.9}$$

The following theorem follows immediately from (1.7) and (1.9).

THEOREM 1. *Let X be a based CW-complex. The duality homomorphism $D:\chi(\tilde{K}_*(X))\to\tilde{K}^{*+1}(X)$ is an isomorphism if and only if $\tilde{H}_*(X)\otimes Q=0$.*

For any group G we denote by BG a classifying space for G, taken as a based CW-complex. As is well known, $\tilde{H}_*(BG)\otimes Q=0$ for any finite group G. We obtain Vick's result [7] as a corollary of the above theorem.

COROLLARY 3 (Vick). *Let G be a finite group. $\tilde{K}^n(BG)$ is isomorphic to the character group of the discrete abelian group $\tilde{K}_{n-1}(BG)$ for each degree $n \in Z_2$.*

§ 2. Eilenberg-MacLane spaces $K(\pi, n)$.

2.1. Let π be an abelian group and n a positive integer. First we recall the Eilenberg-MacLane complex $\mathcal{K}(\pi, n)$ [**4**]. $\mathcal{K}(\pi, n)$ is a semi-simplicial complex equipped with the following structure: q-cells are n-cocycle $\nu \in Z^n(\Delta_q; \pi)$ defined on the standard q-simplexes Δ_q and with coefficients in the abelian group π; i-th face and degeneracy operators

$$F_i\colon Z^n(\Delta_q; \pi) \longrightarrow Z^n(\Delta_{q-1}; \pi)$$

and

$$D_i\colon Z^n(\Delta_q; \pi) \longrightarrow Z^n(\Delta_{q+1}; \pi)$$

are induced by the standard monotonic maps $\varepsilon^i\colon \Delta_{q-1} \to \Delta_q$ and $\eta^i\colon \Delta_{q+1} \to \Delta_q$.

A homomorphism $\phi\colon \pi \to \pi'$ of abelian groups induces a semi-simplicial map

$$\mathcal{K}(\phi, n)\colon \mathcal{K}(\pi, n) \longrightarrow \mathcal{K}(\pi', n).$$

In particular, $\mathcal{K}(\pi, n)$ is a semi-simplicial subcomplex of $\mathcal{K}(\pi', n)$ if π is a subgroup of π'. Clearly $\mathcal{K}(\mathrm{id}, n)$ is the identity map and the composite of $\mathcal{K}(\phi', n)$ and $\mathcal{K}(\phi, n)$ coincides with $\mathcal{K}(\phi' \cdot \phi, n)$, where $\mathrm{id}\colon \pi \to \pi$ is the identity and $\phi\colon \pi \to \pi'$ and $\phi'\colon \pi' \to \pi''$ are homomorphisms of abelian groups.

Let $\mathcal{K}$ be a semi-simplicial complex. We denote by $|\mathcal{K}|$ the geometric realization of $\mathcal{K}$. $|\mathcal{K}|$ has the following properties [**6**].

(2.1) i) *$|\mathcal{K}|$ is a CW-complex having one q-cell corresponding to each non degenerate q-simplex of $\mathcal{K}$.*

ii) *A semi-simplicial map $\mu\colon \mathcal{K} \to \mathcal{K}'$ induces a cellular map $|\mu|\colon |\mathcal{K}| \to |\mathcal{K}'|$.*

ii′) *If $\mathcal{K}$ is a semi-simplicial subcomplex of $\mathcal{K}'$, then $|\mathcal{K}|$ is a subcomplex of $|\mathcal{K}'|$.*

iii) $|\mathrm{id}| = \mathrm{id}$ *and* $|\mu' \cdot \mu| = |\mu'| \cdot |\mu|$ *where* $\mathrm{id}\colon \mathcal{K} \to \mathcal{K}$ *is the identity map and* $\mu\colon \mathcal{K} \to \mathcal{K}'$ *and* $\mu'\colon \mathcal{K}' \to \mathcal{K}''$ *are semi-simplicial maps.*

We shall write $K(\pi, n)$ for the geometric realization $|\mathcal{K}(\pi, n)|$ of the Eilenberg-MacLane complex $\mathcal{K}(\pi, n)$ and $K(\phi, n)$ for $|\mathcal{K}(\phi, n)|$.

Let π and π' be abelian groups and $\phi\colon \pi \to \pi'$ a homomorphism. Take connected based CW-complexes X and Y with homotopy groups $\pi_n(X) \cong \pi$, $\pi_n(Y) \cong \pi'$ and $\pi_i(X) = \pi_i(Y) = 0$ for $i \neq n$, i.e., Eilenberg-MacLane spaces of type (π, n) and (π', n) respectively. Then there exists a continuous map $f\colon X \to Y$ such that

(2.2) $$f_* = \phi : \pi_n(X) \longrightarrow \pi_n(Y).$$

The map f is uniquely determined up to homotopy.

By the aid of the above results we have

PROPOSITION 4. i) *Let π be an abelian group. The geometric realization $K(\pi, n)$ of the Eilenberg-MacLane complex $\mathcal{K}(\pi, n)$ is an Eilenberg-MacLane space of type (π, n), taken as a based CW-complex.*

ii) *A homomorphism $\phi : \pi \to \pi'$ of abelian groups induces a cellular map $K(\phi, n) : K(\pi, n) \to K(\pi', n)$ such that $K(\phi, n)_* = \phi : \pi_n(K(\pi, n)) \to \pi_n(K(\pi', n))$.*

ii′) *If π is a subgroup of π', then $K(\pi, n)$ is a subcomplex of $K(\pi', n)$ with $\pi_n(K(\pi, n)) \subset \pi_n(K(\pi', n))$.*

iii) $K(\mathrm{id}, n) = \mathrm{id}$ *and* $K(\phi', n) \cdot K(\phi, n) = K(\phi' \cdot \phi, n)$ *where* $\mathrm{id} : \pi \to \pi$ *is the identity homomorphism and $\phi : \pi \to \pi'$ and $\phi' : \pi' \to \pi''$ are homomorphisms of abelian groups.*

2.2. Let π be an abelian group and $\mathfrak{U}_\pi = \{\pi^\lambda\}$ be the set of all finitely generated subgroups of π ordered by inclusions. Obviously $\mathfrak{U}_\pi$ is a directed set and $\pi \cong \bigcup_\lambda \pi^\lambda$. Proposition 4 implies that $\mathcal{C}(\pi, n) = \{K(\pi^\lambda, n)\}$ forms a direct system of based subcomplexes of $K(\pi, n)$. The definition of $K(\pi, n)$ and Proposition 4 show that $\bigcup_\lambda K(\pi^\lambda, n)$ becomes an Eilenberg-MacLane space of type (π, n). Therefore,

(2.3) $$\bigcup_\lambda K(\pi^\lambda, n) \quad \textit{is homotopy equivalent to } K(\pi, n).$$

Applying [**2**, Theorem 3] to the direct system $\mathcal{C}(\pi, n)$ we obtain

PROPOSITION 5. *Let h_* be an additive (reduced) homology theory defined on the category of based CW-complexes and π be an abelian group. For each $n \geqq 1$ there is an isomorphism*

$$h_*(K(\pi, n)) \cong \varinjlim_\lambda h_*(K(\pi^\lambda, n))$$

where π^λ runs over all finitely generated subgroups of π.

REMARK. Let h^* be an additive (reduced) cohomology theory. As a dual of the above proposition we have a spectral sequence $\{E_r\}$ associated with $h^*(K(\pi, n))$ such that

$$E_2^{p,q} = \varprojlim_\lambda{}^p h^q(K(\pi^\lambda, n))$$

for each $n \geqq 1$, by means of [**2**, Theorem 2].

2.3. Anderson-Hodgkin [**1**] computed the K^*-groups of certain Eilenberg-MacLane spaces.

THEOREM (Anderson-Hodgkin). *Let π' be a countable abelian group and $q > 1$. Then*

$$\tilde{K}^*(K(\pi', n); Z_q) = 0$$

and the natural homomorphism $\phi:\pi'\rightarrow\pi'\otimes Q$ *induces the isomorphism*

$$K(\phi, n)^* : \tilde{K}^*(K(\pi'\otimes Q, n)) \cong \tilde{K}^*(K(\pi', n))$$

for $n\geqq 3$, *and also for* $n=2$ *in case* π' *being a torsion abelian group.*

PROOF. The second result was obtained in [1]. Therefore it is sufficient to show that

$$\tilde{K}^*(K(\pi'\otimes Q, n)\,;\, Z_q)=0$$

for $n\geqq 1$ and $q>1$, in order to get the first one. $\tilde{H}^*(K(\pi'\otimes Q, n))$ is a Q-module, i. e., $\tilde{H}^*(K(\pi'\otimes Q, n)\,;\, Z_q)=0$. In this case the Atiyah-Hirzebruch spectral sequence of $\tilde{K}^*(K(\pi'\otimes Q, n)\,;\, Z_q)$ collapses, and hence it is strongly convergent by [2, Proposition 9]. So we get the required equalities immediately.

Let π be an arbitrary abelian group and a positive integer n fix. We assume that $n\geqq 2$ if π is a torsion abelian group, and that $n\geqq 3$ if not so. The above theorem combined with Lemma 2 asserts that

$$\tilde{K}_*(K(\pi^\lambda, n)\,;\, Z_q)=0$$

where π^λ is a finitely generated subgroup of π. Applying Proposition 5 we see

$$\tilde{K}_*(K(\pi, n)\,;\, Z_q) \cong \varinjlim_{\lambda} \tilde{K}_*(K(\pi^\lambda, n)\,;\, Z_q)=0 \tag{2.4}$$

and by (1.3)

$$\tilde{K}_*(K(\pi, n)\,;\, Q/Z) \cong \varinjlim_{q} \tilde{K}_*(K(\pi, n)\,;\, Z_q)=0\,.$$

Using the long exact sequence (1.4) we have an isomorphism

$$i:\ \tilde{K}_*(K(\pi, n)) \longrightarrow \tilde{K}_*(K(\pi, n)\,;\, Q)\,.$$

Since $\tilde{K}_*(K(\pi, n)\,;\, Q)\cong \tilde{H}_{**}(K(\pi, n)\,;\, Q)$ by (1.8) there exists an isomorphism

$$\tilde{K}_*(K(\pi, n)) \cong \tilde{H}_{**}(K(\pi, n))\otimes Q\,. \tag{2.5}$$

As a result we have

THEOREM 2. *Let* π *be an arbitrary abelian group and* $q>1$. *Then*

$$\tilde{K}_*(K(\pi, n)\,;\, Z_q)=0 \quad \textit{and} \quad \tilde{K}_*(K(\pi, n)) \cong \tilde{H}_{**}(K(\pi, n))\otimes Q$$

for $n\geqq 3$, *and also for* $n=2$ *in case* π *being a torsion abelian group.*

Finally we dualize results of Theorem 2 using the universal coefficient sequence (1.1).

THEOREM 3. *Let* π *be an arbitrary abelian group and* $q>1$. *Then*

$$\tilde{K}^*(K(\pi, n)\,;\, Z_q)=0 \quad \textit{and} \quad \tilde{K}^{*+1}(K(\pi, n)) \cong \mathrm{Ext}\,(\tilde{H}_{**}(K(\pi, n))\otimes Q, Z)$$

for $n\geqq 3$, *and also for* $n=2$ *in case* π *being a torsion abelian group.*

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