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Riemannian manifolds of nullity index zero and curvature tensor-preserving transformations

Dedicated to Professor S. Sasaki on his 60th birthday

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§ 1. Introduction.

Riemannian manifolds (M, g) of constant nullity μ were studied by Rosenthal [8]: Under certain assumptions (M, g) is a direct product manifold of an $(m-\mu)$-dimensional Riemannian manifold and a μ-dimensional Euclidean space, when $m=\dim M$.

In this paper we study Riemannian manifolds of nullity index zero and give local decompositions in § 2. By R we denote the Riemannian curvature tensor. (M, g) is of nullity index zero on M, if at each point x, for a tangent vector Z at x, $R(X, Y)Z=0$ for any tangent vectors X and Y at x implies $Z=0$. Assume that a Riemannian manifold (M, g) is of nullity index zero and admits a (C^∞-) distribution D, which is invariant by curvature transformations $R(X, Y)$ for any vector fields X and Y, and $1 \leqq \dim D \leqq m-1$. Denote by $D^\perp$ the distribution orthocomplementary to D with respect to the metric g. Then $D^\perp$ is invariant by curvature transformations. In Theorem 2.6 we show that D and $D^\perp$ are parallel. Hence (M, g) is locally a product manifold.

As one of the results related to the equivalence problem in Riemannian geometry, Nomizu and Yano [7] obtained the following: Let (M, g) be an irreducible, locally symmetric Riemannian manifold with $\dim M=m \geqq 3$; then a curvature tensor-preserving transformation of (M, g) onto another Riemannian manifold (M', g') is homothetic. In § 3, we generalize this theorem. A Riemannian manifold (M, g) is locally homogeneous (by definition) if, for any points x and y in M, there is an isometry of some neighborhood of x onto some neighborhood of y which sends x to y.

THEOREM A. *Let (M, g) be an irreducible, locally homogeneous Riemannian manifold of nullity index zero, $m \geqq 3$. Then, a curvature tensor-preserving transformation of (M, g) onto another Riemannian manifold (M', g') is homothetic.*

In particular, we have

THEOREM A′. *Let (M, g) be an irreducible, locally homogeneous Riemannian manifold, $m \geqq 3$. If the Ricci curvature tensor is non-singular (at some point),* *then a curvature tensor-preserving transformation of* (M, g) *onto another* (M', g') *is homothetic.*

In proof, Theorem 2.6 and a theorem of Teleman [15] are applied. Teleman's theorem involves the notion of non-divisibility of the Riemannian curvature tensor. In § 4, we generalize his theorem to pseudo-Riemannian manifolds, since this kind of problem is important also in pseudo-Riemannian geometry. In a pseudo-Riemannian manifold (M, g), the Riemannian curvature tensor R is called non-divisible at x, if the connected subgroup G of $GL(M_x)$ of endomorphisms of M_x whose Lie algebra is generated by $\{R(X, Y), X, Y \in M_x\}$ is irreducible.

THEOREM B. *Let* (M, g) *be a pseudo-Riemannian manifold of signature* (p, q) *such that* $p \neq q$ *and* $m \geqq 3$. *If non-divisible points of* R *is dense in* M, *a curvature tensor-preserving transformation of* (M, g) *onto another pseudo-Riemannian manifold* (M', g') *is homothetic.*

In § 5, applications of Theorem 2.6 to Riemannian manifolds satisfying $R(X, Y)\cdot R=0$ or $R(X, Y)\cdot R_1=0$ are given, where $R(X, Y)$ acts on the tensor algebra at each point as a derivation and R_1 denotes the Ricci curvature tensor.

In § 6, some remarks are given.

In this paper, manifolds are assumed to be connected and of class C^∞. Tensor fields, distributions, etc. are assumed to be of class C^∞, if otherwise stated.

§ 2. Distributions which are invariant by curvature transformations.

Let (M, g) be a Riemannian manifold with (positive definite) metric tensor g. The dimension of M is denoted by m. By ∇ and R we denote the Riemannian connection with respect to g and the Riemannian curvature tensor:

$$R(X, Y)Z=\nabla_{[X,Y]}Z-\nabla_X\nabla_Y Z+\nabla_Y\nabla_X Z$$

for vector fields X, Y and Z on M. Let x be a point of M, and let M_x be the tangent space at x to M. By $Y, Z \in M_x$ we mean that Y and Z are tangent vector at x. We define a subspace N_x of M_x by

$$N_x=\{X\in M_x;\ R(X, Y)Z=0 \quad \text{for all } Y, Z\in M_x\}. \tag{2.1}$$

N_x is called the nullity space at x, and $\dim N_x=\mu(x)$ is called the nullity index at x (or the index of nullity at x) (cf. Chern-Kuiper [1]).

The Riemannian curvature tensor R satisfies

$$R(X, Y)=-R(Y, X)\,, \quad g(R(X, Y)Z, W)=-g(R(X, Y)W, Z)\,, \tag{2.2}$$

$$g(R(X, Y)Z, W)=g(R(Z, W)X, Y)\,, \tag{2.3}$$

$$(2.4)\qquad R(X, Y)Z+R(Y, Z)X+R(Z, X)Y=0,$$

$$(2.5)\qquad (\nabla_X R)(Y, Z)+(\nabla_Y R)(Z, X)+(\nabla_Z R)(X, Y)=0$$

for vector fields (or tangent vectors) X, Y, Z and W.

LEMMA 2.1. *Let D be a distribution on (M, g) which is invariant by curvature transformations at each point x:*

$$R(X, Y)W\in D_x \qquad \textit{for all } W\in D_x,\ X,\ Y\in M_x.$$

Then the distribution $D^{\perp}$ orthocomplementary to D with respect to g is invariant by curvature transformations, too.

PROOF. It is not difficult to see that $D^{\perp}$ is C^{∞}, whenever D is C^{∞}. Let $U\in D_x^{\perp}$ and let $W\in D_x$. By (2.2) we get $g(R(X, Y)U, W)=-g(R(X, Y)W, U)=0$. Therefore $R(X, Y)U\perp D_x$, i.e., $R(X, Y)D^{\perp}\subset D^{\perp}$. q.e.d.

LEMMA 2.2. *We have distributions $D^1, D^2, \cdots, D^k$ which are invariant by curvature transformations such that, at each point x of M,*

$$(2.6)\qquad M_x=D_x^1\oplus D_x^2\oplus\cdots\oplus D_x^k,$$

which is an orthogonal decomposition of M_x, and each D^{α} $(\alpha=1, 2, \cdots, k)$ has no proper subdistribution which is invariant by curvature transformations (at each point) on M.

PROOF. Let D and $D^{\perp}$ be distributions invariant by curvature transformations given in Lemma 2.1. If D has a subdistribution D^1 which is invariant by curvature transformations on M, then $(D^1)^{\perp}$ is also an invariant distribution by curvature transformations on M. $D\cap(D^1)^{\perp}$ is also a distribution on M and is invariant by curvature transformations on M. Continuing this step, we have Lemma 2.2. q.e.d.

LEMMA 2.3. *For $X,\ Y\in D_x^{\alpha}$ and for $\beta\neq\alpha$ $(\alpha, \beta=1, 2, \cdots, k)$*

$$(2.7)\qquad R(X, Y)D_x^{\beta}=0.$$

PROOF. Let $U,\ V\in D_x^{\beta}$. By (2.4) we get

$$g(R(X, Y)U, V)+g(R(Y, U)X, V)+g(R(U, X)Y, V)=0.$$

By $R(Y, U)X\in D_x^{\alpha}$ and $R(U, X)Y\in D_x^{\alpha}$ we have $g(R(X, Y)U, V)=0$. Putting $V=R(X, Y)U$, we have $R(X, Y)U=0$.

LEMMA 2.4. *For $X\in D_x^{\alpha}$ and $U\in D_x^{\beta}$, $\beta\neq\alpha$*

$$(2.8)\qquad R(X, U)=0.$$

PROOF. Let A be an arbitrary tangent vector at x. Put $B=R(X, U)A$. By (2.3) we have $g(R(X, U)A, B)=g(R(A, B)X, U)=0$. Thus we have $R(X, U)A=0$. q.e.d.

Let $(X_i,\ i=1, 2, \cdots, m)$ be a local field of orthonormal frames such that

$$X_1, X_2, \cdots, X_r\in D^1,\ X_{r+1}, \cdots, X_{r+s}\in D^2, \cdots, X_{r+s+\cdots+t}\in D^k,$$

where, in general, $X \in D^{\alpha}$ means that X is a (locally defined) vector field such that at each point x, $X_x \in D^{\alpha}_x$. We put

$$\nabla_{X_i} X_j = \nabla_i X_j = \sum_h B_{ijh} X_h\,, \qquad i, j = 1, 2, \cdots, m\,. \tag{2.9}$$

B_{ijh} is skew-symmetric in j and h; $B_{ijh} = -B_{ihj}$. By $X \wedge Y$ we mean that $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. Let $X_a, X_b, X_c, X_d \in D^{\alpha}$ and $X_u \in D^{\beta}$. We put $g(R(X_a, X_b)X_c, X_d) = R_{dcab}$. Then R_{dcab} is skew-symmetric in d and c, and hence we can put

$$R(X_a, X_b) = \frac{1}{2} \sum_{d,c} R_{dcab} X_d \wedge X_c\,, \tag{2.10}$$

$$R(X_a, X_u) = 0\,. \tag{2.11}$$

LEMMA 2.5. *If (M, g) is of nullity index zero at each point of M, then $D^1, D^2, \cdots, D^k$ are parallel.*

PROOF. To apply (2.5) we calculate

$$\begin{aligned}(\nabla_u R)(X_a, X_b) &= \nabla_u (R(X_a, X_b)) - R(\nabla_u X_a, X_b) - R(X_a, \nabla X_{ub}) \\ &= \frac{1}{2} \sum_{d,c} \nabla_u R_{dcab} X_d \wedge X_c + \frac{1}{2} \sum_{d,c} R_{dcab} [(\nabla_u X_d) \wedge X_c + X_d \wedge (\nabla_u X_c)] \\ &\quad - \frac{1}{2} \sum_{d,c,e} (B_{uae} R_{dceb} + B_{ube} R_{dcae}) X_d \wedge X_c\,,\end{aligned} \tag{2.12}$$

$$\begin{aligned}(\nabla_a R)(X_b, X_u) &= -R(\nabla_a X_b, X_u) - R(X_b, \nabla_a X_u) \\ &= -\frac{1}{2} [\sum_{v,y,w} B_{abv} R_{ywvu} X_y \wedge X_w + \sum_{d,c,e} B_{aue} R_{dcbe} X_d \wedge X_c]\,,\end{aligned} \tag{2.13}$$

$$\begin{aligned}(\nabla_b R)(X_u, X_a) &= -R(\nabla_b X_u, X_a) - R(X_u, \nabla_b X_a) \\ &= -\frac{1}{2} [\sum_{v,y,w} B_{bav} R_{ywuv} X_y \wedge X_w + \sum_{d,c,e} B_{bue} R_{dcea} X_d \wedge X_c]\,,\end{aligned} \tag{2.14}$$

where $X_a, X_b, X_c, X_d, X_e \in D^{\alpha}$ and $X_u, X_v, X_y, X_w \in D^{\beta}$. We put

$$\nabla_u X_d = \sum_e B_{ude} X_e + \sum_v B_{udv} X_v + \sum_\theta B_{ud\theta} X_\theta + \cdots + \sum_\xi B_{ud\xi} X_\xi\,,$$

where $X_\theta \in D^{\gamma}, \cdots, X_\xi \in D^{\delta}$; $(\gamma, \cdots, \delta) = (1, 2, \cdots, k) - (\alpha, \beta)$. By (2.5) and (2.12)$\sim$(2.14), and

$$\frac{1}{2} \sum_{d,c} R_{dcab} [(\nabla_u X_d) \wedge X_c + X_d \wedge (\nabla_u X_c)] = \sum_{d,c} R_{dcab} (\nabla_u X_d) \wedge X_c\,,$$

we get (as coefficients of mixed parts $X_* \wedge X_c$)

$$\sum_d R_{dcab} B_{udv} = \sum_d R_{dcab} B_{ud\theta} = \cdots = \sum_d R_{dcab} B_{ud\xi} = 0\,.$$

For fixed u and v, we put $B_d = B_{udv}$. Then we have a locally defined vector field $B^* = \sum B_d X_d \in D^\alpha$. Let Y, Z be any vector fields. Then, applying Lemmas 2.3 and 2.4, we have

$$R(Y, Z)B^* = R(\sum_a Y^a X_a, \sum_b Z^b X_b)(\sum_d B_d X_d),$$

where (Y^a) and (Z^b) are components of Y and Z with respect to (X_a) in D^α. Then $\sum R_{dcab} B_{udv} = 0$ implies that $R(Y, Z)B^* = 0$. Since (M, g) is of nullity index zero at each point, we have $B^* = 0$, i. e., $B_{udv} = 0$. Similarly we have

$$B_{ud\theta} = \cdots = B_{ud\xi} = 0.$$

This implies that, for $X_u \in D^\beta$, $\beta \neq \alpha$,

$$\nabla_u X_d \in D^\alpha, \qquad \text{i. e., } \nabla_u D^\alpha \subset D^\alpha.$$

Similarly we have $\nabla_\theta D^\alpha \subset D^\alpha, \cdots, \nabla_\xi D^\alpha \subset D^\alpha$. Finally we prove $\nabla_a D^\alpha \subset D^\alpha$ for $X_a \in D^\alpha$. In fact, $B_{udv} = 0$ and $B_{ijh} = -B_{ihj}$ give $B_{uvd} = 0$. Changing D^β and D^α we have $B_{abu} = 0$. This is nothing but $\nabla_a D^\alpha \subset D^\alpha$. Thus, D^α is parallel. Nullity index zero at each point implies $\dim D^\alpha \geqq 2$, $\alpha = 1, \cdots, k$. q. e. d.

Summarizing we have

THEOREM 2.6. *Let (M, g) be a Riemannian manifold of nullity index zero at each point.*

(i) *Let D be a distribution on (M, g), which is invariant by curvature transformations $R(X, Y)$, $X, Y \in M_x$ at each point x of M. Denote by $D^\perp$ the distribution orthocomplementary to D with respect to g. Then $D^\perp$ is also invariant by curvature transformations at each point.*

(ii) *Therefore we have distributions $D^1, D^2, \cdots, D^k$, which are invariant by curvature transformations at each point, such that at each point $x \in M_x$ we have the orthogonal decomposition $M_x = D_x^1 \oplus D_x^2 \oplus \cdots \oplus D_x^k$, and that each D^α has no proper subdistribution which is invariant by curvature transformations.*

(iii) *If $k = 1$, the homogeneous holonomy group is irreducible.*

(iv) *If $k \geqq 2$, $D^1, D^2, \cdots, D^k$ are parallel.*

(v) *Hence, for $k \geqq 2$, (M, g) is locally a product manifold of Riemannian manifolds (W_α, g_α), $\alpha = 1, 2, \cdots, k$.*

In (v) of Theorem 2.6, each (W_α, g_α) is not necessarily irreducible. But, for any fixed α, we have some point x of M such that, in local decomposition of a neighborhood of x, (W_α, g_α) is irreducible.

§ 3. Curvature tensor-preserving transformations.

The Riemannian curvature tensor R of a Riemannian manifold (M, g) is called *regular at* x, if $R(X, Y) \neq 0$ for linearly independent X and Y at x, and R is called *regular* if it is regular at each point (Kowalski [**4**]). Let x be a point of M. Denote by $\mathfrak{R}_x$ the set of curvature transformations at x,

i. e.,

$$\mathfrak{R}_x = \{R(X, Y),\ X, Y \in M_x\}\,, \tag{3.1}$$

which is a subset of $\mathfrak{gl}(M_x)$, more of $\mathfrak{o}(M_x)$ (=the Lie algebra of skew-symmetric endomorphisms of M_x). Let $G(\mathfrak{R}_x)$ be the connected subgroup of $GL(M_x)$ (or $O(M_x)$ = the orthogonal group acting on M_x) whose Lie algebra is generated by $\mathfrak{R}_x$. A Riemannian manifold or the Riemannian curvature tensor R is called *non-divisible at* x, if $G(\mathfrak{R}_x)$ is irreducible, and (M, g) or R is called *non-divisible* if it is non-divisible at each point (Teleman [**15**], p. 109). Regularity at x implies non-divisibility at x (Kowalski [**5**]). Since the Lie algebra generated by $\mathfrak{R}_x$ is contained in the holonomy algebra at x (cf. for example, Kobayashi-Nomizu [**3**]), non-divisibility implies irreducibility of the restricted homogeneous holonomy group.

THEOREM 3.1 (Teleman [**15**], cf. also, Kowalski [**4, 5**]). *Let (M, g) be a Riemannian manifold with $m \geqq 3$ and with non-divisible R (more precisely, the set of non-divisible points of R is dense). Then, a curvature tensor-preserving transformation of (M, g) onto another (M', g') is homothetic.*

We say that R is *C^∞-divisible* on an open set W, if there is a distribution D on W such that $1 \leqq \dim D \leqq m-1$ and

$$R(X, Y)D_x \subset D_x \qquad \text{for all } x \in W,\ X, Y \in M_x\,. \tag{3.2}$$

Then Theorem 2.6 has the following

COROLLARY 3.2. *If a Riemannian manifold (M, g) is of nullity index zero at each point and if R is C^∞-divisible on a connected open set W of M, then $(W, g|W)$ is reducible.*

Analytically Corollary 3.2 implies that C^∞-divisibility (3.2) gives for $s = 0, 1, \cdots,$

$$(\nabla_V^s R)(X, Y)D_x \subset D_x\,, \qquad x \in W\,, \tag{3.3}$$

where $X, Y, V_1, V_2, \cdots, V_s \in M_x$, $\nabla^0 R = R$, and $\nabla_V^s R$ has components:

$$(\nabla_V^s R):\ (V_1^i V_2^j \cdots V_s^l \nabla_i \nabla_j \cdots \nabla_l R^z_{wxy})\,. \tag{3.4}$$

REMARK. An example of irreducible Riemannian manifold whose Riemannian curvature tensor R is C^∞-divisible is given by Takagi [**13**]. In fact, let R_1 be the Ricci curvature tensor. If a 3-dimensional Riemannian manifold (M, g) satisfies $R(X, Y)\cdot R_1 = 0$ and R_1 has rank 2 on an open set W, then we have a local field of orthonormal frames X_1, X_2, X_3 such that $R(X_1, X_2) = KX_1 \wedge X_2$ and $R(X_3, X_1) = R(X_3, X_2) = 0$.

A theorem of Nomizu and Yano is as follows:

THEOREM 3.3 (Nomizu-Yano [**7**]). *Let (M, g) be an irreducible, locally symmetric Riemannian manifold, $m \geqq 3$. Then, a curvature tensor-preserving* *transformation of* (M, g) *onto another* (M', g') *is homothetic.*

If (M, g), $m \geqq 2$, is locally symmetric and irreducible, we see that the nullity index is zero at each point. In fact, $\nabla R = 0$ implies that the nullity distribution $x \to N_x$ is parallel.

If (M, g) is locally symmetric, then it is locally homogeneous.

To give generalization of Theorem 3.3 above, the essential point is the relation between non-divisibility and irreducibility, or by Theorem 2.6, the relation between divisibility and C^∞-divisibility.

Our generalization is as follows:

THEOREM 3.4. *Let* (M, g) *be an irreducible, locally homogeneous Riemannian manifold of nullity index zero,* $m \geqq 3$. *Then, a curvature tensor-preserving transformation of* (M, g) *onto* (M', g') *is homothetic.*

To state a Lemma (due to Singer) we prepare some definition. A Riemannian manifold (M, g) is *curvature homogeneous* if for every x and y in M, there exists an isometry f of the tangent space M_x onto the tangent space M_y such that f preserves the Riemannian curvature tensor, i. e., $f^{-1}R(fX, fY)f = R(X, Y)$, $X, Y \in M_x$. A locally homogeneous Riemannian manifold is curvature homogeneous.

Let $F(M)$ be the bundle of orthonormal frames. For an orthonormal frame $b = (x, e_1, e_2, \cdots, e_m)$ we put $R_{ijkl}(b) = g_x(R(e_k, e_l)e_j, e_i)$.

LEMMA 3.5 (Singer [**12**], § 2). (M, g) *is curvature homogeneous if and only if there exists a principal subbundle of* $F(M)$ *over* M *on which the functions* R_{ijkl} *are constant.*

The fact we need is existence of local cross sections of this subbundle. We denote a local cross section by $(x, X_i, i = 1, \cdots, m)$.

PROOF OF THEOREM 3.4. If the Riemannian curvature tensor R is divisible at some point z, we have subspaces D_z^α, $\alpha = 1, \cdots, k$, of M_z, which are invariant by curvature transformations at z and M_z has the orthogonal decomposition

$$M_z = D_z^1 \oplus D_z^2 \oplus \cdots \oplus D_z^k,$$

where D_z^α has no proper subspace which is invariant by curvature transformations at z. Using a local field of orthonormal frames (x, X_i) given by Lemma 3.5, we take a basis:

$$(\sum a_1^i(X_i)_z, \ \sum a_2^i(X_i)_z, \cdots, \sum a_r^i(X_i)_z)$$

of D_z^1, a_u^i being real numbers. Then

$$(\sum a_1^i X_i, \ \sum a_2^i X_i, \cdots, \sum a_r^i X_i)$$

defines a distribution D^1 on an open set W $(\ni z)$ on which our local cross section (x, X_i) is defined. Hence, we have distributions $D^1, \cdots, D^k$. These distributions are C^{∞} and invariant by curvature transformations, since $g(R(X_k, X_l)X_j, X_i)$ are constant on W. Since (M, g) is locally homogeneous, Theorem 2.6 implies that (M, g) is locally a Riemannian product manifold. Therefore R must be non-divisible at each point of M. By Theorem 3.1, we have Theorem 3.4.

Theorem 3.4 is also stated as follows:

THEOREM 3.4′. *Let (M, g) be an irreducible, locally homogeneous Riemannian manifold of nullity index zero, $m \geqq 3$. For another Riemannian metric g^* on M, if the both Riemannian curvature tensors are identical, then g and g^* are homothetic.*

§ 4. Pseudo-Riemannian manifolds.

Let (M, g) be a pseudo-Riemannian manifold with metric tensor g of signature (p, q). That is, for a fixed point x, we have a local coordinate neighborhood $(W, x^i, i=1, \cdots, m)$ such that

$$g=(dx^1)^2+\cdots+(dx^p)^2-(dx^{p+1})^2-\cdots-(dx^m)^2$$

holds at x. Let $\boldsymbol{R}$ be the field of real numbers.

LEMMA 4.1 (Kobayashi-Nomizu [**3**], p. 277). *Let G be a subgroup of $GL(m, \boldsymbol{R})$ which acts irreducibly on $\boldsymbol{R}^m$. Let A be a linear transformation of $\boldsymbol{R}^m$ which commutes with every elements of G. Then*

$$A=aI_m, \qquad or \qquad A=aI_m+bJ,$$

where a, b are real numbers, I_m the identity transformation of $\boldsymbol{R}^m$ and J a linear transformation such that $J^2=-I_m$.

LEMMA 4.2 (cf. Tanno [**16**]). *Let G be a subgroup of $GL(m, \boldsymbol{R})$ which acts irreducibly on $\boldsymbol{R}^m$. Let g be symmetric, non-degenerate bilinear form with signature (p, q) which is invariant by G. Assume*

$$(1) \qquad [m=odd \text{ or } m=2] \text{ or } [m=even \geqq 4 \text{ and } p \neq q].$$

Then, for a symmetric bilinear form g^* which is invariant by G, we have a real number a such that $g^*=ag$.

PROOF. Define A by $g^*(X, Y)=g(AX, Y)$ for $X, Y \in \boldsymbol{R}^m$. Since g is non-degenerate, A is a well defined linear transformation of $\boldsymbol{R}^m$. Since g and g^* are invariant by G, A commutes with every element of G. By Lemma 4.1, we have $A=aI_m$, or $A=aI_m+bJ$. If $m=2$ we see that $b=0$, and if $b\neq 0$ we see that $p=q$ (cf. Tanno [**16**], p. 246-247). q. e. d.

Let $\mathfrak{L}$ be a set of linear endomorphisms of a vector space V. By $S^2(V)$ we denote the space of all symmetric bilinear forms on V. Put

$$\Theta(\mathfrak{L})=\{h \in S^2(V):\ h(LX, Y)+h(X, LY)=0,\ X, Y \in V,\ L \in \mathfrak{L}\}.$$

By $G(\mathfrak{L})$ we denote the connected subgroup of $GL(V)$ whose Lie algebra is generated by $\mathfrak{L}$. The following Proposition for positive definite case was proved by Kowalski [**5**].

PROPOSITION 4.3. *Let V be a vector space with symmetric, non-degenerate bilinear form g of signature (p, q), and let G be a subgroup of $GL(V)$ which is irreducible and leaves g invariant. Let $\mathfrak{L}$ be a set of linear endomorphisms generating the Lie algebra of G. Assume* (1) *of Lemma* 4.2. *Then,*

(i) $\dim \Theta(\mathfrak{L})=1$, *i.e.*, $\Theta(\mathfrak{L})=(g)$.

(ii) *If $X\in V$ and $LX=0$ for any $L\in\mathfrak{L}$, then $X=0$.*

PROOF. In the proof of the positive definite case, Kowalski [**5**] used Theorem 1 in [**3**], p. 277. If we replace this by Lemma 4.2, the proof is similar to that given in [**5**]. q. e. d.

Also in a pseudo-Riemannian manifold (M, g), R is said to be non-divisible at x, if the connected subgroup G of $GL(M_x)$ whose Lie algebra is generated by $\{R(X, Y),\ X, Y\in M_x\}=\mathfrak{R}_x$ is irreducible.

PROPOSITION 4.4. *Let g, g^* be pseudo-Riemannian metrics on a manifold M with the same curvature tensors $R=R^*$. If,*

$$[m=odd \text{ or } m=2] \text{ or } [m=even\geqq 4 \text{ and the signature } (p, q) \text{ of } g \text{ satisfies } p\neq q],$$

then g and g^* are conformal on the closure of the set of all non-divisible points of R.

PROOF. $R=R^*$ implies $\mathfrak{R}_x=\mathfrak{R}_x^*$. By (2.2) and Proposition 4.3 (where $\mathfrak{R}_x=\mathfrak{L}$) we have $g_x^*=a_x g_x$ for some real number a_x. Since $a=(g^{ij}g_{ij}^*)/m$, a is a C^∞-function on the closure of the set of non-divisible points of R. q. e. d.

Corresponding to Theorem 2′ in [**4**], we have

THEOREM 4.5. *Let (M, g) be a pseudo-Riemannian manifold of signature (p, q), such that $p\neq q$, $m\geqq 3$. If the set of all non-divisible points of R is dense in M, a curvature tensor-preserving transformation of (M, g) onto another (M', g') is homothetic.*

We give an outline of the proof. We denote the induced metric φ^*g' on M by g^*, where $\varphi: M\to M'$ is the given curvature tensor-preserving transformation. By Proposition 4.4, we have $g^*=e^{2\alpha}g$ for some function α on M. Then the classical formula gives:

$$R^{*i}{}_{jkl}=R^i{}_{jkl}+\delta^i_k\beta_{jl}-\delta^i_l\beta_{jk}+\beta^i_k g_{jl}-\beta^i_l g_{jk}\,,$$

where, putting $\alpha_i=\nabla_i\alpha$,

$$\beta_{jl}=\nabla_j\alpha_l-\alpha_j\alpha_l+\frac{1}{2}\alpha_r\alpha^r g_{jl}\,.$$

$R^{*}=R$ and $m\geqq 3$ imply $\beta_{jl}=0$. Then calculating $\nabla_{j}\nabla_{k}\alpha_{l}-\nabla_{k}\nabla_{j}\alpha_{l}$ and using the Ricci identity, we have $R^{r}_{ljk}\alpha_{r}=0$. Non-divisibility (on a dense set) implies $\alpha_{r}=0$. That is, α is constant.

COROLLARY 4.6 (cf. Vranceanu [18]). *Let (M, g) be a pseudo-Riemannian manifold of signature (p, q), $p\neq q$, and $m\geqq 3$. Assume that on a coordinate neighborhood $U(x^{i})$, R is non-divisible. Let g^{*} be another metric on $U(x^{i})$. If the Christoffel's symbols satisfy*

$$\Gamma^{i}_{jk}=\Gamma^{*i}_{jk} \qquad \text{on } U\,,$$

then g and g^{*} are homothetic.

§ 5. The conditions $R(X, Y)\cdot R=0$ and $R(X, Y)\cdot R_{1}=0$.

For tangent vectors X and Y at x, $R(X, Y)$ acts on the tensor algebra at x as a derivation. The condition (*) is

$$(*) \qquad R(X, Y)\cdot R=0 \qquad \text{for any } X, Y\in M_{x},\ x\in M.$$

The condition (*) implies in particular

$$(**) \qquad R(X, Y)\cdot R_{1}=0 \qquad \text{for any } X, Y\in M_{x},\ x\in M.$$

Denoting the Ricci transformation by R^{1}, (**) is equivalent to $R(X, Y)\cdot R^{1}=0$. i.e.,

$$(5.1) \qquad R(X, Y)(R^{1}Z)-R^{1}(R(X, Y)Z)=0\,.$$

LEMMA 5.1. *Assume (**). If R^{1} has a simple eigenvalue λ at x, then $\lambda=0$. In this case, the nullity index at x is 1.*

PROOF. Let λ_{i}, e_{i} be eigenvalues, orthonormal eigenvectors such that $R^{1}e_{i}=\lambda_{i}e_{i}$ at x, $i=1, 2, \cdots, m$. By (**), we have

$$R_{1}(R(e_{i}, e_{j})e_{k}, e_{l})+R_{1}(e_{k}, R(e_{i}, e_{j})e_{l})=0, \text{ i.e.,}$$

$$(\lambda_{l}-\lambda_{k})R_{lkij}=0\,.$$

Let λ_{m} be a simple eigenvalue. Then, we get $R_{mkij}=0$. By $R_{mj}=\sum g^{ki}R_{mkij}=0$, we have $R^{1}e_{m}=0$. Hence, $\lambda_{m}=0$. Let X be in N_{x}. Then $R_{1}(X, Y)=0$ for any $Y\in M_{x}$ (cf. § 6 Remark (3)). Therefore, only eigenvectors corresponding to 0 can be in N_{x}, and $\mu(x)=1$.

LEMMA 5.2. *Assume (**). Let $\lambda^{1}, \lambda^{2}, \cdots, \lambda^{k}$ be distinct eigenvalues of R^{1} at x, and let D^{α}_{x}, $\alpha=1, 2, \cdots, k$, be eigenspaces. Then D^{α}_{x} are invariant by curvature transformations.*

PROOF. Let $(X_{i}, i=1, 2, \cdots, m)$ be an orthonormal basis at x such that

$$X_{1}, \cdots, X_{r}\in D^{1}_{x},\ X_{r+1}, \cdots, X_{r+s}\in D^{2}_{x}, \cdots, \cdots X_{r+s+\cdots+t}\in D^{k}_{x}.$$

Let $X_a \in D_x^\alpha$. Put $R(X_i, X_j) = (1/2)\sum R_{klij} X_k \wedge X_l$. Then, putting $X = X_i$, $Y = X_j$ and $Z = X_a$ in (5.1), we have

$$\lambda^\alpha(\sum_{k,l} R_{klij} X_k \wedge X_l) X_a = R^1((\sum_{k,l} R_{klij} X_k \wedge X_l) X_a)\,, \quad \text{i. e.,}$$

$$\lambda^\alpha \sum_{l=1}^{m} R_{laij} X_l = \lambda^1 \sum_{u=1}^{r} R_{uaij} X_u + \cdots + \lambda^k \sum_{\xi=r+s+\cdots+t-t+1}^{m} R_{\xi aij} X_\xi\,.$$

Since $\lambda^1, \lambda^2, \cdots, \lambda^k$ are distinct, we have

$$R_{uaij} = 0, \cdots, \wedge^\alpha, \cdots, R_{\xi aij} = 0\,,$$

where $\wedge^\alpha$ means that (R_{baij}-part) is removed. That is,

$$R(X, Y) D_x^\alpha \subset D_x^\alpha\,. \qquad \text{q. e. d.}$$

We easily see that non-trivial components in (R_{klij}) are

$$(R_{wxuv}, \cdots, R_{cdab}, \cdots, R_{\kappa\lambda\xi\eta})\,,$$

components with mixed indices being zero. Hence, we have

PROPOSITION 5.3. *Let (M, g) be a Riemannian manifold with* (**).

(i) *If R is non-divisible at x, then R_1 is proportional to g at x.*

(ii) *If R is non-divisible on a dense subset of M, then (M, g) is an Einstein space.*

PROPOSITION 5.4. *Let (M, g) be a Riemannian manifold with* (*) *and $m=4$. If R is non-divisible on a dense subset of M, then (M, g) is locally symmetric.*

PROOF. An Einstein space with (*) and $m = 4$ is locally symmetric by a result of Sekigawa [**10**]. Thus Proposition 5.4 follows from Proposition 5.3.

PROPOSITION 5.5. *Let (M, g) be a Riemannian manifold with* (**) *and with nullity index zero at each point. If R^1 has distinct eigenvalues $\lambda^1 > \lambda^2 \cdots > \lambda^k$ on a connected open set W and if eigenvalues are differentiable on W, then $(W, g|W)$ is locally a product manifold of Einstein spaces.*

PROOF. Since λ^α, $\alpha = 1, 2, \cdots, k$, are distinct, we have continuous distributions D^α on W. To show that D^α are differentiable, for $x \in W$, let $X \in D_x^\alpha$. We extend X to a vector field X^* on W. Then

$$(R^1 - \lambda^1 I)(R^1 - \lambda^2 I) \cdots (\wedge^\alpha) \cdots (R^1 - \lambda^k I) X^*$$

belongs to D^α and differentiable. Thus, D^α is differentiable. Then, Theorem 2.6 shows that D^α are parallel. Each integral manifold of D^α is an Einstein space. By Lemma 5.1, $\dim D^\alpha \geqq 2$.

PROPOSITION 5.6. *Let (M, g) be a 5-dimensional Riemannian manifold with* (**) *and with nullity index zero at each point. Then there is a subset V such that $M - V$ is dense and any point $x \in M - V$ has a neighborhood W which is an Einstein space or a product manifold of Einstein spaces.*

PROOF. Since the multiplicity of each non-zero eigenvalue of R^1 at x is

$\geqq 2$ (by Lemma 5.1) and 0 is not a simple eigenvalue (by Lemma 5.1 and nullity index zero at each point), we have possibilities of eigenvalues of R^1: $(0, 0, 0, \lambda, \lambda)$, $(0, 0, \lambda, \lambda, \lambda)$, $(\gamma, \gamma, \lambda, \lambda, \lambda)$ and $(\lambda, \lambda, \lambda, \lambda, \lambda)$ at x (where in first 3 cases, $\lambda, \gamma \neq 0$; in the last case $\lambda \neq 0$ or $=0$).

(i) The case $(0, 0, 0, \lambda, \lambda)$. By Lemma 5.2 and by the statement just above Proposition 5.3, $(R_{wyuv};\ w, y, u, v=1, 2, 3)$ can be considered as components of a Riemannian curvature tensor of a 3-dimensional Riemannian manifold, algebraically at x. Since a 3-dimensional Riemannian manifold with the vanishing Ricci tensor at x has the vanishing Riemannian curvature tensor at x, we have $R_{wyuv}=0$. Hence, the nullity index at x is 3, and this can not occur.

(ii) The case $(0, 0, \lambda, \lambda, \lambda)$ can not occur, too.

(iii) The case $(\gamma, \gamma, \lambda, \lambda, \lambda)$. Since only two γ and λ are distinct, γ and λ are differentiable on some neighborhood W of x (cf. for example, Ryan [**9**], p. 371). Then we apply Proposition 5.5.

(iv) The case $(\lambda, \lambda, \lambda, \lambda, \lambda)$. If this holds on a neighborhood W of x, then $(W, g|W)$ is an Einstein space. If x has no open neighborhood where $R^1=(\lambda, \lambda, \lambda, \lambda, \lambda)$, then the set V of points of this type is of measure zero, i. e., $M-V$ is dense.

REMARK. For the case $m=3$, or 4, cf. Sekigawa [**11**].

§ 6. Remarks.

(1) Let (M, g) be a conformally flat and non-flat Riemannian manifold. If the restricted homogeneous holonomy group is not the special orthogonal group $SO(m)$, then the Ricci transformation R^1 has just two distinct eigenvalues λ and μ on some open set W (Kurita [**6**]). Denote by D^1 and D^2 the distributions on W defined by

$$D_x^1=\{X\in M_x:\ R^1X=\lambda X\}\,, \tag{6.1}$$
$$D_x^2=\{U\in M_x:\ R^1U=\mu U\}\,.$$

Then D^1 and D^2 are differentiable. If $\dim D^1\geqq 2$ and $\dim D^2 \geqq 2$, we have

$$R(X, Y)=KX\wedge Y\,, \qquad X,\ Y\in D^1\,,$$
$$R(U, V)=-KU\wedge V\,, \qquad U,\ V\in D^2\,,$$
$$R(X, U)=0\,, \qquad X\in D^1,\ U\in D^2\,.$$

Theorem 2.6 is applicable.

(2) Let (M, g, J) be a Kählerian manifold with the vanishing Bochner curvature tensor, where J denotes (the almost) complex structure tensor and

g denotes the Kählerian metric tensor. If the restricted homogeneous holonomy group is not the unitary group $U(n)$, $m=2n$, then the Ricci transformation R^1 has just two distinct eigenvalues λ and μ on some open set (cf. Takagi-Watanabe [**14**]). On this open set we have D^1 and D^2 defined similarly by (6.1). Then

$$R(X, Y)Z=\frac{H}{4}[(X\wedge Y)Z+(JX\wedge JY)Z-2g(JX, Y)JZ],$$

$$R(U, V)W=-\frac{H}{4}[(U\wedge V)W+(JU\wedge JV)W-2g(JU, V)JW],$$

$$R(X, U)=0$$

for $X, Y, Z\in D^1$ and $U, V, W\in D^2$. Theorem 2.6 is applicable: (M, g) is locally a product manifold of two Kählerian manifolds of constant holomorphic sectional curvature H and $-H$.

(3) As for Theorem A$'$ in the introduction, we notice that if R_1 is nonsingular at x, then the nullity index at x is zero. Let $X\in N_x$. For any orthonormal basis (e_i) at x, we have

$$R_1(X, Y)=\sum g(R(X, e_i)Y, e_i)=0$$

for any $Y\in M_x$. Hence, $X=0$.

Bibliography

[1] S. S. Chern and N. H. Kuiper, Some theorems on the isometric imbedding of compact Riemann manifolds in euclidean space, Ann. of Math., **56** (1952), 422-430.

[2] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, 1962.

[3] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, I, II, Intersci. Publ., 1963, 1969.

[4] O. Kowalski, On regular curvature structures, Math. Z., **125** (1972), 129-138.

[5] O. Kowalski, A note on the Riemann curvature tensor, Commentat. Math. Univ. Carolinae, **13** (1972), 257-263.

[6] M. Kurita, On the holonomy group of the conformally flat Riemannian manifold, Nagoya Math. J., **9** (1955), 161-171.

[7] K. Nomizu and K. Yano, Some results related to the equivalence problem in Riemannian geometry, Proc. U. S.-Japan Seminar in Differential Geometry, Kyoto, 1965, 95-100.

[8] A. Rosenthal, Riemannian manifolds of constant nullity, Michigan Math. J., **14** (1967), 469-480.

[9] P. J. Ryan, Homogeneity and some curvature conditions for hypersurfaces, Tôhoku Math. J., **22** (1969), 363-388.

[10] K. Sekigawa, On 4-dimensional connected Einstein space satisfying the condition $R(X, Y)\cdot R=0$, Sci. Rep. Niigata Univ., **7** (1969), 29-31.

[11] K. Sekigawa, Notes on some 3- and 4-dimensional Riemannian manifolds, to

appear.
[12] I. M. Singer, Infinitesimally homogeneous spaces, Comm. Pure Appl. Math., **13** (1960), 685-697.
[13] H. Takagi, An example of Riemannian manifolds satisfying $R(X, Y)\cdot R=0$ but not $\nabla R=0$, Tôhoku Math. J., **24** (1972), 105-108.
[14] H. Takagi and Y. Watanabe, On the holonomy groups of Kählerian manifolds with vanishing Bochner curvature tensor, Tôhoku Math. J., **25** (1973), 185-195.
[15] К. Телеман, Об одной теореме бореля-лихнеровича, Rev. Roumaine Math. Pures Appl., **3** (1958), 107-115.
[16] S. Tanno, Strongly curvature-preserving transformations of pseudo-Riemannian manifolds, Tôhoku Math. J., **19** (1967), 245-250.
[17] S. Tanno, Transformations of pseudo-Riemannian manifolds, J. Math. Soc. Japan, **21** (1969), 270-281.
[18] G. Vranceanu, Sur la représentation géodésique des espaces de Riemann, Rev. Roumaine Math. Pures Appl., **1**, no. 3 (1956), 147-165.

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