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On Thullen domains and Hirzebruch manifolds, I

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Introduction.

Let D denote a bounded domain in N-complex Euclidean space $\boldsymbol{C}^N$; namely, a connected, relatively compact open set of $\boldsymbol{C}^N$, and $\mathrm{Aut}(D)$ the group of all biholomorphic transformations[1] of D onto itself. We consider $\mathrm{Aut}(D)$ as a topological group with respect to the compact-uniform topology; it is then well-known since H. Cartan (see [**1**]) that the connected component containing the identity element, $\mathrm{Aut}^0(D)$ $(=G)$, of $\mathrm{Aut}(D)$ constitutes the structure of a (connected) real Lie group. E. Cartan discovered a few years later a distinguished class of bounded domains—the symmetric bounded domains—; in this case G is a real semi-simple Lie group which acts on D transitively, whence D is a homogeneous bounded domain. On the other hand, the theory of homogeneous bounded domains, which was initiated by H. Cartan around 1930, has been developed extensively by many mathematicians in recent times (see, for instance, Pyatetzki-Shapiro [**10**]).

In this article, we shall be concerned with a class of non-homogeneous bounded domains, which seems to be of interest from the view-points of complex analysis and differential geometry. We will now explain our motivation of studies in the following: In 1931, P. Thullen [**11**] discussed a bounded Reinhardt domain of restricted type of dimension two:

$$D_a=\{(z, w)\in \boldsymbol{C}^2 ;\ |z|^2+|w|^a<1\}, \tag{1}$$

(a designates a positive real number different from 2). He determined explicitly the group G of this domain; actually he showed that G is a four-dimensional reductive Lie group whose semi-simple part is isomorphic to the group of automorphisms of the unit disc $D_{(1)}=\{z\in \boldsymbol{C};\ |z|<1\}$ and the center is a circle group consisting of rotations of the coordinate $w: w\rightarrow e^{i\varphi}\cdot w$ $(\varphi\in \boldsymbol{R})$. It had been wanted by us to generalize Thullen's result to the higher dimensional case, and recently I. Naruki [**9**] succeeded to do so; in fact, he dealt the bounded Reinhardt domains in $\boldsymbol{C}^N$ $(N=m+n)$ of the following type:

1) In this paper, we shall call bi-holomorphic transformations simply as automorphisms.

(2) $$D_{m,a}=\{(z_1,\cdots,z_m,w_1,\cdots,w_n)\in \boldsymbol{C}^N;\ \sum_{i=1}^m |z_i|^2+\sum_{k=1}^n |w_k|^{a_k}<1\},$$

where $a=(a_1,\cdots,a_n)$ designates a column vector consisting of positive real numbers a_k $(1\leqq k\leqq n)$ which are all different from two. The structure of the group $G_{m,a}=\mathrm{Aut}^0(D_{m,a})$ is proved by him to be the direct product of $G^{(m)}$ and an n-dimensional toral group (= the group of rotations with respect to the co-ordinates $w_1,\cdots,w_n$), where $G^{(m)}$ denotes the automorphism group of the m-dimensional hypersphere $D_{(m)}=\{(z_1,\cdots,z_m)\in \boldsymbol{C}^m;\ \sum_{i=1}^m |z_i|^2<1\}$. As a matter of fact, Naruki determined the Lie algebra of $G_{m,a}$ that is the totality of *complete* holomorphic vector fields on $D_{m,a}$ from the considerations of their behaviour near the boundary $\partial D_{m,a}$ of $D_{m,a}$.

On the other hand, our version on $D_{m,a}$ is rather geometric; namely we regard $D_{m,a}$ as a *holomorphic fibre space* over $D_{(m)}$ with the projection map $\pi:(z_1,\cdots,z_m,w_1,\cdots,w_n)\to(z_1,\cdots,z_m)$, and then introduce the *canonical compactification*, in a sense, of such a fibre space. In fact, we will define an algebraic manifold Σ as the total space of a certain holomorphic projective bundle over the complex projective space $P_m(\boldsymbol{C})$ with fibre $P_n(\boldsymbol{C})$; this manifold Σ is a natural generalization of those considered previously by F. Hirzebruch [**7**] and E. Brieskorn [**4**]. To introduce the compactification, we have at first to define the unbounded domain $D'_{m,a}$ in $\boldsymbol{C}^N$ by

(2)′ $$D'_{m,a}=\{(z_1,\cdots,z_m,w_1,\cdots,w_n)\in \boldsymbol{C}^N;\ \sum_{i=1}^m |z_i|^2+\sum_{k=1}^n |w_k|^{-a_k}<1,\ w_k\neq 0\ (1\leqq k\leqq n)\},$$

and put

(2)° $$D^0_{m,a}=\{(z_1,\cdots,z_m,w_1,\cdots,w_n)\in D_{m,a};\ w_k\neq 0\ (1\leqq k\leqq n)\}.$$

Then, through the mapping $(z_1,\cdots,z_m,w_1,\cdots,w_n)\to(z_1,\cdots,z_m,w_1^{-1},\cdots,w_n^{-1})$, the both domains $D'_{m,a}$ and $D^0_{m,a}$ are mutually bi-holomorphically equivalent; whence we see $\mathrm{Aut}(D'_{m,a})\cong\mathrm{Aut}(D^0_{m,a})$ *via* the above mapping. We can further show that $\mathrm{Aut}(D^0_{m,a})=\mathrm{Aut}(D_{m,a})$ (see § 3) by using the continuation theorem of bounded holomorphic functions. By the canonical compactification of the fibre space $\pi: D'_{m,a}\to D_{(m)}$ $(\pi(z_1,\cdots,z_m,w_1,\cdots,w_n)=(z_1,\cdots,z_m))$, we mean the following commutative diagram, with the inclusion mappings $D'_{m,a}\to\Sigma$ and $D_{(m)}\to P_m(\boldsymbol{C})$ which are equivariant with respect to the automorphisms of these domains:

(3) $$\begin{array}{ccc} D'_{m,a} & \xrightarrow{\iota} & \Sigma \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ D_{(m)} & \xrightarrow{\iota} & P_m(\boldsymbol{C})\,. \end{array}$$

However, it is to be noted here that this diagram will be established only for the restricted values of $a=(a_1, \cdots, a_n)$ (see § 2), and also that for the original domain D_a considered by Thullen our Σ turns out to be an algebraic surface studied previously by F. Hirzebruch [7].

To explain the contents of the paper: we shall present briefly in § 1 a natural generalization of E. Brieskorn's works [4], [**4a**] in part. Methods and proofs being almost same with [**4a**], we will not necessarily describe those in detail. In § 2, which is the main part of the paper, we show how to construct the diagram (3). Our aim in this section is primarily to write down explicitly the action of $G_{m,a}$ on $D_{m,a}$ and to explain its meaning. In the following paper we shall be going to discuss the potential theory of the generalized Thullen domains.

§ 1. Generalization of Hirzebruch-Brieskorn manifolds.

1.1. In this section we shall discuss the holomorphic projective bundles of special type over the complex projective space. Let us now denote by m, n, any positive integers and by $P_m(\boldsymbol{C})$ (resp. $P_n(\boldsymbol{C})$) the m-dimensional (resp. n-dimensional) complex projective space. A point of $P_m(\boldsymbol{C})$ will be designated by $\mathfrak{y}=(y_0, \cdots, y_m)$, y_i $(0\leqq 1\leqq m)$ denoting the homogeneous coordinates, while that of $P_n(\boldsymbol{C})$ by $\mathfrak{x}=(x_0, \cdots, x_n)$ in the same sense. The open covering $\{U_i\}$ of $P_m(\boldsymbol{C})$ is defined, as usual, by $U_i=\{\mathfrak{y}\in P_m(\boldsymbol{C})\,;\, y_i\neq 0\}$ $(0\leqq i\leqq m)$; then it yields the holomorphic functions $h_{ij}(\mathfrak{y})=y_i y_j^{-1}$ on $U_i\cap U_j$ with value in $\boldsymbol{C}^*$.

Denoting by $p=(p_1, \cdots, p_n)$ an n-tuple of non-negative integers p_j $(1\leqq j\leqq n)$ such that $p_1\leqq p_2 \cdots \leqq p_n$, we introduce $GL(n+1, \boldsymbol{C})$-valued (in fact, diagonal matrix-valued) holomorphic functions on $U_i\cap U_j$:

$$g_{ij}(\mathfrak{y})=\begin{pmatrix} 1 & & & \\ & h_{ij}(\mathfrak{y})^{p_1} & & 0 \\ & & \ddots & \\ 0 & & & h_{ij}(\mathfrak{y})^{p_n} \end{pmatrix}.$$

We may then consider $g_{ij}(\mathfrak{y})$ as an element of $PGL(n, \boldsymbol{C})$, whence they constitute the transition functions of a holomorphic $P_n(\boldsymbol{C})$-bundle over $P_m(\boldsymbol{C})$. The total space of the bundle thus obtained will be denoted by $\Sigma_{m,p}=\Sigma_{m;p_1,\cdots,p_n}$; it is a simply-connected algebraic manifold, of dimension $m+n$, by a theorem of K. Kodaira. In fact, we can define, as will be shown below, the imbedding ι of $\Sigma_{m,p}$ in the multiple complex projective space $P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C})$; in the second factor $P_{(m+1)n}(\boldsymbol{C})$ it is convenient to designate the homogeneous coordinates by (x_{00}, x_{ik}), $(0\leqq i\leqq m\,;\, 1\leqq k\leqq n)$. We now introduce a projective algebraic manifold $\tilde{\Sigma}_{m,p}$ by

$$\tilde{\Sigma}_{m,p}=\{(\mathfrak{y},\mathfrak{x})\in P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C})\,;\ y_j^{p_k}x_{ik}=y_i^{p_k}x_{jk}\ (1\leqq k\leqq n\,;\ 0\leqq i,j\leqq m)^{[2]}\}\,.$$

This manifold is of dimension $m+n$[3], and we can adopt it as the imbedded manifold $\iota(\Sigma_{m,p})$ of $\Sigma_{m,p}$. To verify this, we define the projection $\tilde{\pi}$ by $\tilde{\pi}(\mathfrak{y}\,;\mathfrak{x})=\mathfrak{y}$ for any $(\mathfrak{y}\,;\mathfrak{x})\in\tilde{\Sigma}_{m,p}$ and the mapping $h_i:\tilde{\pi}^{-1}(U_i)\to U_i\times P_n(\boldsymbol{C})$ by $h_i(\mathfrak{y}\,;\mathfrak{x})=(\mathfrak{y}\,;x_{00},x_{i1},\cdots,x_{in})$ $(0\leqq i\leqq m)$, where we note that

$$\tilde{\pi}^{-1}(U_i\cap U_j)\ni(\mathfrak{y}\,;\mathfrak{x})\rightleftarrows\mathfrak{y}\in U_i\cap U_j \quad\text{and}$$
$$(x_{00},x_{i1},\cdots,x_{in})=(x_{00},x_{j1},\cdots,x_{jn})g_{ij}(\mathfrak{y})\,.$$

Thus we obtain, by putting $h_i(\mathfrak{y})\mathfrak{x}=h_i(\mathfrak{y}\,;\mathfrak{x})$, that $h_i(\mathfrak{y})h_j(\mathfrak{y})^{-1}=g_{ij}(\mathfrak{y})$ for $\mathfrak{y}\in U_i\cap U_j$. Hence the bundle $\tilde{\Sigma}\xrightarrow{\pi}P_m(\boldsymbol{C})$ is equivalent to $\Sigma_{m,p}\xrightarrow{\pi}P_m(\boldsymbol{C})$.

We will identify in what follows $\tilde{\Sigma}_{m,p}$ with $\Sigma_{m,p}$, and $\tilde{\pi}$ with π. It is readily proved that $\Sigma_{m,p}$ is a rational variety; actually it contains $\boldsymbol{C}^{m+n}$ as a generic-point set. As a matter of fact, we are able to define the imbedding ι of $\boldsymbol{C}^{m+n}$ into $\Sigma_{m,p}$, denoting the coordinates in $\boldsymbol{C}^{m+n}$ by $(z_1,\cdots,z_m\,;w_1,\cdots,w_n)$, in the following way:

$$\iota(z_1,\cdots,z_m\,;\ w_1,\cdots,w_n)=(\mathfrak{y}\,;\mathfrak{x})\in P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C})\,,$$

where $\mathfrak{y}=(1,z_1,\cdots,z_m)$, $\mathfrak{x}=(x_{00},x_{ik})$; $x_{00}=1$, $x_{ik}=z_i^{p_k}x_{0k}$ $(1\leqq i\leqq m)$, $x_{0k}=w_k$ $(1\leqq k\leqq n)$. We see then obviously that the image points belong to $\Sigma_{m,p}$; we therefore obtain, combined with the natural imbedding ι of $\boldsymbol{C}^m$ into $P_m(\boldsymbol{C})$ such that $\iota(z_1,\cdots,z_m)=(1,z_1,\cdots,z_m)$, the following commutative diagram:

$$\begin{array}{ccl}\boldsymbol{C}^{m+n} & \longrightarrow & \Sigma_{m,p}\subset P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C})\\ \pi\big\downarrow & & \tilde{\pi}\big\downarrow\\ \boldsymbol{C}^m & \longrightarrow & P_m(\boldsymbol{C})\,.\end{array}\tag{4}$$

1.2. For the rest of this section, we will clarify the structure of the bundle $\Sigma_{m,p}$ and the holomorphic automorphism group of the compact complex manifold $\Sigma_{m,p}$ as a transformation group. Let $\mathrm{Aut}^0(\Sigma_{m,p})$ $(=\tilde{G})$ denote the connected component of the identity of the full holomorphic automorphism group $\mathrm{Aut}(\Sigma_{m,p})$; it is a connected, complex Lie group with the compact-uniform topology. By a theorem of A. Blanchard [3], every element of $\mathrm{Aut}^0(\Sigma_{m,p})$ is fibre-preserving in the fibre bundle discussed above; it induces therefore an element of $\mathrm{Aut}(P_m(\boldsymbol{C}))$. We thus obtain a complex Lie group homomorphism, π, of $\mathrm{Aut}^0(\Sigma_{m,p})$ into $\mathrm{Aut}(P_m(\boldsymbol{C}))$, which is actually surjective

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2) $\boldsymbol{C}$ denotes the complex numbers, $\boldsymbol{R}$ the real numbers and $\boldsymbol{C}^n$ the n-dimensional complex euclidean space.

3) The number of the equations is seemingly $\frac{1}{2}m(m+1)n$; however that of essential ones in every local co-ordinates neighbourhood is readily seen to be mn.

since our bundle is *homogeneous* in the sense of R. Bott (soon later we show this fact in an explicit manner). Hence we get the group extension:

$$1 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow \mathrm{Aut}^0(\Sigma_{m,p}) \xrightarrow{\pi} \mathrm{Aut}(P_m(\boldsymbol{C})) \longrightarrow 1,$$

where $\mathrm{Aut}^0(P_m(\boldsymbol{C}))=PGL(m, \boldsymbol{C})$ is a complex simple Lie group, and $\mathrm{Ker}(\pi)$ is a complex, closed normal subgroup of $\tilde{G}=\mathrm{Aut}^0(\Sigma)$ which is denoted by $\tilde{N}$. We shall now determine the structure of $\tilde{N}$. In fact, we can show, as was done in [**4a**], the following:

LEMMA 1. *$\tilde{N}$ is naturally isomorphic to $\Delta^{(n)}/\boldsymbol{C}^*\subset PGL(n, \boldsymbol{C})$, where $\Delta^{(n)}$ denotes a matrix-group of degree $n+1$ consisting of matrices $\delta(t_1, \cdots, t_m)$ whose components $\delta_{kl}(t_1, \cdots, t_m)$ $(0\leqq k,\ l\leqq n)$ are polynomials of m-indeterminants $t_1, \cdots, t_m$ with complex coefficients and with degree $\leqq p_l-p_k$ for each t_i $(0\leqq k,\ l\leqq n,\ p_0=0)$. Thus,*

$$\mathrm{dim}_{\boldsymbol{C}}\Delta^{(n)}=\Sigma_{0\leqq k,l\leqq n,p_k\leqq p_l}\binom{p_l-p_k+m}{m}^{4)}. \tag{5}$$

We present here the proof for the completeness sake, though it is quite similar to that given in [**4a**]. For the proof, we take an element σ of $\tilde{N}$; then σ carries each fibre onto itself; we denote by σ_i the restriction of σ onto $\Sigma_i=\pi^{-1}(U_i)$. Then, we see that σ_i induces naturally a holomorphic mapping $\tilde{\sigma}_i$ of U_i into $PGL(n, \boldsymbol{C})=GL(n+1, \boldsymbol{C})/\boldsymbol{C}^*$, therefore we write

$$\tilde{\sigma}_i(s)=(\sigma_{i\,:\,k,l}(s)), \qquad (0\leqq k,\ l\leqq n)$$

as matrix-form, where $s=(s_0, \cdots, \hat{s}_i, \cdots, s_m)$ denote the inhomogeneous co-ordinates $y_{i'}/y_i$ $(i'\neq i)$ in U_i. Then, in $U_i\cap U_j$, we have a relation between $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$:

$$\tilde{\sigma}_i(s)\cdot g_{ij}(\mathfrak{y})=g_{ij}(\mathfrak{y})\cdot\tilde{\sigma}_j(t),$$

where $t=(t_1, \cdots, t_m)$ designates the inhomogeneous co-ordinates $y_{j'}/y_j$ $(j'\neq j)$ in U_j. Now, we take $j=0$, then we may put $t_1=y_1/y_0, \cdots, t_m=y_m/y_0$. From the above, putting $\sigma_{0\,;\,kl}=\delta_{kl}$, it follows that

$$\sigma_{i\,;\,kl}(s_0, \cdots, \hat{s}_i, \cdots, s_m)=s_0^{p_l-p_k}\cdot\delta_{kl}(t_1, \cdots, t_m);$$

namely $s_0=y_0/y_i \cdots s_m=y_m/y_i$, and $t_1=s_1/s_0, \cdots, t_i=1/s_0, \cdots, t_m=s_m/s_0$. The left-hand side is a holomorphic function of $s_0, \cdots, s_m$; this means that, if $p_l\geqq p_k$, $\delta_{kl}(t)$ is a polynomial function of $t_1, \cdots, t_m$ with the highest degree at most p_l-p_k, and that, if $p_l<p_k$, $\delta_{kl}(t)$ have to vanish identically. Conversely, if we let $\delta_{kl}(t)$ be a polynomial function with the properties stated above, then we may define the holomorphic functions $\sigma_{i\,;\,kl}(s)$ by using the above

4) The dimension formula for $\Delta^{(n)}$ in Lemma 1 can be derived also from the sheaf-theoretic arguments.

equality in $U_0 \cap U_i$; it is uniquely extended to a holomorphic function in U_i. Thus, the collection $\sigma=(\sigma_i)$ corresponding to $\tilde{\sigma}_i=(\sigma_{i\,;\,kl})$ determines an element of $\tilde{N}$ up to non-zero scalar factors. This proves that $\tilde{N}$ is isomorphic to $\Delta^{(n)}/C^*$ *via* the correspondence defined above.

In the case where $p_1<p_2<\cdots<p_n$, especially when $n=1$, $\tilde{N}$ is solvable: hence $\tilde{N}$ is the *radical* of $\tilde{G}$ and the above extension yields the *Levi-decomposition* of $\tilde{G}$. Now we shall define in general case a canonical cross-section $\tilde{G}_1^{(m)}$ of the homomorphism π in $\tilde{G}$ for the later use:

$$\tilde{G}=\tilde{G}_1^{(m)}\cdot\tilde{N}\,; \qquad \tilde{G}_1^{(m)}\cap\tilde{N}=\{1\}\,,$$

where $\tilde{G}_1^{(m)}$ is locally isomorphic to $PGL(m, \boldsymbol{C})$.

For this sake, we shall first endow on $\Sigma_{m,p}$ a *homogeneous bundle structure* by making use of the imbedded form of $\Sigma_{m,p}$ in $P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C})$. In fact, we now define a holomorphic mapping β of $GL(m+1, \boldsymbol{C})\times P_n(\boldsymbol{C})$ into $P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C})$ as follow: Let $g\in GL(m+1, \boldsymbol{C})$, $\xi=(\xi_0, \cdots, \xi_n)\in P_n(\boldsymbol{C})$ and put $\beta(g, \xi)=(\mathfrak{y}\,;\,\mathfrak{x})$, $\mathfrak{y}=(y_0, \cdots, y_m)$, $\mathfrak{x}=(x_{00}, x_{ik})$ as before; they are to be determined by

$$g\cdot\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}=\begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_0 \end{bmatrix}, \quad \text{or} \quad g=\begin{bmatrix} & & y_1 \\ & * & \vdots \\ & & y_m \\ & & y_0 \end{bmatrix}, \tag{6}$$

and by

$$\begin{bmatrix} x_{00} \\ x_{0k} \\ \vdots \\ x_{mk} \end{bmatrix}=\begin{bmatrix} \xi_0 \\ y_0^{p_k}\cdot\xi_k \\ \vdots \\ y_m^{p_k}\cdot\xi_k \end{bmatrix};$$

namely $x_{00}=\xi_0$ and $x_{ik}=y_i^{p_k}\cdot\xi_k$ $(0\leqq i\leqq m\,;\ 1\leqq k\leqq n)$. Then, as is readily seen, the totality of the image points $(\mathfrak{y}\,;\,\mathfrak{x})$ coincides with our $\Sigma_{m,p}$. Next we take the coset-space form $GL(m+1, \boldsymbol{C})/\tilde{H}$ of $P_m(\boldsymbol{C})$, where $\tilde{H}$ is the totality of $g\in GL(m+1, \boldsymbol{C})$ such that $g\cdot{}^t(0, \cdots, 0, 1)={}^t(0, \cdots, 0, 1)$. Namely $\tilde{H}$ is the subgroup of $GL(m+1, \boldsymbol{C})$ such that

$$\tilde{H}=\left\{\begin{pmatrix} A & 0 \\ * & d \end{pmatrix}\in GL(m+1, \boldsymbol{C})\right\}.$$

The representation of $\tilde{H}$ which we will now denote by ρ is the following one:

$$\rho\begin{pmatrix} A & 0 \\ * & d \end{pmatrix}=\begin{pmatrix} 1 & & & 0 \\ & d^{p_1} & & \\ & & \ddots & \\ 0 & & & d^{p_n} \end{pmatrix}\in PGL(n, \boldsymbol{C})\,.$$

For this, we define an equivalence relation in $GL(m+1, \boldsymbol{C})\times P_n(\boldsymbol{C})$ by

$$(g, \xi)\sim(gh, \xi\cdot\rho(h)), \qquad \text{for} \quad h\in\tilde{H}.$$

Then we infer easily that the mapping β defined above is compatible with this relation and that the set of resulting equivalence classes, denoted by $GL(m+1, \boldsymbol{C})\times_{\tilde{H}}P_n(\boldsymbol{C})$, are mapped biholomorphically into $P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C})$, and hence onto $\Sigma_{m,p}$. Thus we obtain the isomorphism: $GL(m+1, \boldsymbol{C})\times_{\tilde{H}}P_n(\boldsymbol{C}) \cong \Sigma_{m,p}$ under the mapping β.

1.3. Through this identification we are able to settle the actions of $GL(m+1, \boldsymbol{C})$ onto $\Sigma_{m,p}$ as the left translations in the left-hand side. Let us now take $\gamma=\begin{pmatrix} A & \mathfrak{b} \\ \mathfrak{c} & d\end{pmatrix}$ in $GL(m+1, \boldsymbol{C})$ (actually in $PGL(m, \boldsymbol{C})$), where A denotes a square matrix of degree m, $d\in\boldsymbol{C}$ and $\mathfrak{b}$ (resp. $\mathfrak{c}$) an m-column (resp. m-row) vector; for these we put as below

$$\tilde{\gamma}(\mathfrak{y}\,;\mathfrak{x})=(\mathfrak{y}'\,;\mathfrak{x}'), \qquad \text{for} \quad (\mathfrak{y}\,;\mathfrak{x})\in\Sigma_{m,p},$$

where, denoting as $\mathfrak{y}=(y_0, \cdots, y_m)$, $\mathfrak{x}=(x_{00}, x_{ik})$, $\mathfrak{y}'=(y'_0, \cdots, y'_m)$, $\mathfrak{x}'=(x'_{00}, x'_{ik})$,

$$\text{(7)} \qquad \begin{cases} {}^t(y'_1, \cdots, y'_m, y'_0)=\gamma\cdot{}^t(y_1, \cdots, y_m y_0), \\ x'_{00}=x_{00}, \\ x'_{ik}=(y'_i/y_i)^{p_k}x_{ik}, \qquad (0\leqq i\leqq m, \ 1\leqq k\leqq n). \end{cases}$$

For the case $i=0$, in particular, we put under the assumption that $x_{00}\neq 0$, $x'_{00}\neq 0$,

$$w_k=x_{0k}/x_{00}, \qquad w'_k=x'_{0k}/x'_{00}.$$

Our transformation $\tilde{\gamma}$ yields obviously an automorphism[5] of $\Sigma_{m,p}$; thus we denote by $\tilde{G}^{(m)}$ the totality of such $\tilde{\gamma}$ (for every $\gamma\in GL(m+1, \boldsymbol{C})$), then $\pi(\tilde{\gamma})=\gamma$ and the semi-simple part of $\tilde{G}^{(m)}$ is locally isomorphic to $PGL(m, \boldsymbol{C})$ under π except the case where all p_k are equal to 1. So, taking as $\tilde{G}_1^{(m)}$ the semi-simple part of $\tilde{G}^{(m)}$, $\tilde{G}_1^{(m)}$ is considered to be a canonical cross section in the sense of § 1-2.

1.4. Example. As an illustration of the above arguments, we now consider the case where $m=n=1$. In this case we put, for brevity, $x_{00}=x_0$, $x_{11}=x_1$, $x_{01}=x_2$; $z_1=z$, $w_1=w$, $p_1=p(\neq 0)$; then $\Sigma_{1,p}=\Sigma_p$ is a projective line, and is naturally imbedded into $P_1(\boldsymbol{C})\times P_2(\boldsymbol{C})$ as the totality of $(y_0, y_1; x, x_1, x_2)$ such that $y_0^p x_1=y_1^p x_2$. The manifolds Σ_p $(p=1, 2, \cdots)$, imbedded in $P_1(\boldsymbol{C})\times P_2(\boldsymbol{C})$, are no other than the ones discussed by F. Hirzebruch [**6**] and are called

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5) When γ is a diagonal matrix with components d; $\gamma=\text{diag}\,(d, \cdots, d)$, $\tilde{\gamma}$ is given as $y'_i=dy_i$ $(0\leqq i\leqq m)$, $x'_{00}=x_{00}$, $x'_{ik}=d^{p_k}\cdot x_{ik}$. Hence $\tilde{\gamma}$ is the identity transformation if and only if $d^{p_k}=1$ for all k. Hence $G^{(m)}$ acts almost effectively.

the *Hirzebruch surfaces.* In this case, $\tilde{G}$ is the semi-direct product of a 3-dimensional complex simple group and the $(p+2)$-dimensional radical.

§ 2. Generalized Thullen domains.

2.1. We shall consider here the relation between the bounded domain $D_{m,a}$ as introduced in (2) and the generalized Hirzebruch-Brieskorn manifolds $\Sigma_{m,p}$. It is one of our objects to clarify the geometric meaning of the previous works made by P. Thullen [**11**] and I. Naruki [**9**]. For this purpose, we have to confine ourselves to the special type of $D_{m,a}$, because the manifolds $\Sigma_{m,p}$ are countably infinite in number. Now we take a set of integers $p_1, \cdots, p_n$ such that

$$0<p_1\leqq p_2\leqq \cdots \leqq p_n\,;\ p_i\neq 1\ (1\leqq i\leqq n)\,,$$

and we use the symbol as in § 1: $p=(p_1, \cdots, p_n)$. For this we put

$$D_{m;\,p_1,\cdots,p_n}=\{(z_1, \cdots, z_m\,;\, w_1, \cdots, w_n)\in \boldsymbol{C}^{m+n}\,;\ \sum_{i=1}^{m}|z_i|^2+\sum_{k=1}^{n}|w_k|^{2/p_k}<1\}\,. \tag{8}$$

We want to refer to this domain as the *generalized Thullen domain of type* $(m\,;\,p_1, \cdots, p_n)$; for abreviation we write often as $D_{m;\,p_1,\cdots,p_n}=D_{m,p}$. Now we consider here two domains $D^0_{m,p}$ and $D'_{m,p}$ corresponding to $D_{m,p}$:

$$\left\{\begin{aligned} D^0_{m,p}&=\{(z_1, \cdots, z_m, w_1, \cdots, w_n)\in D_{m,p}\,;\ w_k\neq 0\ (1\leqq k\leqq n)\}\,,\\ D'_{m,p}&=\{(z_1, \cdots, z_m, w_1, \cdots, w_n)\in \boldsymbol{C}^{m+n}\,;\ \sum_{i=1}^{m}|z_i|^2+\sum_{k=1}^{n}|w_k|^{-2/p_k}<1\}\,. \end{aligned}\right. \tag{8*}$$

Then, as was stated in the Introduction, $D^0_{m,p}$ is equivalent to $D'_{m,p}$ by the mapping $(z_1, \cdots, z_m, w_1, \cdots, w_n)\rightarrow(z_1, \cdots, z_m, w_1^{-1}, \cdots, w_n^{-1})$; whence we shall henceforth identify the both domains *via* the above mapping: $D^0_{m,p}=D'_{m,p}$. The latter $D'_{m,p}$ is imbedded into $\Sigma_{m,p}$ by ι as was shown in § 1-1 (see (4)), while the former $D^0_{m,p}$ an open set of $D_{m,p}$. Further from a well-known Riemann's continuation theorem for bounded holomorphic function ([**6**], p. 19), we infer immediately that $\mathrm{Aut}\,(D_{m,p})=\mathrm{Aut}\,(D^0_{m,p})$, namely every automorphism of $D^0_{m,p}$ can be uniquely extended to that of $D_{m,p}$. It is noted here as in (4) that we obtain the following diagram, combined with the natural imbedding ι of $D_{(m)}=\{(z_1, \cdots, z_m)\,;\ \sum_{i=1}^{m}|z_i|^2<1\}$ into $P_m(\boldsymbol{C})$ given by $\iota(z_1, \cdots, z_m)=(1, z_1, \cdots, z_m)$, and with the projection $\pi:\pi(\mathfrak{z}, \mathfrak{w})=\mathfrak{z}$.

$$\begin{tikzcd} D^0_{m,p}=D'_{m,p} \arrow[r, "\iota"] \arrow[d, "\pi"] & \Sigma_{m,p}\subset P_m(\boldsymbol{C})\times P_{(m+1)n}(\boldsymbol{C}) \arrow[d, "\pi"] \arrow[dl, phantom, ""] \\ D_{(m)} \arrow[r, "\iota"] & P_m(\boldsymbol{C})\,. \end{tikzcd} \tag{9}$$

2.2. We are now going to prove that the above imbedding ι (in the first line) is equivariant with respect to $\mathrm{Aut}^0(D_{m,p})$. However we show in this section only that a subgroup $G=G^{(m)}\cdot T^n$ of $\tilde{G}=\tilde{G}^{(m)}\cdot\tilde{N}$ is contained in $\mathrm{Aut}^0(D_{m,p})$, and that ι is equivariant with respect to G; while in the subsequent paper we will show that G actually coincides with $\mathrm{Aut}^0(D_{m,p})$, as was already proved in [**9**]. Now, the group $G=G^{(m)}\cdot T^n$ is the *direct* product of $G^{(m)}$ and n-toral group T^n, where $G^{(m)}=\{\gamma\in G_1^{(m)}; \gamma\cdot\iota(D_{m,p})\subset\iota(D_{m,p})\}$ and is locally isomorphic to $\mathrm{Aut}^0(D_{(m)})$ through the homomorphism π, and T^n consists of the rotations: $w_k\to\exp(\sqrt{-1}\cdot\theta_k)w_k$; $\theta_k\in\boldsymbol{R}$ $(1\leqq k\leqq n)$. It is known that $G^{(m)}$ consists of $\gamma=\begin{pmatrix}A & \mathfrak{b}\\ \mathfrak{c} & d\end{pmatrix}$, in the notation of § 1-3, such that

$$ {}^t\bar{A}A-{}^t\mathfrak{c}\bar{\mathfrak{c}}=I_m\,,\qquad {}^t\mathfrak{b}\bar{\mathfrak{b}}-|d|^2=-1\,,\qquad \mathfrak{b}{}^t\bar{A}-\bar{d}\mathfrak{c}=0\,. $$

Now, as was already indicated in § 1-1, we have, through the imbedding ι, $z_i=y_i/y_0$ $(1\leqq i\leqq m)$ and $w_k=x_{0k}/x_{00}$ $(1\leqq k\leqq n)$. From that $w'_k=x'_{0k}/x'_{00}$, $w_k=x_{0k}/x_{00}$ and (7), it follows then that

$$ w'_k=(y'_0/y_0)^{p_k}\cdot w_k\,,\qquad (1\leqq k\leqq n)\,, $$

in the notation adopted in § 1-3 for the domain $D'_{m,p}$. On the other hand, if we write $\gamma=\begin{pmatrix}A & \mathfrak{b}\\ \mathfrak{c} & d\end{pmatrix}$ as in § 1-3, we see that $y'_0=\mathfrak{c}\mathfrak{y}+dy_0$, $\mathfrak{y}={}^t(y_1,\cdots,y_m)$. Hence we infer from $y_i/y_0=z_i$ and $\mathfrak{z}={}^t(z_1,\cdots,z_m)$ that $y'_0/y_0=\mathfrak{c}\mathfrak{z}+d$; namely for the domain $D^0_{m,p}$ we have

$$ w'_k=(\mathfrak{c}\mathfrak{z}+d)^{-p_k}\cdot w_k\,,\qquad (1\leqq k\leqq n)\,. $$

Summing up the arguments made above, we can state our result as follow:

THEOREM 1. *The group $G=G^{(m)}\cdot T^n$ acts on $D_{m,p}$ by the following rule:*

$$ \mathfrak{z}'=\gamma\cdot\mathfrak{z}=(A\mathfrak{z}+\mathfrak{b})(\mathfrak{c}\mathfrak{z}+d)^{-1}\,, $$
$$ w'_k=e^{\sqrt{-1}\cdot\theta_k}(\mathfrak{c}\mathfrak{z}+d)^{-p_k}\cdot w_k\,,\qquad (1\leqq k\leqq n)\,, $$

where $\gamma=\begin{pmatrix}A & \mathfrak{b}\\ \mathfrak{c} & d\end{pmatrix}\in G^{(m)}$; namely $A^*A-\mathfrak{c}^*\mathfrak{c}=I_m$, $\mathfrak{b}^*\mathfrak{b}-\bar{d}d=-1$, $\mathfrak{b}^*A=\bar{d}\mathfrak{c}$, and θ_k $(1\leqq k\leqq n)$ denote real numbers.

2.3. EXAMPLE. We take up the case where $m=n=1$ as in § 1-3; we follow the abreviation adopted there. Our domain $D_{1,p}=D_p$ is then given by $D_p=\{(z,w)\in\boldsymbol{C}^2; |z|^2+|w|^{2/p}<1\}$ $(p\neq 1)$; this case was originally treated by P. Thullen [**11**]. (See also, S. Bergmann [**2**]). We know that $G^{(1)}$ consists of $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL(2,\boldsymbol{C})$ such that

$$ |a|^2-|c|^2=|d|^2-|b|^2=1\,,\qquad a\bar{b}=c\bar{d}\,. $$

Further these conditions imply the following relation:

$$|a|^2-|b|^2=|d|^2-|c|^2=1, \quad a\bar{c}=b\bar{d}, \quad |ad-bc|=1.$$

The transformation γ on D_p is presented by Theorem 1 as

$$z'=(az+b)(cz+d)^{-1}, \qquad w'=(cz+d)^{-p}w.$$

Now we change slightly the notation. From the above relation, we know that $a\neq 0$, $d\neq 0$; so putting $b/a=\alpha$, $c/d=\bar{\alpha}$ and $a/d=e^{i\theta}$ $(i=\sqrt{-1})$, we can rewrite the first equality as

$$z'=e^{i\theta}(z+\alpha)(1+\bar{\alpha}z)^{-1}.$$

As for the second equality, we note that the absolute value of d is given by $(1-|\alpha|^2)^{-1/2}$; we may therefore put that $d^{-p}=e^{i\varphi}(1-|\alpha|^2)^{p/2}$ for some real φ. Then we obtain

$$w'=e^{i\varphi}(1-|\alpha|^2)^{p/2}(1+\bar{\alpha}z)^{-p}\cdot w.$$

These are exactly the results obtained by Thullen [**11**]; it was also a motivation of our study to clarify the *meaning* of the formula appeared above.

The domains under consideration are also meaningful for $p=1$; in this case $D_1=D_{(2)}$, further Σ_1 and the diagram (9) make sense. However the full automorphism group $\mathrm{Aut}(D_1)$ can not be extended to $\mathrm{Aut}^0(\Sigma_1)$. Thus, the case $p=1$ is singular in our arguments.

2.4. As a supplement to the preceding sections, we describe here the unbounded models of generalized Thullen domains; namely we will introduce the so-called *Cayley transformations*[6] for them.

We consider here the domain $D_{m,a}$ for any unrestricted values of a_i $(1\leqq i\leqq n)$ (see, Introduction); for this we define the unbounded domain $H_{m,a}$ by

$$H_{m,a}=\{(z_1,\cdots,z_m,w_1,\cdots,w_n);\ \mathrm{Im}(z_1)-\sum_{i=2}^{m}|z_i|^2-\sum_{k=1}^{n}|w_k|^{a_k}>0\}, \tag{10}$$

where $\mathrm{Im}(z_1)$ designates the imaginary part of z_1. While, let us define:

$$\begin{cases} z_1=(z_1'-\sqrt{-1})(z_1'+\sqrt{-1})^{-1}, \quad z_i=2z_i'(z_1'+\sqrt{-1})^{-1}, \quad (2\leqq i\leqq m) \\ w_k=2^{2/a_k}w_k'(z_1'+\sqrt{-1})^{-2/a_k}, \quad (1\leqq k\leqq n). \end{cases} \tag{11}$$

Then the assignment $(z', w')\to(z, w)$ provides a biholomorphic transformation of $H_{m,a}$ onto $D_{m,a}$; its inverse $\sigma:(z,w)\to(z',w')$ will be called the *Cayley transformation* of $D_{m,a}$, since σ coincides, when all $a_k=2$, with the Cayley transformation of the hypersphere $D_{(m+n)}$ (see [**10**]). When $m=n=1$, in particular, this was already pointed out by E. Cartan [**5**]; in fact, he con-

6) This was suggested by S. Kaneyuki, for whom the author is thankful.

sidered the unbounded domain $H_{1,a}^0 = \{(z, w)\,;\ \mathrm{Im}(z) > |w|^a,\ w \neq 0\}$ and showed that the Lie group $G' = SL(2, \boldsymbol{R}) \times T^1$ acts on $H_{1,a}^0$ by the following rule: For $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e^{\sqrt{-1}\cdot h}\right) \in G'$, let us put

$$g(z, w) = ((az+b)(cz+d)^{-1},\ e^{\sqrt{-1}\cdot h}(cz+d)^{-2/a}w)\,.$$

Now we shall prove as for our Cayley transformation σ of $D_{m,a}$ onto $H_{m,a}$ the analogue of E. Cartan's argument and also Pyatetzki-Shapiro's one are valid in the following sense:

THEOREM 2. *When $a_k = 2/p_k$ $(1 \leqq k \leqq n)$ as before, the Cayley transformation σ is furnished by an element of $\tilde{G}$.*

In fact, we shall now introduce a transformation $\tilde{\sigma}^{-1}$ of $\tilde{G} = \mathrm{Aut}^0(\Sigma_{m,p})$ by the following:

$$(12)\qquad \begin{cases} y_0 = \sqrt{-1}\cdot y_0' + y_1'\,, \\ y_1 = -\sqrt{-1}\, y_0' + y_1'\,, \\ y_i = 2\cdot y_1' \quad (2 \leqq i \leqq m)\,, \\ x_{00} = x_{00}'\,, \\ x_{ik} = 2^{p_k}(y_i/y_i')^{p_k} x_{ik}'\,, \qquad (0 \leqq i \leqq m\,;\ 1 \leqq k \leqq n)\,. \end{cases}$$

Namely, the first three equalities provide the inverse of Cayley transformation $\sigma^{(m)}$ of $D_{(m)}$ onto $H_{(m)} = \{(z_1, \cdots, z_m) \in \boldsymbol{C}^m;\ \mathrm{Im}(z_1) > \sum_{i=2}^{m} |z_i|^2\}$ which is an element of $K^{(m)}$, a maximal compact subgroup of $G^{(m)}$ (i.e. $K^{(m)} \cong PU(m)$). We therefore infer from the formula (7) that $\tilde{\sigma}$ is the product of $\tilde{\sigma}^{(m)}$, the lift $\sigma^{(m)}$, and the element of $\tilde{N}$ that is given as an element of $\Delta^{(n)}$ by

$$\begin{pmatrix} 1 & & & \\ & 2^{-p_1} & & 0 \\ & & \ddots & \\ & 0 & & 2^{-p_n} \end{pmatrix} \in \Delta^{(n)}\,, \qquad (p_k \neq 1)\,.$$

On the other hand, putting $y_i/y_0 = z_i$, $y_i'/y_0' = z_i'$ $(1 \leqq i \leqq m)$, $x_{0k}/x_{00} = w_k$, $x_{0k}'/x_{00}' = w_k'$ $(1 \leqq k \leqq n)$ in $\pi^{-1}(U_0)$, the equalities in (12) imply that

$$\begin{aligned} z_1 &= (z_1' - \sqrt{-1})(z_1' + \sqrt{-1})^{-1}\,, \\ z_i &= 2z_i'(z_1' + \sqrt{-1})^{-1}\,, \qquad (2 \leqq i \leqq m)\,, \\ w_k &= 2^{p_k} w_k' (z_1' + \sqrt{-1})^{-p_k} \qquad (1 \leqq k \leqq n)\,. \end{aligned}$$

Thus we get the inverse of the Cayley transformation σ of $D_{m,a}$ onto $H_{m,a}$.

2.5. REMARK. We consider here a special class of $H_{m,a}$ for which $m=1$ and $a_1 = a_2 = \cdots = a_n = a$ (for any $n \geqq 1$); namely we are concerned with

$$H_{1,a}=\{(z, w_1, \cdots, w_n);\ \mathrm{Im}(z)-\sum_{k=1}^{n}|w_k|^a>0\}.$$

We now put $\Phi(\mathfrak{w})=\sum_{k=1}^{n}|w_k|^a$, for any $\mathfrak{w}=(w_1, \cdots, w_n)$, then $\Phi(\alpha\cdot\mathfrak{w})=|\alpha|^a\Phi(\mathfrak{w})$ for any $\alpha\in \boldsymbol{C}$. Hence, our domain $H_{1,a}$ is a *generalized Siegel domain* in the sense of Kaup, Matsushima and Ochiai [8], as is observed from its special case (B) (see, p. 477 in [8]), and so the original Thullen domain in $\boldsymbol{C}^2$ is holomorphically equivalent to a generalized Siegel domain.

Appendix. Bergmann's kernel functions.

1. As an appendix we shall here calculate the Bergmann's kernel function of the generalized Thullen domain $D_{m,p_1,\cdots,p_n}$; in the special case where $m=n=1$, this has been already done by S. Bergmann himself [2].

Generalized Thullen domains are clearly circular domains (in fact, these are Reinhardt domains). It is well-known, for these domains, the standard way to calculate the Bergmann's kernel functions (see [1]). Namely, let $\Delta_{m,p_1,\cdots,p_n}$ denote the *real representative domain* of $D_{m,p_1,\cdots,p_n}$ which is presented by

$$\Delta_{m,p_1,\cdots,p_n}=\{(x_1, \cdots, x_m, y_1, \cdots, y_n)\in \boldsymbol{R}^{m+n};$$
$$x_i, y_k\geqq 0,\ (1\leqq i\leqq m;\ 1\leqq k\leqq n),\ \sum_{i=1}^{m}x_i^2+\sum_{k=1}^{n}y_k^{2/p_k}<1\}.$$

Then the Bergmann's kernel function $K(\mathfrak{z}, \mathfrak{w})=K(z_1, \cdots, z_m, w_1, \cdots, w_n)$ is:

$$K(\mathfrak{z}, \mathfrak{w})=\sum_{r,s=0}^{\infty}a_{r,s}|z_1|^{2r_1}\cdots|z_m|^{2r_m}|w_1|^{2s_1}\cdots|w_n|^{2s_n},$$

where $r=(r_1, \cdots, r_m)$, $s=(s_1, \cdots, s_n)$ denote integer vectors with $r_i\geqq 0$, $s_k\geqq 0$ and the constant coefficient $a_{r,s}$ are given by

$$a_{r,s}^{-1}=(2\pi)^{m+n}\int_{\Delta}(x_1^{2r_1+1}\cdots x_m^{2r_m+1}y_1^{2s_1+1}\cdots y_n^{2s_n+1})dxdy,$$
$$(dx=dx_1\cdots dx_m,\ dy=dy_1\cdots dy_n).$$

2. Now we put $x_1^{2r_1+1}\cdots x_m^{2r_m+1}=F(r_1, \cdots, r_m; x)$, $y_1^{2s_1+1}\cdots y_n^{2s_n+1}=F(s_1, \cdots, s_n; y)$. Firstly, we have to calculate the integral:

$$I_{r,s}=\int_{\Delta}F(r_1, \cdots, r_m; x)F(s_1, \cdots, s_n; y)dxdy$$
$$=\int_{|x|<1,\, x_i\geqq 0}F(r_1, \cdots, r_m; x)dx\int_{\Sigma y_k^{2/p_k}<1-|x|^2,\, y_k\geqq 0}F(s_1, \cdots, s_n; y)dy,$$

where $|x|=(\sum_{i=1}^{m}x_i^2)^{1/2}$. Then the second factor in the above reduces to $J_n(p, s)(1-|x|^2)^q$, whereby we put

$$J_n(p, s) = \int_{\Sigma y_k^{2/p_k}<1,\, y_k \geqq 0} F(s_1, \cdots, s_n; y)dy,$$

and $q = \sum_{k=1}^{n} q_k$, $q_k = p_k(s_k+1)$ $(1 \leqq k \leqq n)$. We can now write as below:

$$I_{r,s} = J_n(p, s) \cdot J_q(r_1, \cdots, r_m),$$

$$J_q(r_1, \cdots, r_m) = \int_{|x|<1, x_i \geqq 0} F(r_1, \cdots, r_m; n)(1-|x|^2)^q dx.$$

Our problem reduces, therefore, to calculate two integrals $J_n(p, s)$ and $J_q(r_1, \cdots, r_m)$. By a straightforward calculation we have

$$J_n(p, s) = \frac{p_1! \cdots p_n! (q_1-1)! \cdots (q_n-1)!}{2^n(q_1 + \cdots + q_n)!},$$

$$J_q(r_1, \cdots, r_m) = \frac{r_1! \cdots r_m! \, q!}{2^m(N_m+q)!}, \quad N_m = \sum_{i=1}^{m} r_i + m.$$

Hence we get

$$I_{r,s} = \frac{(\prod_{k=1}^{n} p_k)(\prod_{i=1}^{m} r_i!)\{\prod_{k=1}^{n}(q_k-1)!\}}{2^{m+n}(r+q+m)!}; \qquad r = \sum_{i=1}^{m} r_i.$$

Consequently

$$a_{r,s} = \pi^{-(m+n)} \left\{ \frac{(r+q+m)!}{(\prod_{i=1}^{m} r_i!)(\prod_{k=1}^{n}(q_k-1)!\, p_k)} \right\}.$$

3. We are now in a position to calculate the power series:

$$K(\mathfrak{z}, \mathfrak{w}) = \sum_{r,s=0}^{\infty} a_{r,s} x^r y^s; \; x^r = x_1^{r_1} \cdots x_m^{r_m}, \; y^s = y_1^{s_1} \cdots y_n^{s_n},$$

$$(x_i = |z_i|^2, \; y_k = |w_k|^2)$$

$$= \pi^{-(m+n)} \sum_{r,s=0}^{\infty} \left\{ \frac{(r+q+m)!}{(\prod_{i=1}^{m} r_i!)(\prod_{k=1}^{n} p_k(q_k-1)!)} \right\} x^r y^s$$

$$= \pi^{-(m+n)} \left[\sum_{s=0}^{\infty} \frac{y^s}{\prod_{k=1}^{n} p_k(q_k-1)!} \left\{ \sum_{r=0}^{\infty} \frac{(r+q+m)!}{r_1! \cdots r_m!} x^r \right\} \right],$$

where

$$\sum_{r=0}^{\infty} \frac{(r+q+m)!}{r_1! \cdots r_m!} x^r = (m+q)!\,(1 - \sum_{i=1}^{m} x_i)^{-(m+q+1)}.$$

Hence we have

$$K(\mathfrak{z}, \mathfrak{w}) = \pi^{-(m+n)}(\prod_{k=1}^{n} p_k)^{-1} \sum_{s=0}^{\infty} \left\{ \frac{(m+q)!}{\prod_{k=1}^{n}(q_k-1)!} \right\} \left\{ \frac{y_1^{s_1} \cdots y_n^{s_n}}{(1 - \sum_{i=1}^{m} x_i)^{m+q+1}} \right\}.$$

On the other hand, we put $t_k=y_k(1-\sum_{i=1}^{m}x_i)^{-p_k}$ $(1\leqq k\leqq n)$, then $\prod_{k=1}^{n}y_k^{s_k}(1-\sum_{i=1}^{m}x_i)^{-q_k}$ $=(1-\sum_{i=1}^{m}x_i)^{-p}\prod_{k=1}^{n}t_k^{s_k}$ for $p=\sum_{k=1}^{n}p_k$. Hence we obtain:

$$K(\mathfrak{z}, \mathfrak{w})=\pi^{-(m+n)}(\prod_{k=1}^{n}p_k)^{-1}\left\{\sum_{s=0}^{\infty}\frac{(m+q)!}{\prod_{k=1}^{n}(q_k-1)!}\prod_{k=1}^{n}t_k^{s_k}\right\}(1-\sum_{i=1}^{m}x_i)^{-p-m-1}.$$

We note here that

$$(\prod_{k=1}^{n}p_k)^{-1}\left\{\frac{(m+q)!}{\prod_{k=1}^{n}(q_k-1)!}\right\}=\frac{(m+q)!}{q_1!\cdots q_n!}\prod_{k=1}^{n}(s_k+1).$$

Thus to determine $K(\mathfrak{z}, \mathfrak{w})$, it suffices to do so the following power series of n-variables $t_1, \cdots, t_n$:

$$\Phi_m(t_1, \cdots, t_n)=\sum_{s=0}^{\infty}\left\{\frac{(m+q)!(s_1+1)\cdots(s_n+1)}{q_1!\cdots q_n!}\right\}t_1^{s_1}\cdots t_n^{s_n}.$$

This series can be written as

$$\Phi_m(t_1, \cdots, t_n)=\frac{\partial^n}{\partial t_1\cdots\partial t_n}\left[\sum_{s_1,\cdots,s_n=1}\left\{\frac{(m+q)!}{q_1!\cdots q_n!}\right\}t_1^{s_1}\cdots t_n^{s_n}\right],$$

whereby, changing slightly the preceding notation, we designate here as $q_k=p_ks_k$, $q=q_1+\cdots+q_n$. On the other hand, from the formula

$$\sum_{s_1,\cdots,s_n=0}\left\{\frac{(m+s_1+\cdots+s_n)!}{s_1!\cdots s_n!}\right\}\tau_1^{s_1}\cdots\tau_n^{s_n}=m!(1-\sum_{k=1}^{n}\tau_k)^{-m-1},$$

we get

$$\sum_{s_1,\cdots,s_n=1}^{\infty}\left\{\frac{(m+q_1+\cdots+q_n)!}{q_1!\cdots q_n!}\right\}\tau_1^{q_1}\cdots\tau_n^{q_n}=\left(\frac{m!}{p_1\cdots p_n}\right)\sum_{l_k=1}^{p_k}(1-\sum_{k=1}^{n}\zeta_k^{l_k}\tau_k)^{-m-1},$$

where ζ_k denotes a primitive p_k-root of 1[7]. Using this formula, we can now write down the function $K(\mathfrak{z}, \mathfrak{w})$ by using the variables τ_k; $\tau_k^{p_k}=t_k$:

$$(13)\quad \begin{cases} K(\mathfrak{z}, \mathfrak{w})=\pi^{-(m+n)}(1-\sum_{i=1}^{m}|z_i|^2)^{-m-p-1}\Phi_m(t_1, \cdots, t_n), \\ \Phi_m(t_1, \cdots, t_n)=m!(p_1\cdots p_n)^{-1}\frac{\partial^n}{\partial t_1\cdots\partial t_n}[\sum_{l_k=1}^{p_k}(1-\sum_{k=1}^{n}\zeta_k^{l_k}t_k)^{-m-1}]; \end{cases}$$

in this formula we have to replace t_k by $|w_k|^2(1-\sum_{i=1}^{m}|z_i|^2)^{-p_k}$ after differentiation.

4. In the special case where $m=n=1$ $(p_1=p)$, we get from the above result (13) the following formula:

7) For a positive integer $p>1$, we have

$$p\cdot\sum_{s=0}^{\infty}\binom{\alpha}{ps}\tau^{ps}=\sum_{l=1}^{p}(1+\zeta^l\tau)^{\alpha}.$$

$$K(z, w)=\pi^{-2}(1-|z|^2)^{p-2}\{(1-|z|^2)^p-|w|^2\}^{-3}\{(p+1)(1-|z|^2)^p+(p-1)|w|^2\}.$$

This was already obtained in S. Bergmann [2].

References

[1] H. Behnke and P. Thullen, Theorie der Funktionen mehrerer komplexer Veränderlichen, Erg. der Math. Band 3, 1934.

[2] S. Bergmann, Zur Theorie von pseudokonformen Abbildungen, Math. Sbornik. Akad. Nauk SSSR, 1936, 79-96.

[3] A. Blanchard, Sur les variétés analytiques complexes, Ann. Sci. École Norm. Sup., **73** (1956), 157-202.

[4] E. Brieskorn, Über holomorphe P_n-Bündle über P_1, Math. Ann., **157** (1965), 343-357.

[4a] E. Brieskorn, Zur differentialtopologischen und analytischen Klassifizierung gewisser algebraischen Mannigfaltigkeiten, Univ. of Bonn, 1962 (mimeographed).

[5] E. Cartan, Sur la géométrie des pseudo-conforme des hypersurfaces de l'espace de deux variables complexes, I, Oeuvre complètes, Partie II, Volume 2, 1953, 1231-1304.

[6] R. Gunning and H. Rossi, Analytic functions of several complex variables, Prentice-Hall Series in Modern Analysis, 1965.

[7] F. Hirzebruch, Über eine Klasse von einfach-zusammenhängenden komplexen Mannigfaltigkeiten, Math. Ann., **124** (1951), 77-86.

[8] W. Kaup, Y. Matsushima and T. Ochiai, On the automorphisms and equivalences of generalized Siegel domains, Amer. J. Math., **90** (1970), 475-498.

[9] I. Naruki, The holomorphic equivalence problem for a class of Reinhardt domains, Pub. Research Inst. of Math. Sciences, Kyoto Univ., **4** (1968), 527-543.

[10] Pyatetzki-Shapiro, Géométrie des domaines classiques et théorie des fonctions automorphes, Dunod, Paris, 1966.

[11] P. Thullen, Zu den Abbildungen durch analytische Funktionen mehrerer komplexer Veränderlichen. Die Invarianz des Mittelpunktes von Kreiskörper, Math. Ann., **104** (1931), 244-259.

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