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A classification of simple spinnable structures on a 1-connected Alexander manifold

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§ 1. Introduction.

The notion of a spinnable structure on a closed smooth manifold has been introduced by I. Tamura [5] and independently by Winkelnkemper [6] ("open book decomposition" in his term), who obtained necessary and sufficient conditions for existence of it on at least a simply connected closed manifold.

The purpose of the paper is to classify "simple" spinnable structures on a smooth 1-connected closed oriented "Alexander" $(2n+1)$-manifold in terms of their "Seifert matrices".

In the following all things will be considered from the oriented differentiable point of view. A closed oriented $(2n+1)$-manifold is an *Alexander manifold*, if $H_n(M)=H_{n+1}(M)=0$.

In § 2, we shall define a Seifert form $\gamma(\mathcal{S})$ of a simple spinnable structure $\mathcal{S}$ on an Alexander $(2n+1)$-manifold. A matrix $\Gamma(\mathcal{S})$ representing $\gamma(\mathcal{S})$ is called a Seifert matrix. It is shown that $\Gamma(\mathcal{S})$ is unimodular, i.e. $\det \Gamma(\mathcal{S}) = \pm 1$, and determines the intersection matrix of the generator of $\mathcal{S}$ and its n-th monodromy.

The following classification theorem of simple spinnable structures on S^{2n+1} $(n \geqq 3)$ will be proved in §§ 3 and 4.

THEOREM A. *For a unimodular $m \times m$-matrix A, there is a spinnable structure $\mathcal{S}$ on S^{2n+1} with $\Gamma(\mathcal{S})=A$, provided that $n \geqq 3$.*

THEOREM B. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are simple spinnable structures on S^{2n+1} with congruent*) Seifert matrices, then they are isomorphic, provided that $n \geqq 3$**).*

One should notice that Theorem B implies that isolated hypersurface singularities of complex dimension n $(\geqq 3)$ are classified completely by means of Seifert matrices associated with Milnor's spinnable structures.

Based on Theorems A and B, in § 5 we have the following classification theorem of simple spinnable structures on a 1-connected Alexander $(2n+1)$-manifold $(n \geqq 3)$.

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*) Integral matrices A and B are congruent, if there exists a unimodular matrix P such that $A=P^t \cdot B \cdot P$.

**) A. Durfee [7] independently proved Theorems A and B.

THEOREM C. *There is a one to one correspondence of isomorphism classes of simple spinnable structures on a 1-connected Alexander $(2n+1)$-manifold M with congruence classes of unimodular matrices via Seifert matrices, provided that $n \geqq 3$.*

§2. Simple spinnable structures and Seifert forms.

Let F be an m-manifold with boundary ∂F, and $h: F \to F$ a diffeomorphism with $h/U = \text{id.}$ for some open neighborhood U of ∂F in F. Then an $(m+1)$-manifold $T(F, h)$ without boundary is defined as follows; its underlying topological space is obtained from $F \times [0, 1]$ by identifying

$$(x, 1) \text{ with } (h(x), 0) \qquad \text{for all } x \in F$$

and

$$(y, t) \text{ with } (y, 0) \qquad \text{for all } (y, t) \in \partial F \times [0, 1].$$

Note that a part $T(F-\partial F, h/F-\partial F)$ of $T(F, h)$ carries the natural smooth structure as a smooth fiber bundle over S^1 with fiber $F-\partial F$. Taking a small collar $\partial F \times [0, 1)$ of ∂F in $U \subset F$, a coordinate homeomorphism $T(\partial F \times [0, 1), \text{id.}) \to \partial F \times \text{Int}\, D^2$ is defined by sending $(x, s, t) \in (\partial F \times [0, 1)) \times [0, 1]$ to $(x, se^{i2\pi t}) \in \partial F \times \text{Int}\, D^2$. Since those smooth structures are compatible at the intersection, it follows that the smooth manifold $T(F, h)$ is obtained. A *spinnable structure* on a manifold M is a triple $\mathcal{S} = \{F, h, g\}$ which consists of $T(F, h)$ and a diffeomorphism $g: T(F, h) \to M$. The manifold F, the diffeomorphism h and ∂F are called *generator, characteristic diffeomorphism* and *axis* of $\mathcal{S}$, respectively. Spinnable structures $\mathcal{S}$ and $\mathcal{S}'$ on oriented manifolds M and M' are *isomorphic*, if there is an orientation preserving diffeomorphism

$$f: M \longrightarrow M'$$

such that $f \circ g(F \times t) = g'(F' \times t)$ for all $t \in [0, 1]$. By the uniqueness of collar neighborhoods, the isotopy class of a diffeomorphism of F keeping ∂F fixed determines unique isotopy class of a diffeomorphism h of F keeping some open neighborhoods of ∂F in F fixed, which determines unique spinnable structure $\{F, h, \text{id}\}$ on $T(F, h)$ up to isomorphism. Thus, in the following, we shall be concerned with an isotopy class of a characteristic diffeomorphism keeping ∂F fixed. A spinnable structure $\mathcal{S} = \{F, h, g\}$ on an m-manifold M is simple, if F is obtained from a ball by attaching handles of indices $\leqq [m/2]$.

First of all we prove:

PROPOSITION 2.1. *If $\mathcal{S} = \{F, h, g\}$ is a simple spinnable structure on a closed orientable $(2n+1)$-manifold M and $n \geqq 2$, then $g|F \times t: F \times t \to M$ is n-connected, in particular, if $M = S^{2n+1}$, then F is $(n-1)$-connected and hence is of the homotopy type of a bouquet of n-spheres;*

$$F \simeq \bigvee_{i=1}^{m} S_i^n .$$

PROOF. For the proof, putting $F_t = g(F\times t)$, it suffices to show that (M, F_0) is n-connected. We put $W = g(F\times[0, 1/2])$ and $W' = g(F\times[1/2, 1])$. Since $\mathcal{S}$ is simple, it follows from the general position that there is a *PL* embedding $f: K \to \operatorname{Int} W'$ from an n-dimensional compact polyhedron K into $\operatorname{Int} W'$ which is a homotopy equivalence. Since $2n+1 \geqq 5$, $\pi_1(\partial F) \cong \pi_1(F)$ and hence $\partial W' = \partial W$ is a deformation retract of $W' - f(K)$, we have that

$$\begin{aligned}\pi_i(M, F_0) &\cong \pi_i(M, W) = \pi_i(M, M - W') \\ &\cong \pi_i(M, M - f(K)) \\ &= 0 \qquad \text{for} \quad i \leqq n ,\end{aligned}$$

completing the proof.

We shall call a closed oriented $(2n+1)$-manifold M is an *Alexander manifold*, if $H_n(M) = H_{n+1}(M) = 0$. By the Poincaré duality, then $H_{n-1}(M)$ is torsion free and hence if $\mathcal{S}$ is a simple spinnable structure on M, then $H_{n-1}(F)$ and $H_n(F)$ are torsion free. Then a bilinear form, called *Seifert form*;

$$\gamma: H_n(F) \otimes H_n(F) \longrightarrow \boldsymbol{Z}$$

is defined by

$$\gamma(\alpha \otimes \beta) = L(g_\#(\alpha \times t_0), g_\#(\beta \times t_1)) ,$$

where $0 \leqq t_0 < 1/2$, $1/2 \leqq t_1 < 1$, and $L(\xi, \eta)$ stands for the linking number of cycles ξ and η in M so that $L(\xi, \eta) =$ intersection number $\langle \lambda, \eta \rangle$ of chains λ and η in M for some λ with $\partial\lambda = \xi$.

For a basis $\alpha_1, \cdots, \alpha_m$ of a free abelian group $H_n(F)$, a square matrix $(\gamma(\alpha_i \otimes \alpha_j)) = (\gamma_{ij})$ will be called a *Seifert matrix* of $\mathcal{S}$ and denoted by $\Gamma(\mathcal{S})$. It is a routine work to make sure that the congruence class of $\Gamma(\mathcal{S})$ is invariant under the isomorphism class of $(M, \mathcal{S})$. Namely, if $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, then there is a unimodular matrix A such that $A^t \Gamma(\mathcal{S}) A = \Gamma(\mathcal{S}')$.

We have an alternative expression of $\Gamma(\mathcal{S})$ in terms of an isomorphism

$$a: H_n(W) \overset{\partial^{-1}}{\cong} H_{n+1}(M, W) \overset{\mathrm{exc}^{-1}}{\cong} H_{n+1}(W', \partial W') \overset{\mathrm{P.}}{\cong} H^n(W') \overset{\mathrm{D.}}{\cong} H_n(W')$$

which will be called the *Alexander isomorphism*, where P is the Poincaré duality isomorphism and D is the dual isomorphism.

We have homomorphisms

$$\varphi: H_n(W) \overset{\partial^{-1}}{\cong} H_{n+1}(M, W) \overset{\mathrm{exc}^{-1}}{\cong} H_{n+1}(W', \partial W) \overset{\partial}{\longrightarrow} H_n(\partial W)$$

and

$$\varphi': H_n(W') \cong H_{n+1}(M, W') \cong H_{n+1}(W, \partial W) \longrightarrow H_n(\partial W)$$

so that $i_* \circ \varphi = \mathrm{id}.$ and $i'_* \circ \varphi'_* = \mathrm{id}.$ and the following sequences are exact:

$$0 \longrightarrow H_n(W') \xrightarrow{\varphi'} H_n(\partial W) \xrightarrow{i_*} H_n(W) \longrightarrow 0\,,$$

$$0 \longrightarrow H_n(W) \xrightarrow{\varphi} H_n(\partial W) \xrightarrow{i'_*} H_n(W') \longrightarrow 0\,,$$

where $i_*: H_n(\partial W) \to H_n(W)$ and $i'_*: H_n(\partial W) \to H_n(W')$ are homomorphisms induced from the inclusion maps. Let $\alpha_1, \cdots, \alpha_m$ be a basis of $H_n(W)$. Then, putting $\beta_i = a(\alpha_i)$, $i=1, \cdots, m$, we have a basis $\beta_1, \cdots, \beta_m$ of $H_n(W')$. By the definition of the Alexander isomorphism, if we put $\bar{\alpha}_i = \varphi(\alpha_i)$ and $\bar{\beta}_i = \varphi'(\beta_i)$, $i=1, \cdots, m$, then we have that the intersection number in ∂W

$$\langle \bar{\alpha}_i, \bar{\beta}_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{for } i \neq j\,, \\ 1 & \text{for } i = j\,. \end{cases}$$

Let $g_t: F \to M$ be an embedding defined by

$$g_t(x) = g(x, t) \qquad \text{for all} \quad x \in F,\ t \in [0, 1]\,.$$

For a subspace X of M with $g_t(F) \subset X$, we denote the range restriction of g_t to X by $X|g_t: F \to X$;

$$X|g_t(x) = g_t(x) \qquad \text{for all} \quad x \in F\,.$$

We identify a basis $\alpha_1, \cdots, \alpha_m$ of $H_n(W)$ with that of $H_n(F)$ via $(W|g_{1/3})_*$ and a basis $\beta_1, \cdots, \beta_m$ of $H_n(W')$ with that of $H_n(F)$ via $(W|g_{2/3})_*$.

Again by the definition of the Alexander isomorphism, we have that

$$L(\alpha_i, \beta_j) = \delta_{ij} \qquad \text{for} \quad i, j = 1, \cdots, m\,.$$

Since $W|g_{1/3}$ and $W|g_{1/2} = i \circ (\partial W|g_{1/2})$ are homotopic in W and $W'|g_{2/3}$ and $W'|g_{1/2} = i' \circ (\partial W|g_{1/2})$ are homotopic in W', it follows that $(\partial W|g_{1/2})_*(\alpha_i)$ is of a form

$$(\partial W|g_{1/2})_*(\alpha_i) = \bar{\alpha}_i + \sum_{j=1}^{m} a_{ij} \bar{\beta}_j$$

and hence that $(W'|g_{2/3})_*(\alpha_i) = \sum_{j=1}^{m} a_{ij}\beta_j = \sum_{j=1}^{m} a_{ij} a(\alpha_j)$. Therefore, we have that $\gamma_{ij} = L((g_{1/3})_\# \alpha_i, (g_{2/3})_\# \alpha_j) = L(\alpha_i, \sum a_{jk}\beta_k) = a_{ji}$ for $i, j = 1, \cdots, m$. Thus we conclude as follows:

PROPOSITION 2.2. *For a basis* $\alpha_1, \cdots, \alpha_m$ *of* $H_n(F) \overset{(W|g_{1/3})_*}{\cong} H_n(W)$, *the following* (1), (2) *and* (3) *are equivalent.*

(1) $(\partial W|g_{1/2})_*(\alpha_i) = \bar{\alpha}_i + \sum_{j=1}^{m} a_{ij} \bar{\beta}_j$,

(2) $a^{-1} \circ (W'|g_{2/3})_*(\alpha_i) = \sum_{j=1}^{m} a_{ij} \alpha_j$

and

(3) $\Gamma^t = (a_{ij})$.

In particular, the Seifert matrix Γ is unimodular.

Now we determine algebraic structures of simple spinnable structures on an Alexander manifold.

THEOREM 2.3. *Let $\mathcal{S}=\{F, h, g\}$ be a simple spinnable structure on an Alexander manifold M^{2n+1}.*

(1) *The intersection matrix $I=I(F)$ of F and the Seifert matrix $\Gamma=\Gamma(\mathcal{S})$ of $\mathcal{S}$ are related in a formula:*

$$-I=\Gamma+(-1)^n\Gamma^t$$

where Γ^t is the transposed matrix of Γ.

(2) *The n-th monodromy $h_*: H_n(F)\to H_n(F)$ is given by a formula:*

$$h_*=(-1)^{n+1}\Gamma^t\cdot\Gamma^{-1}$$

or

$$h_*-E=(-1)^n I\cdot\Gamma^{-1}\,, \qquad \text{where } E \text{ is the identity matrix.}$$

PROOF. For the proof of (1), we follow Levine [**3**], p. 542. We take chains $d=(-1)^n g_\#(\alpha_i\times[1/3, 2/3])$, e_1 and e_2 in M such that

$$\partial d=g_\#(\alpha_i\times 2/3)-g_\#(\alpha_i\times 1/3)=(g_{2/3})_\#(\alpha_i)-(g_{1/3})_\#(\alpha_i)\,,$$

$$\partial e_1=-(g_{2/3})_\#(\alpha_i)$$

and

$$\partial e_2=(g_{1/3})_\#(\alpha_i)\,.$$

Since $d+e_1+e_2$ is a cycle, we have that

$$\begin{aligned}0&=\langle d+e_1+e_2, (g_{1/2})_\#(\alpha_j)\rangle\\&=\langle d, (g_{1/2})_\#(\alpha_j)\rangle+\langle e_1, (g_{1/2})_\#(\alpha_j)\rangle+\langle e_2, (g_{1/2})_\#(\alpha_j)\rangle\\&=\langle\alpha_i, \alpha_j\rangle+(-1)L((g_{2/3})_\#(\alpha_i), (g_{1/2})_\#\alpha_j)+L((g_{1/3})_\#(\alpha_i), (g_{1/2})_\#\alpha_j)\,.\end{aligned}$$

Since

$$\begin{aligned}L((g_{2/3})_\#(\alpha_i), (g_{1/2})_\#(\alpha_j))&=(-1)^{n+1}L((g_{1/2})_\#(\alpha_j), (g_{2/3})_\#(\alpha_i))\\&=(-1)^{n+1}\gamma(\alpha_j\otimes\alpha_i)\end{aligned}$$

and

$$L((g_{1/3})_\#(\alpha_i), (g_{1/2})_\#(\alpha_j))=\gamma(\alpha_i\otimes\alpha_j)\,,$$

we have that

$$-I=\Gamma+(-1)^n\Gamma^t\,,$$

completing the proof of (1). To prove (2), we take chains $d=(-1)^n g_\#(\alpha_i\times[0,1])$, e_0 and e_1 in M so that $\partial d=g_{1\#}(\alpha_i)-g_{0\#}(\alpha_i)$, $\partial e_0=g_{0\#}(\alpha_i)$ and $\partial e_1=-g_{1\#}(\alpha_i)$ $=-g_{0\#}(h_*(\alpha_i))$. Since $d+e_0+e_1$ is an $(n+1)$-cycle in M, we have that

$$
\begin{aligned}
0 &= \langle d+e_0+e_1, (g_{1/2})_\#(\alpha_j)\rangle \\
&= \langle d, (g_{1/2})_\#(\alpha_j)\rangle + \langle e_0, (g_{1/2})_\#(\alpha_j)\rangle + \langle e_1, (g_{1/2})_\#(\alpha_j)\rangle \\
&= \langle \alpha_i, \alpha_j\rangle + L(g_{0\#}(\alpha_i), (g_{1/2})_\#(\alpha_j)) + (-1)L(g_{0\#}(h_*(\alpha_i)), (g_{1/2})_\#(\alpha_j)) \\
&= \langle \alpha_i, \alpha_j\rangle + \gamma(\alpha_i \otimes \alpha_j) - \gamma(h_*(\alpha_i)\otimes \alpha_j) \\
&= \langle \alpha_i, \alpha_j\rangle + \gamma((\mathrm{id}-h_*)(\alpha_i)\otimes \alpha_j)
\end{aligned}
$$

and hence that

$$-I = (E-h_*)\cdot \Gamma ,$$

where E is the identity matrix (δ_{ij}). Therefore, by making use of (1), we have that

$$
\begin{aligned}
(h_*-E) &= I\cdot \Gamma^{-1} \\
&= -E + (-1)^{n+1}\Gamma^t \cdot \Gamma^{-1} ,
\end{aligned}
$$

or

$$h_* = (-1)^{n+1}\Gamma^t \cdot \Gamma^{-1} ,$$

completing the proof.

§ 3. Proof of Theorem A.

Suppose that we are given an $m\times m$ unimodular matrix $A=(a_{ij})$. Let K denote a bouquet of m n-dimensional spheres; $K=\bigvee_{i=1}^{m} S_i^n$. We have a *PL* embedding $f\colon K\to S^{2n+1}$. Let W be a smooth regular neighborhood of $f(K)$ in $S^{2n+1}=S$ and $W'=S-\mathrm{Int}\, W$. We denote the Alexander isomorphism

$$H_n(W) \cong H^n(S-\mathrm{Int}\, W) = H^n(W') = \mathrm{Hom}\,(H_n(W')) \cong H_n(W')$$

by $a\colon H_n(W)\cong H_n(W')$. Thus we have that W, W' and ∂W are $(n-1)$-connected, and there are splittings

$$
\begin{aligned}
\varphi\colon\ & H_n(W) \cong H_{n+1}(S, W) \cong H_{n+1}(W', \partial W) \longrightarrow H_n(\partial W) , \\
\varphi'\colon\ & H_n(W') \cong H_{n+1}(S, W') \cong H_{n+1}(W, \partial W) \longrightarrow H_n(\partial W)
\end{aligned}
$$

of $i_*\colon H_n(\partial W)\to H_n(W)$ and $i'_*\colon H_n(\partial W)\to H_n(W')$, respectively. Note that the following sequences are exact.

$$0 \longrightarrow H_n(W) \xrightarrow{\ \varphi\ } H_n(\partial W) \xrightarrow{\ i'_*\ } H_n(W') \longrightarrow 0$$

and

$$0 \longrightarrow H_n(W') \xrightarrow{\ \varphi\ } H_n(\partial W) \longrightarrow H_n(W) \longrightarrow 0 .$$

If $\alpha_1, \cdots, \alpha_m$ is a basis of $H_n(K)\cong H_n(W)$ represented by $S_1^n, \cdots, S_m^n$ and we put $a(\alpha_i)=\beta_i$, $\varphi(\alpha_i)=\bar{\alpha}_i$, and $\varphi(\beta_i)=\bar{\beta}_i$, $i=1, \cdots, m$, then we have that the intersection numbers in ∂W $\langle\bar{\alpha}_i, \bar{\alpha}_j\rangle=0$, $\langle\bar{\beta}_i, \bar{\beta}_j\rangle=0$ and $\langle\bar{\alpha}_i, \bar{\beta}_j\rangle=\delta_{ij}$ for $i, j=1, \cdots, m$, and the linking numbers in S $L(\alpha_i, \beta_j)=\delta_{ij}$, $i, j=1, \cdots, m$.

A splitting $s: H_n(W)\to H_n(\partial W)$ of $i_*: H_n(\partial W)\to H_n(W)$ will be called a *non-singular section*, if $i'_*\circ s: H_n(W)\to H_n(W')$ is an isomorphism. Indeed, a section $s: H_n(W)\to H_n(\partial W)$ has to be of a form

$$s(\alpha_i)=\bar{\alpha}_i+\sum_{j=1}^{m} a_{ij}\bar{\beta}_j$$

and hence $i'_*\circ s(\alpha_i)=\sum_{j=1}^{m} a_{ij}\beta_j$. Thus the correspondence $s\mapsto(a_{ij})$ gives rise to a one to one correspondence of non-singular sections $H_n(W)\to H_n(\partial W)$ with unimodular $m\times m$ matrices (a_{ij}). As is found by Winkelnkemper [6] and also Tamura [4] for a non-singular section $s: H_n(W)\to H_n(\partial W)$, there is a PL embedding $f': K^n\to\partial W$, provided that $n\geqq 3$, which is homotopic to $f: K\to W$ and $f'_*(\alpha_i)=s(\alpha_i)$ in ∂W. Moreover, if F is a regular neighborhood of $f'(K)$ in ∂W and $F'=\partial W-\operatorname{Int} F$, then $(W; F, F')$ and $(W'; F', F)$ are relative h-cobordisms, since $s(\alpha_1), \cdots, s(\alpha_m)$ is a basis of $H_n(F)$ as a subgroup of $H_n(\partial W)$ and the inclusion maps induce isomorphisms

$$j_*:\ H_n(F)\cong H_n(W); \qquad j_*(s(\alpha_i))=\alpha_i$$

and

$$j_*:\ H_n(F)\cong H_n(W'); \qquad j'_*(s(\alpha_i))=i'_*\circ s(\alpha_i)=\sum_{j=1}^{m} a_{ij}\beta_j$$

and W, W', F, F' are 1-connected.

It follows that by the h-cobordism theorem, S^{2n+1} admits a spinnable structure $\mathcal{S}_A=\{F, h, g\}$ for a given unimodular matrix $A=(a_{ij})$ such that

$$g(F\times[0, 1/2])=W,$$

$$g(F\times[1/2, 1])=W'$$

and

$$g(x, 1/2) \qquad \text{for all} \quad x\in F.$$

We would like to show that $\Gamma(\mathcal{S}_A)=A^t$. We have seen that $(\partial W|g_{1/2})_*(\alpha_i)$ $=s(\alpha_i)=\bar{\alpha}_i+\sum_{j=1}^{m} a_{ij}\beta_j$. It follows from Proposition 2.2 that $\Gamma(\mathcal{S}_A)=A^t$. Therefore, for a given unimodular matrix A, $\mathcal{S}_{At}$ is the required spinnable structure on S^{2n+1}, completing the proof.

§ 4. Proof of Theorem B.

The crux of the proof of Theorem B is due to J. Levine [2], who proved essentially the following:

PROPOSITION 4.1 (Levine). *Let $\mathcal{S}=\{F, h, g\}$ and $\mathcal{S}'=\{F', h', g'\}$ be spinnable structures on S^{2n+1}. Suppose that $n\geqq 3$. Then two generators F_0 and F'_0 are ambient isotopic in S^{2n+2} if $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{S}')$ are congruent.*

PROOF. By a suitable change of bases, we may assume that $\Gamma(\mathcal{S})=\Gamma(\mathcal{S}')$. The rest of the proof is what Levine has done in his classification of simple knots (Lemma 3, [2], § 14—§ 16, pp. 191-192). His arguments work equally well in our case, completing the proof.

Thus we have a diffeomorphism $f\colon S^{2n+1}\to S^{2n+1}$ such that $f(F_0)=F'_0$, and f is diffeotopic to the identity. By opening out the spinnable structure, we have a diffeomorphism $H\colon F\times[0,1]\to F'\times[0,1]$ such that

$$H(x,0)=(k(x),t) \qquad \text{for all} \quad (x,t)\in\partial F\times[0,1]$$

$$H(x,0)=(k(x),0) \qquad \text{for all} \quad x\in F$$

and

$$H(x,1)=(h'^{-1}\circ k\circ h(x),1) \qquad \text{for all} \quad x\in F,$$

where

$$(k(x),0)=(g')^{-1}\circ f\circ g(x,0) \qquad \text{for all} \quad x\in F.$$

This implies that $(k^{-1}\times\mathrm{id})\circ H\colon F\times[0,1]\to F\times[0,1]$ is an pseudo-diffeotopy from id to $k^{-1}\circ h'^{-1}\circ k\circ h$ keeping ∂F fixed. Since $n\geqq 3$, F and ∂F are 1-connected, it follows from Cerf [1] that the pseudo-diffeotopy is diffeotopic to a diffeotopy $G\colon F\times I\to F\times I$ keeping $\partial(F\times I)$ fixed. This implies that f is diffeotopic to an isomorphism $(S^{2n+1},\mathcal{S})\to(S^{2n+1},\mathcal{S}')$ keeping F_0 fixed. Therefore, $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, completing the proof.

REMARK. As is known from the proof, $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic by an ambient diffeotopy.

§ 5. Proof of Theorem C.

Let M be a 1-connected closed Alexander $(2n+1)$-manifold. A simple spinnable structure $\mathcal{S}=\{F,h,g\}$ on M is *canonical*, if $H^n(F)=0$, that is, F is of the homotopy type of a finite CW-complex of dimension $n-1$. A canonical simple spinnable structure on M is " canonical " in the following sense:

THEOREM D. *There exist canonical simple spinnable structures on a* 1-*connected closed Alexander* $(2n+1)$-*manifold which are unique up to ambient isotopy, provided that* $n\geqq 3$.

PROOF. The existence is proved by the arguments of Winkelnkemper [6] together with the condition that $H^n(M)=0$. The uniqueness is proved by easy isotopy arguments making use of simple engulfing and the h-cobordism theorem for matching generators together with the arguments in the proof of Theorem B, completing the proof of Theorem D.

For simple spinnable structures $\mathcal{S}_1$ and $\mathcal{S}_2$ on Alexander $(2n+1)$-manifolds M_1 and M_2, we have a connected sum $\mathcal{S}_1\#\mathcal{S}_2$ which is simple on an Alexander manifold $M_1\# M_2$. Then we have that the Seifert form $\gamma(\mathcal{S}_1\#\mathcal{S}_2)$ is a direct sum $\gamma(\mathcal{S}_1)\oplus\gamma(\mathcal{S}_2)$. Let $\mathcal{S}_0$ be the canonical simple spinnable structure on a

1-connected Alexander $(2n+1)$-manifold M. If $\mathcal{S}_1$ is a simple spinnable structure on S^{2n+1}, then a connected sum $\mathcal{S}_0 \# \mathcal{S}_1$ is regarded as a simple spinnable structure on M and $\gamma(\mathcal{S}_0 \# \mathcal{S}_1) = \gamma(\mathcal{S}_1)$. This implies that any unimodular matrix can be realized as a Seifert matrix of a simple spinnable structure on M. Further, we have the following decomposition theorem:

THEOREM E (Unique decomposition theorem). *Let M be a 1-connected Alexander $(2n+1)$-manifold with a canonical simple spinnable structure $\mathcal{S}_0$.*

Suppose that $n \geqq 3$.

(Existence) *For a simple spinnable structure $\mathcal{S}$ on M there is a simple spinnable structure $\mathcal{S}_1$ on S^{2n+1} so that $\mathcal{S}$ is isomorphic with $\mathcal{S}_0 \# \mathcal{S}_1$.*

(Uniqueness) *If $\mathcal{S}_0 \# \mathcal{S}_2$ is a second decomposition of $\mathcal{S}$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic.*

PROOF OF THEOREM E. The uniqueness follows from the fact that $\gamma(\mathcal{S}_0 \# \mathcal{S}_1) = \gamma(\mathcal{S}_1)$ and $\gamma(\mathcal{S}_0 \# \mathcal{S}_2) = \gamma(\mathcal{S}_2)$ together with Theorem B. The existence follows from the following together with Theorem D:

LEMMA 5.1. *Let F be a generator of a simple spinnable structure on a 1-connected Alexander $(2n+1)$-manifold.*

Suppose that $n \geqq 3$.

(I) *Then F is diffeomorphic with a boundary connected sum $F_0 \natural F_1$, where F_0 is of the homotopy type of a finite CW-complex of dimension $n-1$ and F_1 is of the homotopy type of a bouquet of n-spheres.*

(II) *A diffeomorphism $h: F \to F$ with $h/\partial F = \mathrm{id}.$ is diffeotopic to a diffeomorphism $h': F \to F$ keeping ∂F fixed such that $h'(F_0) = F_0$, $h'(F_1) = F_1$ and $h'/D^{2n-1} = \mathrm{id}.$, where $D^{2n-1} = F_0 \cap F_1$.*

OUTLINE OF THE PROOF OF LEMMA 5.1. Observe that F is homotopy equivalent to a polyhedron K obtained from a finite CW-complex of dimension $n-1$ and a bouquet of n-spheres by connecting them an arc. By the embedding arguments and the h-cobordism theorem, we can realize K as a spine of F, which implies the conclusion (I). For the proof of (II), we take a mapping cylinder of $h: F \to F$. By making use of the relative h-cobordism theorem on the submapping cylinders of $h/F_0: F_0 \to h(F_0)$ and $h/F_1: F_1 \to h(F_1)$, we have a pseudo-isotopy from h to $h_1: F \to F$ keeping ∂F fixed such that $h_1(F_0) = F_0$ and $h_1(F_1) = F_1$. In particular, we have that $h_1(D^{2n-1}) = D^{2n-1}$ and $h_1/\partial D^{2n-1} = \mathrm{id}.$, and hence $h_2 = h_1/D^{2n-1}$ determines an element α of Γ_{2n}. If we put $\Sigma = T(D^{2n-1}, h_2)$, then Σ is a homotopy $2n$-sphere representing α. The homotopy sphere Σ separates M into two parts. Let Δ be a part containing F_1. Since the inclusion map $F_0 \subset M - \mathrm{Int}\,\Delta$ is n-connected and $\Sigma = \partial\Delta$ is a homotopy $2n$-sphere, it follows that Δ is contractible, and hence Σ is a $2n$-sphere. This implies that h_1/D^{2n-1} is pseudo-isotopic to the identity keeping ∂D^{2n-1} fixed. Thus we may assume that h is pseudo-isotopic to $h': F \to F$ keeping ∂F fixed such that $h'(F_0)=F_0$, $h'(F_1)=F_1$ and $h'/D^{2n-1}=\mathrm{id}$. By Cerf's theorem, h and h' are actually isotopic keeping ∂F fixed, completing the proof.

PROOF OF THEOREM C. Theorem C is an easy consequence of Theorems A, B, D and E, completing the proof.

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