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On deformations of holomorphic maps II

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This paper is the second part of a study of deformations of holomorphic maps. In the first part [4], which will be referred to as Part I, we have proved two fundamental theorems on deformations of non-degenerate holomorphic maps. In § 4, we shall generalize these two theorems to the case in which the holomorphic maps in consideration are not necessarily non-degenerate.

In §§ 5, 6, we shall study deformations of holomorphic maps in the sense iii) in the introduction of Part I. Namely, we fix a family $q:\mathcal{Y}\rightarrow S$ of deformations of complex manifolds, and study deformations of holomorphic maps into the family $q:\mathcal{Y}\rightarrow S$. We shall prove two fundamental theorems and a theorem of stability.

Finally in § 7, we shall study deformations of compositions of holomorphic maps.

Some of the results were announced in [2] and [3].

An application was reported in [3]. Details will appear in [5].

We shall employ the notation of Part I.

§ 4. Deformations of holomorphic maps (general case).

Let Y be a complex manifold. A family $(\mathcal{X}, \Phi, p, M)$ of holomorphic maps into Y consists of a family $p:\mathcal{X}\rightarrow M$ of compact complex manifolds and a holomorphic map $\Phi:\mathcal{X}\rightarrow Y\times M$ such that $pr_2\circ\Phi=p$, where pr_2 denotes the projection onto the second factor (see Definition 1.1, in § 1, Part I). Let 0 be a point on M, $X=X_0$, and let $f=\Phi_0: X\rightarrow Y$ be the holomorphic map induced by Φ. Letting Θ_X and Θ_Y denote the sheaf of germs of holomorphic vector fields on X and Y, respectively, we have a canonical homomorphism $F:\Theta_X\rightarrow f^*\Theta_Y$.

Let $\mathfrak{U}=\{U_i\}$ be a finite Stein covering of X. For any sheaf $\mathcal{F}$ on X, we let $C^q(\mathfrak{U}, \mathcal{F})$ and $\mathcal{Z}^q(\mathfrak{U}, \mathcal{F})$ denote, respectively, the group of q-cochains and the group of q-cocycles with coefficients in $\mathcal{F}$ with respect to the covering $\mathfrak{U}$. We define the coboundary map $\delta: C^q(\mathfrak{U}, \mathcal{F})\rightarrow C^{q+1}(\mathfrak{U}, \mathcal{F})$ as usual (see [1], II. 5.1, for example).

DEFINITION 4.1. We set

$$D_{X/Y}=\frac{\{(\tau,\rho)\in C^0(\mathfrak{U},f^*\Theta_Y)\times Z^1(\mathfrak{U},\Theta_X):\delta\tau=F\rho\}}{\{(Fg,\delta g):g\in C^0(\mathfrak{U},\Theta_X)\}}.$$

LEMMA 4.2. (1) *$D_{X/Y}$ does not depend on the choice of the Stein covering.*

(2) *$D_{X/Y}$ is a finite dimensional vector space.*

(3) *We have two exact sequences:*

$$H^0(X,\Theta_X)\xrightarrow{F}H^0(X,f^*\Theta_Y)\longrightarrow D_{X/Y}\longrightarrow H^1(X,\Theta_X)\xrightarrow{F}H^1(X,f^*\Theta_Y),\tag{4.1}$$

$$0\longrightarrow H^1(X,\Theta_{X/Y})\longrightarrow D_{X/Y}\longrightarrow H^0(X,\mathcal{T}_{X/Y})\longrightarrow H^2(X,\Theta_{X/Y}),\tag{4.2}$$

where $\Theta_{X/Y}$ denotes the sheaf of germs of relative vector fields on X over Y and $\mathcal{T}_{X/Y}$ denotes the cokernel of $F:\Theta_X\to f^*\Theta_Y$.

PROOF. We define a homomorphism $D_{X/Y}\to H^1(X,\Theta_X)$ by sending the class of (τ,ρ) in $D_{X/Y}$ into the cohomology class of ρ. Similarly, we define a homomorphism $H^0(X,f^*\Theta_Y)\to D_{X/Y}$ by sending $\tau\in H^0(X,f^*\Theta_Y)\cong Z^0(\mathfrak{U},f^*\Theta_Y)$ into the class of $(\tau,0)$. We can easily check that the sequence (4.1) is exact.

By the five lemma, it follows that $D_{X/Y}$ does not depend on the choice of the Stein covering. The second assertion also follows from the exact sequence (4.1).

It remains to define the exact sequence (4.2). First note that we have an exact sequence

$$0\longrightarrow\Theta_{X/Y}\xrightarrow{J}\Theta_X\xrightarrow{F}f^*\Theta_Y\xrightarrow{P}\mathcal{T}_{X/Y}\longrightarrow 0.$$

With any $\phi\in Z^1(\mathfrak{U},\Theta_{X/Y})$ we associate the class of $(0,J\phi)$ in $D_{X/Y}$. This defines the first homomorphism.

For any element of $D_{X/Y}$, we take a representative (τ,ρ), $\tau=\{\tau_i\}$. Then the collection $\{P\tau_i\}$ represents an element of $H^0(X,\mathcal{T}_{X/Y})$. This correspondence defines the second homomorphism.

Finally, any element of $H^0(X,\mathcal{T}_{X/Y})$ is represented by $\tau\in C^0(\mathfrak{U},f^*\Theta_Y)$ such that $\delta\tau=F\rho$ with some $\rho\in C^1(\mathfrak{U},\Theta_X)$. Since we have $F(\delta\rho)=\delta(\delta\tau)=0$, $\delta\rho$ can be considered as a 2-cocycle with coefficients in $\Theta_{X/Y}$. This defines the third homomorphism.

It is easy to check that (4.2) is exact. Q.E.D.

COROLLARY. (1) *If f is non-degenerate, we have $D_{X/Y}\cong H^0(X,\mathcal{T}_{X/Y})$.*

(2) *If f is smooth, we have $D_{X/Y}\cong H^1(X,\Theta_{X/Y})$.*

Now we shall define a characteristic map $\tau:T_0(M)\to D_{X/Y}$, where $T_0(M)$ denotes the tangent space of M at 0. Restricting M to a neighborhood of 0 if necessary, we may assume the following:

i) M is an open set in $\boldsymbol{C}^r$ with coordinates $t=(t_1,\cdots,t_r)$ and $0=(0,\cdots,0)$

ii) $\mathcal{X}$ is covered by a finite number of Stein coordinate neighborhoods $\mathcal{U}_i$. Each $\mathcal{U}_i$ is covered by a system of coordinates $(z_i, t)=(z_i^1, \cdots, z_i^n, t_1, \cdots, t_r)$ such that $p(z_i, t)=t$.

iii) $\Phi(\mathcal{U}_i)$ is contained in some coordinate neighborhood $V_i\times M$. Each V_i is covered by a system of coordinates $w_i=(w_i^1, \cdots, w_i^m)$.

iv) Φ is given by $w_i=\Phi_i(z_i, t)$.

v) $(z_i, t)\in\mathcal{U}_i$ coincides with $(z_j, t)\in\mathcal{U}_j$ if and only if $z_i=\phi_{ij}(z_j, t)$.

vi) $w_i\in V_i$ coincides with $w_j\in V_j$ if and only if $w_i=\psi_{ij}(w_j)$.

Then we have

$$\Phi_i(\phi_{ij}(z_j, t), t)=\psi_{ij}(\Phi_j(z_j, t)), \tag{4.3}$$

$$\phi_{ij}(\phi_{jk}(z_k, t), t)=\phi_{ik}(z_k, t). \tag{4.4}$$

We let $U_i=X\cap\mathcal{U}_i$ and let $\mathfrak{U}$ denote the covering $\{U_i\}$ of X. For any element $\frac{\partial}{\partial t}\in T_0(M)$, let

$$\begin{aligned}\tau_i&=\sum_\beta \frac{\partial\Phi_i^\beta}{\partial t}\Big|_{t=0}\frac{\partial}{\partial w_i^\beta}\in\Gamma(U_i, f^*\Theta_Y),\\ \rho_{ij}&=\sum_\alpha \frac{\partial\phi_{ij}^\alpha}{\partial t}\Big|_{t=0}\frac{\partial}{\partial z_i^\alpha}\in\Gamma(U_{ij}, \Theta_X).\end{aligned} \tag{4.5}$$

From the equalities (4.3) and (4.4), we infer that

$$\tau_j-\tau_i=F\rho_{ij}, \qquad \text{on} \quad U_{ij},$$

$$\rho_{jk}-\rho_{ik}+\rho_{ij}=0, \qquad \text{on} \quad U_{ijk}.$$

Hence the pair of $\tau=\{\tau_i\}\in C^0(\mathfrak{U}, f^*\Theta_Y)$ and $\rho=\{\rho_{ij}\}\in\mathcal{Z}^1(\mathfrak{U}, \Theta_X)$ represents an element of $D_{X/Y}$. Thus we define a linear map $\tau: T_0(M)\to D_{X/Y}$. τ is independent of the choice of the systems of coordinates. We shall call τ *the characteristic map* of the family at 0.

THEOREM 4.3. *Let $(\mathcal{X}, \Phi, p, M)$ be a family of holomorphic maps into Y, $0\in M$, $X=X_0$, and $f=\Phi_0: X\to Y$. If the characteristic map $\tau: T_0(M)\to D_{X/Y}$ is surjective, then the family is complete at* 0. (See Definition 1.2 in Part I.)

THEOREM 4.4. *Let $f: X\to Y$ be a holomorphic map of a compact complex manifold X into a complex manifold Y. Assume that the canonical homomorphism $F:\Theta_X\to f^*\Theta_Y$ satisfies the following conditions:*

i) *$F: H^1(X, \Theta_X)\to H^1(X, f^*\Theta_Y)$ is surjective,*

ii) *$F: H^2(X, \Theta_X)\to H^2(X, f^*\Theta_Y)$ is injective.*

Then there exist a family $(\mathcal{X}, \Phi, p, M)$ of holomorphic maps into Y and a point $0\in M$ such that

1) *$X=p^{-1}(0)$ and $\Phi_0: X\to Y$ coincides with $f: X\to Y$,*

2) *$\tau: T_0(M)\to D_{X/Y}$ is bijective.*

REMARK. Theorem 4.3 and Theorem 4.4 are generalizations of Theorem 2.1 and Theorem 3.1 in Part I, respectively. Moreover Theorem 4.4 is an improvement of Theorem 2′ which was stated in [2]. In fact, the vanishing of $H^1(X, \mathfrak{T}_{X/Y})$ and $H^2(X, \Theta_{X/Y})$ implies two conditions i) and ii) in Theorem 4.4.

Proof of Theorem 4.3 is similar to that of Theorem 2.1 in Part I. In fact, it suffices to modify Lemma 2.2 as follows. Let $(t_1, \cdots, t_r)$ be a system of coordinates on M with center at 0. Using the previous notation, $\tau(\partial/\partial t_\lambda)$ is represented by a pair $(\{\tau_{\lambda i}\}, \{\rho_{\lambda ij}\}) \in C^0(\mathfrak{U}, f^*\Theta_Y) \times \mathcal{Z}^1(\mathfrak{U}, \Theta_X)$, where $\tau_{\lambda i}$ and $\rho_{\lambda ij}$ are defined by (4.5) with the aid of $\partial/\partial t_\lambda$. By hypothesis, the classes of $(\{\tau_{\lambda i}\}, \{\rho_{\lambda ij}\})$, $\lambda = 1, \cdots, r$, generate $D_{X/Y}$. In view of the equalities (2.7), (2.8), (2.9) and (2.10), we replace Lemma 2.2 by the following trivial lemma:

LEMMA 4.5. *Suppose that* $\gamma = \{\gamma_i\} \in C^0(\mathfrak{U}, f^*\Theta_Y)$ *and* $\Gamma = \{\Gamma_{ij}\} \in \mathcal{Z}^1(\mathfrak{U}, \Theta_X)$ *satisfying* $\delta\gamma = F\Gamma$ *are given. Then we can find* $t = (t_\lambda) \in \boldsymbol{C}^r$ *and* $g = \{g_i\} \in C^0(\mathfrak{U}, \Theta_X)$ *such that*

$$\Gamma_{ij} = g_j - g_i + \sum_\lambda t_\lambda \rho_{\lambda ij}, \qquad on \quad U_{ij},$$

$$\gamma_i = F g_i + \sum_\lambda t_\lambda \tau_{\lambda i}, \qquad on \quad U_i.$$

Since the other part of the proof of Theorem 2.1 can be applied without change, Lemma 4.5 proves Theorem 4.3.

Similarly, we can prove Theorem 4.4 by a little modification of the proof of Theorem 3.1 in Part I concerning the following two points:

1) Determination of the linear part.

2) Vanishing of obstructions (Lemma 3.2).

PROOF OF THEOREM 4.4. We may assume the following:

i) X is covered by a finite number of coordinate neighborhoods U_i with a system of coordinates $z_i = (z_i^1, \cdots, z_i^n)$ and $U_i = \{z_i \in \boldsymbol{C}^n : |z_i| < 1\}$.

ii) $f(U_i)$ is contained in a coordinate neighborhood V_i on Y. Each V_i is covered by a system of coordinates $w_i = (w_i^1, \cdots, w_i^m)$.

iii) f is given by $w_i = f_i(z_i)$ on U_i.

iv) $z_i \in U_i$ and $w_i \in V_i$ coincide with $z_j \in U_j$ and $w_j \in V_j$, respectively, if and only if $z_i = b_{ij}(z_j)$, $w_i = g_{ij}(w_j)$.

Let $r = \dim D_{X/Y}$, and let $M = \{t \in \boldsymbol{C}^r : |t| < \varepsilon\}$ with a sufficiently small $\varepsilon > 0$.

Our purpose is to construct

i) a differentiable vector (0, 1)-form $\phi(t)$ depending holomorphically on t and

ii) vector valued differentiable functions $\Phi_i(z_i, t)$ on $U_i \times M$ depending holomorphically on t

which satisfy the following equalities:

(4.6) $$\phi(0)=0\,,$$

(4.7) $$\bar\partial\phi-(1/2)[\phi,\phi]=0\,,$$

(4.8) $$\Phi_i(z_i,0)=f_i(z_i)\,,$$

(4.9) $$\bar\partial\Phi_i-\phi\cdot\Phi_i=0\,,$$

(4.10) $$\Phi_i(b_{ij}(z_j),t)=g_{ij}(\Phi_j(z_j,t))\,.$$

First we shall construct $\phi(t)$ and $\Phi_i(t)$ as formal power series in t. Constant terms are determined by (4.6) and (4.8). The linear part of the equations (4.7), (4.9) and (4.10) are given by

(4.11) $$\bar\partial\phi_1=0\,,\qquad \bar\partial\Phi_{i|1}-F\phi_1=0\,,\qquad \Phi^\alpha_{i|1}=\sum_\beta\frac{\partial g^\alpha_{ij}}{\partial w^\beta_j}(f_j)\Phi^\beta_{j|1}\,.$$

Here we introduce the following notation: $\mathcal{A}^{0,q}(\Theta_X)$ denotes the sheaf of germs of differentiable $(0,q)$-forms with coefficients in Θ_X, and we set $A^{0,q}(\Theta_X)=\Gamma(X,\mathcal{A}^{0,q}(\Theta_X))$. Similar notation will be employed for $f^*\Theta_Y$.

LEMMA 4.6. *We have an isomorphism:*

$$D_{X/Y}\cong\frac{\{(\Phi,\phi)\in A^{0,0}(f^*\Theta_Y)\times A^{0,1}(\Theta_X):\ \bar\partial\Phi=F\phi,\ \bar\partial\phi=0\}}{\{(F\xi,\bar\partial\xi):\ \xi\in A^{0,0}(\Theta_X)\}}\,.$$

PROOF. Let $(\tau,\rho)\in C^0(\mathfrak{U},f^*\Theta_Y)\times\mathcal{Z}^1(\mathfrak{U},\Theta_X)$ be a representative of an element of $D_{X/Y}$. Let $\tau=\{\tau_i\}$ and $\rho=\{\rho_{ij}\}$. Since ρ is a cocycle, we can find $\eta_i\in\Gamma(U_i,\mathcal{A}^{0,0}(\Theta_X))$ such that $-\rho_{ij}=\eta_j-\eta_i$ on U_{ij}. We define $\phi\in A^{0,1}(\Theta_X)$ by the formulae:

$$\phi=\bar\partial\eta_i\,,\qquad \text{on}\quad U_i\,.$$

On the other hand, we have $\tau_j-\tau_i=-F\eta_j+F\eta_i$ on U_{ij}. Hence, defining $\Phi\in A^{0,0}(f^*\Theta_Y)$ by the formulae:

$$\Phi=F\eta_i+\tau_i\,,\qquad \text{on}\quad U_i\,,$$

we obtain the equality $\bar\partial\Phi=F\phi$.

Moreover, if we replace η_i by $\eta_i+\xi$ with $\xi\in A^{0,0}(\Theta_X)$, then ϕ and Φ are replaced by $\phi+\bar\partial\xi$ and $\Phi+F\xi$, respectively. In a similar way, we conclude that the class of (Φ,ϕ) does not depend on the choice of the representative.

Conversely, if a pair (Φ,ϕ) satisfying $\bar\partial\Phi=F\phi$ and $\bar\partial\phi=0$ is given, we can find $\eta_i\in\Gamma(U_i,\mathcal{A}^{0,0}(\Theta_X))$ such that $\phi=\bar\partial\eta_i$ on U_i. We set $\rho_{ij}=-\eta_j+\eta_i$ on U_{ij} and $\tau_i=-F\eta_i+\Phi$ on U_i. Then we have the equalities $\bar\partial\rho_{ij}=\bar\partial\tau_i=0$, and $\tau_j-\tau_i=F\rho_{ij}$. Moreover, if we replace η_i by η_i-g_i with $g_i\in\Gamma(U_i,\Theta_X)$, then ρ_{ij} and τ_i are replaced by $\rho_{ij}+g_j-g_i$ and τ_i+Fg_i, respectively.

These correspondences give an isomorphism as desired. Q.E.D.

In order to construct ϕ_1 and $\Phi_{i|1}$ which satisfy (4.11), we take a basis of $D_{X/Y}$, and represent it as

$$\{(\Phi_\lambda, \phi_\lambda) \in A^{0,0}(f^*\Theta_Y) \times A^{0,1}(\Theta_X)\}_{\lambda=1,2,\cdots,r}.$$

We write Φ_λ in the form $\Phi_\lambda = \sum_\alpha \Phi^\alpha_{\lambda i} \frac{\partial}{\partial w^\alpha_i}$ on U_i. Then $\phi_1 = \sum_\lambda \phi_\lambda t_\lambda$ and $\Phi^\alpha_{i|1} = \sum_\lambda \Phi^\alpha_{\lambda i} t_\lambda$ satisfy (4.11).

If we determine ϕ and Φ_i up to degree $\mu-1$, then we meet an obstruction for extending ϕ and Φ_i up to degree μ. The vanishing of the obstruction is assured by the following lemma (compare Lemma 3.2 in Part I).

LEMMA 4.7. *Under the hypotheses of Theorem* 4.4, *suppose that*

$$\Xi' \in A^{0,1}(f^*\Theta_Y), \qquad \xi \in A^{0,2}(\Theta_X)$$

satisfying

$$\bar\partial \Xi' = F\xi, \qquad \bar\partial \xi = 0 \tag{4.12}$$

are given (compare (3.21)). *Then ξ is $\bar\partial$-exact. Moreover, if we take an element $\phi' \in A^{0,1}(\Theta_X)$ satisfying $\bar\partial\phi' = -\xi$, then we can find $\Phi' \in A^{0,0}(f^*\Theta_Y)$ and $\chi \in A^{0,1}(\Theta_X)$ such that*

$$\bar\partial \chi = 0, \qquad \bar\partial \Phi' + F\chi = \Xi' + F\phi' \tag{4.13}$$

(cf. (3.24), (3.25)).

PROOF. Let $\langle\xi\rangle$ denote the cohomology class corresponding to ξ in $H^2(X, \Theta_X)$. Then, by (4.12), $F(\langle\xi\rangle) \in H^2(X, f^*\Theta_Y)$ is zero. Hence, by hypothesis ii), we obtain $\langle\xi\rangle = 0$.

If $\phi' \in A^{0,1}(\Theta_X)$ satisfies the equality $\bar\partial\phi' = -\xi$, then we have $\bar\partial(\Xi' + F\phi') = 0$. By hypothesis i), the cohomology class $\langle \Xi' + F\phi' \rangle \in H^1(X, f^*\Theta_Y)$ is the image of an element $\langle\chi\rangle \in H^1(X, \Theta_X)$. This implies the existence of Φ' and χ which satisfy (4.13).

From Lemma 4.7, it follows that there exist formal power series ϕ and Φ_i in t which satisfy (4.6)-(4.10).

Moreover, the proof of convergence of Theorem 3.1 can be applied to the present situation without any change. This completes the construction of ϕ and Φ_i.

As in the proof of Theorem 3.1, ϕ determines a family $p: \mathcal{X} \to M$ of deformations of X, and the collection $\{\Phi_i\}$ determines a holomorphic map $\Phi: \mathcal{X} \to Y \times M$ over M.

We may assume that $\mathcal{X}$ is covered by a finite number of coordinate neighborhoods $\mathcal{U}_\alpha$. Each $\mathcal{U}_\alpha$ is covered by a system of coordinates $(\eta_\alpha, t) = (\eta^1_\alpha, \cdots, \eta^n_\alpha, t_1, \cdots, t_r)$. Identifying $\mathcal{X}$ with $X \times M$ as a differentiable manifold, we may assume that each $\mathcal{U}_\alpha$ is contained in some $U_i \times M$. With each α we associate an index $\iota(\alpha)$ such that $\mathcal{U}_\alpha \subset U_{\iota(\alpha)} \times M$ and, for simplicity let $U_\alpha = U_{\iota(\alpha)}$, $z_\alpha = z_{\iota(\alpha)}$, and $w_\alpha = w_{\iota(\alpha)}$. Moreover, we may assume that each η^σ_α is a differentiable function $\eta^\sigma_\alpha(z_\alpha, t)$ of z_α and t depending holomorphically on t and $\eta^\sigma_\alpha(z_\alpha, 0) = z^\sigma_\alpha$.

Since η_α^σ are holomorphic with respect to the complex structure $\mathcal{X}$, we have

$$\bar{\partial}\eta_\alpha^\sigma - \phi\cdot\eta_\alpha^\sigma = 0 .$$

Moreover, we have

$$\eta_\alpha^\sigma = \phi_{\alpha\beta}^\sigma(\eta_\beta, t) , \qquad \text{on} \quad \mathcal{U}_{\alpha\beta} ,$$

where $\phi_{\alpha\beta}^\sigma$ are holomorphic functions of (η_β, t) on $\mathcal{U}_{\alpha\beta}$. On the other hand, Φ is given by

$$w_\alpha^\lambda = \Psi_\alpha^\lambda(\eta_\alpha, t) , \qquad \text{on} \quad \mathcal{U}_\alpha ,$$

where Ψ_α^λ are holomorphic functions of (η_α, t). Then we have

$$\eta_\alpha^\sigma(z_\alpha, t) = \phi_{\alpha\beta}^\sigma(\eta_\beta(z_\beta, t), t) , \qquad \text{on} \quad \mathcal{U}_{\alpha\beta} ,$$
$$\Phi_\alpha^\lambda(z_\alpha, t) = \Psi_\alpha^\lambda(\eta_\alpha(z_\alpha, t), t) , \qquad \text{on} \quad \mathcal{U}_\alpha .$$

Let $(\partial/\partial t) \in T_0(M)$ and let "$\cdot$" denote the operation $\left.\frac{\partial}{\partial t}\right|_{t=0}$. Moreover, let $U_\alpha^* = X \cap \mathcal{U}_\alpha$ and $\mathfrak{U}^* = \{U_\alpha^*\}$. With these notations, we have

$$(4.14)\qquad \begin{aligned} &\bar{\partial}\Big(\sum_\sigma \dot{\eta}_\alpha^\sigma \frac{\partial}{\partial z_\alpha^\sigma}\Big) = \dot{\phi} , && \text{on} \quad U_\alpha^* , \\ &\sum_\sigma \dot{\eta}_\alpha^\sigma \frac{\partial}{\partial z_\alpha^\sigma} = \sum_\sigma \dot{\eta}_\beta^\sigma \frac{\partial}{\partial z_\beta^\sigma} + \sum_\sigma \dot{\phi}_{\alpha\beta}^\sigma \frac{\partial}{\partial z_\alpha^\sigma} , && \text{on} \quad U_{\alpha\beta}^* , \\ &\sum_\lambda \dot{\Phi}_\alpha^\lambda \frac{\partial}{\partial w_\alpha^\lambda} = \sum_\lambda \dot{\Psi}_\alpha^\lambda \frac{\partial}{\partial w_\alpha^\lambda} + F\Big(\sum_\sigma \dot{\eta}_\alpha^\sigma \frac{\partial}{\partial z_\alpha^\sigma}\Big) , && \text{on} \quad U_\alpha^* . \end{aligned}$$

By definition, $\tau\Big(\frac{\partial}{\partial t}\Big)$ is represented by

$$\Big(\sum_\lambda \dot{\Psi}_\alpha^\lambda \frac{\partial}{\partial w_\alpha^\lambda}, \sum_\sigma \dot{\phi}_{\alpha\beta}^\sigma \frac{\partial}{\partial z_\alpha^\sigma}\Big) \in C^0(\mathfrak{U}^*, f^*\Theta_Y) \times \mathcal{Z}^1(\mathfrak{U}^*, \Theta_X) .$$

Hence, from the isomorphism in Lemma 4.6, we infer that $\tau : T_0(M) \to D_{X/Y}$ is bijective. This completes the proof of Theorem 4.4.

§ 5. Deformations of holomorphic maps into a family.

Let $(\mathcal{Y}, q, S)$ be a fixed family of complex manifolds. Namely $\mathcal{Y}$ and S are complex manifolds and $q : \mathcal{Y} \to S$ is a surjective smooth holomorphic map.

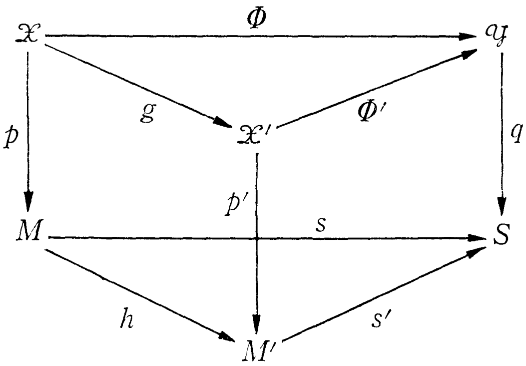
DEFINITION 5.1. By a family of holomorphic maps into $(\mathcal{Y}, q, S)$, we mean a quintuplet $(\mathcal{X}, \Phi, p, M, s)$ of complex manifolds $\mathcal{X}$, M, and holomorphic maps $p : \mathcal{X} \to M$, $\Phi : \mathcal{X} \to \mathcal{Y}$, $s : M \to S$ with the following properties:

1) p is a surjective smooth proper holomorphic map,
2) $s \circ p = q \circ \Phi$.

Two families $(\mathcal{X}, \Phi, p, M, s)$ and $(\mathcal{X}', \Phi', p', M', s')$ are said to be equivalent if there exist complex analytic isomorphisms

$$g: \mathcal{X} \longrightarrow \mathcal{X}', \qquad h: M \longrightarrow M'$$

such that the diagram



commutes.

We define induced families and the concept of completeness in a similar way as in § 1, Part I.

Let $(\mathcal{X}, \Phi, p, M, s)$ be a family of holomorphic maps into $(\mathcal{Y}, q, S)$, $0 \in M$, $X = X_0$, and let $\tilde{f}: X \rightarrow \mathcal{Y}$ be the restriction of Φ to X. Moreover, let $0^* = s(0)$, $Y = Y_{0^*}$, and let $f: X \rightarrow Y$ be the holomorphic map induced by Φ. Let $\tilde{F}: \Theta_X \rightarrow \tilde{f}^*\Theta_{\mathcal{Y}}$ and $F: \Theta_X \rightarrow f^*\Theta_Y$ denote the canonical homomorphisms. We denote by $D_{X/\mathcal{Y}}$ the module defined by Definition 4.1. Namely, if $\mathfrak{U} = \{U_i\}$ is a Stein covering of X, then

$$D_{X/\mathcal{Y}} = \frac{\{(\tilde{\tau}, \rho) \in C^0(\mathfrak{U}, \tilde{f}^*\Theta_{\mathcal{Y}}) \times \mathcal{Z}^1(\mathfrak{U}, \Theta_X) : \delta\tilde{\tau} = \tilde{F}\rho\}}{\{(\tilde{F}g, \delta g) : g \in C^0(\mathfrak{U}. \Theta_X)\}}.$$

As in § 4, we have a characteristic map $\tau: T_0(M) \rightarrow D_{X/\mathcal{Y}}$.

We shall give another expression of $D_{X/\mathcal{Y}}$.

We may assume the following:

i) S is an open set in $C^{r'}$ with a system of coordinates $s = (s^1, \cdots, s^{r'})$ and $0^* = (0, \cdots, 0)$.

ii) X is covered by a finite number of coordinate neighborhoods U_i. Each U_i is covered by a system of coordinates $z_i = (z_i^1, \cdots, z_i^n)$.

iii) $f(U_i)$ is contained in a coordinate neighborhood $\mathcal{V}_i$ of $\mathcal{Y}$. Each $\mathcal{V}_i$ is covered by a system of coordinates $(w_i, s_i) = (w_i^1, \cdots, w_i^m, s_i^1, \cdots, s_i^{r'})$ and q is given by $s = s_i$.

iv) f is given by $w_i = f_i(z_i)$.

v) $z_i \in U_i$ coincides with $z_j \in U_j$ if and only if $z_i = b_{ij}(z_j)$.

vi) $(w_i, s_i) \in \mathcal{V}_i$ coincides with $(w_j, s_j) \in \mathcal{V}_j$ if and only if $w_i = \phi_{ij}(w_j, s_j)$, $s_i = s_j$. We set $g_{ij}(w_j) = \phi_{ij}(w_j, 0)$.

Let $\tau' : T_{0^*}(S) \to H^1(Y, \Theta_Y)$ be the infinitesimal deformation map of the family $(\mathcal{Y}, q, S)$ at 0^*. Then each $f^*\rho'\left(\frac{\partial}{\partial s^\nu}\right) \in H^1(X, f^*\Theta_Y)$ is represented by a 1-cocycle $\tilde{\rho}'_\nu = \{\tilde{\rho}'_{\nu ij}\}$ with $\tilde{\rho}'_{\nu ij} = \sum_\lambda \frac{\partial \phi_{ij}^\lambda}{\partial s_j^\nu}(f_j(z_j), 0)\frac{\partial}{\partial w_i^\lambda}$.

LEMMA 5.1. *Let $\mathfrak{U}$ denote the covering $\{U_i\}$. Then we have an isomorphism*

$$D_{X/\mathcal{Y}} = \frac{\{(\tau, \rho, (\theta^\nu)) \in C^0(\mathfrak{U}, f^*\Theta_Y) \times \mathcal{Z}^1(\mathfrak{U}, \Theta_X) \times \boldsymbol{C}^{r'} : \delta\tau = F\rho - \sum \theta^\nu \tilde{\rho}'_\nu\}}{\{(Fg, \delta g, 0) : g \in C^0(\mathfrak{U}, \Theta_X)\}}.$$

PROOF. Let $(\tilde{\tau}, \rho) \in C^0(\mathfrak{U}, \tilde{f}^*\Theta_{\mathcal{Y}}) \times \mathcal{Z}^1(\mathfrak{U}, \Theta_X)$ be a representative of an element of $D_{X/\mathcal{Y}}$. Let $\tilde{\tau} = \{\tilde{\tau}_i\}$ and $\rho = \{\rho_{ij}\}$. We write each $\tilde{\tau}_i$ explicitly as follows:

$$\tilde{\tau}_i = \sum_\lambda \tau_i^\lambda \frac{\partial}{\partial w_i^\lambda} + \sum_\nu \theta_i^\nu \frac{\partial}{\partial s_i^\nu}, \qquad \text{on } U_i.$$

Note that we have

$$\frac{\partial}{\partial w_j^\beta} = \sum_\lambda \frac{\partial \phi_{ij}^\lambda}{\partial w_j^\beta}\frac{\partial}{\partial w_i^\lambda}, \qquad \frac{\partial}{\partial s_j^\nu} = \sum_\lambda \frac{\partial \phi_{ij}^\lambda}{\partial s_j^\nu}\frac{\partial}{\partial w_i^\lambda} + \frac{\partial}{\partial s_i^\nu}, \qquad \text{on } \mathcal{V}_{ij}.$$

Hence, from the cocycle condition for $(\tilde{\tau}, \rho)$, we infer that

$$\sum_\lambda \tau_j^\lambda \frac{\partial}{\partial w_j^\lambda} - \sum_\lambda \tau_i^\lambda \frac{\partial}{\partial w_i^\lambda} = F\rho_{ij} - \sum_\nu \theta^\nu \tilde{\rho}'_{\nu ij}, \qquad \text{on } U_{ij},$$

$$\theta_j^\nu = \theta_i^\nu, \qquad \text{on } U_{ij} \text{ for each } \nu = 1, 2, \cdots, r'.$$

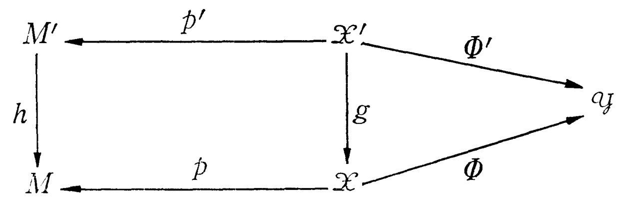
Since X is compact, $\theta^\nu = \theta_i^\nu$ is a constant. Moreover, letting $\tau_i = \sum_\lambda \tau_i^\lambda \frac{\partial}{\partial w_i^\lambda} \in \Gamma(U_i, f^*\Theta_Y)$, we obtain $\tau = \{\tau_i\} \in C^0(\mathfrak{U}, f^*\Theta_Y)$. We can easily check that above correspondence $(\tilde{\tau}, \rho) \to (\tau, \rho, (\theta^\nu))$ gives an isomorphism as desired.

COROLLARY. 1) *We have a canonical homomorphism $\pi : D_{X/\mathcal{Y}} \to H^1(X, \Theta_X)$ such that $\pi \circ \tau : T_0(M) \to H^1(X, \Theta_X)$ gives the infinitesimal deformation map.*

2) *An element $\rho \in H^1(X, \Theta_X)$ is in the image of π if and only if $F\rho \in H^1(X, f^*\Theta_Y)$ is contained in the image of $f^* \circ \rho' : T_{0^*}(S) \to H^1(X, f^*\Theta_Y)$.*

THEOREM 5.2. *Let $(\mathcal{X}, \Phi, p, M, s)$ be a family of holomorphic maps into a family $(\mathcal{Y}, q, S)$, $0 \in M$, $X = X_0$, and let $\tilde{f} : X \to \mathcal{Y}$ be the restriction of Φ to X. If the characteristic map $\tau : T_0(M) \to D_{X/\mathcal{Y}}$ is surjective, then the family is complete at 0.*

PROOF. Let $(\mathcal{X}', \Phi', p', M', s')$ be another family of holomorphic maps into $(\mathcal{Y}, q, S)$. Assume that there exists a point $0' \in M'$ such that the restriction $\tilde{f}' : X'_{0'} \to \mathcal{Y}$ of Φ' to $X'_{0'} = (p')^{-1}(0')$ is equivalent to $\tilde{f}$. By Theorem 4.3, we have holomorphic maps $g : \mathcal{X}' \to \mathcal{X}$ and $h : M' \to M$ such that the diagram



commutes (we replace M' by an open neighborhood of $0'$ if necessary).

It follows that $s\circ h\circ p'=s'\circ p'$. Since p' is surjective, we conclude that $s\circ h=s'$. This proves the assertion.

In order to prove a theorem of existence, we need another expression of $D_{X/\mathcal{Y}}$. For each ν, let $\tilde{\varphi}_\nu\in A^{0,1}(f^*\Theta_Y)$ be a $\bar{\partial}$-closed (0, 1)-form which represents the cohomology class $-f^*\rho'\left(\frac{\partial}{\partial s^\nu}\right)$. Then there exist $\zeta_{\nu i}\in\Gamma(U_i, \mathcal{A}^{0,0}(f^*\Theta_Y))$ such that

$$-\rho'_{\nu ij}=\zeta_{\nu j}-\zeta_{\nu i}\,,\qquad \tilde{\varphi}_\nu=\bar{\partial}\zeta_{\nu i}\,.$$

LEMMA 5.3. *We have an isomorphism*

$$D_{X/\mathcal{Y}}=\frac{\{(\Phi,\phi,(\theta^\nu))\in A^{0,0}(f^*\Theta_Y)\times A^{0,1}(\Theta_X)\times C^{r'}:\ \bar{\partial}\Phi=F\phi-\sum\theta^\nu\tilde{\varphi}_\nu,\ \bar{\partial}\phi=0\}}{\{(F\xi,\bar{\partial}\xi,0):\ \xi\in A^{0,0}(\Theta_X)\}}\,.$$

PROOF. We may assume that $D_{X/\mathcal{Y}}$ is defined by the isomorphism in Lemma 5.1. Let $(\tau,\rho,(\theta^\nu))$ be a representative of an element of $D_{X/\mathcal{Y}}$. Then we can find $\eta_i\in\Gamma(U_i,\mathcal{A}^{0,0}(\Theta_X))$ such that $-\rho_{ij}=\eta_j-\eta_i$ on U_{ij}. We define $\phi\in A^{0,1}(\Theta_X)$ by $\phi=\bar{\partial}\eta_i$ on U_i. On the other hand, we have

$$\tau_j-\tau_i=-F\eta_j+F\eta_i+\sum_\nu\theta^\nu(\zeta_{\nu j}-\zeta_{\nu i})\,,\qquad\text{on}\quad U_{ij}\,.$$

We define $\Phi\in A^{0,0}(f^*\Theta_Y)$ by $\Phi=\tau_i+F\eta_i-\sum_\nu\theta^\nu\zeta_{\nu i}$ on U_i. Then we have $\bar{\partial}\Phi=F\phi-\sum\theta^\nu\tilde{\varphi}_\nu$.

As in Lemma 4.6, this correspondence gives an isomorphism. Q. E. D.

THEOREM 5.4. *Let $(\mathcal{Y},q,S)$ be a family of complex manifolds, $0^*\in S$, $Y=Y_{0^*}$ and let $\rho':T_{0^*}(S)\to H^1(Y,\Theta_Y)$ be the infinitesimal deformation map. Assume that*

i) *$H^1(X,f^*\Theta_Y)$ is generated by the image of $F:H^1(X,\Theta_X)\to H^1(X,f^*\Theta_Y)$ and the image of $f^*\circ\rho':T_{0^*}(S)\to H^1(X,f^*\Theta_Y)$,*

ii) *$F:H^2(X,\Theta_X)\to H^2(X,f^*\Theta_Y)$ is injective.*

Then there exist a family $(\mathcal{X},\Phi,p,M,s)$ of holomorphic maps into $(\mathcal{Y},q,S)$ and a point $0\in M$ such that

1) *$s(0)=0^*$, $X=p^{-1}(0)$ and Φ_0 coincides with f,*

2) *$\tau:T_0(M)\to D_{X/\mathcal{Y}}$ is bijective.*

REMARK. If f is non-degenerate, then the above conditions i) and ii) are reduced to the following: The composition $P\circ f^*\circ\rho':T_{0^*}(S)\to H^1(X,\mathcal{T}_{X/Y})$ is surjective.

PROOF. We take systems of coordinates on X and on $\mathcal{Y}$ as introduced before Lemma 5.1. Moreover we assume that $U_i=\{z_i\in \boldsymbol{C}^n : |z_i|<1\}$.

Let $r=\dim D_{X/\mathcal{Y}}$ and $M=\{t\in \boldsymbol{C}^r : |t|<\varepsilon\}$ with a sufficiently small $\varepsilon>0$.

We shall construct

i) a differentiable vector (0, 1)-form $\phi(t)$ depending holomorphically on t,

ii) differentiable functions $\Phi_i : U_i\times M\to \boldsymbol{C}^m$ depending holomorphically on t, and

iii) a holomorphic map $h : M\to \boldsymbol{C}^{r'}$,

which satisfy

$$h(0)=0\,, \tag{5.0}$$

$$\phi(0)=0\,, \tag{5.1}$$

$$\bar{\partial}\phi-(1/2)[\phi, \phi]=0\,, \tag{5.2}$$

$$\Phi_i(z_i, 0)=f_i(z_i)\,, \tag{5.3}$$

$$\bar{\partial}\Phi_i-\phi\cdot\Phi_i=0\,, \tag{5.4}$$

$$\Phi_i(b_{ij}(z_j), t)=\psi_{ij}(\Phi_j(z_j, t), h(t))\,. \tag{5.5}$$

I) Existence of formal solutions.

Using the notation of Part I, we shall construct formal power series $\phi(t)$, $\Phi_i(t)$, and $h(t)$ in t.

In view of (5.0), (5.1) and (5.3), we set

$$\phi_0=0\,, \qquad \Phi_{i|0}=f_i(z_i)\,, \qquad h_0=0\,. \tag{5.6}$$

Clearly (5.2), (5.4) and (5.5) are equivalent to the following systems of congruences:

$$\bar{\partial}\phi^{\mu}-(1/2)[\phi^{\mu}, \phi^{\mu}]\underset{\mu}{\equiv}0\,, \tag*{(5.7)$_\mu$}$$

$$\bar{\partial}\Phi_i^{\mu}-\phi^{\mu}\cdot\Phi_i^{\mu}\underset{\mu}{\equiv}0\,, \tag*{(5.8)$_\mu$}$$

$$\Phi_i^{\mu}(b_{ij}(z_j), t)\underset{\mu}{\equiv}\psi_{ij}(\Phi_j^{\mu}(z_j, t), h^{\mu}(t))\,, \tag*{(5.9)$_\mu$}$$

for $\mu=1, 2, \cdots$.

We construct solutions of (5.0)-(5.5) by induction on μ. We suppose that $\phi^{\mu-1}$, $\Phi_i^{\mu-1}$ and $h^{\mu-1}$, satisfying $(5.7)_{\mu-1}$, $(5.8)_{\mu-1}$ and $(5.9)_{\mu-1}$, have been already determined.

We define homogeneous polynomials $\xi_\mu\in A^{0,2}(\Theta_X)$ (by this we mean that ξ_μ is a homogeneous polynomial of degree μ with coefficients in $A^{0,2}(\Theta_X)$), $\Xi_{i|\mu}\in \Gamma(U_i, \mathcal{A}^{0,1}(f^*\Theta_Y))$ and $\Gamma_{ij|\mu}\in \Gamma(U_{ij}, \mathcal{A}^{0,0}(f^*\Theta_Y))$ by the following congruences:

$$(5.10)\qquad \xi_\mu \underset{\mu}{\equiv} \phi^{\mu-1}-(1/2)[\phi^{\mu-1}, \phi^{\mu-1}],$$

$$(5.11)\qquad -\Xi_{i|\mu} \underset{\mu}{\equiv} (\Phi_i^{\mu-1}-\phi^{\mu-1}\cdot\Phi_i^{\mu-1})\cdot\frac{\partial}{\partial w_i},$$

$$(5.12)\qquad \Gamma_{ij|\mu} \underset{\mu}{\equiv} (\Phi_i^{\mu-1}-\psi_{ij}(\Phi_j^{\mu-1}, h^{\mu-1}))\cdot\frac{\partial}{\partial w_i}.$$

(For the notation, see the proof of Theorem 2.1 in Part I.)

Then we have the following equalities:

$$(5.13)\qquad \bar{\partial}\xi_\mu=0, \qquad \text{in } \Gamma(X, \mathcal{A}^{0,3}(\Theta_X)),$$

$$(5.14)\qquad \bar{\partial}\Xi_{i|\mu}=F\xi_\mu, \qquad \text{in } \Gamma(U_i, \mathcal{A}^{0,2}(f^*\Theta_Y)),$$

$$(5.15)\qquad \Xi_{j|\mu}-\Xi_{i|\mu}=\bar{\partial}\Gamma_{ij|\mu}, \qquad \text{in } \Gamma(U_{ij}, \mathcal{A}^{0,1}(f^*\Theta_Y)),$$

$$(5.16)\qquad \Gamma_{jk|\mu}-\Gamma_{ik|\mu}+\Gamma_{ij|\mu}=0, \qquad \text{in } \Gamma(U_{ijk}, \mathcal{A}^{0,0}(f^*\Theta_Y)).$$

The proofs are similar to those of (3.13)-(3.16) in Part I.

Our purpose is to determine

$$\phi^\mu=\phi^{\mu-1}+\phi_\mu, \qquad \Phi_i^\mu=\Phi_i^{\mu-1}+\Phi_{i|\mu}, \qquad h^\mu=h^{\mu-1}+h_\mu,$$

which satisfy $(5.7)_\mu$, $(5.8)_\mu$ and $(5.9)_\mu$.

$(5.7)_\mu$, $(5.8)_\mu$ and $(5.9)_\mu$ are equivalent to the following equalities:

$$(5.17)\qquad \bar{\partial}\phi_\mu=-\xi_\mu,$$

$$(5.18)\qquad \Xi_{i|\mu}=\bar{\partial}\Phi_{i|\mu}-F\phi_\mu,$$

$$(5.19)\qquad \Gamma_{ij|\mu}=\Phi_{j|\mu}-\Phi_{i|\mu}+\sum_\nu h_\mu^\nu\tilde{\rho}'_{\nu ij},$$

where we denote by the same letter Φ_i the section $\sum_\lambda \Phi_i^\lambda\frac{\partial}{\partial w_i^\lambda}$ of $f^*\Theta_Y$, and we set $\tilde{\rho}'_{\nu ij}=\sum_\lambda\frac{\partial\psi_{ij}^\lambda}{\partial s_j^\nu}(f_j(z_j), 0)\frac{\partial}{\partial w_i^\lambda}$. The proofs are similar to those of the corresponding assertions in § 3, Part I.

LEMMA 5.5. *Under the hypotheses of Theorem* 5.4, *we can find* $\phi_\mu\in A^{0,1}(\Theta_X)$, $\Phi_{i|\mu}\in\Gamma(U_i, \mathcal{A}^{0,0}(f^*\Theta_Y))$ *and* $h_\mu=(h_\mu^\nu)\in \boldsymbol{C}^{\tau'}$ *which satisfy* (5.17), (5.18) *and* (5.19).

PROOF. In virtue of the equality (5.16), we can find $\Gamma_{i|\mu}\in\Gamma(U_i, \mathcal{A}^{0,0}(f^*\Theta_Y))$ such that

$$(5.20)\qquad \Gamma_{ij|\mu}=\Gamma_{j|\mu}-\Gamma_{i|\mu}.$$

From the equalities (5.15) and (5.20), we infer that

$$(5.21)\qquad \Xi'_\mu=\Xi_{i|\mu}-\bar{\partial}\Gamma_{i|\mu}$$

determines a global section $\Xi'_\mu\in A^{0,1}(f^*\Theta_Y)$. From (5.14), it follows that $\bar{\partial}\Xi'_\mu=F\xi_\mu$. Hence by hypothesis ii), ξ_μ is $\bar{\partial}$-exact. Take any $\phi'_\mu\in A^{0,1}(\Theta_X)$ which satisfies

(5.22) $$\bar{\partial}\phi'_{\mu}=-\xi_{\mu}.$$

Then it follows that

(5.23) $$\bar{\partial}(\Xi'_{\mu}+F\phi'_{\mu})=0.$$

By hypothesis i), we can find $\chi_{\mu}\in A^{0,1}(\Theta_X)$, $\Phi'_{\mu}\in A^{0,0}(f^*\Theta_Y)$, and $h_{\mu}=(h^{\nu}_{\mu})\in \boldsymbol{C}^{r'}$ such that

(5.24) $$\bar{\partial}\chi_{\mu}=0,$$

(5.25) $$\Xi'_{\mu}+F\phi'_{\mu}=F\chi_{\mu}+\sum_{\nu}h^{\nu}_{\mu}\tilde{\psi}_{\nu}+\bar{\partial}\Phi'_{\mu},$$

where each $\tilde{\psi}_{\nu}\in A^{0,1}(f^*\Theta_Y)$ denotes a $\bar{\partial}$-closed (0, 1)-form which represents the cohomology class $-f^*\rho'\left(\frac{\partial}{\partial s^{\nu}}\right)$. We take $\zeta_{\nu i}\in\Gamma(U_i, \mathcal{A}^{0,0}(f^*\Theta_Y))$ such that

$$-\tilde{\rho}'_{\nu ij}=\zeta_{\nu j}-\zeta_{\nu i}, \qquad \tilde{\psi}_{\nu}=\bar{\partial}\zeta_{\nu i}.$$

We set

(5.26) $$\phi_{\mu}=\phi'_{\mu}-\chi_{\mu},$$

(5.27) $$\Phi_{i|\mu}=\Phi'_{\mu}+\Gamma_{i|\mu}+\sum_{\nu}h^{\nu}_{\mu}\zeta_{\nu i}.$$

Then (5.17) follows from (5.26), (5.22) and (5.24). (5.18) follows from (5.27), (5.25), (5.26) and (5.21). Finally, (5.19) follows from (5.27) and (5.20). This proves Lemma 5.5.

For $\mu=1$ we determine ϕ_1, $\Phi_{i|1}$ and h_1 as follows: Take

$$(\Phi'_{1\lambda}, \phi_{1\lambda}, (\theta^{\nu}_{\lambda}))\in A^{0,0}(f^*\Theta_Y)\times A^{0,1}(\Theta_X)\times\boldsymbol{C}^{r'} \qquad (\lambda=1, 2, \cdots, r),$$

which represent a basis of $D_{X/\mathcal{Y}}$ via the isomorphism in Lemma 5.3. Then we have $\bar{\partial}\Phi'_{1\lambda}=F\phi_{1\lambda}-\sum_{\nu}\theta^{\nu}_{\lambda}\tilde{\psi}_{\nu}$. We set

$$\phi_1=\sum_{\lambda}\phi_{1\lambda}t_{\lambda}, \qquad \Phi_{i|1}=\sum_{\lambda}(\Phi'_{1\lambda}+\sum_{\nu}\theta^{\nu}_{\lambda}\zeta_{\nu i})t_{\lambda}, \qquad h^{\nu}_1=\sum_{\lambda}\theta^{\nu}_{\lambda}t_{\lambda}.$$

It is easy to check that ϕ_1, $\Phi_{i|1}$ and $h_1=(h^{\nu}_1)$ satisfy $(5.7)_1$, $(5.8)_1$ and $(5.9)_1$.

Once we have determined linear parts, we can extend them to formal power series in t satisfying (5.0)-(5.5), as we have already seen.

II) Proof of convergence.

In a similar way as in the proof of Theorem 3.1 in Part I, we can show that the solution $\phi(t)$, $\Phi_i(t)$ and $h(t)$ of (5.0)-(5.5) can be chosen so that $\phi(t)$ and $\Phi_i(t)$ converge in the norm $|\ |_{k+\alpha}$ (k: an integer$\geqq 2$, $0<\alpha<1$) and that $h(t)$ converges absolutely for sufficiently small $|t|$.

III) Final step.

By the same argument as in the proof of Theorem 3.1 in Part I, we obtain

1) a family $p: \mathcal{X} \to M$ of deformations of $X = X_0$,

2) holomorphic maps $\Phi: \mathcal{X} \to \mathcal{Y}$ and $s: M \to S$ such that $q \circ \Phi = s \circ p$.

By construction, we have $s(0) = 0^*$ and Φ induces f on X.

It remains to show that τ is bijective. For this purpose, we take systems of holomorphic coordinates $\{\mathcal{U}_\alpha, (\eta_\alpha, t)\}$ on $\mathcal{X}$ as in the last part of the proof of Theorem 4.4. We have $\eta_\alpha = \phi_{\alpha\beta}(\eta_\beta, t)$, and Φ is given by $w_\alpha = \Psi_\alpha(\eta_\alpha, t)$, $s = h(t)$. Moreover, we have three equalities (4.14).

On the other hand, $\tau(\partial/\partial t)$ is given by the class of $(\tilde{\tau}, \rho) \in C^0(\mathfrak{U}^*, \tilde{f}^*\Theta_{\mathcal{Y}}) \times \mathcal{Z}^1(\mathfrak{U}^*, \Theta_X)$ with

$$\tilde{\tau}_\alpha = \sum_\lambda \dot{\Psi}^\lambda_\alpha \frac{\partial}{\partial w^\lambda_\alpha} + \sum_\nu \dot{h}^\nu \frac{\partial}{\partial s^\nu_\alpha}, \qquad \rho_{\alpha\beta} = \sum \dot{\phi}^\sigma_{\alpha\beta} \frac{\partial}{\partial z^\sigma_\alpha}.$$

By the isomorphisms in Lemmas 5.1 and 5.3, $(\tilde{\tau}, \rho)$ corresponds to $(\dot{\Phi}'_1, \dot{\phi}_1)$, where $\Phi'_1 = \sum_\lambda \Phi_{1\lambda} t_\lambda$. This proves that τ is bijective.

§ 6. Stability of manifolds over Y.

In this section, we shall prove the following theorem.

THEOREM 6.1. *Let $f: X \to Y$ be a holomorphic map of a compact complex manifold X into a complex manifold Y. Assume that*

i) *$F: H^1(X, \Theta_X) \to H^1(X, f^*\Theta_Y)$ is surjective,*

ii) *$F: H^2(X, \Theta_X) \to H^2(X, f^*\Theta_Y)$ is injective.*

Then for any family $q: \mathcal{Y} \to M$ of complex manifolds such that $Y = q^{-1}(0)$ for some point $0 \in M$, there exist

1) *an open neighborhood N of 0,*

2) *a family $p: \mathcal{X} \to N$ of deformations of $X = p^{-1}(0)$,*

3) *a holomorphic map $\Phi: \mathcal{X} \to \mathcal{Y}|_N$ over N which induces f over $0 \in N$.*

REMARKS. 1) If f is non-degenerate, then two conditions i) and ii) are reduced to the vanishing of $H^1(X, \mathcal{T}_{X/Y})$.

2) In the case where f is an embedding, Theorem 6.1 has been proved by Kodaira (see [6] Theorem 1).

PROOF. We shall copy the proof of Theorem 5.4.

We may assume that $M = \{t = (t_\lambda) \in \mathbf{C}^r : |t| < \varepsilon\}$ with a sufficiently small $\varepsilon > 0$. Our purpose is to find a solution ϕ, Φ_i of (5.1)-(5.5) with $h(t) = t$.

Using the notation of the proof of Theorem 5.4, we identify s and t. For each $\nu = 1, 2, \cdots, r$, let $\tilde{\varphi}_\nu \in A^{0,1}(f^*\Theta_Y)$ be a $\bar{\partial}$-closed form which represents the cohomology class of $-\{\tilde{\rho}'_{\nu ij}\}$ where $\tilde{\rho}'_{\lambda ij} = \sum_\alpha \frac{\partial \varphi^\alpha_{ij}}{\partial t_\lambda}(f_j(z_j), 0) \frac{\partial}{\partial w^\alpha_i}$. Then we can find $\zeta_{\lambda i} \in \Gamma(U_i, \mathcal{A}^{0,0}(f^*\Theta_Y))$ such that

$$-\tilde{\rho}'_{\lambda ij} = \zeta_{\lambda j} - \zeta_{\lambda i}, \qquad \tilde{\varphi}_\lambda = \bar{\partial} \zeta_{\lambda i}.$$

By hypothesis i), we can find $\Phi'_{1\lambda} \in A^{0,0}(f^*\Theta_Y)$ and $\phi_{1\lambda} \in A^{0,1}(\Theta_X)$ such that

$$\bar{\partial}\Phi'_{1\lambda}=F\phi_{1\lambda}-\tilde{\phi}_{\lambda}.$$

Then $\phi_1=\sum_{\lambda}\phi_{1\lambda}t_\lambda$, $\Phi_{i|1}=\sum_{\lambda}(\Phi'_1+\zeta_{\lambda i})t_\lambda$ and $h_1^\lambda=t_\lambda$ satisfy $(5.7)_1$, $(5.8)_1$ and $(5.9)_1$.

For $\mu\geqq 2$, we can solve (5.17), (5.18) and (5.19) with $h_\mu^\nu=0$, by virtue of the stronger hypothesis i) of Theorem 6.1. This completes the proof of Theorem 6.1.

§7. Deformation of compositions of holomorphic maps.

In this section, we shall prove two propositions on deformations of compositions of holomorphic maps.

PROPOSITION 7.1. *Let $f: X\rightarrow Y$, $g: Y\rightarrow Z$ and $h=g\circ f$ be holomorphic maps of complex manifolds. We assume that*

i) *X and Y are compact,*

ii) *g is non-degenerate and the canonical homomorphism $f^*G: f^*\Theta_Y\rightarrow h^*\Theta_Z$ is injective, where G denotes the homomorphism: $\Theta_Y\rightarrow g^*\Theta_Z$,*

iii) *there exist a family $(\mathcal{Y}, \Psi, q, N)$ of holomorphic maps into Z and a point $0'\in N$ such that $Y=q^{-1}(0')$ and such that Ψ induces g on Y,*

iv) *the composition $f^*\circ\tau: T_{0'}(N)\rightarrow H^0(X, f^*\mathfrak{T}_{Y/Z})$ is surjective, where $\tau: T_{0'}(N)\rightarrow H^0(\mathcal{Y}, \mathfrak{T}_{Y/Z})$ is the characteristic map of $(\mathcal{Y}, \Psi, q, N)$ at $0'$, and $f^*: H^0(Y, \mathfrak{T}_{Y/Z})\rightarrow H^0(X, f^*\mathfrak{T}_{Y/Z})$ is the pull-back homomorphism.*

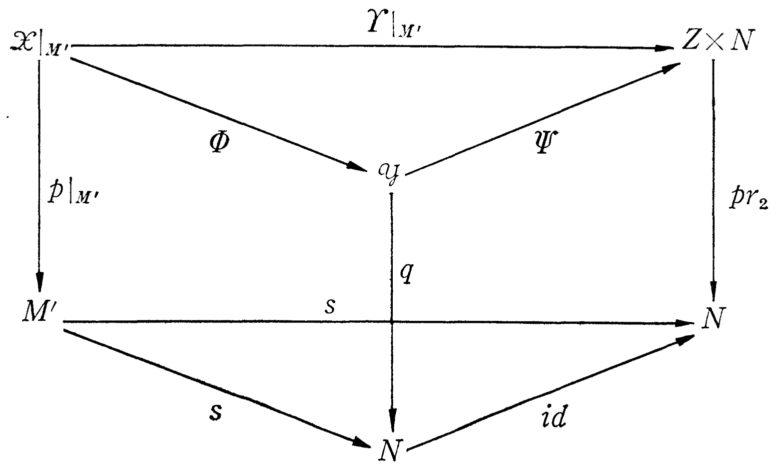
Let $(\mathcal{X}, \Upsilon, p, M)$ be a family of holomorphic maps into Z and let 0 be a point on M such that $X=p^{-1}(0)$ and that Υ induces h on X. Then there exist

1) *an open neighborhood M' of 0,*

2) *a holomorphic map $s: M'\rightarrow N$,*

3) *a holomorphic map $\Phi: \mathcal{X}|_{M'}\rightarrow\mathcal{Y}$,*

such that $s(0)=0'$, Φ induces f on X, and the diagram



commutes.

PROOF. We may assume the following:

i) $M=\{t\in C^{r}: |t|<\varepsilon\}$, $0=(0, \cdots, 0)$, $N=\{s\in C^{r'}: |s|<1\}$, and $0'=(0, \cdots, 0)$, where ε denotes a sufficiently small positive number.

ii) $\mathcal{X}$ (or $\mathcal{Y}$) is covered by a finite number of coordinate neighborhoods $\mathcal{U}_i$ $(i\in I)$ (or $\mathcal{V}_i$ $(i\in J)$) and each $\mathcal{U}_i$ (or $\mathcal{V}_i$) is covered by a system of coordinates (z_i, t) (or (w_i, s)) such that $p(z_i, t)=t$ (or $q(w_i, s)=s$). Moreover, each $\mathcal{U}_i$ is a polydisc $\{(z_i, t): |z_i|<1, |t|<\varepsilon\}$. We set $U_i=\mathcal{U}_i\cap X$ and $V_i=\mathcal{V}_i\cap Y$.

iii) I is a subset of J and $f(U_i)$ is contained in V_i for each $i\in I$. f is given by $w_i=f_i(z_i)$ on U_i.

iv) For each $i\in J$, there exists a coordinate neighborhood W_i on Z such that $\Upsilon(\mathcal{U}_i)\subset W_i\times M$ for $i\in I$ and $\Psi(\mathcal{V}_i)\subset W_i\times N$ for $i\in J$.

v) Each W_i is covered by a system of coordinates y_i.

vi) Υ and Ψ are, respectively, given by $y_i=\Upsilon_i(z_i, t)$ and $y_i=\Psi_i(w_i, s)$.

vii) $(z_i, t)\in\mathcal{U}_i$ and $(w_i, s)\in\mathcal{V}_i$ coincide with $(z_j, t)\in\mathcal{U}_j$ and $(w_j, s)\in\mathcal{V}_j$, respectively, if and only if $z_i=\phi_{ij}(z_j, t)$ and $w_i=\psi_{ij}(w_j, s)$.

viii) $y_i\in W_i$ coincides with $y_j\in W_j$ if and only if $y_i=e_{ij}(y_j)$.

We set $g_i(w_i)=\Psi_i(w_i, 0)$, $h_i(z_i)=\Upsilon_i(z_i, 0)$, $b_{ij}(z_j)=\phi_{ij}(z_j, 0)$ and $c_{ij}(w_j)=\psi_{ij}(w_j, 0)$.

Now we shall construct holomorphic functions

$$s=(s^{\nu}): M\longrightarrow C^{r'}, \qquad \Phi_i: \mathcal{U}_i\longrightarrow C^{m} \qquad (i\in I)$$

such that

$$s(0)=0, \quad \Phi_i(z_i, 0)=f_i(z_i), \qquad \text{on } U_i\ (i\in I), \tag{7.0}$$

$$\Phi_i(\phi_{ij}(z_j, t), t)=\psi_{ij}(\Phi_j(z_j, t), s(t)), \qquad \text{on } \mathcal{U}_{ij}\ (i, j\in I), \tag{7.1}$$

$$\Upsilon_i(z_i, t)=\Psi_i(\Phi_i(z_i, t), s(t)), \qquad \text{on } \mathcal{U}_i\ (i\in I). \tag{7.2}$$

First we prove the existence of solutions s and Φ_i as formal power series in t.

Let $(7.1)_\mu$ and $(7.2)_\mu$ denote the congruences mod $t^{\mu+1}$ derived from (7.1) and (7.2) by replacing s and Φ_i by s^{μ} and Φ_i^{μ}, respectively.

We set $s_0=0$, $\Phi_{i|0}=f_i(z_i)$. We shall construct s^{μ} and Φ_i^{μ} by induction on μ. Suppose that we have already determined $s^{\mu-1}$ and $\Phi_i^{\mu-1}$ which satisfy $(7.1)_{\mu-1}$ and $(7.2)_{\mu-1}$.

We define homogeneous polynomials $\Gamma_{ij|\mu}\in\Gamma(U_{ij}, f^*\Theta_Y)$, $\gamma_{i|\mu}\in\Gamma(U_i, h^*\Theta_Z)$ of degree μ by the following congruences:

$$\Gamma_{ij|\mu}\underset{\mu}{\equiv}(\Phi_i^{\mu-1}(\phi_{ij}, t)-\psi_{ij}(\Phi_j^{\mu-1}, s^{\mu-1}))\cdot\frac{\partial}{\partial w_i}, \tag{7.3}$$

$$\gamma_{i|\mu}\underset{\mu}{\equiv}(\Upsilon_i-\Psi_i(\Phi_i^{\mu-1}, s^{\mu-1}))\cdot\frac{\partial}{\partial y_i}. \tag{7.4}$$

Then we have

(7.5) $$\Gamma_{jk|\mu}-\Gamma_{ik|\mu}+\Gamma_{ij|\mu}=0\,,$$

(7.6) $$\gamma_{j|\mu}-\gamma_{i|\mu}=(f^*G)\Gamma_{ij|\mu}\,.$$

PROOF. (7.5) is proved in [7] Lemma 2. (7.6) can be proved as follows: For simplicity, we omit the indices $\mu-1$.

$$\begin{aligned}\gamma^{\alpha}_{i|\mu}(z_i, t) &\underset{\mu}{\equiv} \gamma^{\alpha}_{i|\mu}(\phi_{ij}(z_j, t), t)\\ &\underset{\mu}{\equiv} \Upsilon^{\alpha}_{i|\mu}(\phi_{ij}, t)-\Psi^{\alpha}_i(\Phi_i(\phi_{ij}, t), s)\\ &\underset{\mu}{\equiv} e^{\alpha}_{ij}(\Upsilon_j, t)-\Psi^{\alpha}_i(\psi_{ij}(\Phi_j, s)+\Gamma_{ij|\mu}, s)\\ &\underset{\mu}{\equiv} e^{\alpha}_{ij}(\Psi_j(\Phi_j, s)+\gamma_{j|\mu})-\Psi^{\alpha}_i(\psi_{ij}(\Phi_j, s), s)-\sum_{\beta}\frac{\partial g^{\alpha}_i}{\partial w^{\beta}_i}(f_i)\Gamma^{\beta}_{ij|\mu}\\ &\underset{\mu}{\equiv} \sum_{\beta}\frac{\partial e^{\alpha}_{ij}}{\partial y^{\beta}_j}(h_j)\gamma^{\beta}_{j|\mu}-\sum_{\beta}\frac{\partial g^{\alpha}_i}{\partial w^{\beta}_i}(f_i)\Gamma^{\beta}_{ij|\mu}\,.\end{aligned}$$

Q. E. D.

We set $\rho'_{\nu ij}=\sum_{\lambda}\frac{\partial\phi^{\lambda}_{ij}}{\partial s^{\nu}}\Big|_{s=0}\frac{\partial}{\partial w^{\lambda}_i}\in\Gamma(V_{ij}, \Theta_Y)$ and $\tau'_{\nu i}=\sum_{\alpha}\frac{\partial\Psi^{\alpha}_i}{\partial s^{\nu}}\Big|_{s=0}\frac{\partial}{\partial y^{\alpha}_i}\in\Gamma(V_i,$ $g^*\Theta_Z)$. Then we have

(7.7) $$\tau'_{\nu j}-\tau'_{\nu i}=G\rho'_{\nu ij}\,,\qquad \text{on}\quad V_{ij}\,.$$

We can easily show that $(7.1)_\mu$ and $(7.2)_\mu$ are equivalent to the following:

(7.8) $$\Gamma_{ij|\mu}=\Phi_{j|\mu}-\Phi_{i|\mu}+\sum_{\nu}s^{\nu}_{\mu}f^*\rho'_{\nu ij}\,,$$

(7.9) $$\gamma_{i|\mu}=(f^*G)\Phi_{i|\mu}+\sum_{\nu}s^{\nu}_{\mu}f^*\tau'_{\nu i}\,,$$

where $\Phi_{i|\mu}$ denote $\sum_{\lambda}\Phi^{\lambda}_{i|\mu}\frac{\partial}{\partial w^{\lambda}_i}$.

LEMMA 7.2. *Under the hypotheses of Proposition* 7.1, *we can find* $\Phi_{i|\mu}\in\Gamma(U_i, f^*\Theta_Y)$ *and* $s_\mu=(s^{\nu}_{\mu})\in C^{r'}$ *which satisfy* (7.8) *and* (7.9).

PROOF. We have an exact sequence

$$0\longrightarrow f^*\Theta_Y\xrightarrow{f^*G} h^*\Theta_Z\longrightarrow f^*\mathfrak{T}_{Y/Z}\longrightarrow 0\,,$$

by hypotheis ii).

By the equality (7.6), the collection $\{\gamma_{i|\mu}\}$ represents an element of $H^0(X, f^*\mathfrak{T}_{Y/Z})$. Hence, by hypothesis iv), we can find $s_\mu=(s^{\nu}_{\mu})$ such that $\{\gamma_{i|\mu}\}$ and $\{\sum_{\nu}s^{\nu}_{\mu}\tau'_{\nu i}\}$ represent the same element in $H^0(X, f^*\mathfrak{T}_{Y/Z})$. This implies the existence of $\Phi_{i|\mu}$ which satisfy (7.9). By hypothesis ii), (7.8) follows from (7.9) and (7.7). This proves Lemma 7.2.

Lemma 7.2 completes the inductive construction of Φ^{μ}_i and s^{μ}.

By a similar argument as in the proof of Theorem 2.1 in Part I, we can show that Φ_i and s converge absolutely and uniformly for sufficiently small $|t|$, if we choose solutions $\Phi_{i|\mu}$ and s_μ of (7.8) and (7.9) properly in each step of the above construction. Q. E. D.

PROPOSITION 7.3. *Let $f: X\to Y$, $g: Y\to Z$ and $h=g\circ f$ be holomorphic maps of complex manifolds. Let $p:\mathcal{X}\to M$, $q:\mathcal{Y}\to M$ and $\pi:\mathcal{Z}\to M$ be families of complex manifolds such that $X=p^{-1}(0)$, $Y=q^{-1}(0)$ and $Z=\pi^{-1}(0)$ for some point $0\in M$. Moreover, let $\Phi:\mathcal{X}\to\mathcal{Y}$ and $\Upsilon:\mathcal{X}\to\mathcal{Z}$ be holomorphic maps over M which induces f and h over $0\in M$, respectively. Assume that*

0) *p and q are proper,*

i) *$f^*: H^0(Y, g^*\Theta_Z)\to H^0(X, h^*\Theta_Z)$ is surjective,*

ii) *$f^*: H^1(Y, g^*\Theta_Z)\to H^1(X, h^*\Theta_Z)$ is injective.*

Then there exist an open neighborhood N of 0, and a holomorphic map $\Psi:\mathcal{Y}|_N\to\mathcal{Z}|_N$ over N such that $\Upsilon|_N=\Psi\circ(\Phi|_N)$.

PROOF. We may assume the following:

i) $M=\{t\in \boldsymbol{C}^r: |t|<\varepsilon\}$, with a sufficiently small $\varepsilon>0$, and $0=(0,\cdots,0)$.

ii) $\mathcal{X}$ (or $\mathcal{Y}$) is covered by a finite number of coordinate neighborhoods $\mathcal{U}_i$ $(i\in I)$ (or $\mathcal{V}_i$ $(i\in J)$) and each $\mathcal{U}_i$ (or $\mathcal{V}_i$) is covered by a system of coordinates (z_i, t) (or (w_i, t)) such that $p(z_i, t)=t$ (or $q(w_i, t)=t$). Moreover, each $\mathcal{V}_i$ is a polydisc $\{(w_i, t): |w_i|<1, |t|<\varepsilon\}$. We set $U_i=\mathcal{U}_i\cap X$ and $V_i=\mathcal{V}_i\cap Y$.

iii) I is a subset of J and $\Phi(\mathcal{U}_i)$ is contained in $\mathcal{V}_i$ for each $i\in I$. Φ is given by $w_i=\Phi_i(z_i, t)$ and we set $f_i(z_i)=\Phi_i(z_i, 0)$.

iv) For each $i\in J$, there exists a coordinate neighborhood $\mathcal{W}_i$ on $\mathcal{Z}$ such that $\Phi(\mathcal{U}_i)\subset\mathcal{W}_i$ for $i\in I$ and $g(V_i)\subset\mathcal{W}_i\cap Z$ for $i\in J$.

v) Each $\mathcal{W}_i$ is covered by a system of coordinates (y_i, t) such that $\pi(y_i, t)=t$.

vi) Υ and g are given, respectively, by $y_i=\Upsilon_i(z_i, t)$ and $y_i=g_i(w_i)$. We set $h_i(z_i)=\Upsilon_i(z_i, 0)$.

vii) $(z_i, t)\in\mathcal{U}_i$, $(w_i, t)\in\mathcal{V}_i$ and $(y_i, t)\in\mathcal{W}_i$ coincide with $(z_j, t)\in\mathcal{U}_j$, $(w_j, t)\in\mathcal{V}_j$ and $(y_j, t)\in\mathcal{W}_j$, respectively, if and only if $z_i=\phi_{ij}(z_j, t)$, $w_i=\psi_{ij}(w_j, t)$ and $y_i=\theta_{ij}(y_j, t)$. We set $e_{ij}(y_j)=\theta_{ij}(y_j, 0)$.

Now we shall construct holomorphic functions $\Psi_i:\mathcal{V}_i\to\boldsymbol{C}^l$ $(l=\dim Z)$ such that

(7.10) $\qquad \Psi_i(w_i, 0)=g_i(z_i) \quad$ on V_i, $\quad$ for $i\in I$,

(7.11) $\qquad \Psi_i(\psi_{ij}, t)=\theta_{ij}(\Psi_j, t) \quad$ on $\mathcal{V}_{ij}$, $\quad$ for $i, j\in J$,

(7.12) $\qquad \Upsilon_i(z_i, t)=\Psi_i(\Phi_i, t) \quad$ on $\mathcal{U}_i$, $\quad$ for $i\in I$.

First we prove the existence of functions Ψ_i as formal power series in t.

Let $(7.11)_\mu$ and $(7.12)_\mu$ denote, respectively, the congruence mod $t^{\mu+1}$ derived from (7.11) and (7.12) by replacing Ψ_i by Ψ^μ_i, for each $\mu=1, 2, \cdots$.

We set $\Psi_{i|0}=g_i(z_i)$. We construct Ψ_i^{μ} by induction on μ. Suppose that we have already determined $\Psi_i^{\mu-1}$ which satisfy $(7.11)_{\mu-1}$ and $(7.12)_{\mu-1}$.

We define homogeneous polynomials $\Gamma_{ij|\mu}\in\Gamma(V_{ij}, g^*\Theta_Z)$ $(i, j\in J)$ and $\gamma_{i|\mu}\in\Gamma(U_i, h^*\Theta_Z)$ $(i\in I)$ of degree μ by the following congruences:

$$\Gamma_{ij|\mu}\underset{\mu}{\equiv}(\Psi_i^{\mu-1}(\phi_{ij}, t)-\theta_{ij}(\Psi_j^{\mu-1}, t))\cdot\frac{\partial}{\partial y_i}, \tag{7.13}$$

$$\gamma_{i|\mu}\underset{\mu}{\equiv}(\Upsilon_i-\Psi_i^{\mu-1}(\Phi_i, t))\cdot\frac{\partial}{\partial y_i}. \tag{7.14}$$

Then we have

$$\Gamma_{jk|\mu}-\Gamma_{ik|\mu}+\Gamma_{ij|\mu}=0 \quad \text{on } V_{ijk}, \quad \text{for } i, j, k\in J, \tag{7.15}$$

$$\gamma_{j|\mu}-\gamma_{i|\mu}=f^*\Gamma_{ij|\mu} \quad \text{on } U_{ij}, \quad \text{for } i, j\in I. \tag{7.16}$$

PROOF. (7.15) has been proved in [7] Lemma 2. (7.16) can be proved as follows: For simplicity, we omit the indices $\mu-1$. We have

$$\begin{aligned}\gamma_{i|\mu}^{\alpha}(z_i, t)&\underset{\mu}{\equiv}\gamma_{i|\mu}^{\alpha}(\phi_{ij}(z_j, t), t)\\&\underset{\mu}{\equiv}\Upsilon_i^{\alpha}(\phi_{ij}, t)-\Psi_i^{\alpha}(\Phi_i(\phi_{ij}, t), t)\\&\underset{\mu}{\equiv}\theta_{ij}^{\alpha}(\Upsilon_j, t)-\Psi_i^{\alpha}(\phi_{ij}(\Phi_j, t), t).\end{aligned}$$

Moreover,

$$\begin{aligned}\Psi_i^{\alpha}(\phi_{ij}(\Phi_j, t), t)&\underset{\mu}{\equiv}\theta_{ij}^{\alpha}(\Psi_j(\Phi_j, t), t)+\Gamma_{ij|\mu}^{\alpha}(f_j)\\&\underset{\mu}{\equiv}\theta_{ij}^{\alpha}(\Upsilon_j, t)-\sum_{\beta}\frac{\partial e_{ij}^{\alpha}}{\partial y_j^{\beta}}(h_j)\gamma_{j|\mu}^{\beta}+\Gamma_{ij|\mu}^{\alpha}(f_j).\end{aligned}$$

This proves (7.16). Q. E. D.

We can easily show that $(7.11)_{\mu}$ and $(7.12)_{\mu}$ are equivalent to the following:

$$\Gamma_{ij|\mu}=\Psi_{j|\mu}-\Psi_{i|\mu} \quad \text{on } V_{ij}, \quad \text{for } i, j\in J, \tag{7.17}$$

$$\gamma_{i|\mu}=f^*\Psi_{i|\mu} \quad \text{on } U_i, \quad \text{for } i\in I, \tag{7.18}$$

where $\Psi_{i|\mu}$ denote $\sum_{\alpha}\Psi_{i|\mu}^{\alpha}\frac{\partial}{\partial y_i^{\alpha}}\in\Gamma(V_i, g^*\Theta_Z)$.

LEMMA 7.4. *Under the hypothesis of Proposition* 7.3, *we can find* $\Psi_{i|\mu}\in\Gamma(V_i, g^*\Theta_Z)$ *which satisfy* (7.17) *and* (7.18).

PROOF. From (7.16) and the hypothesis ii), the 1-cocycle $\{\Gamma_{ij|\mu}\}$ is cohomologous to 0. Hence, we can find $\Psi'_{i|\mu}\in\Gamma(V_i, g^*\Theta_Z)$ such that

$$\Gamma_{ij|\mu}=\Psi'_{j|\mu}-\Psi'_{i|\mu} \quad \text{on } V_{ij}. \tag{7.19}$$

Then $\{\gamma_{i|\mu}-f^*\Psi'_{i|\mu}\}$ represents a homogeneous polynomial with coefficients in $H^0(X, h^*\Theta_Z)$. Hence, by hypothesis i), we can find $\chi_{\mu}\in H^0(Y, g^*\Theta_Z)$ such that

$$\gamma_{i|\mu}-f^*\Psi'_{i|\mu}=f^*\chi_{\mu} \quad \text{on } U_i, \quad \text{for } i\in I. \tag{7.20}$$

We set $\Psi_{i|\mu}=\Psi'_{i|\mu}+\chi_{\mu}$. From (7.19) and (7.20), we infer that $\Psi_{i|\mu}$ satisfy (7.17) and (7.18). This proves Lemma 7.4.

Lemma 7.4 completes the inductive construction of Ψ^{μ}_{i}.

By a similar argument as in the proof of Theorem 2.1 in Part I, we can show that Ψ_i converge absolutely and uniformly for sufficiently small $|t|$ if we choose solutions $\Psi_{i|\mu}$ of (7.17) and (7.18) properly in each step of the above construction.

The following lemma gives a sufficient condition for $f: X\to Y$ to satisfy the hypotheses of Proposition 7.3.

LEMMA 7.5. *Let $f: X\to Y$ be a holomorphic map of compact complex manifolds. Let $\mathcal{O}_X$ and $\mathcal{O}_Y$ denote the structure sheaves, respectively, on X and Y. Moreover let E be a locally free sheaf on Y and let*

$$f_q^*: H^q(Y, E) \longrightarrow H^q(X, f^*E)$$

denote the canonical homomorphism for each $q=0, 1, \cdots$.

Assume that

$$f_*\mathcal{O}_X=\mathcal{O}_Y\,, \qquad R^1f_*\mathcal{O}_X=0\,.$$

Then f_q^* is bijective for $q=0, 1$, and is injective for $q=2$. If moreover f satisfies

$$f_*\mathcal{O}_X=\mathcal{O}_Y\,, \qquad R^qf_*\mathcal{O}_X=0 \qquad \text{for} \quad q>0\,,$$

then f_q^* is bijective for any $q=0, 1, 2, \cdots$.

PROOF. From hypotheses, we infer that

$$f_*f^*E=E \quad \text{and} \quad R^1f_*(f^*E)=0\,.$$

The assertion for $q=0$ follows immediately.

From the spectral sequence

$$E_2^{p,q}=H^p(Y, R^qf_*(f^*E)) \Longrightarrow H^n(X, f^*E)\,, \tag{7.21}$$

we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(Y, E) \longrightarrow H^1(X, f^*E) \longrightarrow H^0(Y, R^1f_*(f^*E)) \\ \longrightarrow H^2(Y, E) \longrightarrow H^2(X, f^*E) \end{aligned}$$

(see [**1**], I. 4.5.1). Hence, the assertions for $q=1, 2$ follow from the vanishing of $R^1f_*(f^*E)$.

If moreover f satisfies $R^qf_*\mathcal{O}_X=0$, for $q>0$, then we have $R^qf_*(f^*E)=0$ for $q>0$. Hence the spectral sequence (7.21) degenerates (see [**1**], I. 4.4). This implies that f_q^* is bijective for any $q=0, 1, 2, \cdots$. Q. E. D.

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