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On Veronese manifolds

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From differential geometric point of view, a Veronese surface may be considered as a minimal immersion of a 2-dimensional sphere of curvature 1/3 into a 4-dimensional unit sphere. S. S. Chern, M. doCarmo and S. Kobayashi [**1**] gave a local characterization of a Veronese surface. The author [**4**] also characterized a Veronese surface by non-zero constant normal curvature.

We call an isometric immersion (a submanifold) an *isotropic immersion* (*isotropic*) if all its normal curvature vectors have the same length at each point. Let $P^n(c)$ (resp. $P_n(c)$) be an n-dimensional real (resp. complex) projective space of curvature c and $S^m(c)$ be an m-dimensional sphere of curvature c. B. O'Neill [**10**] proved the following results:

(A) *There exists a non-umbilic isotropic minimal imbedding* $\psi: P^n(c)\rightarrow S^{n+p}(\tilde{c})$ *where* $c=\dfrac{n\tilde{c}}{2(n+1)}$ *and* $p=\dfrac{1}{2}n(n+1)-1$.

(B) *There exists a Kaehler imbedding* $\psi: P_n(c)\rightarrow P_{n+p}(\tilde{c})$, *where* $2c=\tilde{c}$ *and* $p=\dfrac{1}{2}n(n+1)$.

Taking account of [**8**], we may call these submanifolds *the Veronese submanifolds*, in particular, we may call the former *the real Veronese submanifold* and the latter *the complex Veronese submanifold.* M. doCarmo and N. Wallach [**2**] characterized a real Veronese submanifold. The author and K. Ogiue ([**6**], [**7**]) also gave some characterizations of a real Veronese submanifold in terms of isotropic immersion. K. Ogiue [**9**] gave characterizations of a complex Veronese submanifold.

The purpose of the present paper is to characterize Veronese manifolds by means of geometric invariant functions on submanifolds.

§ 1. Real submanifolds in real space forms.

Let M^n be an n-dimensional submanifold immersed in an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}$ of constant curvature $\tilde{c}$, (i. e., Riemannian submani-

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fold with induced Riemannian metric). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation on M^n (resp. $\tilde{M}^{n+p}$), then *the second fundamental form* (*the shape operator*) σ of the immersion is given by

$$\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y\,, \quad \text{where } X \text{ and } Y \text{ are vector fields on } M^n\,,$$

and it satisfies $\sigma(X, Y)=\sigma(Y, X)$. We choose a local field of orthonormal frames $e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ in $\tilde{M}^{n+p}$ in such a way that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n (and, consequently, the remaining vectors are normal to $\tilde{M}^{n+p}$). With respect to the frame field of $\tilde{M}^{n+p}$ chosen above, let $\tilde{\omega}_1, \cdots, \tilde{\omega}^{n+p}$ be the field of dual frames. Then the structure equations of M^{n+p} are given by(*)

$$\text{(1.1)} \qquad d\tilde{\omega}_A=\sum \tilde{\omega}_{AB}\wedge\tilde{\omega}_B\,, \qquad \tilde{\omega}_{AB}+\tilde{\omega}_{BA}=0\,,$$

$$\text{(1.2)} \qquad d\tilde{\omega}_{AB}=\sum \tilde{\omega}_{AC}\wedge\tilde{\omega}_{CB}-\tilde{c}\tilde{\omega}_A\wedge\tilde{\omega}_B\,.$$

Restricting these forms to M^n, we have the structure equations of the immersion:

$$\text{(1.3)} \qquad \omega_\alpha=0\,,$$

$$\text{(1.4)} \qquad \omega_{i\alpha}=\sum h_{ij}^\alpha\omega_j\,, \qquad h_{ij}^\alpha=h_{ji}^\alpha\,,$$

$$\text{(1.5)} \qquad d\omega_i=\sum \omega_{ij}\wedge\omega_j\,, \qquad \omega_{ij}+\omega_{ji}=0\,,$$

$$\text{(1.6)} \qquad d\omega_{ij}=\sum \omega_{ik}\wedge\omega_{kj}-\Omega_{ij}\,, \qquad \Omega_{ij}=\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l\,,$$

$$\text{(1.7)} \qquad R_{ijkl}=\tilde{c}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum(h_{ik}^\alpha h_{jl}^\alpha-h_{il}^\alpha h_{jk}^\alpha)\,,$$

$$\text{(1.8)} \qquad d\omega_{\alpha\beta}=\sum \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\Omega_{\alpha\beta}\,, \qquad \Omega_{\alpha\beta}=\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j\,,$$

$$\text{(1.9)} \qquad R_{\alpha\beta ij}=\sum(h_{ik}^\alpha h_{kj}^\beta-h_{jk}^\alpha h_{ki}^\beta)\,.$$

Then, the second fundamental form (the shape operator) σ can be written as

$$\sigma(X, Y)=\sum h_{ij}^\alpha\omega_i(X)\omega_j(Y)e_\alpha \qquad \text{or} \qquad \sigma(e_i, e_j)=\sum h_{ij}^\alpha e_\alpha\,.$$

If we define h_{ijk}^α by

$$\sum h_{ijk}^\alpha\omega_k := dh_{ij}^\alpha+\sum h_{ik}^\alpha\omega_{kj}+\sum h_{kj}^\alpha\omega_{ki}+\sum h_{ij}^\beta\omega_{\beta\alpha}\,,$$

then, from (1.2), (1.3) and (1.4), we have $h_{ijk}^\alpha=h_{ikj}^\alpha$. The second fundamental form σ is said to be *parallel* if $h_{ijk}^\alpha=0$ for all α, i, j, k.

Now, we consider the *following non-negative functions* on M^n:

(*) We use the following convention on the range of indices unless otherwise stated; $A, B, C=1, 2, \cdots, n+p$, $i, j, k, l=1, 2, \cdots, n$, $\alpha, \beta, \gamma=n+1, \cdots, n+p$, and we agree that repeated indices under a summation sign without indication are summed over the respective range.

$$S:=\|\sigma\|^2=\sum h_{ij}^{\alpha}h_{ij}^{\alpha}\,, \qquad ([\mathbf{1}])\,,$$

(1.10) $$K_N:=\sum R_{\alpha\beta ij}R_{\alpha\beta ij}=\sum\Big(\sum_k (h_{ik}^{\alpha}h_{kj}^{\beta}-h_{jk}^{\alpha}h_{ki}^{\beta})\Big)^2\,, \qquad ([\mathbf{5}])\,,$$

(1.11) $$L_N:=\sum\langle\sigma(e_i, e_j),\ \sigma(e_k, e_l)\rangle^2=\sum\Big(\sum_{ij} h_{ij}^{\alpha}h_{ij}^{\beta}\Big)^2=\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ij}^{\beta}h_{kl}^{\beta}\,,$$

where $\langle\,,\,\rangle$ is the inner product in the normal space to M^n. We know that S is *the square of the length of the second fundamental form* and K_N is *the square of the length of curvature tensor of the normal bundle*, which is called the *normal scalar curvature* of M^n in $\tilde{M}$ defined in [**5**]. Set $S_{\alpha\beta}:=\sum h_{ij}^{\alpha}h_{ij}^{\beta}$, the $(p\times p)$-matrix $(S_{\alpha\beta})$ is symmetric and can be diagonal for a suitable choice of frames $e_{n+1},\cdots,e_{n+p}$. Then, setting $S_{\alpha}:=S_{\alpha\alpha}=\sum(h_{ij}^{\alpha})^2$, we have $S=\sum S_{\alpha}$ and $L_N=\sum S_{\alpha}^2$. Since S_{α} is the square of the length of the second fundamental form for the direction e_{α}, we may consider L_N as *the sum of the 4-th power of the length of the second fundamental forms for suitable frames.*

We know that if M^n is minimal, i.e., $\sum h_{ii}^{\alpha}=0$ for any α, then the functions above satisfy a differential equation, that is,

LEMMA 1 ([**1**]). *Let Δ be the Laplacian. Then we have*

(1.12) $$\frac{1}{2}\Delta S=n\tilde{c}S-K_N-L_N+\sum(h_{ijk}^{\alpha})^2 \qquad on \quad M^n\,.$$

M^n is said to be *λ-isotropic at a point $x\in M^n$* if $\sigma(X, X)$ has the same length λ for any unit tangent vector X to M^n at x. We say M^n is *isotropic* if it is isotropic at each point of M^n. Then we have the following

LEMMA 2 ([**10**]). *M^n is λ-isotropic at a point $x\in M^n$ if and only if setting $A_{ij}=\sigma(e_i, e_j)$, for any orthonormal frames of M^n at x, we have*

(1.13) $$\begin{gathered}\|A_{jj}\|=\lambda\,, \qquad \langle A_{ii}, A_{ij}\rangle=0\,, \qquad \langle A_{ii}, A_{jj}\rangle+2\|A_{ij}\|^2=\lambda^2\,,\\ \langle A_{ii}, A_{jk}\rangle+2\langle A_{ij}, A_{ik}\rangle=\langle A_{ij}, A_{kl}\rangle+\langle A_{ik}, A_{jl}\rangle+\langle A_{il}, A_{jk}\rangle=0\,,\end{gathered}$$

where $\langle\,,\,\rangle$ denotes the inner product in the normal space N_x at x and all indices are different from each other.

§ 2. Real Veronese submanifolds.

We first prove the following

PROPOSITION 1. *Let M^n be an n-dimensional submanifold immersed in an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}$ of constant curvature $\tilde{c}$. Then we get*

(2.1) $$K_N\leqq nL_N \qquad everywhere\ on\ M^n\,,$$

where the equality holds at a point $x\in M^n$ if and only if M^n is isotropic and minimal at x and the sectional curvature of M^n at x is constant for all tangent *planes to* M^n *at* x.

PROOF. Denoting $A_{ij}:=\sigma(e_i, e_j)=\sum h_{ij}^{\alpha}e_{\alpha}$, from (1.10) and (1.11) we get

$$K_N=2\sum_{i,j}(\sum_k\langle A_{ik}, A_{kj}\rangle)^2-2\sum\langle A_{ik}, A_{jl}\rangle\langle A_{il}, A_{kj}\rangle \tag{2.2}$$
$$=2\sum_{\neq}\langle A_{ik}, A_{kj}\rangle\langle A_{il}, A_{lj}\rangle+4\sum_{\neq}\langle A_{ii}, A_{ij}\rangle\langle A_{jj}, A_{ji}\rangle$$
$$+8\sum_{\neq}\langle A_{ii}, A_{ij}\rangle\langle A_{ik}, A_{kj}\rangle-2\sum_{\neq}\langle A_{ik}, A_{jl}\rangle\langle A_{il}, A_{kj}\rangle$$
$$-8\sum_{\neq}\langle A_{ij}, A_{ik}\rangle\langle A_{ii}, A_{kj}\rangle-4\sum_{\neq}\langle A_{ii}, A_{ij}\rangle^2-2\sum_{\neq}\langle A_{ik}, A_{kj}\rangle^2$$
$$+2\sum_{\neq}\|A_{ij}\|^2\cdot\|A_{ik}\|^2+4\sum_{\neq}\|A_{ii}\|^2\cdot\|A_{ij}\|^2-4\sum_{\neq}\|A_{ij}\|^2\cdot\langle A_{ii}, A_{jj}\rangle\,,$$

$$L_N=\sum\langle A_{ij}, A_{kl}\rangle^2=\sum_{\neq}\langle A_{ij}, A_{kl}\rangle^2+4\sum_{\neq}\langle A_{ij}, A_{ik}\rangle^2+4\sum_{\neq}\langle A_{ii}, A_{ij}\rangle^2 \tag{2.3}$$
$$+2\sum_{\neq}\langle A_{ii}, A_{kl}\rangle^2+\sum_{\neq}\langle A_{ii}, A_{jj}\rangle^2+2\sum_{\neq}\|A_{ij}\|^4+\sum\|A_{ii}\|^4\,,$$

where $\sum_{\neq}$ denotes the summation over different indices.

From these equations we have

$$nL_N-K_N=\sum_{\neq}\{\langle A_{ik}, A_{kj}\rangle-\langle A_{il}, A_{lj}\rangle\}^2+2\sum_{\neq}\{\langle A_{ii}, A_{ij}\rangle-\langle A_{jj}, A_{ji}\rangle\}^2 \tag{2.4}$$
$$+\sum_{\neq}\{\langle A_{ij}, A_{kl}\rangle+\langle A_{ik}, A_{jl}\rangle\}^2+(n-2)\sum_{\neq}\langle A_{ij}, A_{kl}\rangle^2$$
$$+4\sum_{\neq}\{\langle A_{ij}, A_{ik}\rangle+\langle A_{ii}, A_{kj}\rangle\}^2+2n\sum_{\neq}\langle A_{ik}, A_{kj}\rangle^2$$
$$+4\sum_{\neq}\{\langle A_{ii}, A_{ij}\rangle-\langle A_{ik}, A_{kj}\rangle\}^2+2(n-2)\sum_{\neq}\langle A_{ii}, A_{jk}\rangle^2$$
$$+n\sum_{\neq}\{\langle A_{ii}, A_{jj}\rangle+2\|A_{ij}\|^2/n\}^2+8\sum_{\neq}\langle A_{ii}, A_{ij}\rangle^2$$
$$+\frac{4(n-1)}{n}\sum_{\neq}\{\|A_{ij}\|^2-n\|A_{ii}\|^2/2(n-1)\}^2$$
$$+\sum_{\neq}\{\|A_{ij}\|^2-\|A_{ik}\|^2\}^2\geqq 0\,,$$

where the equality holds at $x\in M^n$ if and only if for any orthonormal frames $e_1, \cdots, e_{n+p}$ in $\tilde{M}$ such that $e_1, \cdots, e_n$ are tangent to M^n, we have

$$\begin{cases} \langle A_{ii}, A_{ij}\rangle=\langle A_{ij}, A_{ik}\rangle=\langle A_{ij}, A_{kl}\rangle=\langle A_{ii}, A_{jk}\rangle=0\,, \\ \|A_{ij}\|^2=n\|A_{ii}\|^2/2(n-1)\,, \qquad n\langle A_{ii}, A_{jj}\rangle+2\|A_{ij}\|^2=0\,, \\ \|A_{ij}\|=\|A_{ik}\|\,, \end{cases} \tag{2.5}$$

where different indices indicate different numbers. From (2.5) we have

$$\|A_{ii}\|=\|A_{jj}\| \qquad \text{for all } i,\ j\,,$$

and so

$$\|\sum A_{ii}\|^2=n\|A_{ii}\|^2+\sum_{\neq}\langle A_{ii}, A_{jj}\rangle=0\,,$$

which implies that the immersion is minimal at x.

We assume that the equality of (2.1) holds at a point $x\in M$. Then, using (1.7) and (2.5), we easily have

$$R_{ijkl}=\left(\tilde{c}-\frac{S}{n(n-1)}\right)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),$$

which implies that the sectional curvature of M^n at x is constant for all tangent planes to M^n at x. We get (1.13) from (2.5), so the immersion is isotropic at x.

Coversely, we assume that the immersion is isotropic and minimal at $x\in M^n$ and the sectional curvature of M^n at x is constant c for all tangent planes to M^n at x. It follows from (1.7) that we have

$$\langle A_{ik}, A_{jl}\rangle-\langle A_{il}, A_{jk}\rangle=(\tilde{c}-c)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}),$$

from which we have

$$\text{(2.6)}\qquad \begin{cases} \langle A_{ij}, A_{jl}\rangle=\langle A_{il}, A_{jk}\rangle & \text{if at least three indices are different} \\ \|A_{ij}\|^2-\langle A_{ii}, A_{jj}\rangle=\tilde{c}-c, & (i\neq j), \\ \langle A_{ii}, A_{jk}\rangle=\langle A_{ij}, A_{ik}\rangle, & (j\neq k). \end{cases}$$

Since the immersion is minimal at x, we easily get (2.5) from (1.13) and (2.6), so the equality of (2.1) holds at x. Q. E. D.

Now, using this proposition 1, we can prove the following

THEOREM 1. *Let M^n be an n-dimensional, compact, oriented and connected submanifold minimally immersed in an $(n+p)$-dimensional sphere $S^{n+p}(\tilde{c})$ of curvature $\tilde{c}$. If the immersion is full and the following inequality holds everywhere on M^n;*

$$(n+1)L_N\leqq n\tilde{c}S$$

where L_N is the function on M^n defined by (1.11) *and S is the square of the length of the second fundamental form, then $p=0$ or $p=\frac{1}{2}n(n+1)-1$ and M^n is of constant curvature $\frac{n\tilde{c}}{2(n+1)}$, i.e., M^n is a real Veronese submanifold in $S^{n+p}(\tilde{c})$.*

PROOF. It follows from Lemma 1, (2.1) and our assumption that we have

$$\text{(2.7)}\qquad \frac{1}{2}\Delta S=\sum(h^{\alpha}_{ijk})^2+n\tilde{c}S-K_N-L_N\geqq n\tilde{c}S-(n+1)L_N\geqq 0.$$

Since M^n is compact and oriented, we have $\int_{M^n}\Delta S dV=0$, where dV is the volume element of M^n. Then from (2.7) we have

$$\text{(2.8)}\qquad h^{\alpha}_{ijk}=0 \quad \text{for all } i, j, k \text{ and } \alpha,$$

$$(2.9)\qquad n\tilde{c}S=(n+1)L_n=\frac{n+1}{n}K_n \quad \text{everywhere on } M^n.$$

It follows from Proposition 1 that M^n is isotropic. Next, we shall show that M^n is of constant curvature c and $p=0$ or $p=\frac{1}{2}n(n+1)-1$. Setting $\lambda:=\|A_{ii}\|$, from (2.5) we have

$$(2.10)\qquad S=\frac{1}{2}n(n+2)\lambda^2.$$

Since S is constant by (2.8), λ is so on M^n. From (1.7) we have

$$(2.11)\qquad c=\tilde{c}-\frac{S}{n(n-1)}.$$

It follows from (2.5), (2.9) and (2.10) that we have

$$(2.12)\qquad S=0 \quad \text{or} \quad S=\frac{n(n-1)(n+2)\tilde{c}}{2(n+1)} \quad \text{on } M^n.$$

If $S=0$ on M^n, then from (2.11) we have $c=\tilde{c}$ and M^n is totally geodesic (i.e., $p=0$). If $S>0$ on M^n, from (2.11) and (2.12) we get $c=\frac{n\tilde{c}}{2(n+1)}$. In this case, from (2.5) we easily see that the dimension of the vector space generated by $\{\sigma(e_i, e_i)\}_{1\leqq i\leqq n}$ and $\{\sigma(e_i, e_j)\}_{1\leqq i<j\leqq n}$ is $\frac{1}{2}n(n+1)-1$, that is, the dimension of the first normal space of M^n is $\frac{1}{2}n(n+1)-1$ at each point of M^n. On the other hand, it follows from (2.8) that the second fundamental form σ is parallel everywhere on M^n. Therefore, by Theorem in [3], we see that $p=\frac{1}{2}n(n+1)-1$.

By means of Theorem 1 in [6], we see that M^n is a real Veronese submanifold in $S^{n+p}(\tilde{c})$. Q.E.D.

§ 3. Kaehler submanifolds of complex space forms.

Let $\tilde{M}_{n+p}(\tilde{c})$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$ and M_n be an n-dimensional Kaehler submanifold immersed in $\tilde{M}_{n+p}(\tilde{c})$ (i.e., complex submanifold with the induced Kaehler structure). Let J (resp. $\tilde{J}$) be the complex structure of M_n (resp. $\tilde{M}_{n+p}(\tilde{c})$) and let g (rssp. $\tilde{g}$) be the Kaehler metric of M_n (resp. $\tilde{M}_{n+p}(\tilde{c})$). We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. $\tilde{g}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y, \quad \text{for vector fields } X \text{ and } Y \text{ on } M_n,$$

and it satisfies

$$(3.1)\qquad \sigma(X, Y)=\sigma(Y, X), \qquad \sigma(JX, Y)=\sigma(X, JY)=\tilde{J}\sigma(X, Y).$$

We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{1*}=\tilde{J}e_1, \cdots, e_{n*}=\tilde{J}e_n, e_{\tilde{1}}, \cdots, e_{\tilde{p}}, e_{\tilde{1}*}=\tilde{J}e_{\tilde{1}}, \cdots, e_{\tilde{p}*}=\tilde{J}e_{\tilde{p}}$ in $\tilde{M}_{n+p}(\tilde{c})$ in such a way that, restricted to M_n, $e_1, \cdots, e_n, e_{1*}, \cdots, e_{n*}$ are tangent to M_n[(#)]. With respect to the frame field of $\tilde{M}_{n+p}(\tilde{c})$ chosen above, let $\omega_1, \cdots, \omega_n, \omega_{1*}, \cdots, \omega_{n*}, \omega_{\tilde{1}}, \cdots, \omega_{\tilde{p}}, \omega_{\tilde{1}*}, \cdots, \omega_{\tilde{p}*}$ be the field of dual frames. Then the Kaehler metric can be expressed locally as $g=\sum \omega_j\omega_j$ and $\tilde{g}=\sum \omega_I\omega_I$. The structure equations of $\tilde{M}_{n+p}(\tilde{c})$ are given by

$$d\omega_I=\sum \omega_{IJ}\wedge\omega_J\,, \qquad \omega_{IJ}+\omega_{JI}=0\,, \tag{3.2}$$

$$\begin{aligned} &\omega_{ab}=\omega_{a*b*}\,, \qquad \omega_{\alpha\beta}=\omega_{\alpha*\beta*}\,, \qquad \omega_{a\beta}=\omega_{a*\beta*}\,, \\ &\omega_{ab*}=\omega_{ba*}\,, \qquad \omega_{\alpha\beta*}=\omega_{\beta\alpha*}\,, \qquad \omega_{a\beta*}=\omega_{\beta a*}\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &d\omega_{IJ}=\sum \omega_{IK}\wedge\omega_{KJ}-\tilde{\Omega}_{IJ}\,, \qquad \tilde{\Omega}_{IJ}=\frac{1}{2}\sum \tilde{R}_{IJKL}\omega_K\wedge\omega_L\,, \\ &\tilde{R}_{IJKL}=\frac{\tilde{c}}{4}(\delta_{IK}\delta_{JL}-\delta_{IL}\delta_{JK}+\tilde{J}_{IK}\tilde{J}_{JL}-\tilde{J}_{IL}\tilde{J}_{JK}+2\tilde{J}_{IJ}\tilde{J}_{KL}) \end{aligned} \tag{3.4}$$

where

$$(J_{KL})=\begin{pmatrix} 0 & -I_n & & \mathbf{0} \\ I_n & 0 & & \\ & & 0 & -I_p \\ \mathbf{0} & & I_p & 0 \end{pmatrix}, \quad I_s \text{ being the identity matrix of degree } s.$$

Restricting these forms to M_n, we have the structure equations of M_n:

$$\begin{aligned} &\omega_\mu=0\,, \qquad \omega_{i\mu}=\sum h_{ij}^\mu\omega_j\,, \qquad h_{ij}^\mu=h_{ji}^\mu\,, \\ &d\omega_i=\sum \omega_{ij}\wedge\omega_j\,, \qquad \omega_{ij}+\omega_{ji}=0\,, \\ &d\omega_{ij}=\sum \omega_{ik}\wedge\omega_{kj}-\Omega_{ij}\,, \qquad \Omega_{ij}=\frac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l\,, \\ &\Omega_{ij}=\tilde{\Omega}_{ij}-\sum \omega_{i\mu}\wedge\omega_{\mu j}\,. \end{aligned} \tag{3.5}$$

We can easily see that the equation of Gauss is written as

$$R_{ijkl}=\frac{\tilde{c}}{4}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}+J_{ik}J_{jl}-J_{il}J_{jk}+2J_{ij}J_{kl})+\sum(h_{ik}^\mu h_{jl}^\mu-h_{il}^\mu h_{jk}^\mu)\,. \tag{3.6}$$

(#) We use the following convention on the range of indices unless otherwise stated:

$$A, B, C, D=1, \cdots, n, \tilde{1}, \cdots, \tilde{p}\,; \qquad a, b, c, d=1, \cdots, n\,;$$
$$I, J, K, L=1, \cdots, n, 1^*, \cdots, n^*, \tilde{1}, \cdots, \tilde{p}, \tilde{1}^*, \cdots, \tilde{p}^*\,;$$
$$i, j, k, l=1, \cdots, n, 1^*, \cdots, n^*\,;$$
$$\mu, \nu, \cdots=1, \cdots, p, \tilde{1}^*, \cdots, \tilde{p}^*\,; \qquad \alpha, \beta, \gamma=\tilde{1}, \cdots, \tilde{p}\,.$$

and we agree that repeated indices under a summation sign without indication are summed over the respective range.

Now, we can consider the following non-negative functions on M_n:

$$K_N := \sum(\sum_k (h^\mu_{ik} h^\nu_{kj} - h^\mu_{jk} h^\nu_{ki}))^2\,, \tag{3.7}$$

$$L_N := \sum h^\mu_{ij} h^\nu_{ij} h^\mu_{kl} h^\nu_{kl}\,, \qquad \text{(see Lemma 3.4 in [9])}. \tag{3.8}$$

Then we know the following useful result.

LEMMA 3 ([**9**]). *Let Δ denote the Laplacian. Then we have*

$$\frac{1}{2}\Delta S = \sum(h^\mu_{ijk})^2 - K_N - L_N + \frac{n+2}{2}\tilde{c}S\,, \tag{3.9}$$

where S is the square of the length of the second fundamental form.

§ 4. Complex Veronese submanifolds.

We first show the following

PROPOSITION 2. *Let M_n be an n-dimensional Kaehler submanifold immersed in an $(n+p)$-dimensional complex space form M_{n+p}. Then we have*

$$K_N \leqq (n+1)L_N \qquad \text{everywhere on } M_n \tag{4.1}$$

where the equality of (4.1) *holds at a point $x \in M_n$ if and only if the holomorphic sectional curvature of M_n at x is constant.*

PROOF. Denoting $A_{ij} := \sigma(e_i, e_j) = \sum h^\mu_{ij} e_\mu$, from (3.1) we see

$$A_{a^*b^*} = -A_{ab}\,, \qquad A_{ab^*} = A_{a^*b} = \tilde{J}A_{ab}\,. \tag{4.2}$$

By means of (2.2), (2.3) and (4.2), we have

$$\begin{aligned} K_N = {} & 2\sum_{\neq}\langle A_{ik}, A_{kj}\rangle\langle A_{il}, A_{lj}\rangle + 4\sum_{\neq}\langle A_{ii}, A_{ij}\rangle\langle A_{jj}, A_{ji}\rangle \\ & + 8\sum_{\neq}\langle A_{ii}, A_{ij}\rangle\langle A_{ik}, A_{kj}\rangle - 8\sum_{\neq}\langle A_{ij}, A_{ik}\rangle\langle A_{ii}, A_{kj}\rangle \\ & - 4\sum_{\neq}\langle A_{ii}, A_{ij}\rangle^2 - 2\sum_{\neq}\langle A_{ik}, A_{kj}\rangle^2 + 16\sum_{\neq}\|A_{ab}\|^4 \\ & - 2\sum_{\neq}{}'\langle A_{ik}, A_{jl}\rangle\langle A_{il}, A_{kj}\rangle + 32\sum_{\neq}\|A_{aa}\|^2\cdot\|A_{ab}\|^2 \\ & + 16\sum_{\neq}\|A_{ab}\|^2\cdot\|A_{ac}\|^2 + 16\sum\|A_{aa}\|^4\,, \end{aligned} \tag{4.3}$$

$$\begin{aligned} L_N = {} & \sum_{\neq}{}'\langle A_{ij}, A_{kl}\rangle^2 + 4\sum_{\neq}\langle A_{ij}, A_{ik}\rangle^2 + 4\sum_{\neq}\langle A_{ii}, A_{ij}\rangle^2 + 2\sum_{\neq}\langle A_{ii}, A_{kl}\rangle^2 \\ & + 4\sum_{\neq}\langle A_{aa}, A_{bb}\rangle^2 + 16\sum_{\neq}\|A_{ab}\|^4 + 8\sum\|A_{aa}\|^4\,, \end{aligned} \tag{4.4}$$

where $\sum_{\neq}$ denotes the summation over different indices and $\sum_{\neq}{}'$ denotes the summation over different indices except the case $k=i^*$ and $l=j^*$. From these equations we have

$$(4.5)\quad \begin{aligned}&(n+1)L_N-K_N\\ &=\sum_{\neq}'\{\langle A_{ij}, A_{kl}\rangle+\langle A_{ik}, A_{jl}\rangle\}^2+(n-1)\sum_{\neq}'\langle A_{ij}, A_{kl}\rangle^2\\ &\quad+\sum_{\neq}\{\langle A_{ik}, A_{kj}\rangle-\langle A_{il}, A_{lj}\rangle\}^2+2\sum_{\neq}\{\langle A_{ii}, A_{ij}\rangle-\langle A_{jj}, A_{ji}\rangle\}^2\\ &\quad+4\sum_{\neq}\{\langle A_{ij}, A_{ik}\rangle+\langle A_{ii}, A_{jk}\rangle\}^2+2(n+1)\sum_{\neq}\langle A_{ik}, A_{jk}\rangle^2\\ &\quad+4\sum_{\neq}\{\langle A_{ii}, A_{ij}\rangle-\langle A_{ik}, A_{kj}\rangle\}^2+2(n-1)\sum_{\neq}\langle A_{ii}, A_{jk}\rangle^2\\ &\quad+4(n+1)\sum_{\neq}\langle A_{aa}, A_{bb}\rangle^2+8\sum_{\neq}\{\|A_{ab}\|^2-\|A_{ac}\|^2\}^2\\ &\quad+8\sum_{\neq}\{\|A_{aa}\|^2-2\|A_{ab}\|^2\}^2\geqq 0,\end{aligned}$$

where the equality holds at a point $x\in M_n$ if and only if for any above orthonormal frames of M_n we have

$$(4.6)\quad \begin{aligned}&\langle A_{ik}, A_{kj}\rangle=\langle A_{ii}, A_{ij}\rangle=\langle A_{ii}, A_{jk}\rangle=0,\\ &\langle A_{ij}, A_{kl}\rangle=0, \qquad (k\neq i^*,\ l\neq j^*)\\ &\|A_{ab}\|^2=\|A_{ac}\|^2=\frac{1}{2}\|A_{aa}\|^2, \qquad \langle A_{aa}, A_{bb}\rangle=0,\end{aligned}$$

where different indices indicate different numbers.

From (4.6) we easily see that the holomorphic sectional curvature of M_n at x is constant.

When the holomorphic sectional curvature of M_n at x is constant c, we easily obtain $K_N=(n+1)L_N=n(n+1)^2(\tilde{c}-c)^2$. Q.E.D.

Using this result, we have the following

THEOREM 2. *Let M_n be an n-dimensional compact Kaehler submanifold immersed in an $(n+p)$-dimensional complex projective space $P_{n+p}(\tilde{c})$ of curvature $\tilde{c}$. If the immersion is full and the following inequality holds*

$$2L_N\leqq\tilde{c}S \qquad \text{everywhere on } M_n,$$

where $S=\|\sigma\|^2$ and L_N is the function on M_n defined by (3.8), *then $p=0$ or $p=\frac{1}{2}n(n+1)$ and M_n is of constant holomorphic sectional curvature $\frac{\tilde{c}}{2}$, i.e., M_n is a complex Veronese submanifold in $P_{n+p}(\tilde{c})$.*

PROOF. From Lemma 3, (4.1) and our assumption we have

$$(4.7)\quad \frac{1}{2}\Delta S=\sum(h^{\mu}_{ijk})^2+\frac{n+2}{2}\tilde{c}S-K_N-L_N\geqq\frac{n+2}{2}(\tilde{c}S-2L_N)\geqq 0.$$

Since M_n is compact, we easily see that all equalities of (4.7) hold everywhere on M_n. Therefore we have

$$(4.8)\quad h^{\mu}_{ijk}=0 \qquad \text{for all } i, j, k \text{ and } \mu,$$

(4.9) $$2K_N=2(n+1)L_N=(n+1)\tilde{c}S \quad \text{everywhere on } M_n.$$

It follows from (3.6) and Proposition 2 that the holomorphic sectional curvature H is given by $H=\tilde{c}-S/\{n(n+1)\}$. Since S is constant by (4.8), H is so, i. e., M_n is of constant holomorphic sectional curvature c. Using (4.6) and (4.8), we easily see that the dimension of the first normal space of M_n is not greater than $\frac{1}{2}n(n+1)$. Then, by Theorem in [**3**], $p\leqq\frac{1}{2}n(n+1)$. Therefore, by means of Theorem 4.4 in [**9**], we complete the proof. Q. E. D.

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