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The Cauchy problem for an involutive system of partial differential equations in two independent variables

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§ 0. Introduction.

Let Φ be an involutive system of partial differential equations. The existence of (local) *analytic* solutions of Φ is guaranteed by the Cartan-Kähler theorem. However the existence of *differentiable* solutions of Φ has not been shown as yet. Suppose that Φ has only one unknown function of two independent variables. The structure of such Φ has been investigated in detail by the author [5]. In particular he introduced the characteristic polynomial of Φ. The purpose of this paper is to prove the existence of C^∞ solutions of Φ under the condition that the characteristic polynomial of Φ possesses real and distinct roots; We prove this by showing that the Cauchy problem for Φ with initial data on a non-characteristic curve possesses a unique C^∞ solution. Since our existence proof may be a clue to showing the existence of C^∞ solutions of a general involutive system (satisfying some appropriate condition), we have a good reason to discuss the existence of C^∞ solutions of such systems as above.

In the case of a single hyperbolic equation of the second order with two independent variables, it is H. Lewy [6] who proved the result for the first time. This result was extended to the case of a single hyperbolic equation of higher order by H. Lewy himself and K. O. Friedrichs [7] and completed by M. Cinquini-Cibrario [1]. These results were proved by reducing the Cauchy problem to that for a system of equations in characteristic form. They can also be proved quite differently by reducing a given equation to a hyperbolic system of the first order with several unknown functions (cf. Friedrichs [3], Courant and Hilbert [2], Chapter V). By applying the theory developed by the author [5] and the existence theorem of M. Cinquini-Cibrario [1], we prove our result for an involutive system by the method given by K. O. Friedrichs and H. Lewy [7]. From the proof it also follows that the value of the solution

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of the Cauchy problem at a point depends only upon the data on the finite segment of the initial curve (the domain of dependence); it is given by the segment of the initial curve cut out by the both outer characteristic curves passing that point.

Every notions appearing in this paper are assumed to be in the category of infinite differentiability although this assumption can be refined.

§1. The Cauchy problem.

Let Φ be a system of partial differential equations of order m. Suppose that it has a single unknown function of two independent variables. Let (M, N, ρ) be a differentiable fibered manifold where ρ is the projection from M onto the base space N such that $\dim M=3$ and $\dim N=2$, and let $J^m(M, N, \rho)$ be the space of m-jets of cross-sections of (M, N, ρ). We denote by $\mathcal{E}(J^m)$ the sheaf of germs of differentiable functions on $J^m(M, N, \rho)$. Φ is defined to be a locally finitely generated subsheaf of ideals in $\mathcal{E}(J^m)$. The mapping ρ_k^m from $J^m(M, N, \rho)$ to $J^k(M, N, \rho)$ is defined by $\rho_k^m(j_a^m(f))=j_a^k(f)$ $(0\leqq k\leqq m)$. We shall denote $\rho\circ\rho_0^m$ by ρ_{-1}^m. Let (x, y, z) be a coordinate system on a neighbourhood U of M such that there exists a coordinate system (x', y') on ρU satisfying $x=x'\circ\rho$, $y=y'\circ\rho$. Then a coordinate system of $J^m(M, N, \rho)$ on $(\rho_0^m)^{-1}U$ is given by $(x, y, z, p_{i,k};\ 1\leqq i+k\leqq m)$, where $p_{i,k}=\partial^{i+k}z(x, y)/\partial x^i\partial y^k$, $z(x, y)$ being z-coordinate of a cross-section of (M, N, ρ). For convenience we shall write $p_{0,0}=z$.

The set of the integral points of Φ is denoted by $I\Phi$. Let X_0 be an integral point of Φ. Suppose that $\Phi=0$ gives a regular local equation of $I\Phi$ around X_0, and that one can take a system of local generators

$$F_\alpha(x, y, z, \cdots, p_{i,k}, \cdots)\ (1\leqq\alpha\leqq r), \qquad f_\alpha(x, y, z, \cdots, p_{i,k}, \cdots)\ (1\leqq\alpha\leqq t)$$

of Φ on an open neighbourhood $\mathcal{U}$ of X_0 in such a manner that the rank of Jacobian matrix $\partial(F_1, \cdots, F_r)/\partial(p_{m,0}, \cdots, p_{0,m})$ at X_0 is equal to r and that $\partial f_\alpha/\partial p_{m-\beta,\beta}=0$ $(1\leqq\alpha\leqq t,\ 0\leqq\beta\leqq m)$ on a neighbourhood of X_0 in $I\Phi$.

The prolongation $p\Phi$ of Φ, defined on $\tilde{\mathcal{U}}=(\rho_m^{m+1})^{-1}\mathcal{U}$, is by definition the locally finitely generated subsheaf of ideals in $\mathcal{E}(\tilde{\mathcal{U}})$ generated by $F_\alpha(1\leqq\alpha\leqq r)$, f_α $(1\leqq\alpha\leqq t)$ and

$$\partial_x F_\alpha,\ \partial_y F_\alpha\ (1\leqq\alpha\leqq r), \qquad \partial_x f_\alpha,\ \partial_y f_\alpha\ (1\leqq\alpha\leqq t),$$

where

$$\partial_x=\frac{\partial}{\partial x}+\sum_{i+k=0}^{\infty}p_{i+1,k}\frac{\partial}{\partial p_{i,k}}, \qquad \partial_y=\frac{\partial}{\partial y}+\sum_{i+k=0}^{\infty}p_{i,k+1}\frac{\partial}{\partial p_{i,k}}.$$

Φ is said to be p-closed at X if $(p\Phi)_{\tilde{X}}\cap\mathcal{E}_X(J^m)$ is contained in Φ_X, where $\tilde{X}$ is a point of $J^{m+1}(M, N, \rho)$ satisfying $\rho_m^{m+1}\tilde{X}=X$.

We shall introduce an integer which indicates, roughly speaking, the number of independent equations exactly of order $m+1$ in $p\Phi$;

$$r_{m+1}(X)=\operatorname{rank}\partial(\partial_x F_1, \partial_y F_1, \cdots, \partial_x F_r, \partial_y F_r)/\partial(p_{m+1,0}, p_{m,1}, \cdots, p_{0,m+1})(X).$$

The criterion of involution given by the author ([5], Theorem I) is stated as follows.

I. *Under the above conditions, Φ is involutive at X_0 if and only if the following two conditions are satisfied:*

(i) $r_{m+1}(X)=r+1$ *on a neighbourhood of X_0 in $I\Phi$.*

(ii) Φ *is p-closed at X_0.*

The *characteristic polynomial* of Φ at $X\in I\Phi$ is defined to be the highest common factor of the polynomials

$$F_\alpha^{(0)}dy^m - F_\alpha^{(1)}dy^{m-1}dx + \cdots + (-1)^m F_\alpha^{(m)}dx^m \qquad (\alpha=1, 2, \cdots, r), \tag{1}$$

where $F_\alpha^{(\beta)}=\partial F_\alpha(X)/\partial p_{m-\beta,\beta}$ (Kakié [5], §3). Let $\mu_0 dy-\mu_1 dx$ be a real linear factor of the characteristic polynomial of Φ at X (μ_0, μ_1 are supposed to be real numbers). The direction of the line in $T_a(N)$ ($a=\rho_{-1}^m X$) spanned by $\mu_0\,\partial/\partial x+\mu_1\,\partial/\partial y$ is called a characteristic direction of Φ at X. One of the fundamental results concerning the involutive systems discussed here is the following one (Kakié [5], Theorem II).

II. *Suppose that Φ is involutive at X. Then the characteristic polynomial of Φ at X is of degree $m+1-r$.*

Φ can be represented by the following differential system on $\mathcal{U}\subset J^m(M, N, \rho)$:

$$(\Sigma(\Phi))\begin{cases} f_\alpha=0 \quad (\alpha=1, 2, \cdots, t), & F_\alpha=0 \quad (\alpha=1, 2, \cdots, r), \\ df_\alpha=0 \quad (\alpha=1, 2, \cdots, t), & dF_\alpha=0 \quad (\alpha=1, 2, \cdots, r), \\ \tilde{\omega}_{i,k}=dp_{i,k}-p_{i+1,k}dx-p_{i,k+1}dy=0 & (0\leqq i+k<m). \end{cases}$$

Remark that when Φ is involutive at each point on $\mathcal{U}$, the equations $df_\alpha=0$ are consequences of the other equations in $\Sigma(\Phi)$. A solution of Φ is canonically identified with a two-dimensional integral manifold $\mathcal{M}$ of $\Sigma(\Phi)$ satisfying $\dim(\rho_{-1}^m)_*T(\mathcal{M})=2$. An integral curve $\mathcal{J}$ of $\Sigma(\Phi)$ such that $\dim(\rho_{-1}^m)_*T(\mathcal{J})=1$ is said to be *non-characteristic* if and only if the direction of the line $(\rho_{-1}^m)_*T_X(\mathcal{J})$ is not a characteristic one of Φ for each $X\in\mathcal{J}$.

The Cauchy problem for Φ can be transformed into the corresponding one for $\Sigma(\Phi)$. We shall now state our main result.

THEOREM. *Suppose that Φ is involutive at X_0 and that the roots of the characteristic polynomial of Φ at X_0 are real and distinct. Then for any non-characteristic integral curve $\mathcal{J}$ of $\Sigma(\Phi)$ satisfying* $\dim(\rho_{-1}^m)_*T(\mathcal{J})=1$, *there exists* *a unique two-dimensional integral manifold* $\mathcal{M}$ *of* $\Sigma(\Phi)$ *passing through* $\mathcal{I}$ *such that* $\dim(\rho_{-1}^{m})_{*}T(\mathcal{M})=2$.

A point on the integral manifold $\mathcal{M}$ *is uniquely determined only by the finite segment of the initial curve* $\mathcal{I}$ *cut out by the two outer characteristic curves on* $\mathcal{M}$ *passing the point.*

Suppose that Φ is involutive at X_0. We shall denote by ν the degree of the characteristic polynomial of Φ at X_0; $\nu=m+1-r$. When $\nu=0$, Φ is completely integrable at X_0; for each point $X\in I\Phi$ sufficiently near X_0, one can obtain the two-dimensional integral manifold of $\Sigma(\Phi)$ passing through X by integrating $\Sigma(\Phi)$ which is completely integrable. Hence our Theorem is obviously valid. Note that the last assertion in it is vacuous. When $\nu=1$, the differential system $\Sigma(\Phi)$ has one-dimensional Cauchy's characteristics; These characteristic curves are obtained by integrating a system of ordinary differential equations (cf. Kakié [5], § 7). Hence the existence of the solution of the Cauchy problem stated above can be derived from that for a system of ordinary differential equations. Therefore in this case our Theorem is again valid. Thus it remains only to prove our Theorem in the case when $\nu\geqq 2$. We shall prove it in the remaining sections.

§ 2. Monge characteristic systems.

We shall devote this section to recalling Monge characteristic systems of an involutive system and deducing some propositions needed later on.

Let us consider Φ which will be always assumed to satisfy the assumptions of Theorem in the preceding section. Without loss of generality, we may assume that the ν distinct characteristic directions of Φ in a neighbourhood $\mathcal{V}$ of X_0 in $I\Phi$ are defined respectively by $\partial/\partial x+\lambda_1\partial/\partial y, \cdots, \partial/\partial x+\lambda_\nu\partial/\partial y$, where the λ_j's are differentiable functions on $\mathcal{V}$. Besides we shall assume that Φ is involutive at each point on $\mathcal{V}$.

The characteristic system (of order m) of Φ corresponding to the characteristic direction defined by $\partial/\partial x+\lambda_l\partial/\partial y$ is defined to be the Pfaffian system on $\mathcal{V}$ which defines at each point $X\in\mathcal{V}$ the annihilator in $T_X^*(J^m)$ of the space

$$\Big\{\Big(\frac{d}{dx}+\sum_{\beta=0}^{m}\tilde{\varphi}_\beta\frac{\partial}{\partial p_{m-\beta,\beta}}\Big)+\lambda_l\Big(\frac{d}{dy}+\sum_{\beta=0}^{m}\tilde{\psi}_\beta\frac{\partial}{\partial p_{m-\beta,\beta}}\Big);\ \varphi_\beta=\tilde{\varphi}_\beta,\ \psi_\beta=\tilde{\psi}_\beta$$

annihilate the following functions of $2(m+1)$ variables φ_β, ψ_β

$$(**)\qquad \left[\begin{array}{l}\varphi_\beta-\psi_{\beta-1}\ (0<\beta\leqq m),\\ \dfrac{dF_\alpha}{dx}(X)+\sum\limits_{\beta=0}^{m}F_\alpha^{(\beta)}\varphi_\beta,\ \dfrac{dF_\alpha}{dy}(X)+\sum\limits_{\beta=0}^{m}F_\alpha^{(\beta)}\psi_\beta\quad(\alpha=1,2,\cdots,r)\end{array}\right\},$$

where

$$\frac{d}{dx}=\frac{\partial}{\partial x}+\sum_{i+k=0}^{m-1}p_{i+1,k}\frac{\partial}{\partial p_{i,k}}, \quad \frac{d}{dy}=\frac{\partial}{\partial y}+\sum_{i+k=0}^{m-1}p_{i,k+1}\frac{\partial}{\partial p_{i,k}}$$

(see Kakié [**5**], § 4 (in particular Proposition 5) and § 5). For brevity we shall write $[F_\alpha]_x=dF_\alpha(X)/dx$, $[F_\alpha]_y=dF_\alpha(X)/dy$. The characteristic system corresponding to the characteristic direction defined by $\partial/\partial x+\lambda_l\partial/\partial y$ is given by

$$(2)_l \qquad \begin{cases} dy-\lambda_l dx\,, \qquad \omega_l=\sum_{\beta=0}^{m} e_l^\beta dp_{m-\beta,\beta}+a_l dx+b_l dy\,, \\ dF_\alpha\ (\alpha=1, 2, \cdots, r)\,, \qquad \tilde{\omega}_{i,k}\ (0\leqq i+k<m)\,, \end{cases}$$

where $(e_l^0, \cdots, e_l^m)$ is a system of differentiable functions on $\mathcal{V}$ satisfying

$$(3)_l \qquad \begin{cases} \sum_{\beta=0}^{m} e_l^\beta\varphi_\beta+\lambda_l\sum_{\beta=0}^{m} e_l^\beta\psi_\beta\equiv 0 \\ \qquad\qquad \left(\bmod\begin{bmatrix} \varphi_\beta-\psi_\beta & (0<\beta\leqq m)\,, & \\ \sum_{\beta=0}^{m}F_\alpha^{(\beta)}\varphi_\beta, & \sum_{\beta=0}^{m}F_\alpha^{(\beta)}\psi_\beta & (1\leqq\alpha\leqq r)\end{bmatrix}\right), \\ \operatorname{rank}(F_\alpha^{(\beta)}, e_l^\beta\,;\ \beta=0, 1, \cdots, m\downarrow\,,\ \alpha=1, 2, \cdots, r\rightarrow)=r+1\,, \end{cases}$$

and $a_l=-\sum_{\beta=0}^{m}e_l^\beta\varphi_\beta^0$, $b_l=-\sum_{\beta=0}^{m}e_l^\beta\psi_\beta^0$ in which φ_β^0, ψ_β^0 are a system of functions on $\mathcal{V}$ annihilating (**) (see Kakié [**5**]). From the definition we immediately obtain

$$(4) \qquad (\sum_{\beta=0}^{m}e_l^\beta\varphi_\beta+a_l)+\lambda_l(\sum_{\beta=0}^{m}e_l^\beta\psi_\beta+b_l)\equiv 0 \qquad (\bmod\,(**))\,.$$

Let us recall the following Pfaffian forms, which appear in the classical theory of characteristic systems (cf. Goursat [**4**], Friedrichs and Lewy [**7**]):

$$\Omega_\alpha^{(x)}(\lambda_l)=\sum_{\beta=0}^{m-1}(-1)^\beta\Delta_\alpha^\beta(\lambda_l)dp_{m-\beta,\beta}+[F_\alpha]_x dx\,,$$

$$\Omega_\alpha^{(y)}(\lambda_l)=\sum_{\beta=0}^{m-1}(-1)^\beta\Delta_\alpha^\beta(\lambda_l)dp_{m-\beta-1,\beta+1}+[F_\alpha]_y dx\,,$$

where

$$\Delta_\alpha^\beta(\lambda_l)=F_\alpha^{(0)}\lambda_l^\beta-F_\alpha^{(1)}\lambda_l^{\beta-1}+\cdots+(-1)^\beta F_\alpha^{(\beta)} \qquad (0\leqq\beta\leqq m)\,.$$

We obtain at once the following formulae:

$$(5) \qquad dF_\alpha\equiv\Omega_\alpha^{(x)}(\lambda_l)+\lambda_l\Omega_\alpha^{(y)}(\lambda_l) \qquad (\bmod\ dy-\lambda_l dx,\ \tilde{\omega}_{i,k}\,;\ 0\leqq i+k<m)\,.$$

The rank of the Pfaffian forms $\Omega_\alpha^{(x)}(\lambda_l)$, $\Omega_\alpha^{(y)}(\lambda_l)$ $(1\leqq\alpha\leqq r)$ is equal to $r+1$. In fact, since Φ is p-closed at each point on $\mathcal{V}$, the mapping ρ_m^{m+1} from $I(p\Phi)\cap(\rho_m^{m+1})^{-1}\mathcal{V}$ to $\mathcal{V}$ is surjective. That is, the following system of linear equations with the unknown quantities $p_{m+1-\beta,\beta}$ $(0\leqq\beta\leqq m+1)$ has actually solutions:

$$\sum_{\beta=0}^{m} F_{\alpha}^{(\beta)} p_{m+1-\beta,\beta}+[F_{\alpha}]_x=0\,, \quad \sum_{\beta=0}^{m} F_{\alpha}^{(\beta)} p_{m-\beta,\beta+1}+[F_{\alpha}]_y=0 \quad (\alpha=1, 2, \cdots, r)\,.$$

Hence the following is valid:

$$\text{(6)} \qquad \operatorname{rank}\begin{bmatrix} F_1^{(0)} & F_1^{(1)} \cdots F_1^{(m)} & 0 & [F_1]_x \\ 0 & F_1^{(0)} \cdots F_1^{(m-1)} & F_1^{(m)} & [F_1]_y \\ \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots \\ F_r^{(0)} & F_r^{(1)} \cdots F_r^{(m)} & 0 & [F_r]_x \\ 0 & F_r^{(0)} \cdots F_r^{(m-1)} & F_r^{(m)} & [F_r]_y \end{bmatrix} = \operatorname{rank} D \quad (\text{on } \mathcal{V})\,,$$

where D is the matrix composed of the first $m+2$ columns of the matrix on the left side of (6). Multiply the first column-vector of D by $-\lambda_l$ and add it to its second column. Next, multiply by $-\lambda_l$ the second column-vector of the matrix thus obtained and add it to its third column. Proceed step by step in the same way, then we have a matrix whose components of the last column vanish. We shall denote by $D(\lambda_l)$ the obtained matrix deleted the last column. $D(\lambda_l)$ is nothing else than the matrix composed of coefficients of $dp_{m-\beta,\beta}$ $(0\leqq\beta\leqq m)$ of the above $2r$ Pfaffian forms Ω's. From the construction of $D(\lambda_l)$ it follows that $\operatorname{rank} D(\lambda_l)=\operatorname{rank} D$ (on $\mathcal{V}$). On the other hand, since $\operatorname{rank} D$ at X is equal to $r_{m+1}(X)$, $\operatorname{rank} D=r+1$ (see (ii) in I). Therefore equality (6) implies that the assertion is valid.

III. *The following two Pfaffian systems* (i) *and* (ii) *defined on* $\mathcal{V}$ *are equivalent to each other in consequence of* $dy-\lambda_l dx=0$, $\tilde{\omega}_{i,k}=0$ $(0\leqq i+k<m)$:

(i) ω_l, dF_α $(\alpha=1, 2, \cdots, r)$.

(ii) $\Omega_\alpha^{(x)}(\lambda_l)$, $\Omega_\alpha^{(y)}(\lambda_l)$ $(\alpha=1, 2, \cdots, r)$.

This fact clarifies the link between our definition of characteristic systems (i) and the classical one (ii) (cf. Goursat [**4**], Friedrichs and Lewy [**7**]).

PROOF. As we have already shown, the Pfaffian systems (i) and (ii) have the same rank $r+1$. Hence on account of (5), in order to complete the proof, it suffices to show

$$\text{(7)} \qquad \omega_l\equiv 0 \quad (\operatorname{mod} dy-\lambda_l dx,\ \Omega_\alpha^{(x)}(\lambda_l),\ \Omega_\alpha^{(y)}(\lambda_l)\,;\ 1\leqq\alpha\leqq r)\,.$$

This is derived from (4) as follows. If we replace the variables φ_β and ψ_β in (4) respectively by

$$\varphi_\beta-\lambda_l\varphi_{\beta+1}+\cdots+(-\lambda_l)^{m-\beta}\varphi_m+(-\lambda_l)^{m-\beta+1}\psi_m$$

and

$$\psi_\beta-\lambda_l\psi_{\beta+1}+\cdots+(-\lambda_l)^{m-\beta}\psi_m\,,$$

then the expression on the left side becomes

(8) $$\{\sum_{\beta=0}^{m}(-1)^{\beta}\Gamma_{l}^{\beta}(\lambda_{l})\varphi_{\beta}+a_{l}\}+\lambda_{l}\{\sum_{\beta=0}^{m-1}(-1)^{\beta}\Gamma_{l}^{\beta}(\lambda_{l})\psi_{\beta}+b_{l}\},$$

where

$$\Gamma_{l}^{\beta}(\lambda)=e_{l}^{0}\lambda^{\beta}-e_{l}^{1}\lambda^{\beta-1}+\cdots+(-1)^{\beta}e_{l}^{\beta}\qquad(0\leqq\beta\leqq m).$$

After replacing φ_{β} and ψ_{β} in (8) by φ_{β}/ξ and ψ_{β}/ξ respectively where ξ is an indeterminate, multiply the expression by ξ. Let us substitute $dp_{m-\beta,\beta}$, $dp_{m-\beta-1,\beta+1}$ and dx into φ_{β}, ψ_{β} and ξ in the obtained expression respectively. Since we have the relations

$$\Gamma_{l}^{0}(\lambda_{l})=e_{l}^{0},\quad(-1)^{\beta}\Gamma_{l}^{\beta}(\lambda_{l})+\lambda_{l}(-1)^{\beta-1}\Gamma_{l}^{\beta-1}(\lambda_{l})=e_{l}^{\beta}\quad(1\leqq\beta\leqq m),$$

the expression becomes the Pfaffian form ω_{l} if we take into account $dy=\lambda_{l}dx$. Besides, since $\Delta_{\alpha}^{m}(\lambda_{l})=0$ $(1\leqq\alpha\leqq r)$, we find at once that, by the same procedure as above, the expressions in (**) become $\Omega_{\alpha}^{(x)}(\lambda_{l})$, $\Omega_{\alpha}^{(y)}(\lambda_{l})$ $(1\leqq\alpha\leqq r)$. Therefore we obtain (7). Q. E. D.

Let us associate with $\Gamma_{l}^{m}(\lambda)$ the polynomial

$(9)_l$ $$e_{l}^{0}dy^{m}-e_{l}^{1}dy^{m-1}dx+\cdots+(-1)^{m}e_{l}^{m}dx^{m}.$$

We find it convenient to denote by $G_{1},\cdots,G_{r}$ and R_{l} the r polynomials (1) and $(9)_{l}$ respectively in which dy, dx are replaced by ξ, η respectively. Then it follows from $(3)_{l}$ that

(10) $$\xi R_{l}-\lambda_{l}\eta R_{l}\equiv 0\qquad(\operatorname{mod}\xi G_{1},\eta G_{1},\cdots,\xi G_{r},\eta G_{r}).$$

From the Theorem I it follows that, using the same notation as in Appendix,

$$\dim\langle\xi G_{1},\eta G_{1},\cdots,\xi G_{r},\eta G_{r}\rangle=\dim\langle G_{1},\cdots,G_{r}\rangle+1.$$

Hence (10) implies that

$$\dim\langle\xi G_{1},\eta G_{1},\cdots,\xi G_{r},\eta G_{r},\xi R_{l},\eta R_{l}\rangle=\dim\langle G_{1},\cdots,G_{r},R_{l}\rangle+1.$$

Since $\dim\langle G_{1},\cdots,G_{r},R_{l}\rangle=r+1$ (see $(3)_{l}$), applying Lemma in Appendix, we find that the highest common factor of $G_{1},\cdots,G_{r},R_{l}$ is of degree $\nu-1$ $(\nu=m+1-r)$. Moreover (10) can be written in the form

$$(\xi-\lambda_{l}\eta)R_{l}\equiv 0\qquad(\operatorname{mod}G_{1},\cdots,G_{r}).$$

Noting that $\xi-\lambda_{l}\eta$ is a common linear factor of $G_{1},\cdots,G_{r}$, we conclude from this that the highest common factor of $G_{1},\cdots,G_{r},R_{l}$ is obtained from that of $G_{1},\cdots,G_{r}$ by excluding the linear factor $\xi-\lambda_{l}\eta$. This fact enables us to deduce the following result.

IV. *For each integer* $h=1,2,\cdots,\nu$,

$$\operatorname{rank}\left(\begin{array}{c|l} F_{\alpha}^{(\beta)} & \alpha=1,2,\cdots,r\downarrow \\ e_{l}^{\beta} & l=1,2,\cdots,h\downarrow \end{array}\;;\;\beta=0,1,\cdots,m\rightarrow\right)=r+h\quad(\text{on }\mathcal{V}).$$

Proof. Since by assumption λ_l $(1\leqq l\leqq h)$ are distinct from one another, the fact above explained implies that the highest common factor of $r+h$ polynomials $G_1, \cdots, G_r, R_1, \cdots, R_h$ is of degree $\nu-h$; it is obtained from the highest common factor of $G_1, \cdots, G_r$ by dividing it by the factor $(\xi-\lambda_1\eta)\cdots(\xi-\lambda_h\eta)$ of degree h. On the other hand, we can prove that

$$\begin{aligned}\dim\langle\xi G_\alpha, \eta G_\alpha, \xi R_j, \eta R_j;\ 1\leqq\alpha\leqq r,\ 1\leqq j\leqq h\rangle\\ =\dim\langle G_\alpha, R_j;\ 1\leqq\alpha\leqq r,\ 1\leqq j\leqq h\rangle+1\end{aligned}$$

by induction on h. Hence, applying Lemma in Appendix to the $r+h$ polynomials G_α, R_j, we obtain the desired result. Q. E. D.

We finally give the following result needed later on.

V. *For any λ_j which is distinct from λ_l, the following is valid:*

$$\begin{aligned}\{\sum_{\beta=0}^{m-1}(-1)^\beta\Gamma_l^\beta(\lambda_j)dp_{m-\beta,\beta}+a_l dx\}+\lambda_l\{\sum_{\beta=0}^{m-1}(-1)^\beta\Gamma_l^\beta(\lambda_j)dp_{m-\beta,\beta+1}+b_l dx\}\\ \equiv 0 \qquad (\mathrm{mod}\ \Omega_\alpha^{(x)}(\lambda_j),\ \Omega_\alpha^{(y)}(\lambda_j);\ 1\leqq\alpha\leqq r) \qquad \text{on } \mathcal{V}.\end{aligned}$$

Proof. The fact already shown indicates that λ_j is a root of the equation

$$\Gamma_l^m(\lambda)=e_l^0\lambda^m-e_l^1\lambda^{m-1}+\cdots+(-1)^m e_l^m=0.$$

Hence by repeating the same argument as in the proof of III, we can deduce from (4) the desired result. Q. E. D.

§ 3. Reduction the Cauchy problem for Φ to that for a system of equations in characteristic form.

From now on we shall always assume that $\nu\geqq 2$ (see the last part of § 1). Let $\mathcal{M}$ be the solution of the Cauchy problem for Φ with non-characteristic initial curve $\mathcal{I}$; $\mathcal{M}$ is the two-dimensional integral manifold of $\Sigma(\Phi)$ satisfying $\dim(\rho_{-1}^m)_*T(\mathcal{M})=2$ and passing through $\mathcal{I}$. It will be supposed that $\mathcal{M}$ is contained in the neighbourhood $\mathcal{V}$ of X_0 (see § 2). Such integral manifold $\mathcal{M}$ is generated by each of the ν distinct families of one-parameter characteristic curves; Those families are defined respectively by $dy-\lambda_1 dx=0, \cdots, dy-\lambda_\nu dx=0$. Let us consider the tangent line to the initial curve $\mathcal{I}$ at a point X on $\mathcal{I}$; more precisely, consider the line $(\rho_{-1}^m)_*T_X(\mathcal{I})$ in $T_a(N)$ $(a=\rho_{-1}^m X)$. If the line is rotated by straight angle, its direction coincides successively with ν characteristic directions of Φ. We shall call the two characteristic directions with which the direction of that line coincides firstly and lastly the outer ones (with respect to $\mathcal{I}$), and the two corresponding characteristic curves (on $\mathcal{M}$) the outer ones.

Instead of the coordinate system (x, y), we introduce a coordinate system (σ, τ) on $\mathcal{M}$ such that the both outer characteristic curves on $\mathcal{M}$ are given by $\sigma=$ const. and $\tau=$ const. respectively. We shall suppose that the outer characteristic curves are defined by $dy-\lambda_\nu dx=0$ and $dy-\lambda_1 dx=0$ respectively. When x, y are considered as functions of σ, τ, they satisfy the equations

$$\frac{\partial y}{\partial \sigma}-\lambda_1\frac{\partial x}{\partial \sigma}=0, \qquad \frac{\partial y}{\partial \tau}-\lambda_\nu\frac{\partial x}{\partial \tau}=0,$$

where λ_1, λ_ν denote their values on $\mathcal{M}$ which are regarded as the functions of x, y.

The manifold $\mathcal{M}$ can be represented by

$$(x(\sigma, \tau), y(\sigma, \tau), z(\sigma, \tau), p_{i,k}(\sigma, \tau);\ 1\leqq i+k\leqq m) \tag{11}$$

where σ, τ varies in a domain in two-dimensional space. Let $\mathcal{I}$ be defined by the equation $\tau=g(\sigma)$; that is, $\mathcal{I}$ is defined by (11) in which τ is replaced by $g(\sigma)$. From the definition of the coordinate system (σ, τ) we have $g'(\sigma)<0$.

We shall denote by the superscript σ or τ the derivative with respect to the parameter σ or τ respectively. Moreover, for each λ_l $(1\leqq l\leqq\nu)$, we set

$$\frac{d}{dt_l}=A_l\frac{\partial}{\partial\sigma}+B_l\frac{\partial}{\partial\tau}, \quad \text{where}\quad A_l=\frac{\partial y}{\partial\tau}-\lambda_l\frac{\partial x}{\partial\tau}, \quad B_l=-\Big(\frac{\partial y}{\partial\sigma}-\lambda_l\frac{\partial x}{\partial\sigma}\Big),$$

and we write $f^{tl}=df/dt_l$ for a function f of σ, τ. It should be noted that the derivative d/dt_l has the definite meaning only when $x, y, z, p_{i,k}$ $(1\leqq i+k\leqq m)$ are given as functions of σ, τ, and that if we consider d/dt_l on the manifold $\mathcal{M}$ represented as above, d/dt_l defines the vector field on $\mathcal{M}$ along the characteristic curves defined by $dy-\lambda_l dx=0$.

We are now in a position to deduce equations satisfied by (11) defining the integral manifold $\mathcal{M}$. Since $dF_\alpha=0$ $(1\leqq\alpha\leqq r)$ on $\mathcal{M}$ and $p^\sigma_{i,k}-p_{i+1,k}x^\sigma-p_{i,k+1}y^\sigma=0$ $(0\leqq i+k<m)$, we have

$$\sum_{\beta=0}^{m}F^{(\beta)}_\alpha p^\sigma_{m-\beta,\beta}+[F_\alpha]_x x^\sigma+[F_\alpha]_y y^\sigma=0 \qquad (\alpha=1, 2, \cdots, r).$$

If we replace φ_β and ψ_β in (4) by $\partial p_{m-\beta,\beta}/\partial x$ and $\partial p_{m-\beta,\beta}/\partial y$ respectively and furthermore λ_l in it by y^{tl}/x^{tl}, then the functions obtained from (**) vanish on $\mathcal{M}$, and hence we obtain

$$\sum_{\beta=0}^{m}e^\beta_l p^{tl}_{m-\beta,\beta}+a_l x^{tl}+b_l y^{tl}=0 \qquad (l=1, 2, \cdots, \nu).$$

Consequently, (11) defining the integral manifold $\mathcal{M}$ of $\Sigma(\Phi)$ must be a solution of the following system of differential equations:

$$(12)\qquad \begin{cases} y^\sigma-\lambda_1 x^\sigma=0\,, \qquad y^\tau-\lambda_\nu x^\tau=0\,, \\ {}^\sigma U_{i,k}=p^\sigma_{i,k}-p_{i+1,k}x^\sigma-p_{i,k+1}y^\sigma=0 \qquad (0\leqq i+k<m)\,, \\ \sum\limits_{\beta=0}^{m} F^{(\beta)}_\alpha p^\sigma_{m-\beta,\beta}+[F_\alpha]_x x^\sigma+[F_\alpha]_y y^\sigma=0 \qquad (\alpha=1, 2, \cdots, r)\,, \\ \sum\limits_{\beta=0}^{m} e^\beta_l p^{tl}_{m-\beta,\beta}+a_l x^{tl}+b_l y^{tl}=0 \qquad (l=1, 2, \cdots, \nu)\,. \end{cases}$$

We shall from now on regard this system as that defined on an open set $\mathcal{U}$ in $J^m(M, N, \rho)$ by extending the functions e_l, a_l, b_l on $\mathcal{V}$ to those on $\mathcal{U}$.

Conversely, let us consider the Cauchy problem for (12). Assume that a curve in the (σ, τ)-space is given and it is defined by the equation $\tau=g(\sigma)$ $(\sigma_0<\sigma<\sigma_1)$ with $g'(\sigma)\neq 0$. Let a system of functions defined on $\sigma_0<\sigma<\sigma_1$ be given as follows:

$$(13)\qquad (X(\sigma),\ Y(\sigma),\ Z(\sigma),\ P_{i,k}(\sigma)\,;\ 1\leqq i+k\leqq m)\,.$$

We assume that (13) defines a curve in $\mathcal{U}$. The curve defined by (13) is an integral curve of $\Sigma(\Phi)$ if and only if F_α $(1\leqq\alpha\leqq r)$ and f_α $(1\leqq\alpha\leqq t)$ vanish on it and (13) satisfies the strip conditions

$$P'_{i,k}(\sigma)-P_{i+1,k}(\sigma)X'(\sigma)-P_{i,k+1}(\sigma)Y'(\sigma)=0 \qquad (0\leqq i+k<m)\,.$$

Besides it is a non-characteristic integral curve of $\Sigma(\Phi)$ if and only if

$$Y'(\sigma)-\lambda_l X'(\sigma)\neq 0 \qquad (l=1, 2, \cdots, \nu)\,,$$

where the variables in the λ_l's are supposed to be replaced by their values given by (13).

By the Cauchy problem for (12) with initial data (13) we mean the problem of finding the solution of (12) such that its value on the curve $\tau=g(\sigma)$ coincides with (13); $x(\sigma, g(\sigma))=X(\sigma)$, $y(\sigma, g(\sigma))=Y(\sigma)$, and so on. The Cauchy problem for Φ can be reduced to that for (12). In fact, combining the method of Friedrichs and Lewy [7] with the results we have deduced, we have the following proposition.

VI. *The solution of the Cauchy problem for* (12) *with initial data* (13) *defining an integral curve of* $\Sigma(\Phi)$ *gives the two-dimensional integral manifold* $\mathcal{M}$ *satisfying* $\dim(\rho^m_{-1})_*T(\mathcal{M})=2$ *passing through the curve defined by* (13).

PROOF. What we must prove is the following two facts:

(a) The functions F_α $(1\leqq\alpha\leqq r)$ and f_α $(1\leqq\alpha\leqq t)$ vanish for the solution.

(b) The solution satisfies the strip conditions

$$\tilde{\omega}_{i,k}=dp_{i,k}-p_{i+1,k}dx-p_{i,k+1}dy=0 \qquad (0\leqq i+k<m)\,.$$

We first prove (a). The equations in (12) imply that the derivative, with respect to the parameter σ, of each function F_α in which the variables are replaced by the solution vanishes. Since the F_α's vanish for the initial data, this implies that the F_α's vanish for the solution everywhere. Now the fact that Φ is p-closed at each point on $\mathcal{V}$ implies that

$$df_\alpha \equiv 0 \quad (\mathrm{mod}\, \tilde{\omega}_{i,k}\ (0 \leqq i+k < m),\ f_1, \cdots, f_t,\ F_1, \cdots, F_r) \quad (\alpha = 1, \cdots, t).$$

Hence for the solution we have

$$f_\alpha^\sigma \equiv 0 \qquad (\mathrm{mod}\, f_1, \cdots, f_t) \qquad (\alpha = 1, 2, \cdots, t).$$

Since $f_\alpha = 0$ for the initial data, the f_α's vanish everywhere for the solution by virtue of the well-known uniqueness theorem concerning linear homogeneous ordinary differential equations.

Let us now prove (b). Since the solution satisfies ${}^\sigma U_{i,k} = 0$ $(0 \leqq i+k < m)$, it suffices to prove that the solution satisfies

$${}^\tau U_{i,k} = p_{i,k}^\tau - p_{i+1,k} x^\tau - p_{i,k+1} y^\tau = 0 \qquad (0 \leqq i+k < m).$$

We shall denote by ${}^\tau U_{i,k}^\sigma$ the derivative of ${}^\tau U_{i,k}$ with respect to σ and by ${}^\sigma U_{i,k}^\tau$ that of ${}^\sigma U_{i,k}$ with respect to τ. For simplicity we shall write $\lambda = \lambda_\nu$. By simple calculation we obtain the relations

$$(14) \qquad {}^\tau U_{i,k}^\sigma - {}^\sigma U_{i,k}^\tau = p_{i+1,k}^\tau x^\sigma + p_{i,k+1}^\tau y^\sigma - p_{i+1,k}^\sigma x^\tau - p_{i,k+1}^\sigma y^\tau \quad (0 \leqq i+k < m).$$

Since ${}^\sigma U_{i,k} = 0$ for the solution, (14) implies that the following relations are valid for the solution of (12):

$$(15) \qquad p_{m-\beta,\beta}^\sigma + \lambda p_{m-1-\beta,\beta+1}^\sigma = -\frac{1}{x^\tau}\, {}^\tau U_{m-1-\beta,\beta}^\sigma + \frac{1}{x^\tau}(p_{m-\beta,\beta}^\tau x^\sigma + p_{m-1-\beta,\beta+1}^\tau y^\sigma)$$
$$(\beta = 0, 1, \cdots, m-1).$$

Furthermore, using the relations similar to (14), we obtain the following equalities valid for the solution of (12):

$$(16) \qquad p_{m-\beta,\beta}^{t_l} + \lambda p_{m-1-\beta,\beta+1}^{t_l} = -\frac{c_l}{x^\tau}\, {}^\tau U_{m-1-\beta,\beta}^\sigma + \frac{1}{x^\tau}(p_{m-\beta,\beta}^\tau x^{t_l} + p_{m-1-\beta,\beta+1}^\tau y^{t_l})$$
$$(l = 1, 2, \cdots, \nu;\ \beta = 0, 1, \cdots, m-1),$$

where

$$c_l = (x^{t_l} y^\tau - x^\tau y^{t_l})/(x^\sigma y^\tau - x^\tau y^\sigma) \qquad (\neq 0).$$

From equalities (14) it follows that for the solution of (12)

$$(17) \qquad {}^\tau U_{j,h}^\sigma \equiv 0 \qquad (\mathrm{mod}\, {}^\tau U_{i,k};\ 0 \leqq i+k < m) \qquad (0 \leqq j+h \leqq m-2).$$

Replacing $p_{m-\beta,\beta}^\sigma$ by the expressions given by (15) successively from $\beta = 0$ to $\beta = m$ gives rise to the equalities satisfied for the solution

$$\sum_{\beta=0}^{m} F_{\alpha}^{(\beta)} p_{m-\beta,\beta}^{\sigma}+[F_{\alpha}]_x x^{\sigma}+[F_{\alpha}]_y y^{\sigma}=-\frac{1}{x^{\tau}}\{\sum_{\beta=0}^{m-1}(-1)^{\beta}\varDelta_{\alpha}^{\beta}(\lambda)^{\tau}U_{m-1-\beta,\beta}^{\sigma}\}$$
$$+\frac{1}{x^{\tau}}[x^{\sigma}\{\sum_{\beta=0}^{m-1}(-1)^{\beta}\varDelta_{\alpha}^{\beta}(\lambda)p_{m-\beta,\beta}^{\tau}+[F_{\alpha}]_x x^{\tau}\}$$
$$+y^{\sigma}\{\sum_{\beta=0}^{m-1}(-1)^{\beta}\varDelta_{\alpha}^{\beta}(\lambda)p_{m-1-\beta,\beta}^{\tau}+[F_{\alpha}]_y x^{\tau}\}] \qquad (\alpha=1, 2, \cdots, r).$$

Hence the solution satisfies

$$\sum_{\beta=0}^{m-1}(-1)^{\beta}\varDelta_{\alpha}^{\beta}(\lambda)^{\tau}U_{m-1-\beta,\beta}^{\sigma}=x^{\sigma}\tilde{\Omega}_{\alpha}^{(x)}(\lambda)+y^{\sigma}\tilde{\Omega}_{\alpha}^{(y)}(\lambda) \quad (\alpha=1, 2, \cdots, r), \tag{18}$$

where $\tilde{\Omega}$'s represent the values of the Pfaffian forms Ω's in which $dp_{m-\beta,\beta}$, dx, dy are replaced by $p_{m-\beta,\beta}^{\tau}$, x^{τ}, y^{τ} respectively.

On the other hand, by III and (a) already shown, it is easily deduced that the following are valid for the solution

$$\tilde{\Omega}_{\alpha}^{(x)}(\lambda),\ \ \tilde{\Omega}_{\alpha}^{(y)}(\lambda)\equiv 0 \quad (\mathrm{mod}\ {}^{\tau}U_{i,k}\,;\ 0\leqq i+k<m) \quad (\alpha=1, 2, \cdots, r). \tag{19}$$

Hence from (18) we have

$$\sum_{\beta=0}^{m-1}(-1)^{\beta}\varDelta_{\alpha}^{\beta}(\lambda)^{\tau}U_{m-1-\beta,\beta}^{\sigma}\equiv 0 \quad (\mathrm{mod}\ {}^{\tau}U_{i,k}\,;\ 0\leqq i+k<m) \quad (\alpha=1, 2, \cdots, r). \tag{20}$$

In the similar manner, from another group of equations in (12) we can deduce the following equalities satisfied for the solution

$$\sum_{\beta=0}^{m-1}(-1)^{\beta}\Gamma_{l}^{\beta}(\lambda)^{\tau}U_{m-1-\beta,\beta}^{\sigma}=\frac{1}{c_l}[x^{t_l}\{\sum_{\beta=0}^{m-1}(-1)^{\beta}\Gamma_{l}^{\beta}(\lambda)p_{m-\beta,\beta}^{\tau}+a_l x^{\tau}\}$$
$$+y^{t_l}\{\sum_{\beta=0}^{m-1}(-1)^{\beta}\Gamma_{l}^{\beta}(\lambda)p_{m-1-\beta,\beta+1}^{\tau}+b_l x^{\tau}\}] \quad (l=1, 2, \cdots, \nu-1).$$

By virtue of V, we find that the terms on the right side can be expressed as linear combinations of $\tilde{\Omega}_{\alpha}^{(x)}(\lambda)$, $\tilde{\Omega}_{\alpha}^{(y)}(\lambda)$ $(1\leqq\alpha\leqq r)$. Therefore, on account of (19), we conclude that the following are valid for the solution:

$$\sum_{\beta=0}^{m-1}(-1)^{\beta}\Gamma_{l}^{\beta}(\lambda)^{\tau}U_{m-1-\beta,\beta}^{\sigma}\equiv 0 \quad (\mathrm{mod}\ {}^{\tau}U_{i,k}\,;\ 0\leqq i+k<m) \quad (l=1, 2, \cdots, \nu-1). \tag{21}$$

The matrix composed of the coefficients on the left side in the equations (20) and (21) is constructed from the following matrix in the same manner as in constructing $D(\lambda)$ from D (see the argument below (6) in § 2):

$$\left(\begin{array}{l|l} F_{\alpha}^{(\beta)} & \alpha=1, 2, \cdots, r\downarrow \\ e_{l}^{\beta} & l=1, 2, \cdots, \nu-1\downarrow \end{array}\ ;\ \beta=0, 1, \cdots, m\rightarrow\right).$$

Therefore IV indicates that both matrices have the rank m.

The system obtained by adjoining (17), (20) and (21) forms a system of linear homogeneous ordinary differential equations with unknown functions ${}^{\tau}U_{i,k}$ $(0 \leqq i+k < m)$; The matrix of its coefficients has non-zero determinant. The initial values of the ${}^{\tau}U_{i,k}$'s vanish, for the ${}^{\sigma}U_{i,k}$'s vanish and the strip conditions toward the direction tangent to the initial curve $\tau = g(\sigma)$ are satisfied. Consequently the uniqueness theorem asserts that ${}^{\tau}U_{i,k}$ $(0 \leqq i+k < m)$ vanish everywhere. Thus we have proved (b). Q. E. D.

§ 4. The existence theorem.

In completing the proof of our Theorem, we shall use the existence theorem established by M. Cinquini-Cibrario [**1**]. Let ρ, μ, ρ_i $(1 \leqq i \leqq q)$ and a_{js} $(3 \leqq j \leqq n$, $1 \leqq s \leqq n)$ be differentiable functions defined on a domain W in the space of n variables $u_1, u_2, \cdots, u_n$. We shall consider a system of non-linear differential equations of the first order with n unknown functions $u_1(\sigma, \tau), \cdots, u_n(\sigma, \tau)$ of two independent variables σ, τ:

$$
(22) \qquad \left\{
\begin{array}{ll}
\dfrac{\partial u_1}{\partial \sigma} = \rho \dfrac{\partial u_2}{\partial \sigma}, \quad \dfrac{\partial u_2}{\partial \tau} = \mu \dfrac{\partial u_1}{\partial \tau}, & \\[2ex]
\sum\limits_{s=1}^{n} a_{js} \dfrac{\partial u_s}{\partial \sigma} = 0 & (j = 3, 4, \cdots, h), \\[2ex]
\sum\limits_{s=1}^{n} a_{js} \dfrac{\partial u_s}{\partial \tau} = 0 & (j = h+1, h+2, \cdots, k), \\[2ex]
\sum\limits_{s=1}^{n} a_{js} \dfrac{du_s}{dt_1} = 0 & (j = k+1, k+2, \cdots, l_1), \\[2ex]
\sum\limits_{s=1}^{n} a_{js} \dfrac{du_s}{dt_2} = 0 & (j = l_1+1, l_2+2, \cdots, l_2), \\[1ex]
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\[1ex]
\sum\limits_{s=1}^{n} a_{js} \dfrac{du_s}{dt_q} = 0 & (l_{q-1}+1, \cdots, n),
\end{array}
\right.
$$

where h, k, l_i are integers such that $h \leqq k \leqq l_1 \leqq \cdots \leqq l_{q-1} \leqq n$ and the derivatives d/dt_i are defined by

$$
\frac{d}{dt_i} = A_i \frac{\partial}{\partial \sigma} + B_i \frac{\partial}{\partial \tau},
$$

where

$$
(23) \qquad A_i = \frac{\partial u_2}{\partial \tau} - \rho_i \frac{\partial u_1}{\partial \tau}, \qquad B_i = -\Big(\frac{\partial u_2}{\partial \sigma} - \rho_i \frac{\partial u_1}{\partial \sigma}\Big).
$$

Note that the derivatives d/dt_i have the definite meaning only when the u_s's are given as functions of σ, τ.

Writing $\Delta = \det(a_{js}; j = 3, 4, \cdots, n, s = 3, 4, \cdots, n)$, we shall suppose that the following conditions are satisfied:

(24) $\qquad \Delta \neq 0, \quad 1-\rho\mu \neq 0, \quad 1-\rho_i\mu \neq 0, \quad \rho-\rho_i \neq 0, \quad \rho_i-\rho_j \neq 0 \quad (i \neq j)$

on $W \quad (i, j=1, 2, \cdots, q)$.

Consider the curve γ in the (σ, τ)-space defined by $\tau=g(\sigma)$ with $g'(\sigma) \neq 0$ $(\sigma_0<\sigma<\sigma_1)$. Let $U_1(\sigma), U_2(\sigma), \cdots, U_n(\sigma)$ be n functions defined on $\sigma_0<\sigma<\sigma_1$. Suppose that each point $(U_1(\sigma), \cdots, U_n(\sigma))$ belongs to W. Moreover suppose that the following conditions are satisfied:

$$\left\{\begin{array}{l} U_1'(\sigma)-\rho U_2'(\sigma) \neq 0, \qquad U_2'(\sigma)-\mu U_1'(\sigma) \neq 0, \\ U_1'(\sigma)-\rho_i U_2'(\sigma) \neq 0 \qquad (i=1, 2, \cdots, q) \qquad \text{for } \sigma_0<\sigma<\sigma_1, \end{array}\right. \tag{25}$$

in which $u_1, \cdots, u_n$ in ρ, μ, ρ_i are supposed to be replaced by $U_1(\sigma), \cdots, U_n(\sigma)$ respectively.

The Cauchy problem for the system (22) is the problem of finding the n functions $u_1(\sigma, \tau), \cdots, u_n(\sigma, \tau)$ which satisfy (22) and which also satisfy the condition on γ

$$u_1(\sigma, g(\sigma))=U_1(\sigma), \cdots, u_n(\sigma, g(\sigma))=U_n(\sigma). \tag{26}$$

By differentiating the first two equalities in (26), we have

$$\frac{\partial u_1}{\partial \sigma}+g'(\sigma)\frac{\partial u_1}{\partial \tau}=U_1'(\sigma), \qquad \frac{\partial u_2}{\partial \sigma}+g'(\sigma)\frac{\partial u_2}{\partial \tau}=U_2'(\sigma) \qquad \text{on } \gamma.$$

On account of the second inequality in (24), these two equations and the first two equations in (22) determine the values $\partial u_1/\partial\sigma$, $\partial u_1/\partial\tau$, $\partial u_2/\partial\sigma$, $\partial u_2/\partial\tau$ on γ, and hence the A_i and B_i defined by (23) are determined on γ by the given data as known functions of σ; in particular, the ratios $A_i : B_i$ are also determined on γ. Each of such ratios determines at each point on γ a direction in the (σ, τ)-space. We shall assume furthermore that the following condition is satisfied:

(27) *At each point on the curve γ, all the q directions defined by q values ρ_i $(1 \leqq i \leqq q)$ are situated on the same corner formed by the two lines parallel to the σ, τ axis through the point and the tangent line to γ is situated on the supplementary corner.*

M. Cinquini-Cibrario [1] proved the following theorem by the method of successive approximation.

EXISTENCE THEOREM. *Under the conditions* (24), (25) *and* (27), *the Cauchy problem for* (22) *with initial data* (26) *on γ possesses a unique solution $u_1(\sigma, \tau)$, $\cdots$, $u_n(\sigma, \tau)$ in a sufficiently small neighbourhood of γ. Moreover the values of $u_1(\sigma, \tau), \cdots, u_n(\sigma, \tau)$ at a point depend only upon the initial data on the segment of γ cut out by the two lines parallel to the σ, τ axis through the point.*

REMARK. Although M. Cinquini-Cibrario proved the existence of C^2 solutions whose second derivatives are Lipschitz continuous, the proof can be completed in order to show the existence of C^{∞} solutions.

We are now in a position to complete the proof of our Theorem. By the fact VI, it is sufficient to prove that the above existence theorem can be applied to the Cauchy problem for the system (12). The system (12) is essentially of the same type as the system (22); If we denote the unknown functions $x, y, z, p_{i,k}$ $(1 \leqq i+k \leqq m)$ and the functions $\lambda_1, \lambda_2, \cdots, \lambda_{\nu-1}, \lambda_\nu$ respectively by $u_1, u_2, \cdots, u_n$ and $\rho, \rho_1, \cdots, \rho_q, 1/\mu$ $(n=2+(m+1)(m+2)/2,\ q=\nu-2)$, the system (12) becomes a system of the form (22). The condition corresponding to (25) is satisfied, for it is nothing else than the condition that the curve defined by the initial data is non-characteristic (cf. the argument around (13) in § 3). For any given initial curve of the Cauchy problem for $\Sigma(\Phi)$, we can choose such a coordinate system (σ, τ) that the condition corresponding to (27) is satisfied (cf. § 3). Hence it remains only to show that the condition corresponding to (24) is satisfied. Using proposition V in which $h=\nu$, we readily see that the determinant corresponding to Δ does not vanish. The other conditions in (24) are obviously satisfied, for $\lambda_1, \lambda_2, \cdots, \lambda_\nu$ are distinct from one another. Thus we have proved our Theorem.

Appendix. An algebraic lemma.

Let K be a field. Let us consider r homogeneous polynomials in two indeterminates ξ, η:

$$P_\alpha(\xi, \eta) = a_0^{(\alpha)}\xi^m + a_1^{(\alpha)}\xi^{m-1}\eta + \cdots + a_m^{(\alpha)}\eta^m \qquad (\alpha = 1, 2, \cdots, r),$$

where $a_i^{(\alpha)} \in K$. We shall denote by $\langle P_1, P_2, \cdots, P_r\rangle$ the vector space spanned by the polynomials $P_1, P_2, \cdots, P_r$ over K in the polynomial ring $K[\xi, \eta]$ and by $(P_1, P_2, \cdots, P_r)$ the highest common factor of the polynomials $P_1, P_2, \cdots, P_r$.

Under a certain condition, we can give a method of calculating the degree of $(P_1, P_2, \cdots, P_r)$.

LEMMA. *Assume that*

$$\dim \langle \xi P_1, \eta P_1, \cdots, \xi P_r, \eta P_r\rangle = \dim \langle P_1, \cdots, P_r\rangle + 1 .$$

Then

$$\deg (P_1, \cdots, P_r) = m+1-\dim \langle P_1, \cdots, P_r\rangle .$$

(For the proof, see Kakié [5], § 3, Lemma 2.)

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