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Extremizations and Dirichlet integrals on Riemann surfaces

Dedicated to Professor Leo Sario on his 60th birthday

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Consider a subregion W of a Riemann surface R with an analytic relative boundary ∂W, compact or noncompact. We denote by $H(W;\partial W)$ the class of continuous functions v on R harmonic on W and vanishing on $R-W$. The operator μ from a domain in $H(W;\partial W)$ into $H(R)$ given by

(1) $$\mu v=\lim_{\Omega\to R} H_v^{\Omega}$$

is referred to as the *extremization* relative to (R, W) in the Kuramochi terminology, where $\{\Omega\}$ is the directed net of regular subregions of R and H_v^{Ω} is the harmonic function on Ω with boundary values v on $\partial\Omega$. Let $HX(W;\partial W)$ be the subclass of $H(W;\partial W)$ consisting of members with the property $X=D$ or BD where D means the finiteness of the *Dirichlet integral*

(2) $$D_R(v)=\int_R dv\wedge{}^*dv\ ,$$

B the boundedness, and BD both B and D. As a consequence of the Dirichlet principle the domain of μ contains $HX(W;\partial W)$ and the range of $\mu_X=\mu|HX(W;\partial W)$ is contained in $HX(R)$ ($X=D$ and BD), i.e.

(3) $$\mu_D: HD(W;\partial W)\longrightarrow HD(R)$$

and also

(4) $$\mu_{BD}: HBD(W;\partial W)\longrightarrow HBD(R)$$

are linear operators, which are *injective*, positive, and isometric with respect to the supremum norm on R; and the former is an extension of the latter. Here we recall the following theorem of Royden: the classes $HBD(W;\partial W)$ and $HBD(R)$ are *dense* in $HD(W;\partial W)$ and $HD(R)$, respectively, with respect to the Dirichlet seminorm $D_R(\cdot)^{1/2}$ and the supremum seminorms $\sup_K|\cdot|$ for all compact subsets K of R. In view of this we naturally come up with the following

QUESTION. *Does the surjectiveness of μ_{BD} imply that of μ_D?*

The answer is trivially in the affirmative for surfaces R with $HD(R)=HBD(R)$ and the question should be asked for surfaces R not in this degenerate category. The primary purpose of this paper is to show that the answer to the above question is in general in the negative. Namely we shall prove the following

MAIN THEOREM. *On any Riemann surface R not in the class $U_{\widetilde{HD}}$ with $HD(R)-HBD(R)\neq\emptyset$, there exists a subregion W with an analytic relative boundary such that μ_{BD} is surjective and yet μ_D is not.*

Here the Constantinescu-Cornea class $U_{\widetilde{HD}}$ of Riemann surfaces R is defined as follows: Let $\widetilde{HD}(R)$ be the class of nonnegative harmonic functions u on R such that there exists a decreasing sequence $\{u_n\}$ in $HD(R)$ with $u=\lim_{n\to\infty} u_n$ on R. A function u in $\widetilde{HD}(R)$ is said to be *minimal* if $u>0$ and $u\geqq v\geqq 0$ for any v in $\widetilde{HD}(R)$ implies the existence of a constant c_v in $[0, 1]$ such that $v=c_v u$. Then $U_{\widetilde{HD}}$ is defined to be the class of *hyperbolic* Riemann surfaces R such that $\widetilde{HD}(R)$ contains a minimal function. Planer hyperbolic Riemann surfaces or hyperbolic Riemann surfaces of finite genus are examples of R such that $R\notin U_{\widetilde{HD}}$ and $HD(R)-HBD(R)\neq\emptyset$. Although the restriction $HD(R)-HBD(R)\neq\emptyset$ is essential for the validity of the conclusion of the above theorem, we do not know whether any additional restriction is really needed. From the proof in the sequel we see that the condition '$R\notin U_{\widetilde{HD}}$ and $HD(R)-HBD(R)\neq\emptyset$' can be weakened as 'there exists an $h\in HD(R)-HBD(R)$ and a sequence $\{z_n^*\}$ of distinct points in the Royden harmonic boundary such that each z_n^* has vanishing harmonic measure and $\lim_{n\to\infty} h(z_n^*)=\infty$'. A sufficient condition for this is 'the existence of an infinite sequence of distinct points with vanishing capacities in the Royden harmonic boundary'. The *program* of the proof is as follows. First in nos. 1-7 we develope a general theory. Necessary and sufficient conditions on W for μ_{BD} to be surjective is given in no. 2. Under the assumption that μ_{BD} is surjective, we shall show in no. 7 that the condition

$$-\int_{\partial W} h^2 {}^*d\omega<\infty \tag{5}$$

is necessary and sufficient for a nonnegative $h\in HD(R)$ to belong to the range $\mu_D(HD(W;\partial W))$, where ω is the relative harmonic measure of W. This result is derived as a result of more general assertion in no. 4: A general $h\in HD(R)$ belongs to $\mu_D(HD(W;\partial W))$ if and only if $D_R(\omega h)<\infty$, where μ_{BD} is supposed to be surjective. The significance of the condition (5) reveals itself if one observes the Parreau inclusion $HD<HM_2$. Based on this general theory the construction of the required W in our theorem will be carried over in nos. 8-14. The subregion W has the form $R-\bigcup_{j=1}^{\infty}\overline{X}_j$ where $\{X_j\}$ is a sequence of

closure disjoint regular subregions of R.

As an application of the main theorem and actually of the construction of W we give an example of a *density* P on R, i.e. a nonnegative locally Hölder continuous second order differential $P=P(z)dxdy$ $(z=x+iy)$ on a Riemann surface R. We denote by $PX(R)$ the class of solutions v of $\Delta v=Pv$ (i.e. $d^*dv=vP$) on R with the property $X=D$, BD, E or BE, where E means the finiteness of the *energy integral* $E_R(u)=\int_R(du\wedge{}^*du+u^2P)$. The *reduction operator* T is an operator from a domain of the solution space $P(R)$ of $\Delta v=Pv$ on R into $H(R)$ given, as (1), by

$$Tv=\lim_{\Omega\to R}H_v^\Omega. \tag{6}$$

The domain of T contains $PX(R)$ and $T_X=T|PX(R)$ is an operator from $PX(R)$ into $HX(R)$ $(X=D, BD, E, BE; E=D$ for $H)$, which is linear, *injective*, positive, and isometric with respect to the supremum norm on R. Since $PBY(R)$ is dense in $PY(R)$ as in the case of harmonic functions, there arises the *question* whether the surjectiveness of T_{BY} implies that for T_Y $(Y=D, E)$. We shall give a *negative* answer to this question. A density is said to be *Green energy finite* (*finite*, resp.) on R if

$$\int_{R\times R}G(z,\zeta)P(z)P(\zeta)dxdyd\xi d\eta<\infty \quad \left(\int_R P(z)dxdy<\infty,\ \text{resp.}\right) \tag{7}$$

where $G(z,\zeta)$ is the harmonic Green's function on R and $\zeta=\xi+i\eta$. The condition (7) is known to imply the surjectiveness of T_{BD} (T_{BE}, resp.). Using the subregion $W=R-\bigcup_{j=1}^{\infty}\bar{X}_j$ in the main theorem we shall construct in nos. 15-19 a density P on R with $P=0$ on W and with the following property:

COROLLARY. *On any Riemann surface R not in the class $U_{\widetilde{HD}}$ with $HD(R)-HBD(R)\neq\emptyset$, there exists a both Green energy finite and finite density P on R such that T_D and T_E are not surjective.*

1. Throughout this paper we assume that R is *hyperbolic*. Otherwise $HD(R)=HBD(R)=\boldsymbol{R}$ (the real numbers) and $HD(W;\partial W)=HBD(W;\partial W)=\boldsymbol{R}$ or $\{0\}$, which are of no interest from our present view point. We say that an open subset Z of R has a piecewise analytic (analytic, resp.) relative boundary ∂Z if for each $z\in\partial Z$ there exists a parametric disk V about z such that $(\partial Z)\cap V$ is a piecewise analytic (analytic, resp.) simple arc connecting two distinct points of ∂V in V. An open subset Z of R is said to be *normal* if $\partial Z=\partial(R-\bar{Z})$ is piecewise analytic. An open subset Z of R is said to be *regular* if Z is relatively compact and $\partial Z=\partial(R-\bar{Z})$ is analytic. For convenience we say in this paper that an open subset Z of R is *stuffed* if $R-Z$ has no compact component. We use this notion in such contexts as stuffed normal and stuffed regular open subsets. We need to consider the *Royden compactification* R^* of R. Hereafter we use the bar $\bar{A}$ to mean the closure of a subset $A \subset R^*$ with respect to R^* but by the notation ∂A for a subset $A \subset R^*$ we mean the relative boundary $\partial(A \cap R)$ of $A \cap R$ with respect to R. We denote by Δ the Royden *harmonic boundary* of R. A neighborhood U^* of a point $z^* \in \Delta$ is said to be *normal* if $U = U^* \cap R$ is a normal subset of R. If moreover $R - U$ has no compact component (i.e. U is stuffed), then we say that U^* is *stuffed normal*. The set of normal neighborhoods of $z^* \in \Delta$ forms a base of neighborhoods of z^*. We denote by $\tilde{M}(R)$ the class of continuous Tonelli functions on R with finite Dirichlet integrals over R. Each function in $\tilde{M}(R)$ is a continuous mapping of R^* into $[-\infty, \infty]$. Let Z be R or a normal open subset of R. We set $\tilde{M}_{\Delta \cup K}(R) = \{f \in \tilde{M}(R) ; f | \Delta \cup K = 0\}$ where $K = \overline{R - \bar{Z}}$. Then $\tilde{M}_{\Delta \cup K}(R)$ is complete with respect to the convergence in the Dirichlet seminorm $D_R(\cdot)^{1/2}$ and supremum seminorms on each compact subset of R. We denote by $HD(Z; R)$ the subclass of $\tilde{M}(R)$ consisting of those f which are harmonic on Z. The *orthogonal decomposition* of $\tilde{M}(R)$ states that

$$\tilde{M}(R) = HD(Z; R) \oplus \tilde{M}_{\Delta \cup K}(R) \tag{8}$$

where $HD(Z; R) \cap \tilde{M}_{\Delta \cup K}(R) = \{0\}$ and $HD(Z; R) \perp \tilde{M}_{\Delta \cup K}(R)$ (orthogonal) in the Dirichlet inner product $D_R(u, v) = \int_R du \wedge {}^*dv$. The bounded members in $\tilde{M}(R)$ form the Royden algebra $M(R)$. On setting $HBD(Z; R) = HD(Z; R) \cap M(R)$ and $M_{\Delta \cup K}(R) = \tilde{M}_{\Delta \cup K}(R) \cap M(R)$, (8) takes the form

$$M(R) = HBD(Z; R) \oplus M_{\Delta \cup K}(R). \tag{9}$$

Another fact we shall frequently make use of is the *maximum principle*: Let s be a superharmonic function on an open subset Z bounded from below. If $\liminf_{z \to z^*} s(z) \geqq m$ for each $z^* \in (\partial Z) \cup (\bar{Z} \cap \Delta)$, then $s \geqq m$ on Z. In particular any $u \in HD(Z; R)$ takes its maximum and minimum on $(\overline{\partial Z}) \cap (\bar{Z} \cap \Delta)$. For the detail of the theory of Royden compactification and, in particular, the orthogonal decomposition and the maximum principle, we refer to the monograph of Sario-Nakai [7, Chap. III].

2. Hereafter till no. 7 we denote by W a general normal subregion of R with an analytic relative boundary ∂W. We discuss in this no. 2 the conditions to assure the surjectiveness of μ_{BD}: $HBD(W; \partial W) \to HBD(R)$. The *relative harmonic measure* ω of the ideal boundary of W is defined by

$$\omega = \lim_{\Omega \to R} H_{1_{W \cap \Omega}}^{W \cap \Omega} \tag{10}$$

where $1_{W\cap\Omega}$ is the boundary function on $\partial(W\cap\Omega)$ such that $1_{W\cap\Omega}=1$ on $(\partial\Omega)\cap W$ and $1_{W\cap\Omega}=0$ on $(\partial W)\cap\bar{\Omega}$. We extend ω to a continuous function on R by setting $\omega=0$ on $R-W$. We maintain the following

THEOREM. *The following five conditions are equivalent by pairs:* (a) *The extremization* $\mu_{BD}: HBD(W;\partial W)\to HBD(R)$ *is surjective;* (b) *There exists a* v *in* $HBD(W;\partial W)$ *with* $\mu_{BD}v=1$; (c) *The relative harmonic measure* ω *belongs to* $HBD(W;\partial W)$ *and* $\mu_{BD}\omega=1$; (d) *There exists a continuous Tonelli potential* p *on* R *with* $D_R(p)<\infty$ *such that* $p\geqq 1$ *on* $R-W$; (e) *The closure* $\overline{W}$ *is a neighborhood of the harmonic boundary* Δ.

The implication from (a) to (b) is trivial. Next let v be the function in (b) and observe that $v=\mu_{BD}v+(v-\mu_{BD}v)$ is the decomposition in (9) with $Z=R$, i.e. $v-\mu_{BD}v=v-1\in M_{\Delta}(R)$ or $v=1$ on Δ. On the other hand, since $0\leqq\omega<1$ on R,

$$\liminf_{z\to z^*}(v(z)-\omega(z))\geqq 0$$

for every z^* in $(\partial W)\cup(\overline{W}\cap\Delta)$ and a fortiori the maximum principle yields $v\geqq\omega$ on R. Repeating the same argument for $1-v$, we also see that $1\geqq v$ on R. By (10) we deduce that $\omega\geqq v$ on R. Therefore $\omega=v$ on R and the condition (c) is valid. To derive (d) from (c), let $p=1-\omega$, which is a continuous Tonelli superharmonic function on R with $D_R(p)<\infty$. Observe that $H_p^{\Omega}=1-H_{\omega}^{\Omega}$ for every regular subregion Ω of R. Therefore $\lim_{\Omega\to R}H_p^{\Omega}=1-\mu_{BD}\omega=0$, i, e. p is a potential on R. Clearly $p=1\geqq 1$ on $R-W$. Next assume (d). Since $p\in\tilde{M}_{\Delta}(R)$, p is $[0,\infty]$-valued continuous on R^* and $p=0$ on Δ. Set $U^*=\{z^*\in R;\, p(z^*)<1\}$, which is a neighborhood of Δ and $U^*\cap R\subset W$. A fortiori $U^*\subset\overline{W}$ and $\overline{W}$ is a neighborhood of Δ, i.e. (e) is derived. The final step is the deduction of (a) from (e). Let u be an arbitrary element in $HBD(R)$. There exists an $f\in M(R)$ such that $f=u$ on Δ and $f=0$ on $R^*-\overline{W}$. Let $f=v+g$ be the decomposition in (9) with $Z=W$. Then clearly $v=f=u$ on Δ and $v=f=0$ on $R^*-\overline{W}$. Therefore $v\in HBD(W;\partial W)$. Again observe that $v=\mu_{BD}v+(v-\mu_{BD}v)$ is the decomposition in (9) with $Z=R$. In particular, $\mu_{BD}v=v=u$ on Δ. The maximum principle applied to $\mu_{BD}v-u$ on R implies that $\mu_{BD}v=u$, i.e. (a) is deduced.

3. The next purpose is to characterize the image $\mu_D(HD(W;\partial W))$ in $HD(R)$. We first prove the existence of *universal* constants a and b in $(0,\infty)$ such that

$$D_R(\omega v)\leqq aD_R(v)\,,\qquad D_R(\omega(v-\mu_D v))\leqq bD_R(v) \tag{11}$$

for every v in $HD(W;\partial W)$. Here we stress that no assumption is made on the relative harmonic measure ω. Intuitively we feel that constants a and b depend on the relative shape of (R, W); for this reason we are especially interested in the fact that a and b are completely free from (R, W). The above inequalities as well as the theorem in the next no. are motivated by works of Singer [8, 9].

For the proof we set $w=1-\omega$. Let $\{R_n\}$ be an exhaustion of R by regular subregions and w_n be the harmonic function on $W\cap R_n$ with boundary values 1 on $(\partial W)\cap R_n$ and 0 on $(\partial R_n)\cap W$. We extend w_n to R_n by setting $w_n=1$ on $R_n-W\cap R_n$. Then $w=\lim_{n\to\infty} w_n$ uniformly on each compact subset of R. We denote by $D_n(\cdot)$ the Dirichlet integral over R_n. Although w_n is not of finite Dirichlet integral, we have $D_n(w_n v)<\infty$. It is not difficult to check this directly but the following indirect method may also be one of the simplest. Let f_k be a harmonic function on $W\cap R_n$ with boundary values 0 on $(\partial R_n)\cap W$ and a C^1 function φ_k on $(\partial W)\cap R_n$ such that $0\leqq\varphi_k\leqq 1$, $\varphi_k=0$ in a neighborhood of $(\partial W)\cap(\partial R_n)$ on $(\partial W)\cap R_n$, and $\lim_{k\to\infty}\varphi_k=1$ on $(\partial W)\cap R_n$. Then $D_{W\cap R_n}(f_k)<\infty$ $(k=1, 2, \cdots)$ and $w_n=\lim_{k\to\infty} f_k$ uniformly on each compact subset of $W\cap R_n$. By the Stokes formula

$$\begin{aligned} D_{W\cap R_n}(f_k v) &= -\int_{W\cap R_n} f_k v d(*d(f_k v)) \\ &= -2\int_{W\cap R_n} f_k v df_k \wedge *dv \\ &= -2\int_{W\cap R_n} f_k(d(f_k v)-f_k dv)\wedge *dv \\ &= -2\int_{W\cap R_n} f_k d(f_k v)\wedge *dv + 2\int_{W\cap R_n} f_k^2 dv\wedge *dv\,. \end{aligned}$$

By $0<f_k<1$ on $W\cap R_n$ and the Schwarz inequality

$$D_{W\cap R_n}(f_k v)\leqq 2D_{W\cap R_n}(f_k v)^{1/2}\cdot D_n(v)^{1/2}+2D_n(v)\,.$$

This inequality assures that $\limsup_{k\to\infty} D_{W\cap R_n}(f_k v)<+\infty$. The Fatou lemma yields

$$D_n(w_n v)=D_{W\cap R_n}(w_n v)\leqq \liminf_{k\to\infty} D_{W\cap R_n}(f_k v)<\infty\,.$$

Once $w_n v$ is seen to be of finite Dirichlet integral we can repeat the same procedure replacing $f_k v$ by $w_n v$ to conclude that

$$D_n(w_n v)\leqq 2D_n(w_n v)^{1/2}\cdot D_n(v)^{1/2}+2D_n(v)\,.$$

This implies that $D_n(w_n v)^{1/2}\leqq(1+\sqrt{3})D_n(v)^{1/2}$. On letting $n\to\infty$, the Fatou lemma yields $D_R(wv)^{1/2}\leqq(1+\sqrt{3})D_R(v)^{1/2}$. Thus the first of (11) is valid with $a=(2+\sqrt{3})^2$.

The proof for the second of (11) is similar to the above. Let $v_n=H_v^{R_n}$ and set $g_n=v-v_n$. Then $\lim_{n\to\infty} g_n=v-\mu_D v$ uniformly on each compact subset of R and $\lim_{n\to\infty} D_n(g_n)=D_R(v-\mu_D v)$. Again $D_n(\omega_n)=\infty$ but $D_n(\omega_n g_n)<\infty$ where $\omega_n=1-w_n$. This can be seen as in the case of $D_n(w_n v)<\infty$ (see also the computation below). The Stokes formula yields

$$\begin{aligned} D_n(\omega_n g_n) &= D_{W\cap R_n}(\omega_n g_n) \\ &= -\int_{W\cap R_n} \omega_n g_n d(^*d(\omega_n g_n)) \\ &= -2\int_{W\cap R_n} \omega_n g_n d\omega_n \wedge {}^*dg_n \\ &= -2\int_{W\cap R_n} \omega_n(d(\omega_n g_n)-\omega_n dg_n)\wedge {}^*dg_n \\ &= -2\int_{W\cap R_n} \omega_n d(\omega_n g_n)\wedge {}^*dg_n + 2\int_{W\cap R_n} \omega_n^2 dg_n\wedge {}^*dg_n\,. \end{aligned}$$

By $0<\omega_n<1$ on $W\cap R_n$ and the Schwarz inequality

$$D_n(\omega_n g_n) \leqq 2D_n(\omega_n g_n)^{1/2}\cdot D_n(g_n)^{1/2}+2D_n(g_n)\,.$$

Then $D_n(\omega_n g_n)^{1/2}\leqq(1+\sqrt{3})D_n(g_n)^{1/2}$ and $D_R(\omega(v-\mu_D v))^{1/2}\leqq(1+\sqrt{3})D_R(v-\mu_D v)^{1/2}$, i.e. the second of (11) is valid for $b=(1+\sqrt{3})^2$.

4. In this no. 4 we assume that μ_{BD} is surjective. By the theorem in no. 2, the assumption is equivalent to that $D_R(\omega)<\infty$ and $\mu_{BD}\omega=1$. As an application of (11) we first obtain the following

THEOREM. *Under the assumption that μ_{BD} is surjective, a function u in the class $HD(R)$ belongs to the image $\mu_D(HD(W;\partial W))$ if and only if*

$$D_R(\omega\mu)<\infty\,. \tag{12}$$

That the condition (12) is necessary follows instantly from (11) since $u=v-(v-\mu_D v)$ for a $v\in HD(W;\partial W)$ with $u=\mu_D v$. Conversely assume the condition (12) is valid. Let $\omega u=v+f$ be the orthogonal decomposition of ωu in (8) with $Z=W$. Once more apply (8) with $Z=R$ to v to deduce $v=\mu_D v+g$. Then $\omega u=\mu_D v+(f+g)$ is the decomposition in (8) with $Z=R$ and a fortiori $\omega u=\mu_D v$ on Δ. Since $\omega=1$ on Δ, $u=\mu_D v$ on Δ, which implies with the maximum principle that $u=\mu_D v$ on R, i.e. $u\in\mu_D(HD(W;\partial W))$.

5. We next try to reformulate the above theorem to a more manageable form for the practical application. For this reason we need to select a convenient sequence approximating the function $w=1-\omega$. We fix an exhaustion

$\{R_n\}$ as in no. 3 and we still assume that $D_R(w)=D_R(\omega)<\infty$. The approximating sequence $\{w_n\}$ of w in no. 3 has a drawback $D_{W\cap R_n}(w_n)=\infty$ even if $D_R(w)<\infty$. Our main modification is to make the approximating functions Dirichlet finite.

To each pair (m, n) of positive integers $m<n$ we associate the harmonic function $w_{m,n}$ on $W\cap R_n$ with boundary values $w_{m,n}=1$ on $(\partial W)\cap R_m$, $*dw_{m,n}=0$ on $(\partial W)\cap(R_n-\bar{R}_m)$, and $w_{m,n}=0$ on $(\partial R_n)\cap W$. We extend $w_{m,n}$ to $W\cup\partial W$ by setting $w_{m,n}=0$ on $(W\cup\partial W)\cap(R-R_n)$. For each fixed m, $\{w_{m,n}\}$ $(n=1, 2, \cdots)$ forms an increasing sequence which converges to a harmonic function $w_{m,\infty}$ on W with boundary values $w_{m,\infty}=1$ on $(\partial W)\cap R_m$ and $*dw_{m,\infty}=0$ on $(\partial W)\cap(R-\bar{R}_m)$. The property $0<w_{m,n}<1$ on W is inherited by $w_{m,\infty}$, and therefore $\{w_{m,\infty}\}$ $(m=1, 2, \cdots)$ is again an increasing sequence of harmonic functions on W. Since $w_m\leqq w_{m,n}\leqq w_n$, $w_m\leqq w_{m,\infty}\leqq w$ and thus we have

$$w=\lim_{m\to\infty} w_{m,\infty}=\lim_{m\to\infty}(\lim_{n\to\infty} w_{m,n}) \tag{13}$$

uniformly on each compact subset of $W\cup\partial W$. Observe that

$$\begin{aligned} D_W(w_{m,n}, w_{m,n+k}) &= D_{W\cap R_{n+k}}(w_{m,n}, w_{m,n+k}) \\ &= \int_{\partial(W\cap R_{n+k})} w_{m,n}{}^*dw_{m,n+k} \\ &= \int_{\partial(W\cap R_{n+k})} w_{m,n+k}{}^*dw_{m,n+k} \\ &= D_{W\cap R_{n+k}}(w_{m,n+k})=D_W(w_{m,n+k}) \end{aligned}$$

and similary

$$\begin{aligned} D_W(w_{m,n}, w_{m+k,n}) &= \int_{\partial(W\cap R_n)} w_{m+k,n}{}^*dw_{m,n} \\ &= \int_{\partial(W\cap R_n)} w_{m,n}{}^*dw_{m,n} \\ &= D_W(w_{m,n}). \end{aligned}$$

In view of this and $D_W(w_{m,n}-w_{m,n+k})=D_W(w_{m,n})+D_W(w_{m,n+k})-2D_W(w_{m,n}, w_{m,n+k})$ we deduce that

$$D_W(w_{m,n}-w_{m,n+k})=D_W(w_{m,n})-D_W(w_{m,n+k}) \tag{14}$$

and similarly

$$D_W(w_{m,n}-w_{m+k,n})=D_W(w_{m+k,n})-D_W(w_{m,n}). \tag{15}$$

From (14) it follows that

$$\lim_{n\to\infty} D_W(w_{m,n}-w_{m,\infty})=0\,. \tag{16}$$

Using this we deduce from (15) that

$$D_W(w_{m,\infty}-w_{m+k,\infty})=D_W(w_{m+k,\infty})-D_W(w_{m,\infty})\,.$$

On the other hand, on approximating w by f_n in $M(R)$ in $D_R(\cdot)^{1/2}$ with $f_n=1$ on $(\partial W)\cap R_m$ and $f_n=0$ on $R-R_n$ $(n=1, 2, \cdots)$ (cf. [7], Chap. III), we see that $D_W(w, w_{m,\infty})=D_W(w_{m,\infty})$. The Schwarz inequality yields

$$D_W(w_{m,\infty})\leqq D_W(w)\,.$$

Therefore we conclude that

$$\lim_{m\to\infty} D_W(w_{m,\infty}-w)=\lim_{m\to\infty}(\lim_{n\to\infty} D_W(w_{m,n}-w))=0\,. \tag{17}$$

Since $D_W(w_{m,n}, w)=\int_{(\partial W)\cap R_n} w_{m,n}{}^*dw$, we have $D_W(w_{m,\infty}, w)=\int_{\partial W} w_{m,\infty}{}^*dw$. By the Lebesgue-Fatou theorem we conclude that

$$D_W(w)=\int_{\partial W}{}^*dw \qquad \left(\text{i.e. } D_W(\omega)=-\int_{\partial W}{}^*d\omega\right). \tag{18}$$

In particular, $^*dw$ or $-{}^*d\omega$ is a finite positive mass distribution on ∂W in the case $D_R(\omega)<\infty$.

6. We still assume that $D_R(\omega)=D_R(w)<\infty$. For any $u\in HBD(R)$ the Stokes formula implies that

$$\begin{aligned} D_W(w_{m,n}u) &= D_{W\cap R_n}(w_{m,n}u) \\ &= \int_{\partial(W\cap R_n)} w_{m,n}u^*d(w_{m,n}u)-\int_{W\cap R_n} w_{m,n}ud(^*d(w_{m,n}u)) \\ &= \int_{\partial(W\cap R_n)} w_{m,n}u^*d(w_{m,n}u)-2\int_{W\cap R_n} w_{m,n}udw_{m,n}\wedge{}^*du\,. \end{aligned}$$

Here $\int_{\partial(W\cap R_n)} w_{m,n}u^*d(w_{m,n}u)$ is the sum of

$$\int_{\partial(W\cap R_n)} w_{m,n}u^{2*}dw_{m,n}=\int_{(\partial W)\cap R_m} u^{2*}dw_{m,n}$$

and

$$\begin{aligned} \int_{\partial(W\cap R_n)} w_{m,n}^2u^*du &= \int_{W\cap R_n} d(w_{m,n}^2u^*du) \\ &= 2\int_{W\cap R_n} w_{m,n}udw_{m,n}\wedge{}^*du+\int_{W\cap R_n} w_{m,n}^2du\wedge{}^*du\,. \end{aligned}$$

Putting these together we obtain

$$D_W(w_{m,n}u)=\int_{(\partial W)\cap R_m} u^2{}^*dw_{m,n}+\int_{W\cap R_n} w_{m,n}^2 du\wedge{}^*du\,.$$

Since u is bounded in addition to $D_W(u)<\infty$, (16) implies that $D_W(w_{m,n}u-w_{m,\infty}u)\to 0$ as $n\to\infty$. Thus the Lebesgue-Fatou theorem applied to the above as $n\to\infty$ yields

$$D_W(w_{m,\infty}u)=\int_{(\partial W)\cap R_m} u^2{}^*dw_{m,\infty}+\int_W w_{m,\infty}^2 du\wedge{}^*du\,.$$

Similarly, as above, we deduce by letting $m\to\infty$ that

$$D_W(wu)=\int_{\partial W} u^2{}^*dw+\int_W w^2 du\wedge{}^*du \tag{19}$$

for every $u\in HBD(R)$ if $D_R(w)<\infty$. Here the proof for

$$\lim_{m\to\infty}\int_{(\partial W)\cap R_m} u^2{}^*dw_{m,\infty}=\int_{\partial W} u^2{}^*dw$$

may be in order. Let $u^2\leqq K$ on ∂W. Observe that ${}^*dw_{m,\infty}\geqq{}^*dw\geqq 0$ on $(\partial W)\cap R_m$ and therefore

$$\begin{aligned}
A_m &\equiv\left|\int_{(\partial W)\cap R_m} u^2{}^*dw_{m,\infty}-\int_{\partial W} u^2{}^*dw\right| \\
&\leqq \int_{(\partial W)\cap R_m} u^2({}^*dw_{m,\infty}-{}^*dw)+\int_{(\partial W)\cap(R-\bar{R}_m)} u^2{}^*dw \\
&\leqq K\int_{(\partial W)\cap R_m}({}^*dw_{m,\infty}-{}^*dw)+K\int_{(\partial W)\cap(R-\bar{R}_m)}{}^*dw \\
&= K\int_{\partial(W\cap R_n)} w_{m,n}{}^*dw_{m,\infty}-K\int_{\partial W}{}^*dw+2K\int_{(\partial W)\cap(R-\bar{R}_m)}{}^*dw \\
&= K(D_W(w_{m,n}\,w_{m,\infty})-D_W(w))+2K\int_{(\partial W)\cap(R-\bar{R}_m)}{}^*dw
\end{aligned}$$

for every $n>m$. On letting $n\to\infty$

$$A_m\leqq K(D_W(w_{m,\infty})-D_W(w))+2K\int_{(\partial W)\cap(R-\bar{R}_m)}{}^*dw\,.$$

By (17) and (18), the last two terms and therefore A_m tend to zero as $m\to\infty$.

7. For two functions f and g on R we denote by $f\cup g$ ($f\cap g$, resp.) the pointwise maximum (minimum, resp.) of f and g on R. We also denote by $u\vee v$ ($u\wedge v$, resp.) the least harmonic majorant (the greatest harmonic minorant, resp.) of two harmonic functions u and v on R or on W if it exists. The classes $HX(R)$ and $HX(W;\partial W)$ ($X=D$, BD) form vector lattices with lattice operations $\vee$ and $\wedge$ (cf. e.g. Sario-Nakai [7, Chap. III]). We assume that μ_{BD} is surjective. Then $\mu_D(HBD(W;\partial W))=\mu_{BD}(HBD(W;\partial W))=HBD(R)$ and a fortiori the range of μ_D contains $HBD(R)$. For u and v in $HX(R)$ or $HX(W;\partial W)$ $(X=D, BD)$, $u\vee v$ and $u\wedge v$ are characterized by

$$(u\vee v)|\Delta=(u|\Delta)\cup(v|\Delta), \qquad (u\wedge v)|\Delta=(u|\Delta)\cap(v|\Delta) \tag{20}$$

(cf. [7, p. 177]). This with the identity $v=\mu_D v$ on Δ for every $v\in HD(W;\partial W)$ assures that $\mu_D(HD(W:\partial W))$ is a *vector sublattice* of $HD(R)$ and μ_D is a vector lattice isomorphism from $HD(W;\partial W)$ onto $\mu_D(HD(W;\partial W))$. In particular we see that a function u in $HD(R)$ belongs to the range of μ_D if and only if both the positive part $u^+=u\vee 0$ and the negative part $u^-=u\wedge 0$ of u belong to the range of μ_D. We maintain

THEOREM. *Under the assumption that μ_{BD} is surjective, a nonnegative function u in $HD(R)$ belongs to the range $\mu_D(HD(W;\partial W))$ if and only if*

$$-\int_{\partial W} u^2 {}^*d\omega<\infty . \tag{21}$$

We are not successful in proving the above theorem for not necessarily nonnegative u. In other words we do not know whether the integrability of $(u^+ + u^-)^2$ on ∂W with respect to $-{}^*d\omega$ is equivalent to (21) when u changes its sign on R. First we prove the sufficiency of (21). Let $u_n=u\wedge n$. Since $D_R(u_n)\leqq D_R(u\cap n)\leqq D_R(u)$, we use (19) to deduce

$$D_W(wu_n)=\int_{\partial W} u_n^2 {}^*dw+\int_W w^2 du_n\wedge {}^*du_n$$

$$\leqq\int_{\partial W} u^2 {}^*dw+D_R(u) .$$

Thus $D_W(wu)\leqq \liminf_{n\to\infty} D_W(wu_n)<\infty$ and $D_R(\omega u)^{1/2}\leqq D_R(u)^{1/2}+D_R(wu)^{1/2}\leqq 2D_R(u)^{1/2}+D_W(wu)^{1/2}<\infty$. By the theorem in no. 4 we deduce that u belongs to the range of μ_D. Conversely assume that u belongs to the range μ_D, i.e. $u=\mu_D v$ for a $v\in HD(W;\partial W)$. Let $u_n=u\wedge n$ as above and $v_n=v\wedge(n\omega)$. Then u_n (v_n, resp.) converges increasingly to u (v, resp.) and $D_R(u_n-u)$ ($D_R(v_n-v)$, resp.) $\to 0$ as $n\to\infty$. Observe that $\mu_D v_n=u_n$, $D_R(u_n)\leqq D_R(u)$, and $D_R(v_n)\leqq 2D_R(v)$ for large n. By (11) we see that

$$D_R(\omega u_n)^{1/2}\leqq D_R(\omega v_n)^{1/2}+D_R(\omega(v_n-\mu_D v_n))^{1/2}\leqq cD_R(v_n)^{1/2}$$

where $c=a^{1/2}+b^{1/2}$. Therefore $D_R(wu_n)^{1/2}\leqq(1+c)D_R(v_n)^{1/2}\leqq 2(1+c)D_R(v)^{1/2}$. Again by (19) applied to u_n, we have

$$\int_{\partial W} u_n^2 {}^*dw = D_W(wu_n) - \int_W w^2 du_n \wedge {}^*du_n$$

$$\leqq D_W(wu_n) + D_W(u_n)$$

$$\leqq 4(1+c)^2 D_R(v) + D_R(u) \equiv K < \infty .$$

By the Lebesgue-Fatou theorem

$$\int_{\partial W} u^2 {}^*dw = \lim_{n\to\infty} \int_{\partial W} u_n^2 {}^*dw \leqq K < \infty ,$$

i.e. (21) is deduced.

8. Having finished the general discussions we proceed to the proof of the main theorem stated in the introduction. We need to prepare an elementary lemma. Let Y be a stuffed regular open subset and F be a regular subregion of R such that $\bar{Y} \subset F$. We give the orientation to ∂Y positively with respect to Y. Let

$$\mathfrak{F} = \mathfrak{F}(F, Y) = \{u \in H(F - \bar{Y}) \cap C(\bar{F})\,;\; u \,|\, \bar{Y} = 0\} .$$

Then there exists a finite positive constant $c = c(F, Y)$ such that

$$\sum_{j=1}^{k} \left| \int_{-\partial Y_j} {}^*du \right| \leqq c \max_{\partial F} |u| \tag{22}$$

for every $u \in \mathfrak{F}$ where $Y = \bigcup_{j=1}^{k} Y_j$ is the decomposition of Y into closure disjoint regular subregions Y_j. For the proof let $G(z, \zeta) = G_{F-\bar{Y}}(z, \zeta)$ be the harmonic Green's function on $F - \bar{Y}$. Then

$$u(z) = -\frac{1}{2\pi} \int_{\partial F} u(\zeta) {}^*d_\zeta G(z, \zeta)$$

for every $u \in \mathfrak{F}$ and a fortiori

$${}^*d_z u(z) = -\frac{1}{2\pi} \int_{\partial F} u(\zeta) {}^*d_\zeta ({}^*d_z G(z, \zeta))$$

for every $z \in \partial Y$. Therefore

$$\left| \int_{\partial Y_j} {}^*d_z u(z) \right| = \frac{1}{2\pi} \left| \int_{\partial F} u(\zeta) {}^*d_\zeta \Big(\int_{\partial Y_j} {}^*d_z G(z, \zeta) \Big) \right|$$

$$\leqq \left\{ \frac{1}{2\pi} \int_{\partial F} \left| {}^*d_\zeta \Big(\int_{\partial Y_j} {}^*d_z G(z, \zeta) \Big) \right| \right\} \max_{\partial F} |u|$$

and a fortiori

$$c = \sum_{j=1}^{k} \frac{1}{2\pi} \int_{\partial F} \left| {}^*d_\zeta \Big(\int_{\partial Y_j} {}^*d_z G(z, \zeta) \Big) \right|$$

is the required constant $c(F, Y)$.

9. Let U^* be a stuffed normal neighborhood of a point $z^* \in \Delta$. We fix an exhaustion $\{R_n\}$ of R such that $U \cap R_n$ is a stuffed normal open subset of R for each $n=1, 2, \cdots$, where $U=U^* \cap R$. Let Y be a stuffed normal open subset of R which is the union $\bigcup_{j=1}^{k} Y_j$ of a finite number k of closure disjoint regular subregions Y_j and such that $\bar{Y} \cap U^*=\emptyset$. We denote by $w_n=w(\cdot, Y \cup (U \cap R_n))$ the continuous function on R^* such that $w_n=1$ on $\bar{Y} \cup (\overline{U \cap R_n})$, $w_n=0$ on Δ, and harmonic on $R-\bar{Y} \cup (\overline{U \cap R_n})$. Thus $w_n \in M_\Delta(R)$. We maintain

$$\lim_{n\to\infty} \int_{-\partial(U\cap R_n)} {}^*dw(\cdot, Y \cup (U \cap R_n)) = \infty . \tag{23}$$

For the proof let g_m be the continuous function on R^* such that $g_m=1$ on $\bar{Y} \cup (\overline{U \cap R_n})$, $g_m=0$ on R^*-R_m, and harmonic on $R_m-\bar{Y} \cup (\overline{U \cap R_n})$ $(m>n)$. Then $w_n=\lim_{m\to\infty} g_m$ uniformly on each compact subset of R and $\lim_{m\to\infty} D_R(w_n-g_m)=0$ (see e.g. [**7**, p. 162]). By the Stokes formula

$$D_R(g_m)=\int_{(-\partial Y)\cup(-\partial(U\cap R_n))} {}^*dg_m$$

and on letting $m\to\infty$ we deduce

$$D_R(w_n)=\int_{(-\partial Y)\cup(-\partial(U\cap R_n))} {}^*dw_n .$$

Let F be a regular subregion of R such that $\bar{Y} \subset F$ and $\bar{F} \cap \bar{U}=\emptyset$. Since $\{w_n\}$ is increasing on R, $w_\infty=\lim_{n\to\infty} w_n$ exists on R, $w_\infty=1$ on $\bar{Y} \cup (\bar{U} \cap R)$, and harmonic on $R-\bar{Y} \cup \bar{U}$. In particular, $w_\infty-w_n$ converges uniformly to zero on $\bar{F}$ and $w_\infty-w_n \in \mathcal{F}(F, Y)$. Therefore

$$\begin{aligned}\int_{-\partial(U\cap R_n)} {}^*dw_n &= D_R(w_n)+\int_{\partial Y} {}^*dw_n \\ &= D_R(w_n)+\int_{\partial Y} {}^*dw_\infty+\int_{\partial Y} {}^*d(w_n-w_\infty) \\ &\geqq D_R(w_n)+\int_{\partial Y} {}^*dw_\infty-\sum_{j=1}^{k}\left|\int_{\partial Y_j} {}^*d(w_n-w_\infty)\right|.\end{aligned}$$

By using the constant $c=c(F, Y)$ in (22) we deduce

$$\int_{-\partial(U\cap R_n)} {}^*dw_n \geqq D_R(w_n)+\int_{\partial Y} {}^*dw_\infty-c\max_{\partial F}|w_n-w_\infty| .$$

Therefore to prove (23) we only have to show that $\lim_{n\to\infty} D_R(w_n)=\infty$. By the

Dirichlet principle, $\{D_R(w_n)\}$ is increasing as $n\to\infty$ (cf. e.g. [7, p. 162]). Contrary to the assertion assume that $\lim_{n\to\infty} D_R(w_n)<\infty$. Since $w_\infty=\lim_{n\to\infty} w_n$ uniformly on each compact subset of R and $w_n\in M_\Delta(R)$, the Kawamura lemma (cf. [7, p. 153]) implies that $w_\infty\in M_\Delta(R)$ and a fortiori w_∞ is continuous on R^* and $w_\infty=0$ on Δ. In particular, $w_\infty(z^*)=0$. On the other hand, $w_\infty=1$ on U and the continuity of w_∞ implies that $w_\infty=1$ on U^* and hence $w_\infty(z^*)=1$, a contradiction.

10. We shall next show that for any given positive numbers α and β with $\alpha<\beta$ there exists a stuffed regular open subset X with $\bar{X}\subset U$ such that

$$\alpha<\int_{-\partial X} {}^*dw(\cdot,\ Y\cup X)<\beta. \tag{24}$$

Here Y may be empty. First we use (23) to find an $X_0=U\cup R_n$ such that $\int_{-\partial X_0} {}^*dw(\cdot,\ Y\cup X_0)>\beta$. Let X_0 consist of a finite number l of closure disjoint relatively compact stuffed normal subregions X_{0j}. Fix a point $z_j\in X_{0j}$ for each $j=1, \cdots, l$ and consider a function $G_0(z)$ on X_0 such that $G_0|X_{0j}$ is the Green's function on X_{0j} with pole z_j for each $j=1, \cdots, l$. Let

$$X_t=\{z\in X_0\,;\ G_0(z)>t\}$$

for $t\in[0, \infty)$, which is a stuffed regular open subset except possibly for a finite number of t in $(0, \infty)$. Consider the function

$$f(t)=\int_{-\partial X_t} {}^*dw(\cdot,\ Y\cup X_t)\,.$$

We set $w_t=w(\cdot,\ Y\cup X_t)$. As in no. 9 we deduce

$$f(t)=D_R(w_t)+\int_{\partial Y} {}^*dw_t$$

for $t\in[0, \infty)$ and therefore

$$\begin{aligned}|f(t)-f(t_0)| &\leqq |D_R(w_t)-D_R(w_{t_0})|+\left|\int_{-\partial Y} {}^*d(w_t-w_{t_0})\right|\\ &\leqq |D_R(w_t)-D_R(w_{t_0})|+c\max_{\partial F}|w_t-w_{t_0}|\end{aligned}$$

for t and t_0 in $[0, \infty)$, where F and c are as in no. 9. Observe that $w_t\to w_{t_0}$ as $t\to t_0$ uniformly on each compact subset of R. By the Dirichlet principle, $D_R(w_t)\to D_R(w_{t_0})$ $(t\to t_0)$ (cf. [7, p. 162]). Therefore $\lim_{t\to t_0} f(t)=f(t_0)$, i.e. f is continuous on $[0, \infty)$. Let $w=w(\cdot,\ Y)$. Since $D_R(w)=\int_{\partial Y} {}^*dw$, we have, as above,

$$|f(t)| \leqq |D_R(w_t) - D_R(w)| + c \max_{\partial F} |w_t - w|.$$

Since $\lim_{t\to\infty} w_t = w$ uniformly on each compact subset of $R - \{z_1, \cdots, z_l\}$ and $\lim_{t\to\infty} D_R(w_t) = D_R(w)$, we conclude that $f(\infty) = \lim_{t\to\infty} f(t) = 0$. Therefore $f(t)$ is continuous on $[0, \infty]$ with $f(0) > \beta$ and $f(\infty) = 0$. The intermediate value theorem yields the existence of a $t_1 \in (0, \infty)$ such that $f(t_1) \in (\alpha, \beta)$ and $X = X_{t_1}$ is a stuffed regular open subset of R.

11. We assume that $R \notin U_{\widetilde{HD}}$ and therefore any component of Δ contains at least two points (cf. [**7**, Chap. III]). In this no. we start with fixing a stuffed regular open subset $Y = \bigcup_{j=1}^{l} Y_j$ of R such that Y_j $(j = 1, \cdots, l)$ are closure disjoint subregions of R. We also fix a point $z^* \in \Delta$. We shall prove that for any positive number ε there exists a stuffed normal neighborhood U^* of z^* with $\bar{Y} \cap \bar{U}^* = \emptyset$ and

$$\sum_{j=1}^{k} \left| \int_{-\partial Y_j} {}^*dw(\cdot, Y \cup X) - \int_{-\partial Y_j} {}^*dw(\cdot, Y) \right| < \varepsilon \tag{25}$$

for every stuffed regular open subset X with $\bar{X} \subset U = U^* \cap R$. In the proof of this assertion the assumption $R \notin U_{\widetilde{HD}}$ is essentially made use of. Actually we only use the fact that the harmonic measure of z^* is zero. For the proof let $P(z, \zeta^*)$ be the harmonic kernel and μ be the harmonic measure on Δ (cf. [**7**, p. 171]). Since $R \notin U_{\widetilde{HD}}$ is characterized by $\mu(\{\zeta^*\}) = 0$ for every $\zeta^* \in \Delta$ (cf. [**7**, p. 187]), we obtain

$$\lim_{V^* \to \{z^*\}} \int_{\Delta \cap V^*} P(z, \eta^*) d\mu(\zeta^*) = 0$$

uniformly on each compact subset of R where $\{V^*\}$ is the directed net of normal neighborhoods V^* of z^* with $\bar{Y} \cap \bar{V}^* = \emptyset$ such that V^* does not contain any dividing cycle of R, which can be assumed by the fact that any component of Δ contains at least two points. If $R - V^* \cap R$ has compact component E_j $(1 \leqq j < p \leqq \infty)$, then $V^* \cup (\bigcup_{j=1}^{p} E_j)$ is open in R^* for every $q < p$ and thus $U^* = V^* \cup (\bigcup_{j=1}^{p} E_j) = \bigcup_{q=1}^{p} (V^* \cup (\bigcup_{j=1}^{q} E_j))$ is again open. Hence U^* is a stuffed normal neighborhood of z^* with $\bar{Y} \cap \bar{U}^* = \emptyset$ and $\Delta \cap U^* = \Delta \cap V^*$. Therefore

$$\lim_{U^* \to \{z^*\}} \int_{\Delta \cap U^*} P(z, \zeta^*) d\mu(\zeta^*) = 0$$

uniformly on each compact subset of R where $\{U^*\}$ is the directed net of stuffed normal neighborhoods U^* of z^* with $\bar{Y} \cap \bar{U}^* = \emptyset$. We can thus find a decreasing sequence $\{U_n^*\}$ in $\{U^*\}$ such that

$$\lim_{n\to\infty}\int_{\Delta\cap U_n^*} P(z, \zeta^*)d\mu(\zeta^*)=0 .$$

Note that, however, $(\bigcap_{n=1}^{\infty} U_n^*)\cap\Delta\supsetneqq\{z^*\}$ (cf. e.g. [7, p. 156]). Let $k_{n,m}$ be the continuous function on $R^*-(R^*-R)\cap\partial\bar{U}_n$ $(U_n=U_n^*\cap R)$ such that $k_{n,m}=1$ on $\bar{U}_n\cap(R-R_m)$, $k_{n,m}=0$ on $\Delta-\bar{U}_n$, and harmonic on $R-\bar{U}_n\cap(R-R_m)$. Then

$$\lim_{m\to\infty} k_{n,m}=\int_{\Delta\cap U_n^*} P(\cdot, \zeta^*)d\mu(\zeta^*)$$

uniformly on each compact subset of R. Thus $\lim_{n\to\infty}(\lim_{m\to\infty} k_{n,m})=0$ uniformly on each compact subset of R. We can thus find an increasing sequence $\{m(n)\}$ $(n=1, 2, \cdots)$ of positive integers such that $\lim_{n\to\infty} k_n=0$ $(k_n=k_{n,m(n)})$ uniformly on each compact subset of R. By the maximum principle,

$$w(\cdot, Y)\leqq w(\cdot, Y\cup X)\leqq w(\cdot, Y)+k_n$$

for every stuffed regular open subset X with $\bar{X}\subset U_n\cap(R-\bar{R}_{m(n)})$. Let F be a regular subregion of R with $\bar{Y}\subset F$ and $\bar{F}\cap(\overline{U_1\cap(R-\bar{R}_{m(1)})})=\emptyset$. Then

$$\sum_{j=1}^{l}\left|\int_{-\partial Y_j} {}^*dw(\cdot, Y\cup X)-\int_{-\partial Y_j} {}^*dw(\cdot, Y)\right|\leqq c\sup_{\partial F} k_n$$

for every stuffed regular open subset X with $\bar{X}\subset U_n\cap(R-\bar{R}_{m(n)})$, where $c=c(F, Y)$ is the constant in (22). Since $\sup_{\partial F} k_n\to 0$ as $n\to\infty$, we can choose the required U^* in (25) to be $U_n^*\cap(R^*-\bar{R}_{m(n)})$ such that $c\sup_{\partial F} k_n<\varepsilon$.

12. In addition to $R\notin U_{\widetilde{HD}}$ we assume that $HD(R)-HBD(R)\neq\emptyset$. Since $HD(R)$ forms a lattice, we can find an h in $HD(R)-HBD(R)$ such that $h>0$. This function h will be fixed throughout the proof. By the maximum principle we can select a sequence $\{z_n^*\}$ of distinct points in Δ and a sequence $\{\alpha_n\}$ of positive numbers with $n<\alpha_n$ and $3\alpha_n<\alpha_{n+1}$ $(n=1, 2, \cdots)$ such that $h(z_n^*)=2\alpha_n$ $(n=1, 2, \cdots)$. We can choose a sequence $\{V_n^*\}$ of stuffed normal neighborhood V_n^* of z_n^* $(n=1, 2, \cdots)$ such that $\bar{V}_n^*\cap\bar{V}_m^*=\emptyset$ $(n\neq m)$, $h>\alpha_n$ on V_n^* $(n=1, 2, \cdots)$, and $V_n=V_n^*\cap R$ does not contain any dividing cycle of $R\nsim 0$. This can be found by induction. First let V^* be a normal neighborhood of z_1^* such that $\alpha_1<h<3\alpha_1$ on $\bar{V}^*$ and V^* does not contain $\{z_n^*\}$ $(n\geqq 2)$ and V^* does not contain any dividing cycle $\nsim 0$ of R. If $V=V^*\cap R$ has a compact component E in its complement $R-V$, then E is the closure of a regular subregion E^0 and $3\alpha_1>h>\alpha_1$ on ∂E^0. By the maximum principle, we must have $3\alpha_1>h>\alpha_1$

on E. Thus V^* can be replaced by $V^*\cup E$ and repeating this process we can deform V^* to a stuffed normal neighborhood V_1^* with the required property. Assume that $V_1^*, \cdots, V_n^*$ have already been chosen as required and $(\bigcup_{j=1}^{n}\bar{V}_j^*)\cap\{z_j^*\}_{j\geq n+1}=\emptyset$. Let V^* be a normal neighborhood of z_{n+1}^* such that $3\alpha_{n+1}>h>\alpha_{n+1}$ on $\bar{V}^*$ and $\bar{V}^*\cap(\bigcup_{j=1}^{n}\bar{V}_j^*)\cup\{z_j^*\}_{j\geq n+2})=\emptyset$ and V^* does not contain any dividing cycle $\not\sim 0$ of R. Let E be a compact component of $R-V^*\cap R$, if there exists any. Then as above $3\alpha_{n+1}>h>\alpha_{n+1}$ on E. By the choice of $\{\alpha_n\}$, $E\cap\bar{V}_j^*=\emptyset$ $(j=1, \cdots, n)$ and thus the deformed stuffed normal neighborhood V_{n+1}^* by adding all E's to V^* satisfies the required condition.

We shall next construct a sequence $\{X_n\}$ of stuffed regular open subsets X_n of R $(n=1, 2, \cdots)$ such that $\bar{X}_n\cap\bar{X}_m=\emptyset$ $(n\neq m)$,

$$\alpha_j^{-2}<\int_{-\partial X_j}{}^*dw(\cdot, \bigcup_{n=1}^{k}X_n)<2\alpha_j^{-2} \qquad (j=1, \cdots, k) \tag{26}$$

for every $k=1, 2, \cdots$, $\bar{X}_n\subset V_n=V_n^*\cap R$, and in particular

$$h|\bar{X}_n>\alpha_n \tag{27}$$

for every $n=1, 2, \cdots$. The construction of $\{X_n\}$ is again by induction. First let X_1 be a stuffed regular open subset of R with $\bar{X}_1\subset V_1=V_1^*\cap R$ such that

$$\alpha_1^{-2}<\int_{-\partial X_1}{}^*dw(\cdot, X_1)<2\alpha_1^{-2}.$$

The existence is assured by (24). Suppose that $X_1, \cdots, X_k$ have already been chosen. We set $Y=\bigcup_{j=1}^{k}X_j$. Let U^* be as in no. 11 for $z^*=z_{k+1}^*$ and for $\varepsilon>0$ less than $\min_{1\leq j\leq k}(\min(2\alpha_j^{-2}-\beta_j, \beta_j-\alpha_j^{-2}))$ where

$$\beta_j=\int_{-\partial X_j}{}^*dw(\cdot, \bigcup_{n=1}^{k}X_n)$$

for $j=1, \cdots, k$. Set $U_{k+1}^*=U^*\cap V_{k+1}^*$ and $U_{k+1}=U_{k+1}^*\cap R$. Then U_{k+1}^* $(\subset V_{k+1}^*)$ is a stuffed normal neighborhood of z_{k+1}^* and (25) implies that

$$\alpha_j^{-2}<\int_{-\partial X_j}{}^*dw(\cdot, (\bigcup_{n=1}^{k}X_n)\cup X)<2\alpha_j^{-2} \qquad (j=1, \cdots, k)$$

for every stuffed regular open subset X with $X\subset U_n$. By (24) we can choose a stuffed regular open subset $X=X_{k+1}$ with $\bar{X}_{k+1}\subset U_n$ such that

$$\alpha_{k+1}^{-2}<\int_{-\partial X_{k+1}}{}^*dw(\cdot, (\bigcup_{n=1}^{k}X_n)\cup X_{k+1})<2\alpha_{k+1}^{-2}.$$

This completes the induction.

13. We are coming to the final stage of the proof of the main theorem stated in the introduction. As a required subregion W we take

$$W=R-\bigcup_{n=1}^{\infty}\bar{X}_n . \tag{28}$$

Since $\{w(\cdot, \bigcup_{n=1}^{k} X_n)\}$ $(k=1, 2, \cdots)$ is increasing as can be seen by the maximum principle, we can define

$$w(\cdot, \bigcup_{n=1}^{\infty} X_n)=\lim_{k\to\infty} w(\cdot, \bigcup_{n=1}^{k} X_n) \tag{29}$$

where the convergence is uniform on each compact subsets of R. The function (29) is 1 on $\bigcup_{n=1}^{\infty}\bar{X}_n$, continuous superharmonic on R, and harmonic on W. For simplicity we set $w_\infty=w(\cdot, \bigcup_{n=1}^{\infty} X_n)$ and $w_k=w(\cdot, \bigcup_{n=1}^{k} X_n)$. Then using the fact that $w_k\in M_\Delta(R)$ $(k=1, 2, \cdots)$ (cf. no. 9) we deduce

$$\begin{aligned} D_R(w_{k+p}-w_k) &= \sum_{j=k+1}^{k+p}\left\{\int_{-\partial X_j}(1-w_k)^*d(w_{k+p}-w_k)+D_{X_j}(w_k)\right\} \\ &= \sum_{j=k+1}^{k+p}\int_{-\partial X_j}(1-w_k)^*dw_{k+p} . \end{aligned}$$

Since $0<1-w_k<1$ and ${}^*dw_{k+p}>0$ on $-\partial X_j$, (26) implies that

$$D_R(w_{k+p}-w_k)\leqq 2\sum_{j=k+1}^{k+p}\alpha_j^{-2}\leqq 2\sum_{j=k+1}^{k+p} j^{-2} .$$

On letting $p\to\infty$ and by using the Fatou lemma we conclude that

$$D_R(w_\infty-w_k)\leqq 2\sum_{j=k+1}^{\infty} j^{-2}$$

for every k. Therefore

$$\lim_{k\to\infty} D_R(w(\cdot, \bigcup_{n=1}^{\infty} X_n)-w(\cdot, \bigcup_{n=1}^{k} X_n))=0 . \tag{30}$$

Similarly, as above, we have

$$D_R(w_k)=\sum_{j=1}^{k}\int_{-\partial X_j}{}^*dw_k\leqq 2\sum_{j=1}^{k}\alpha_j^{-2}\leqq 2\sum_{j=1}^{k} j^{-2}$$

and therefore

$$D_R(w_\infty)\leqq 2\sum_{j=1}^{\infty} j^{-2} .$$

By (29), (30), and $w_k\in M_\Delta(R)$, we see that $w_\infty\in M_\Delta(R)$. Therefore $1-w_\infty\in HBD(W; \partial W)$ and $\mu_{BD}(1-w_\infty)=1-w_\infty=1$ on Δ. By Theorem in no. 2, μ_{BD} is *surjective* and

(31) $$w=1-\omega=w(\cdot,\ \bigcup_{n=1}^{\infty} X_n)$$

where ω is the relative harmonic measure of the ideal boundary of W.

14. The last task to complete the proof is to show that μ_D is not surjective. We shall show that $h\notin\mu_D(HD(W;\partial W))$, where h is the one introduced in no. 12. By the theorem in no. 7, we have to show that

(32) $$\int_{\partial W} h^2 {}^*dw(\cdot,\ \bigcup_{n=1}^{\infty} X_n)=\infty$$

which is equivalent to that $h\notin\mu_D(HD(W;\partial W))$. As a consequence of (26) we have

(33) $$\alpha_j^{-2}\leqq\int_{-\partial X_j} {}^*dw(\cdot,\ \bigcup_{n=1}^{\infty} X_n)\leqq 2\alpha_j^{-2}\qquad (j=1,2,\cdots).$$

These two relations will also be used in the application. To prove (32), by (27) and (33), we proceed as follows:

$$\begin{aligned}\int_{\partial W} h^2 {}^*dw_\infty &= \sum_{j=1}^{\infty}\int_{-\partial X_j} h^2 {}^*dw_\infty\\ &\geqq \sum_{j=1}^{\infty}\alpha_j^2\int_{-\partial X_j} {}^*dw_\infty\\ &\geqq \sum_{j=1}^{\infty}\alpha_j^2\cdot\alpha_j^{-2}=\infty\,.\end{aligned}$$

The proof of the main theorem in the introduction is herewith complete.

15. We proceed to the proof of the corollary stated in the introduction. Royden [**6**] proved that the finiteness of P implies the surjectiveness of the reduction operator $T_{BE}: PBE(R)\rightarrow HBD(R)$. That the converse of this is 'almost true' is shown by Glasner-Katz [**1**]. Precisely, T_{BE} is surjective if and only if

$$\int_{R-E} P(z)dxdy<\infty$$

for a BD-negligible subset $E\subset R$, i.e. the closed subset E of R such that there exists a subregion W with analytic relative boundary ∂W with $E\subset R-W$ and $\mu_{BD}: HBD(W;\partial W)\rightarrow HBD(R)$ is surjective. That the Green energy finiteness of P implies the surjectiveness of $T_{BD}: PBD(R)\rightarrow HBD(R)$ is shown in [**2**]. Actually T_{BD} is surjective if and only if

(35) $$\int_{(R-E)\times(R-E)} G(z,\zeta)P(z)p(\zeta)dxdyd\xi d\eta<\infty$$

for a BD-negligible set $E\subset R$ (cf. e.g. [3]). Singer [9] showed the existence of a density P on $R=\{|z|<1\}$ with (35) and nonsurjective T_D. Similarly it was shown in [4] the existence of a finite (i.e. (34) with $E=\emptyset$) density P on $R=\{|z|<1\}$ with nonsurjective T_E. Moreover, as a generalization of the above two assertions, the existence of a both finite and Green energy finite density P on $R=\{|z|<1\}$ with nonsurjective T_E and T_D was shown in [5]. Our present corollary stated in the introduction is a generalization of the above three works. The proof is given in nos. 16-19.

16. The construction of a density P on R as stated in the corollary is immediate if we use h and $W=R-\bigcup_{j=1}^{\infty}\bar{X}_j$ determined in nos. 12 and 13 for the proof of the main theorem. We use the function $w_\infty=w(\cdot, \bigcup_{j=1}^{\infty}\bar{X}_j)$. Since w_∞ is harmonic on W and $w_\infty|\partial W=1$, $w_\infty|W$ can be continued harmonically to a region $W^\sim$ containing $W\cup\partial W$. We denote by $w_\infty^\sim$ the harmonic extension of $w_\infty|W$ to $W^\sim$. Let ε_j be chosen in $(0, 1/2)$ so small that the level line

$$l_\eta=\{z\in X_j\cap W^\sim;\, w_\infty^\sim(z)=1+\eta\}$$

consists of a finite number of mutually disjoint analytic Jordan curves in X_j and l_η is homologous to ∂X_j in X_j for each $\eta\in(0, 2\varepsilon_j]$ $(j=1, 2, \cdots)$. We denote by Y_j (Z_j, resp.) the stuffed regular open set of R bounded by l_{ε_j} ($l_{2\varepsilon_j}$, resp.). Then $\bar{Z}_j\subset Y_j$ and $\bar{Y}_j\subset X_j$ $(j=1, 2, \cdots)$. Let $f=w_\infty^\sim$ on $R-\bigcup_{j=1}^{\infty}\bar{Y}_j$ and $f=1+\varepsilon_j$ on $\bar{Y}_j$ $(j=1, 2, \cdots)$. By choosing ε_j small enough we can assume that $D_R(f)<\infty$. The function f is superharmonic on R and harmonic on $R-\bigcup_{j=1}^{\infty}\bar{Y}_j$. By applying the regularization (cf. e.g. Tsuji [10], Yosida [11], [7, p. 150]) to f on each $X_j-\bar{Z}_j$ $(j=1, 2, \cdots)$, we obtain a C^∞ superharmonic function g on R such that

$$g|W\cup(\bigcup_{j=1}^{\infty}\bar{Z}_j)=f.$$

We can also make $D_{X_j-\bar{Z}_j}(g-f)$ as small as we wish by choosing the regularization g close enough to f (see [7, p. 150]) in each $X_j-\bar{Z}_j$ $(j=1, 2, \cdots)$ and thus we can assume that $D_R(g)<\infty$. The property of w_∞ being in $M_\Delta(R)$ is inherited by $g: g\in M_\Delta(R)$, and a fortiori $g|\Delta=0$. Since $g=f=w_\infty$ on W, the following is identical with (33):

$$\alpha_j^{-2}\leqq\int_{-\partial X_j}{}^*dg\leqq 2\alpha_j^{-2}\qquad (j=1, 2, \cdots). \tag{36}$$

17. As the final step of our construction we consider the function

(37) $$e(z)=1-g(z)/2$$

on R. Then $e(z)$ is a Dirichlet finite C^{∞} subharmonic function on R and

(38) $$1/4\leqq e(z)<1$$

on R. Since $g|\Delta=0$, $e|\Delta=1$. The inequality (36) yields

(39) $$2\alpha_j^{-2}\leqq\int_{\partial X_j}{}^*de\leqq 4\alpha_j^{-2}\qquad(j=1, 2, \cdots).$$

Finally we give the required density P on R as follows:

(40) $$P(z)dxdy=(\Delta e(z)/e(z))dxdy,$$

which vanishes on W. We ascertain that the P is a *finite* density on R:

$$\int_R P(z)dxdy\leqq 4\int_R\Delta e(z)dxdy$$
$$=4\sum_{j=1}^{\infty}\int_{X_j}\Delta e(z)dxdy=4\sum_{j=1}^{\infty}\int_{\partial X_j}{}^*de$$
$$\leqq 4\sum_{j=1}^{\infty}4\alpha_j^{-2}\leqq 16\sum_{j=1}^{\infty}j^{-2}<\infty.$$

Since $e\in M(R)$, we can apply the orthogonal decomposition (9) with $Z=R$ to e to obtain $e=k+q$ where $k\in HBD(R)$ and $q\in M_{\triangle}(R)$. Here $k|\Delta=e|\Delta=1$ and thus by the maximum principle, $k=1$ on R, i.e. $e=1+q$. Then $-q=1-e>0$ is superharmonic on R and vanishes on Δ. Again by the maximum principle, $-q$ is a potential on R. The Riesz theorem implies that

$$-q=\frac{1}{2\pi}\int_R G(\cdot, \zeta)(-\Delta_{\zeta}(-q(\zeta))d\xi d\eta=\frac{1}{2\pi}\int G(\cdot, \zeta)P(\zeta)e(\zeta)d\xi d\eta,$$

where $G(z, \zeta)$ is the harmonic Green's function on R. Thus

(41) $$e(z)=1-\frac{1}{2\pi}\int_R G(z, \zeta)P(\zeta)e(\zeta)d\xi d\eta.$$

In view of (38) we in particular see that

(42) $$\int_R G(z, \zeta)P(\zeta)d\xi d\eta<8\pi$$

for every $z\in R$. Therefore

$$\int_{R\times R}G(z, \zeta)P(z)P(\zeta)dxdyd\xi d\eta$$
$$=\int_R\Big(\int_R G(z, \zeta)P(\zeta)\,d\xi d\eta\Big)P(z)dxdy$$
$$\leqq 8\pi\int_R P(z)dxdy<\infty,$$

i.e. we have shown that the P is a *Green energy finite* density on R.

18. The proof of the corollary will be complete if we show the nonsurjectiveness of $T_D: PD(R)\to HD(R)$ and $T_E: PE(R)\to HD(R)$. Since $PD(R)\supset PE(R)$ and T_D is an extension of T_E, we only have to prove that T_D is not surjective. Let e_Ω be the solution of $\Delta u=Pu$ on a regular subregion Ω of R with boundary values 1 on $\partial\Omega$. The directed net $\{e_\Omega\}$ is increasing and $\lim_{\Omega\to R} e_\Omega$ is a solution of $\Delta u=Pu$ on R which is referred to as the *P-unit* on R. We maintain that $e(z)$ in (37) is the P-unit on R, i.e.

$$e(z)=\lim_{\Omega\to R} e_\Omega(z) \tag{43}$$

uniformly on each compact subset of R. Let $G_\Omega(z, \zeta)$ be the harmonic Green's function on Ω. Since $1-e_\Omega$ is a potential on Ω, we deduce, as we deduced (41), that

$$e_\Omega(z)=1-\frac{1}{2\pi}\int_\Omega G_\Omega(z, \zeta)P(\zeta)e_\Omega(\zeta)d\xi d\eta\,.$$

In view of (42), $e_\Omega<1$, and that $G_\Omega(z, \zeta)$ converges to $G(z, \zeta)$ increasingly as $\Omega\to R$, we can apply the Lebesgue convergence theorem to the above identity as $\Omega\to R$ to deduce

$$u(z)=1-\frac{1}{2\pi}\int_R G(z, \zeta)P(\zeta)u(\zeta)d\xi d\eta$$

when $u=\lim_{\Omega\to R} e_\Omega$. Then, on setting $v=e-u$, the subtraction of the above from (41) gives

$$v(z)=-\frac{1}{2\pi}\int_R G(z, \zeta)P(\zeta)v(\zeta)d\xi d\eta$$

and by the Schwarz inequality with (42), we deduce

$$v^2\leqq\frac{2}{\pi}\int_R G(\cdot, \zeta)P(\zeta)v^2(\zeta)d\xi d\eta\,. \tag{44}$$

Here $\Delta v=Pv$ and hence $\Delta v^2=2(Pv^2+|\mathrm{grad}\, v|^2)\geqq 0$, i.e. v^2 is subharmonic on R. The right hand side of (44) is a potential majorizing a nonnegative subharmonic function on R and therefore we must have $v^2=0$ on R, i.e. $e=u$, proving (43).

19. We need to recall the Singer P-unit criterion [**8**]: $u\in T_D(PD(R))$ implies that $D_R(eu)<\infty$. The proof of this part is rather simple but the converse, which is also shown by Singer [**9**], is not easy to prove. However we only need the implication of $D_R(eu)<\infty$ from $u\in T_D(PD(R))$. We claim the nonsurjectiveness of T_D by showing the $h\notin T_D(PD(R))$, i.e.

$$D_R(eh)=\infty\,, \tag{45}$$

where h is the positive function in $HD(R)-HBD(R)$ fixed in no. 12. Recall that $e=1-w_\infty/2$ on W and $w_\infty=1-\omega$, where ω is the relative harmonic measure of the ideal boundary of W. Hence

$$\begin{aligned} D_W(eh)^{1/2} &\geqq \frac{1}{2}D_W(w_\infty h)^{1/2}-D_W(h)^{1/2} \\ &\geqq \frac{1}{2}D_W(\omega h)^{1/2}-\frac{3}{2}D_W(h)^{1/2} \\ &\geqq \frac{1}{2}D_R(\omega h)^{1/2}-\frac{3}{2}D_R(h)^{1/2}\,. \end{aligned}$$

We have seen in no. 14 that $h\in\mu_D(HD(W\,;\partial W))$. By the theorem in no. 4 we must have $D_R(\omega h)=\infty$. Therefore by the above inequality we see that $D_W(eh)=\infty$. A fortiori $D_R(eh)\geqq D_W(eh)$ implies (45). The proof of the corollary is herewith complete.

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