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A class of Markov chains related to selection in polulation genetics

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§ 1. Introduction.

Let $\boldsymbol{Z}_+^2$ be the set of 2-dimensional lattice points with nonnegative coordinates and let $\{Z^{(N)}(n);\ n=0, 1, 2, \cdots\}$, $N=1, 2, \cdots$, be a sequence of time homogeneous Markov chains taking values in $\boldsymbol{Z}_+^2$, each of which is a generalization of direct product branching process. Let $Z^{(N)}(n)=(Z_1^{(N)}(n), Z_2^{(N)}(n))$ and let

$$P_{jk}^{(N)}=P(Z^{(N)}(n+1)=(k, N-k)\mid Z^{(N)}(n)=(j, N-j),\ Z_1^{(N)}(n+1)+Z_2^{(N)}(n+1)=N)$$

for $j, k=0, 1, \cdots, N$. Consider a Markov chain $\{X^{(N)}(n);\ n=0, 1, 2, \cdots\}$ on $\{0, 1, \cdots, N\}$ with one-step transition probability $(P_{jk}^{(N)})$, which we call the induced Markov chain. When $\{Z^{(N)}(n)\}$ is a direct product branching process, $\{X^{(N)}(n)\}$ is the Markov chain introduced by Karlin and McGregor [**3**]. The main purpose of this paper is to show that, for an appropriate class of $\{Z^{(N)}(n)\}$, the sequence of the induced Markov chains (suitably normalized) converges to a diffusion process on the interval [0, 1] having a backward Kolmogorov equation of the form

$$\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}x(1-x)\frac{\partial^2 u}{\partial x^2}+\gamma(x)x(1-x)\frac{\partial u}{\partial x}\,. \tag{1.1}$$

The convergence is in the sense of weak convergence of probability measures in the space of continuous sample functions. Equations of the form (1.1) appear in diffusion approximation to Markov chain models with selection in population genetics (cf. [**1**]). Especially the following four cases are important in genetics:

(i) $\gamma(x)$ is a constant function,

(ii) $\gamma(x)=\alpha_1 x$,

(iii) $\gamma(x)=-\alpha_2(1-x)$,

(iv) $\gamma(x)=\alpha_1 x-\alpha_2(1-x)$,

where α_1, α_2 are constants. Consider $Z_1^{(N)}(n)$ and $Z_2^{(N)}(n)$ as the numbers of individuals of types 1 and 2 (alleles A_1 and A_2), respectively. Existence of selection force corresponds to difference in fertility. In Case (i) we choose $\{Z^{(N)}(n)\}$ as a direct product branching process, making fertility differ between types 1 and 2 (see [2] for examples), but, in other cases, we make fertility depend on the existing proportion of types, so that $\{Z^{(N)}(n)\}$ is not a branching process.

We will give our results in general d type cases. When $d \geqq 3$, however, we have the proof of convergence to diffusion processes only in some special case (this is the case in which only one type has different fertility from the other types). A difficulty arises in connection with the uniqueness of the solution of the martingale problem of the limiting diffusion. This is the same circumstance that we encounter in [6] when we deal with convergence of the Markov chains induced by direct product branching processes with mutation and migration. In one dimension (that is, $d=2$) the uniqueness theorem of Yamada and S. Watanabe [8] applies, since the square root of the diffusion coefficient is Hölder continuous with exponent 1/2 and since we assume the drift coefficient to be Lipschitz continuous. But, in higher dimensions, they show [8] that the same continuity moduli do not imply the uniqueness in general.

In Section 2 we will precisely define our class of $\{Z^{(N)}(n)\}$ and state our results. A lemma on asymptotic estimate of some integrals is given in Section 3. After this lemma is established, the proof in Section 4 of our results is analogous to the discussion in [6]. It would not be difficult to extend the results of this paper to the case involving selection, mutation and migration, using the methods in [6].

A work having some relation with this paper is Kushner's [4], in which he gives an invariance principle in the space D with an application to a genetics model with selection.

§ 2. Results.

Let $\{Z^{(N)}(n); n=0, 1, 2, \cdots\}$, $N=1, 2, \cdots$, be a sequence of time homogeneous Markov chains taking values in $\boldsymbol{Z}_+^d$, the set of d-dimensional lattice points with nonnegative coordinates, and let $f_{N,j}(s_1, \cdots, s_d)$ be the generating function of the one-step transition probability from $j=(j_1, \cdots, j_d)$:

$$f_{N,j}(s_1, \cdots, s_d) = \sum_{k \in \boldsymbol{Z}_+^d} P(Z^{(N)}(n+1)=k \mid Z^{(N)}(n)=j) s_1^{k_1} \cdots s_d^{k_d}. \tag{2.1}$$

Let $|j| = \sum_{p=1}^{d} j_p$ for $j=(j_1, \cdots, j_d)$ and let $\boldsymbol{J}(N)$ be the set of points $j \in \boldsymbol{Z}_+^d$ such that $|j|=N$. We make the following assumptions.

ASSUMPTION 2.1. (i) *If N is sufficiently large, then, for each $j \in \boldsymbol{J}(N)$, $f_{N,j}$*

is of the form

$$f_{N,j}(s_1, \cdots, s_d)=\prod_{p=1}^{d}\Big(\sum_{n=0}^{\infty} c_n s_p^n\Big)^{j_p(1+N^{-1}\gamma_p(|j|^{-1}j))} \tag{2.2}$$

for $0\leqq s_p\leqq 1$ $(p=1, \cdots, d)$, *where* c_n *and* γ_p *satisfy the following conditions.*

(ii) $\{c_n\}$ *is a probability distribution on the nonnegative integers independent of* N, j *and* p *with* $c_0>0$ *and with maximum span* 1 (*that is, there is no pair of* $\gamma>1$ *and* δ *such that* $\sum_n c_{n\gamma+\delta}=1$). *Let* $a=\sum_{n=0}^{\infty} nc_n$ (mean), $f(w)=\sum_{n=0}^{\infty} c_n w^n$ (generating function), $M(w)=\sum_{n=0}^{\infty} c_n e^{nw}$ (moment generating function), $F(w)=M(w)e^{-w}$, $b=\sup\{w;\, M(w)<\infty\}$. *Then, one of the following holds:*

(a) $1<a\leqq+\infty$;

(b) $a=1$ *and* $b>0$;

(c) $a<1$ *and* $\lim_{w\uparrow b} F'(w)>0$, *where* $F'(w)=\dfrac{dF}{dw}$.

(iii) $\gamma_p(x), p=1, \cdots, d$, *are continuous functions of* x *defined on* $\{x=(x_1, \cdots, x_d)\in \boldsymbol{R}^d;\ x_1\geqq 0, \cdots, x_d\geqq 0, \sum_{p=1}^{d} x_p=1\}$.

The above assumption implies that, for big N, individuals of a type p in a generation reproduce their children of the same type p independently of each other according to a common distribution, but the distribution may vary with types and with the composition of the present generation. Fertility may thus vary.

Condition (ii) is common with [**5**] and [**6**], and hence the following can be proved: 1° $b>0$ in Case (c). 2° If $a<1$, $b>0$ and $\lim_{w\uparrow b} M(w)=\infty$, then (c) holds. 3° There exists a unique $\beta\in(-\infty, b)$ such that $F'(\beta)=0$. 4° β is negative, zero, positive in Cases (a), (b), (c), respectively. Let $K(w)=\log M(w)$ for $w<b$. 5° $K'(\beta)=1$ and $K''(\beta)>0$. Let $\sigma=\sqrt{K''(\beta)}$. 6° The associated distribution $\{\hat{c}_n\}$ of $\{c_n\}$ defined by $\hat{c}_n=c_n e^{n\beta}/M(\beta)$ has mean 1 and variance σ^2.

For j, $k\in \boldsymbol{J}(N)$, let

$$P_{jk}^{(N)}=P(Z^{(N)}(n+1)=k \mid Z^{(N)}(n)=j,\ Z^{(N)}(n+1)\in \boldsymbol{J}(N)), \tag{2.3}$$

which can be defined for sufficiently large N by the following lemma.

LEMMA 2.1. *If* N *is large enough, then*

$$P(Z^{(N)}(n+1)\in \boldsymbol{J}(N) \mid Z^{(N)}(n)=j)>0 \quad \textit{for all} \quad j\in \boldsymbol{J}(N).$$

For a fixed N, let $\{X^{(N)}(n)=(X_1^{(N)}(n), \cdots, X_d^{(N)}(n));\, n=0, 1, \cdots\}$ be a Markov chain defined on a probability space $(\Omega^{(N)}, Q^{(N)})$, taking values in $\boldsymbol{J}(N)$ with one-step transition probability $(P_{jk}^{(N)})$ and arbitrary initial distribution. We call $\{X^{(N)}(n)\}$ the induced Markov chain. The state space is essentially $(d-1)$-dimensional, since the sum of components is N. Let

$$Y^{(N)}(t)=(N^{-1}X_1^{(N)}(n), \cdots, N^{-1}X_{d-1}^{(N)}(n)) \quad \text{for} \quad t=N^{-1}n$$

and make linear interpolation:

$$Y^{(N)}(t)=(n+1-Nt)Y^{(N)}(N^{-1}n)+(Nt-n)Y^{(N)}(N^{-1}(n+1))$$
$$\text{for} \quad N^{-1}n \leqq t \leqq N^{-1}(n+1).$$

Let $\boldsymbol{K}$ be the set of points $x=(x_1, \cdots, x_{d-1}) \in \boldsymbol{R}^{d-1}$ such that $x_1 \geqq 0, \cdots, x_{d-1} \geqq 0$, $1-\sum_{l=1}^{d-1} x_l \geqq 0$. Then $Y^{(N)}(t) \in \boldsymbol{K}$ for any t and sample functions of $Y^{(N)}(t)$ are broken lines. Let Ω be the space of continuous paths $\omega: [0, \infty) \rightarrow \boldsymbol{K}$ with the usual topology (that is, the topology of uniform convergence on compact sets of $[0, \infty)$). Let $\mathcal{M}$ be the topological σ-algebra of Ω and let $\mathcal{M}_t$ be the σ-algebra generated by $x(s, \omega)=\omega(s)$, $s \leqq t$. The process $(Y^{(N)}(t), \Omega^{(N)}, Q^{(N)}; 0 \leqq t < \infty)$ induces a probability measure $P^{(N)}$ on $(\Omega, \mathcal{M})$.

Let $a(x)=(a_{pq}(x))_{p,q=1,\cdots,d-1}$ and $b(x)=(b_p(x))_{p=1,\cdots,d-1}$ be $(d-1)\times(d-1)$-matrix and $(d-1)$-vector, respectively, defined on $\boldsymbol{K}$ by

$$a_{pp}(x)=\sigma^2 x_p(1-x_p), \tag{2.4}$$

$$a_{pq}(x)=-\sigma^2 x_p x_q \qquad (p \neq q), \tag{2.5}$$

$$b_p(x)=x_p\{\gamma_p(x)-\sum_{l=1}^{d-1} x_l \gamma_l(x)-(1-\sum_{l=1}^{d-1} x_l)\gamma_d(x)\}, \tag{2.6}$$

where

$$\gamma_r(x)=\gamma_r(x_1, \cdots, x_{d-1}, 1-\sum_{l=1}^{d-1} x_l) \qquad \text{for} \quad r=1, \cdots, d.$$

For each $\theta \in \boldsymbol{R}^{d-1}$ let

$$M_\theta(t, \omega)=\exp\Big\{\langle\theta, x(t, \omega)-x(0, \omega)\rangle-\int_0^t \langle\theta, b(x(u, \omega))\rangle du -\frac{1}{2}\int_0^t \langle\theta, a(x(u, \omega))\theta\rangle du\Big\}. \tag{2.7}$$

As in [6], we call the following problem the martingale problem $(\boldsymbol{K}, a, b, x)$: to find a probability measure P_x on $(\Omega, \mathcal{M})$ such that $P_x(x(0)=x)=1$ and $(M_\theta(t), \mathcal{M}_t, P_x; 0 \leqq t < \infty)$ is a martingale for each $\theta \in \boldsymbol{R}^{d-1}$.

We will prove the following results.

LEMMA 2.2. *For any set of σ^2 and γ_l, $l=1, \cdots, d$, the martingale problem $(\boldsymbol{K}, a, b, x)$ has a solution for each $x \in \boldsymbol{K}$.*

Let $Y^{(N)}(0)=x^{(N)}$, non-random.

THEOREM 2.1. *Let $d-1=1$ and suppose that $(\gamma_1(x_1, 1-x_1)-\gamma_2(x_1, 1-x_1))$ $x_1(1-x_1)$ is Lipschitz continuous. Then, for each x, the solution P_x of the martingale problem $(\boldsymbol{K}, a, b, x)$ is unique. If $x^{(N)} \rightarrow x$, then the sequence of proba-*

bility measures $\{P^{(N)}\}$ *weakly converges to* P_x *as* $N\to\infty$.

THEOREM 2.2. *Suppose that* $\gamma_2(x)=\gamma_3(x)=\cdots=\gamma_d(x)$ *and that* $(\gamma_1(x)-\gamma_2(x))$ $x_1(1-x_1)$ *is a function of* x_1 *alone and is Lipschitz continuous. Then, the solution* P_x *of the martingale problem* $(\boldsymbol{K}, a, b, x)$ *is unique for each* $x\in\boldsymbol{K}$. *If* $x^{(N)}\to x$, *then* $P^{(N)}$ *weakly converges to* P_x *as* $N\to\infty$.

THEOREM 2.3. *If the solution* P_x *of the martingale problem* $(\boldsymbol{K}, a, b, x)$ *is unique, then* $P^{(N)}$ *weakly converges to* P_x *as* $N\to\infty$, *provided that* $x^{(N)}\to x$.

However, we do not know whether the uniqueness holds for general a, b given by (2.4)—(2.6).

If the uniqueness holds, then $(x(t), \Omega, \mathcal{M}, P_x; x\in\boldsymbol{K})$ is a strong Markov process (see Stroock and Varadhan [7]). Let $d-1=1$ and consider the case of Theorem 2.1. Then, the backward Kolmogorov equation of the limiting diffusion is

$$\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}x_1(1-x_1)\frac{\partial^2 u}{\partial x_1^2}+(\gamma_1(x_1, 1-x_1)-\gamma_2(x_1, 1-x_1))x_1(1-x_1)\frac{\partial u}{\partial x_1}, \quad 0\leqq x_1\leqq 1. \tag{2.8}$$

The boundaries 0 and 1 are of pure exit type, acting as traps. The case where γ_1 and γ_2 are constant functions corresponds to gametic selection; $\gamma_1>\gamma_2$ (resp. $\gamma_1<\gamma_2$) means that type 1 has greater (resp. smaller) fertility than type 2. The case where $\gamma_1-\gamma_2$ is a linear function corresponds to zygotic selection. Namely, if genotypes A_1A_1, A_1A_2, A_2A_2 have selective advantages $1+N^{-1}\lambda_1$, $1+N^{-1}\lambda_2$, $1+N^{-1}\lambda_3$, respectively, then, let $\gamma_1(x_1, 1-x_1)=\lambda_1x_1+\lambda_2(1-x_1)$ and $\gamma_2(x_1, 1-x_1)=\lambda_2x_1+\lambda_3(1-x_1)$. It is natural to use these γ_1 and γ_2 in (2.2), because the exponents in (2.2) become

$$1+N^{-1}\gamma_1(|j|^{-1}j)=(1+N^{-1}\lambda_1)|j|^{-1}j_1+(1+N^{-1}\lambda_2)|j|^{-1}j_2,$$

$$1+N^{-1}\gamma_2(|j|^{-1}j)=(1+N^{-1}\lambda_2)|j|^{-1}j_1+(1+N^{-1}\lambda_3)|j|^{-1}j_2,$$

which can be considered as the fertilities of an A_1-individual and an A_2-individual, respectively, in a population consisting of j_1 A_1-individuals and j_2 A_2-individuals. The drift coefficient in (2.8) is now $\{(\lambda_1-\lambda_2)x_1-(\lambda_3-\lambda_2)(1-x_1)\}$ $x_1(1-x_1)$, which gives an interpretation to the cases (ii), (iii), (iv) in Section 1.

§ 3. A lemma on asymptotic estimate of some integral.

The following lemma is an extension of [5] Lemma 3.1 and [6] Lemma 4.3. Assumption 2.1 (ii) is essential to these lemmas. The moment generating function $M(w)$ extends to an analytic function $M(z)=\sum_{n=0}^{\infty}c_ne^{nz}$ of complex z with $\mathrm{Re}\, z<b$. $K(w)$ extends to a function analytic in a neighborhood of β.

Let

$$K(z)=\sum_{n=0}^{\infty}\kappa_n(z-\beta)^n .$$

$(n!)\kappa_n$ is the semi-invariant of order n of the associated distribution $\{\hat{c}_n\}$. We have $\kappa_0=K(\beta)$, $\kappa_1=1$, $\kappa_2=\sigma^2/2$.

LEMMA 3.1. *Let Ξ be a bounded set of real numbers such that, if N is sufficiently large, then, for each $\xi\in\Xi$, the function $f(w)^{N+\xi}$, $0\leqq w\leqq 1$, is the generating function of some distribution on $\boldsymbol{Z}^1_+$. For every large N, let*

$$\tilde{A}(N,\xi)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N+\xi-r}\tilde{M}(z)e^{-Nz}dz\,,\qquad \xi\in\Xi\,, \tag{3.1}$$

where the integral is along the line segment from $\beta-i\pi$ to $\beta+i\pi$, r is a fixed integer, and $\tilde{M}(z)$ is a bounded continuous function on the segment. Suppose that $\tilde{M}(z)$ is analytic in a neighborhood of β and let

$$M(z)^{-r}\tilde{M}(z)=\sum_{n=0}^{\infty}\tilde{\rho}_n(z-\beta)^n$$

there. Then, as $N\to\infty$,

$$\tilde{A}(N,\xi)=\Delta_N e^{\xi K(\beta)}(\tilde{\rho}_0+N^{-1}\tilde{a}(\xi)+O(N^{-2}))\qquad \textit{uniformly in } \xi\in\Xi\,, \tag{3.2}$$

where

$$\Delta_N=\sigma^{-1}(2\pi N)^{-1/2}e^{N(K(\beta)-\beta)}\,, \tag{3.3}$$

$$\tilde{a}(\xi)=\tilde{\rho}_0\Big(\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}-\frac{\xi}{2}-\frac{\xi^2}{2\sigma^2}+\frac{3\xi\kappa_3}{\sigma^4}\Big)+\tilde{\rho}_1\Big(\frac{3\kappa_3}{\sigma^4}-\frac{\xi}{\sigma^2}\Big)-\tilde{\rho}_2\frac{1}{\sigma^2}. \tag{3.4}$$

REMARK. It follows from the assumption on Ξ and Assumption 2.1 (ii) that, for any large N, $M(w)^{N+\xi}=f(e^w)^{N+\xi}$ can be extended to an analytic function of complex z with $\operatorname{Re} z<b$. We denote the extension by $M(z)^{N+\xi}$. We have used $M(z)^{N+\xi-r}$ in (3.1) in this sense.

PROOF. Since $M(z)=e^{K(z)}$ near $z=\beta$, we have, for small $\varepsilon>0$,

$$\begin{aligned}\tilde{A}(N,\xi)=\frac{e^{N(K(\beta)-\beta)}}{2\pi}\int_{-\varepsilon}^{\varepsilon}&\exp\big(N\sum_{n=2}^{\infty}\kappa_n(iy)^n+\xi K(\beta+iy)\big)\\&\cdot M(\beta+iy)^{-r}\tilde{M}(\beta+iy)dy+J\,,\end{aligned} \tag{3.5}$$

$$J=\frac{1}{2\pi i}\Big(\int_{\beta-i\pi}^{\beta-i\varepsilon}+\int_{\beta-i\varepsilon}^{\beta+i\pi}\Big)M(z)^{N+\xi-r}\tilde{M}(z)e^{-Nz}dz\,,$$

using $\kappa_1=1$. Let $\varphi(N)=N^{-1/2}\log N$ and write the integral in (3.5) as I_1+I_2, where I_1 is the integral over $|y|<\varphi(N)$ and I_2 is the integral over $\varphi(N)\leqq|y|<\varepsilon$. Denoting by B any function bounded uniformly in N and $\xi\in\Xi$, we have

$$I_1=\int_{|y|<\varphi(N)} e^{-N\sigma^2y^2/2}\exp\Big\{N\sum_{n=3}^{6}\kappa_n(iy)^n+\xi\sum_{n=0}^{4}\kappa_n(iy)^n\Big\}\cdot\Big\{\sum_{n=0}^{4}\tilde{\rho}_n(iy)^n+B\frac{(\log N)^7}{N^{5/2}}\Big\}dy$$

$$=\frac{e^{\xi K(\beta)}}{\sigma\sqrt{N}}\int_{|u|<\sigma\log N} e^{-u^2/2}g_1(u)g_2(u)g_3(u)du\,,$$

where

$$g_1(u)=\exp\Big\{\sum_{n=3}^{6}\frac{\kappa_n}{N^{(n-2)/2}}\Big(\frac{iu}{\sigma}\Big)^n\Big\},\qquad g_2(u)=\exp\Big\{\xi\sum_{n=1}^{4}\frac{\kappa_n}{N^{n/2}}\Big(\frac{iu}{\sigma}\Big)^n\Big\},$$

$$g_3(u)=\sum_{n=0}^{4}\frac{\tilde{\rho}_n}{N^{n/2}}\Big(\frac{iu}{\sigma}\Big)^n+B\frac{(\log N)^7}{N^{5/2}}.$$

Noting that $|u|<\sigma\log N$, we get

$$g_1(u)=\sum_{n=0}^{4}\frac{1}{N^{n/2}}P_n\Big(\frac{iu}{\sigma}\Big)+B\frac{(\log N)^{15}}{N^{5/2}}\,,$$

$$P_0(v)=1\,,\qquad P_1(v)=\kappa_3v^3\,,\qquad P_2(v)=\kappa_4v^4+2^{-1}\kappa_3^2v^6\,,\ \cdots.$$

If n is odd (resp. even), P_n is a polynomial having odd (resp. even) order terms only. Similarly,

$$g_2(u)=\sum_{n=0}^{4}\frac{\theta_n}{N^{n/2}}\Big(\frac{iu}{\sigma}\Big)^n+B\frac{(\log N)^5}{N^{5/2}},$$

$$\theta_0=1\,,\qquad \theta_1=\xi\kappa_1\,,\qquad \theta_2=\xi\kappa_2+(2\,!)^{-1}\xi^2\kappa_1^2\,,$$

$$\theta_3=\xi\kappa_3+(2\,!)^{-1}2\xi^2\kappa_1\kappa_2+(3\,!)^{-1}\xi^3\kappa_1^3\,,$$

$$\theta_4=\xi\kappa_4+(2\,!)^{-1}\xi^2(2\kappa_1\kappa_3+\kappa_2^2)+(3\,!)^{-1}3\xi^3\kappa_1^2\kappa_2+(4\,!)^{-1}\xi^4\kappa_1^4\,.$$

Hence,

$$g_1(u)g_2(u)g_3(u)=\sum_{n=0}^{4}\frac{1}{N^{n/2}}\sum_{m=0}^{n}P_{n-m}\Big(\frac{iu}{\sigma}\Big)\Big(\frac{iu}{\sigma}\Big)^m\sum_{r=0}^{m}\theta_{m-r}\tilde{\rho}_r+B\frac{(\log N)^{\nu}}{N^{5/2}}$$

with some integer ν. By integration the terms with odd n vanish, since they are odd functions of u. Since we have, for any fixed k,

$$\int_{-\infty}^{\infty}e^{-u^2/2}u^{2k}du=(2k-1)\,!!\sqrt{2\pi}\,,\qquad (2k-1)\,!!=(2k-1)(2k-3)\cdots 3\cdot 1\,,$$

$$\int_{|u|>\sigma\log N}e^{-u^2/2}u^{2k}du=o\Big(\frac{1}{N^n}\Big)\qquad\text{for any } n\,,$$

we get

$$I_1=(2\pi)^{1/2}\sigma^{-1}N^{-1/2}e^{\xi K(\beta)}(\tilde{\rho}_0+N^{-1}\tilde{a}(\xi)+O(N^{-2}))$$

with

$$\tilde{a}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-u^2/2}\Big\{\tilde{\rho}_0P_2\Big(\frac{iu}{\sigma}\Big)+(\theta_1\tilde{\rho}_0+\tilde{\rho}_1)P_1\Big(\frac{iu}{\sigma}\Big)\frac{iu}{\sigma}$$

$$+(\theta_2\tilde{\rho}_0+\theta_1\tilde{\rho}_1+\tilde{\rho}_2)\Big(\frac{iu}{\sigma}\Big)^2\Big\}du\,.$$

Here $O(N^{-2})$ is uniform in $\xi\in\Xi$, since Ξ is bounded. The above $\tilde{a}(\xi)$ is identical with (3.4).

Estimation of I_2 is simple because

$$|I_2|\leqq C_1\int_{\varphi(N)<|y|<\varepsilon}\exp\{\mathrm{Re}\,(N\sum_{n=2}^{\infty}\kappa_n(iy)^n)\}\,dy\leqq C_2\int_{|y|>\varphi(N)}e^{-N\sigma^2y^2/4}dy$$
$$=o(N^{-n})\qquad\text{for any}\quad n\,,$$

if ε is chosen small enough. Here C_1 and C_2 are constants independent of $\xi\in\Xi$. Let us examine J. For large N we have

$$|M(\beta+iy)^{N+\xi}|\leqq M(\beta)^{N+\xi}\,,\qquad \xi\in\Xi\,.$$

The assumption that the distribution $\{c_n\}$ has maximum span 1 implies that, for each $\varepsilon>0$, there is an $\eta>0$ such that

$$|M(\beta+iy)|\leqq M(\beta)(1-\eta)\qquad\text{for}\quad \varepsilon\leqq|y|\leqq\pi\,.$$

Fix N_0 sufficiently large. Then, for $N\geqq N_0$,

$$|M(z)^{N+\xi-r}\tilde{M}(z)e^{-Nz}|=|M(z)^{N-N_0}M(z)^{N_0+\xi-r}\tilde{M}(z)e^{-Nz}|$$
$$\leqq C_3(M(\beta)(1-\eta))^{N-N_0}M(\beta)^{N_0+\xi-r}e^{-N\beta}$$
$$\leqq C_4e^{N(K(\beta)-\beta)}(1-\eta)^{N-N_0}$$

for $z=\beta+iy$ such that $\varepsilon\leqq|y|\leqq\pi$. Here C_3 and C_4 are constants independent of ξ. Hence

$$J=e^{N(K(\beta)-\beta)}o(N^{-n})\qquad\text{for any}\quad n\,,$$

and the proof is complete.

§4. Proof of results.

Given N and $j\in\boldsymbol{J}(N)$, let $x=(x_1,\cdots,x_d)=|j|^{-1}j=N^{-1}j=(N^{-1}j_1,\cdots,N^{-1}j_d)$, $\gamma_p=\gamma_p(x)=\gamma_p(N^{-1}j)$, $y=\sum_{p=1}^{d}x_p\gamma_p=N^{-1}\sum_{p=1}^{d}j_p\gamma_p(N^{-1}j)$. N is supposed to be large so that (2.2) holds. Thus

$$f_{N,j}(s_1,\cdots,s_d)=\prod_{p=1}^{d}f(s_p)^{j_p+x_p\gamma_p}\,.$$

We use the following functions:

$$h_1(w)=f(w)^{N+y}\,,$$
$$h_2(w)=f(w)^{N+y-1}f'(w)w\,,$$
$$h_3(w)=f(w)^{N+y-2}f'(w)^2w^2\,,$$

$$h_4(w)=f(w)^{N+y-1}f''(w)w^2,$$

$$h_5(w)=f(w)^{N+y-3}f'(w)^3w^3,$$

$$h_6(w)=f(w)^{N+y-2}f'(w)f''(w)w^3,$$

$$h_7(w)=f(w)^{N+y-1}f'''(w)w^3,$$

$$h_8(w)=f(w)^{N+y-4}f'(w)^4w^4,$$

$$h_9(w)=f(w)^{N+y-3}f'(w)^2f''(w)w^4,$$

$$h_{10}(w)=f(w)^{N+y-2}f''(w)^2w^4,$$

$$h_{11}(w)=f(w)^{N+y-2}f'(w)f'''(w)w^4,$$

$$h_{12}(w)=f(w)^{N+y-1}f''''(w)w^4.$$

These are power series of w. Let

$$A_\nu(N)=\text{coefficient of } w^N \text{ in } h_\nu(w).$$

Since

(4.1) $$P(Z^{(N)}(n+1)\in \boldsymbol{J}(N) \mid Z^{(N)}(n)=j)=A_1(N),$$

we have

$$P_{jk}^{(N)}=A_1(N)^{-1}(\text{coefficient of } s_1^{k_1}\cdots s_d^{k_d} \text{ in } f_{N,j}(s_1,\cdots,s_d))$$

$$=A_1(N)^{-1}(\text{coefficient of } w^N s_1^{k_1}\cdots s_d^{k_d} \text{ in } \Phi(w,s_1,\cdots,s_d)),$$

where

$$\Phi(w,s_1,\cdots,s_d)=\prod_{p=1}^{d} f(ws_p)^{j_p+x_p\gamma_p}.$$

Let

$$G(s_1,\cdots,s_d)=\sum_{k\in\boldsymbol{J}(N)} P_{jk}^{(N)}s_1^{k_1}\cdots s_d^{k_d},$$

$$C_{p_1\cdots p_m}=D_{p_1\cdots p_m}G(1,\cdots,1),$$

where

$$D_{p_1\cdots p_m}=\frac{\partial^m}{\partial s_{p_1}\cdots\partial s_{p_m}}.$$

We have

$$G(s_1,\cdots,s_d)=A_1(N)^{-1}(\text{coefficient of } w^N \text{ in } \Phi(w,s_1,\cdots,s_d))$$

and

(4.2) $$C_{p_1\cdots p_m}=A_1(N)^{-1}(\text{coefficient of } w^N \text{ in } D_{p_1\cdots p_m}\Phi(w,1,\cdots,1)).$$

We will use

$$C^*_{p_1\cdots p_m}=\sum_{k\in\boldsymbol{J}(N)} k_{p_1}\cdots k_{p_m}P_{jk}^{(N)}$$

also. What we would like to estimate are

$$b_p^{(N)}(x)=b_p^{(N)}\left(\frac{j}{N}\right)=N\sum_{k\in \boldsymbol{J}(N)}\left(\frac{k_p}{N}-\frac{j_p}{N}\right)P_{jk}^{(N)},$$

$$a_{pq}^{(N)}(x)=a_{pq}^{(N)}\left(\frac{j}{N}\right)=N\sum_{k\in \boldsymbol{J}(N)}\left(\frac{k_p}{N}-\frac{j_p}{N}\right)\left(\frac{k_q}{N}-\frac{j_q}{N}\right)P_{jk}^{(N)},$$

$$e_p^{(N)}(x)=e_p^{(N)}\left(\frac{j}{N}\right)=N\sum_{k\in \boldsymbol{J}(N)}\left(\frac{k_p}{N}-\frac{j_p}{N}\right)^4 P_{jk}^{(N)}.$$

By Assumption 2.1, $h_\nu(e^w)$ has an analytic extension $h_\nu(e^z)$ for complex z with $\operatorname{Re} z<b$ and it is easy to prove the following lemma.

LEMMA 4.1. *For each* ν

$$A_\nu(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}h_\nu(e^z)e^{-Nz}dz \tag{4.3}$$

and this expression satisfies the condition for (3.1) *in Lemma* 3.1.

PROOF OF LEMMA 2.1. By Lemma 3.1 and 4.1,

$$A_1(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N+y}e^{-Nz}dz=\Delta_N e^{yK(\beta)}\left(1+\frac{a_1}{N}+O\left(\frac{1}{N^2}\right)\right), \tag{4.4}$$

$$a_1=\frac{3\kappa_4}{\sigma^4}-\frac{15\kappa_3^2}{2\sigma^6}-\frac{y}{2}-\frac{y^2}{2\sigma^2}+\frac{3y\kappa_3}{\sigma^4} \tag{4.5}$$

uniformly in y. Hence Lemma 2.1 follows from (4.1).

LEMMA 4.2.

$$C_p=j_p+x_p(\gamma_p-y)+O(N^{-1}) \qquad \textit{uniformly in } j\in \boldsymbol{J}(N), \tag{4.6}$$

$$b_p^{(N)}(x)=x_p(\gamma_p-y)+O(N^{-1}) \qquad \textit{uniformly in } j\in \boldsymbol{J}(N). \tag{4.7}$$

What we mean by (4.6) is

$$\limsup_{N\to\infty}\sup_{j\in \boldsymbol{J}(N)} N|C_p-(j_p+x_p(\gamma_p-y))|<\infty.$$

We use the phrase *uniformly in* $j\in \boldsymbol{J}(N)$ in this way.

PROOF. We have

$$D_p\Phi(w, s_1, \cdots, s_d)$$
$$=f(ws_1)^{j_1+x_1\gamma_1}\cdots\{(j_p+x_p\gamma_p)f(ws_p)^{j_p+x_p\gamma_p-1}f'(ws_p)w\}\cdots f(ws_d)^{j_d+x_d\gamma_d},$$

$$D_p\Phi(w, 1, \cdots, 1)=(j_p+x_p\gamma_p)h_2(w).$$

Hence by (4.2)

$$C_p=(j_p+x_p\gamma_p)A_1(N)^{-1}A_2(N). \tag{4.8}$$

By Lemma 4.1,

$$A_2(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N+y-1}f'(e^z)e^z e^{-Nz}dz\,.$$

Since $M(z)^{-1}f'(e^z)e^z=M(z)^{-1}M'(z)=K'(z)=1+\sigma^2(z-\beta)+3\kappa_3(z-\beta)^2+\cdots$ near $z=\beta$, we get

$$A_2(N)=\Delta_N e^{yK(\beta)}(1+N^{-1}a_2+O(N^{-2}))\,,\qquad a_2=a_1-y \tag{4.9}$$

by Lemma 3.1 and (4.5). It follows from (4.4), (4.8), (4.9) that

$$C_p=(j_p+x_p\gamma_p)(1-N^{-1}y+O(N^{-2}))=j_p+x_p(\gamma_p-y)+O(N^{-1})$$

uniformly in $j\in\boldsymbol{J}(N)$. Since $b_p^{(N)}(x)=C_p-j_p$, we get (4.7).

LEMMA 4.3.

$$C_{pq}=j_pj_q+j_px_q\gamma_q+j_qx_p\gamma_p-N^{-1}j_pj_q(\sigma^2+2y)+O(1)\qquad \textit{for}\quad p\neq q\,, \tag{4.10}$$

$$C_{pp}=j_p^2+j_p(\sigma^2-1-x_p(\sigma^2+2y-2\gamma_p))+O(1)\,, \tag{4.11}$$

$$a_{pq}^{(N)}(x)=-\sigma^2x_px_q+O(N^{-1})\qquad \textit{for}\quad p\neq q\,, \tag{4.12}$$

$$a_{pp}^{(N)}(x)=\sigma^2x_p(1-x_p)+O(N^{-1})\,. \tag{4.13}$$

All O signs here are uniform in $j\in\boldsymbol{J}(N)$.

PROOF. Let $p\neq q$. We have

$$D_{pq}\Phi(w,s_1,\cdots,s_d)=f(ws_1)^{j_1+x_1\gamma_1}\cdots\{(j_p+x_p\gamma_p)f(ws_p)^{j_p+x_p\gamma_p-1}f'(ws_p)w\}$$
$$\cdots\{(j_q+x_q\gamma_q)f(ws_q)^{j_q+x_q\gamma_q-1}f'(ws_q)w\}\cdots f(ws_d)^{j_d+x_d\gamma_d}$$

for $p<q$ and a similar expression for $p>q$. Hence

$$D_{pq}\Phi(w,1,\cdots,1)=(j_p+x_p\gamma_p)(j_q+x_q\gamma_q)h_3(w)\,,$$

$$C_{pq}=(j_p+x_p\gamma_p)(j_q+x_q\gamma_q)A_1(N)^{-1}A_3(N)\,. \tag{4.14}$$

By Lemmas 3.1, 4.1 and (4.5),

$$A_3(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta-i\pi}M(z)^{N+y-2}f'(e^z)^2e^{2z}e^{-Nz}dz \tag{4.15}$$
$$=\Delta_N e^{yK(\beta)}(1+N^{-1}a_3+O(N^{-2}))\,,\qquad a_3=a_1-2y-\sigma^2\,,$$

since $M(z)^{-2}f'(e^z)^2e^{2z}=K'(z)^2=1+2\sigma^2(z-\beta)+(6\kappa_3+\sigma^4)(z-\beta)^2+\cdots$. It follows from (4.4), (4.14), (4.15) that

$$C_{pq}=(j_pj_q+j_px_q\gamma_q+j_qx_p\gamma_p)(1-N^{-1}(2y+\sigma^2))+O(1)\,,$$

which is (4.10). In order to get (4.11), we have

$$D_{pp}\Phi(w, s_1, \cdots, s_d)=f(ws_1)^{j_1+x_1\gamma_1}\cdots\{(j_p+x_p\gamma_p)_2 f(ws_p)^{j_p+x_p\gamma_p-2}f'(ws_p)^2w^2$$
$$+(j_p+x_p\gamma_p)f(ws_p)^{j_p+x_p\gamma_p-1}f''(ws_p)w^2\}\cdots f(ws_d)^{j_d+x_d\gamma_d},$$
$$D_{pp}\Phi(w, 1, \cdots, 1)=(j_p+x_p\gamma_p)_2h_3(w)+(j_p+x_p\gamma_p)h_4(w),$$

$$C_{pp}=(j_p+x_p\gamma_p)_2A_1(N)^{-1}A_3(N)+(j_p+x_p\gamma_p)A_1(N)^{-1}A_4(N). \tag{4.16}$$

Here we are using the notation $(x)_n=x(x-1)\cdots(x-n+1)$ for real x. We have

$$A_4(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N+y-1}f''(e^z)e^{2z}e^{-Nz}dz=\Delta_N e^{yK(\beta)}(\sigma^2+O(N^{-1})) \tag{4.17}$$

since $M(z)^{-1}f''(e^z)e^{2z}=M^{-1}(M''-M')=K'^2+K''-K'=\sigma^2$ at $z=\beta$. It follows from (4.15), (4.16), (4.17) that

$$C_{pp}=(j_p^2-j_p+2j_px_p\gamma_p)(1-N^{-1}(2y+\sigma^2))+j_p\sigma^2+O(1),$$

which is (4.11). Since $a_{pq}^{(N)}(x)=N^{-1}(C_{pq}^*-j_pC_q^*-j_qC_p^*+j_pj_q)$ for $p=q$ or $p\neq q$, we have

$$a_{pq}^{(N)}(x)=N^{-1}(C_{pq}-j_pC_q-j_qC_p+j_pj_q) \qquad \text{for} \quad p\neq q,$$
$$a_{pp}^{(N)}(x)=N^{-1}(C_{pp}+C_p-2j_pC_p+j_p^2).$$

Hence (4.12) and (4.13) follow from (4.6), (4.10), (4.11).

LEMMA 4.4.

$$C_{ppp}=j_p^3+3j_p^2(\sigma^2-1-x_p(\sigma^2+y-\gamma_p))+O(N) \quad \textit{uniformly in } j\in \boldsymbol{J}(N). \tag{4.18}$$

PROOF. Since

$$D_{ppp}\Phi(w, 1, \cdots, 1)=(j_p+x_p\gamma_p)_3h_5(w)+3(j_p+x_p\gamma_p)_2h_6(w)+(j_p+x_p\gamma_p)h_7(w),$$

we have

$$C_{ppp}=(j_p+x_p\gamma_p)_3\frac{A_5(N)}{A_1(N)}+3(j_p+x_p\gamma_p)_2\frac{A_6(N)}{A_1(N)}+(j_p+x_p\gamma_p)\frac{A_7(N)}{A_1(N)} \tag{4.19}$$
$$=(j_p^3-3j_p^2+3j_p^2x_p\gamma_p)\frac{A_5(N)}{A_1(N)}+3j_p^2\frac{A_6(N)}{A_1(N)}+O(N).$$

By Lemmas 3.1, 4.1 and (4.5),

$$A_5(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N+y-3}f'(e^z)e^{3z}e^{-Nz}dz \tag{4.20}$$
$$=\Delta_N e^{yK(\beta)}(1+N^{-1}a_5+O(N^{-2})), \qquad a_5=a_1-3y-3\sigma^2,$$

$$A_6(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N+y-2}f'(e^z)f''(e^z)e^{3z}e^{-Nz}dz \tag{4.21}$$
$$=\Delta_N e^{yK(\beta)}(\sigma^2+O(N^{-1})),$$

because

$$M(z)^{-3}f'(e^z)^3e^{3z}=K'(z)^3=1+3\sigma^2(z-\beta)+(9\kappa_3+3\sigma^4)(z-\beta)^2+\cdots$$

and

$$M(z)^{-2}f'(e^z)f''(e^z)e^{3z}=M^{-2}M'(M''-M')=K'(K'^2+K''-K')=\sigma^2+\cdots.$$

(4.4), (4.19), (4.20) and (4.21) imply

$$C_{ppp}=(j_p^3-3j_p^2+3j_p^2x_p\gamma_p)(1-N^{-1}(3y+3\sigma^2))+3j_p^2\sigma^2+O(N)$$

and hence (4.18).

LEMMA 4.5.

$$C_{pppp}=j_p^4+j_p^3(6\sigma^2-6-x_p(6\sigma^2+4y-4\gamma_p))+O(N^2) \tag{4.22}$$

uniformly in $j\in \boldsymbol{J}(N)$.

PROOF. This time we have

$$D_{pppp}\Phi(w,1,\cdots,1)=(j_p+x_p\gamma_p)_4h_8(w)+6(j_p+x_p\gamma_p)_3h_9(w)$$
$$+3(j_p+x_p\gamma_p)_2h_{10}(w)+4(j_p+x_p\gamma_p)_2h_{11}(w)+(j_p+x_p\gamma_p)h_{12}(w),$$

and hence, by (4.2),

$$C_{pppp}=(j_p+x_p\gamma_p)_4\frac{A_8(N)}{A_1(N)}+6(j_p+x_p\gamma_p)_3\frac{A_9(N)}{A_1(N)}+3(j_p+x_p\gamma_p)_2\frac{A_{10}(N)}{A_1(N)} \tag{4.23}$$
$$+4(j_p+x_p\gamma_p)_2\frac{A_{11}(N)}{A_1(N)}+(j_p+x_p\gamma_p)\frac{A_{12}(N)}{A_1(N)}$$
$$=(j_p^4-6j_p^3+4j_p^3x_p\gamma_p)\frac{A_8(N)}{A_1(N)}+6j_p^3\frac{A_9(N)}{A_1(N)}+O(N^2).$$

As before, we have

$$A_8(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta+i\pi}M(z)^{N+y-4}f'(e^z)^4e^{4z}e^{-Nz}dz$$
$$=\Delta_Ne^{yK(\beta)}(1+N^{-1}a_8+O(N^{-2})),\qquad a_8=a_1-4y-6\sigma^2,$$
$$A_9(N)=\frac{1}{2\pi i}\int_{\beta-i\pi}^{\beta-i\pi}M(z)^{N+y-3}f'(e^z)^2f''(e^z)e^{4z}e^{-Nz}dz$$
$$=\Delta_Ne^{yK(\beta)}(\sigma^2+O(N^{-1})),$$

because

$$M(z)^{-4}f'(e^z)^4e^{4z}=K'(z)^4=1+4\sigma^2(z-\beta)+(12\kappa_3+6\sigma^4)(z-\beta)^2+\cdots$$

and

$$M(z)^{-3}f'(e^z)^2f''(e^z)e^{4z}=M^{-3}M'^2(M''-M')=K'^2(K'^2+K''-K')=\sigma^2+\cdots.$$

Hence $A_1(N)^{-1}A_8(N)=1-N^{-1}(4y+6\sigma^2)+O(N^{-2})$ and $A_1(N)^{-1}A_9(N)=\sigma^2+O(N^{-1})$.

Thus (4.22) follows from (4.23).

LEMMA 4.6.

$$e_p^{(N)}(x)=O(N^{-1}) \qquad \textit{uniformly in } j\in \boldsymbol{J}(N). \tag{4.24}$$

PROOF. Using $e_p^{(N)}(x)=N^{-3}(C_{pppp}^*-4j_pC_{ppp}^*+6j_p^2C_{pp}^*-4j_p^3C_p^*+j_p^4)$ and expressing $C_p^*, \cdots, C_{pppp}^*$ by $C_p, \cdots, C_{pppp}$, we get

$$e_p^{(N)}(x)=N^{-3}\{C_{pppp}+(-4j_p+6)C_{ppp}+(6j_p^2-12j_p+7)C_{pp}$$
$$+(-4j_p^3+6j_p^2-4j_p+1)C_p+j_p^4\}\,.$$

Use (4.6), (4.11), (4.18) and (4.22) of the preceding four lemmas in this expression. Then all terms cancel except terms of $O(N^{-1})$.

PROOF OF LEMMA 2.2. By the definitions (2.4)—(2.6) of $a(x)$ and $b(x)$, Lemmas 4.2 and 4.3 imply that, for any p and q

$$b_p^{(N)}\Big(\frac{j}{N}\Big)=b_p\Big(\frac{j}{N}\Big)+O\Big(\frac{1}{N}\Big), \qquad a_{pq}^{(N)}\Big(\frac{j}{N}\Big)=a_{pq}\Big(\frac{j}{N}\Big)+O\Big(\frac{1}{N}\Big) \tag{4.25}$$

uniformly in $j\in\boldsymbol{J}(N)$. Here we have identified $(j_1, \cdots, j_d)$ in $\boldsymbol{J}(N)$ with $(j_1, \cdots, j_{d-1})$ and denoted them by the same letter j. By an invariance principle in [**6**], Theorem 3.1, these estimates combined with Lemma 4.6 prove that the sequence of the probability measures $P^{(N)}$ on $(\Omega, \mathcal{M})$ is relatively compact and that the limit of any convergent subsequence is a solution of the martingale problem $(\boldsymbol{K}, a, b, x)$, provided that $Y^{(N)}(0)=x^{(N)}\to x$. In order to show the existence of a solution of the martingale problem $(\boldsymbol{K}, a, b, x)$ for any set of $\sigma^2>0$ and continuous γ_p, $p=1, \cdots, d$, we can assume $\sigma^2=1$ without loss of generality. Note that if $\alpha=\text{const}>0$, then a solution of the problem $(\boldsymbol{K}, \alpha a, \alpha b, x)$ is easily obtained from a solution of the problem $(\boldsymbol{K}, a, b, x)$ by change of time scale. Suppose that γ_p, $p=1, \cdots, d$, are arbitrary continuous functions and let C be the bound of $|\gamma_p|$, $p=1, \cdots, d$. Let $\{c_n\}$ be an arbitrary Poisson distribution. Then $\sigma^2=1$ and the right-hand side of (2.2) is the generating function of a distribution on $\boldsymbol{Z}_+^d$, provided that $N\geqq C$. Hence a solution of the problem $(\boldsymbol{K}, a, b, x)$ exists.

PROOF OF THEOREM 2.3. This is a consequence of (4.25) and Lemma 4.6 in view of [**6**], Theorem 3.2.

PROOF OF THEOREMS 2.1 AND 2.2. By Theorem 2.3, it is enough to see the uniqueness of the solution of the martingale problem $(\boldsymbol{K}, a, b, x)$ for each $x\in\boldsymbol{K}$. This is proved in the same way as in Section 5 of [**6**]. That is, observe first that the uniqueness in question is equivalent to the uniqueness, in the sense of law, of the solution of some stochastic differential equation, and then, in case of $d-1=1$, use the result of Yamada and S. Watanabe [**8**]. In case of $d-1\geqq 2$ in Theorem 2.2, the uniqueness is obtained by some reduction to essentially one-dimensional cases, which is due to S. Watanabe. Note that, under the assumption in Theorem 2.2, $b_1(x)=(\gamma_1(x)-\gamma_2(x))x_1(1-x_1)$, a function of x_1 alone, and, for $p\geqq 2$, $b_p(x)=-(\gamma_1(x)-\gamma_2(x))x_1x_p$ a function of x_1 and x_p.

Notes added on April 12, 1976.

1. Ethier [**A2**] proved uniqueness of the solution of the martingale problem $(\boldsymbol{K}, a, b, x)$ when a is defined by (2.4), (2.5) and b is of class C^5 on $\boldsymbol{K}$, satisfying some conditions at the boundary of $\boldsymbol{K}$. Thus the uniqueness was proved if b is of the form (2.6) and of class C^5. Combining this result with Theorem 2.3, we can now say that $P^{(N)}$ weakly converges to P_x as $N\to\infty$ provided that $x_N\to x$, if b is of class C^5. His paper [**A2**] contains also another proof of the existence of the solution of the martingale problem, which uses the existence theorem in the whole Euclidean space.

2. We indicate that the induced Markov chains in this paper include multi-allele Wright models as special cases. Suppose that the distribution $\{c_n\}$ is Poisson with abitrary mean. It satisfies (ii) of Assumption 2.1, since $b=\infty$. It is easy to see that $\sigma=1$. Let

$$\gamma_p(x_1, \cdots, x_d)=\sum_{q=1}^{d}\lambda_{pq}x_q\,, \qquad p=1, \cdots, d\,, \tag{1}$$

where λ_{pq} are real constants. Let N be so large that $1+N^{-1}\gamma_p(x_1, \cdots, x_d)\geqq 0$. Then the function $f_{N,j}(s_1, \cdots, s_d)$ defined by (2.2) is a generating function. We get, for $|j|=|k|=N$,

$$P_{jk}^{(N)}=N\,!\prod_{p=1}^{d}(k_p\,!)^{-1}u_p^{k_p}\,, \tag{2}$$

where

$$u_p=\Big(\sum_{l,q=1}^{d}\Big(1+\frac{\lambda_{lq}}{N}\Big)\frac{j_l}{N}\frac{j_q}{N}\Big)^{-1}\sum_{q=1}^{d}\Big(1+\frac{\lambda_{pq}}{N}\Big)\frac{j_p}{N}\frac{j_q}{N}\,.$$

If $\lambda_{pq}=\lambda_{qp}$, this is the multi-allele Wright model with zygotic selection. If we assume (1) (even when the distribution $\{c_n\}$ is general), then (2.2) has

$$1+N^{-1}\gamma_p(|j|^{-1}j)=\sum_{q=1}^{d}(1+N^{-1}\lambda_{pq})|j|^{-1}j_q$$

as the exponents, and it is natural to consider $1+N^{-1}\lambda_{pq}$ as the relative advantage of A_p when it is found in the genotype A_pA_q. The drift coefficients of the limiting diffusion are then

$$b_p(x)=x_p\Big(\sum_{q=1}^{d}\lambda_{pq}x_q-\sum_{l,q=1}^{d}\lambda_{lq}x_lx_q\Big) \qquad \text{where } x_d=1-\sum_{l=1}^{d-1}x_l\,.$$

Likewise, if $\gamma_p(x_1, \cdots, x_d)$ is a constant λ_p for each p, then it is natural to consider $1+N^{-1}\lambda_p$ as the relative fertility of gene of allele A_p. The drift coefficients are

$$b_p(x)=x_p(\lambda_p-\sum_{q=1}^{d}\lambda_q x_q) \qquad \text{where } x_d=1-\sum_{l=1}^{d-1}x_l\,.$$

In this case, as is observed by Karlin and McGregor [3], we get (2) with

$$u_p=\Big(\sum_{q=1}^{d}\Big(1+\frac{\lambda_q}{N}\Big)\frac{j_q}{N}\Big)^{-1}\Big(1+\frac{\lambda_p}{N}\Big)\frac{j_p}{N},$$

if $\{c_n\}$ is Poisson. This is the multi-allele Wright model with gametic selection.

3. As for the convergence of Wright models for $d=2$, there are several papers (Norman [**A4**] for example) which deal with the convergence of finite-dimensional distributions, and Guess [**A3**] should also be mentioned for the weak convergence in the space D. For general d, convergence of finite-dimensional distributions of Wright models and estimate of the speed of the convergence are treated by Ethier [**A1**]. He proves also the weak convergence in the space D.

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