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On irreducible unitary characters of a certain group extension of $GL(2, \boldsymbol{C})$

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Introduction.

0-1. Let $G=GL(2, \boldsymbol{C})$ be the complex general linear group of order 2. Denote by $\langle\sigma\rangle$ a group of automorphisms of G generated by the complex conjugation σ. Let $G^{\sim}$ be the semi-direct product of G with $\langle\sigma\rangle$. More precisely, $G^{\sim}$ is the group whose underlying set is $G\times\langle\sigma\rangle$ and whose composition law is given by $(g, \tau)(g', \tau')=(g^{\tau'}g', \tau\tau')$. Then $G^{\sim}$ is a disconnected Lie group which has G as a connected component of the identity element. Let T be an irreducible unitary representation of $G^{\sim}$. Then the restriction of T to G is either an irreducible representation of G or the direct sum of two mutually inequivalent irreducible representations of G. Accordingly, T is said to be *of the first* or *the second kind*. In the following, we assume T to be of the first kind. For each smooth and compactly supported function f on G, it is known that the operator $\int_{G}f(g)T(g, \sigma)dg$ is a trace operator acting on the representation space of T (dg is an invariant measure on G). Moreover it is shown that there exists a locally integrable function trace $T(g, \sigma)$ on G such that

$$\operatorname{trace}\int_{G}f(g)T(g, \sigma)dg=\int_{G}f(g)\operatorname{trace}T(g, \sigma)dg\,.$$

On the other hand, set $G_{R}=GL(2, \boldsymbol{R})$. It is known that, for any irreducible unitary representation r of G_{R}, there exists a locally summable class function trace $r(x)$ on G_{R} such that

$$\operatorname{trace}\int_{G_R}\varphi(x)r(x)dx=\int_{G_R}\varphi(x)\operatorname{trace}r(x)dx$$

for any smooth and compactly supported function φ on G_{R} (dx is an invariant measure on G_{R}). We extend a class function trace r on G_{R} to a class function on G_{C} by setting

$$\operatorname{trace}r(g)=\begin{cases}\operatorname{trace}r(x) & \text{if } g \text{ is conjugate to } x\in G_{R} \text{ in } G_{C}\,,\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 1[1]. *Notations being as above, for each irreducible unitary representation T of the first kind of G, there exists an irreducible unitary representation r of $G_{\boldsymbol{R}}$ such that*

$$\operatorname{trace} T(g, \sigma) = \varepsilon \operatorname{trace} r(g^{\sigma} g) \qquad (\forall g \in G),$$

where $\varepsilon = \pm 1$ does not depend upon g.

REMARK 1. An analogous result for finite general linear groups was given in [**5**].

0-2. This paper consists of two sections. In § 1, we study orbits and orbital integrals on $G_1 = SL(2, \boldsymbol{C})/\pm 1$, with respect to the "$\sigma$-twisted adjoint action": $x \mapsto g^{\sigma} x g^{-1}$ of G_1. We also determine a general form of distributions on G_1, invariant under the above G_1-action and whose supports are concentrated to the following "σ-twisted singular set": $\{g \in G;\ \operatorname{trace} g^{\sigma} g = \pm 2\}$. Results obtained are quite similar to those on orbital integrals on $SL(2, \boldsymbol{R})$ with respect to the adjoint action.

In § 2, we compute irreducible unitary characters of the first kind of the semi-direct product $G_1^{\sim}$ of G_1 with $\langle \sigma \rangle$. Then, Theorem 1 is proved.

Notations.

As usual, $\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Z}$ denote the field of complex numbers, the field of real numbers and the ring of integers respectively.

The group of non-zero complex (resp. real) numbers is denoted by $\boldsymbol{C}^{\times}$ (resp. $\boldsymbol{R}^{\times}$).

For a complex matrix g, g^{σ} means the complex conjugation of g. We use such standard notations in the theory of linear groups as $SL(2, \boldsymbol{C})$, $SL(2, \boldsymbol{R})$, $SU(2)$, $PU(2) = SU(2)/\pm 1$ and $SO(2)$.

For a smooth manifold X, $C_0^{\infty}(X)$ is the space of smooth functions on X with compact support.

Let G be a Lie group and H be a closed subgroup of G. If normalizations of invariant measures of G and H are prescribed, the invariant measure on G/H (if any) is normalized to be the quotient of the invariant measure of G by that of H.

§ 1.

1 Set $G_1 = PSL(2, \boldsymbol{C}) = SL(2, \boldsymbol{C})/\pm 1$. Denote by $G_1^{\sim}$ the semidirect product of G_1 and the Galois group $\langle \sigma \rangle$ of $\boldsymbol{C}$ with respect to $\boldsymbol{R}$ generated by the complex conjugation σ. More precisely, $G_1^{\sim}$ is the group with the underlying set $G_1 \times \langle \sigma \rangle$ whose composition rule is given by $(g, \tau)(g', \tau') = (g^{\tau'} g', \tau\tau')$ $(g, g' \in G_1, \tau, \tau' \in \langle \sigma \rangle)$. Two elements g and g' of G_1 is said to be *σ-twistedly conjugate*

(1) The Theorem was reported by the author at the annual meeting of the Mathematical Society of Japan on fall, 1974.

in G_1 if, for a suitable element x of G_1 $g'=x^{\sigma}gx^{-1}=\bar{x}gx^{-1}$. It is easy to see that (g, σ) and (g', σ) are conjugate in $\tilde{G_1}$ if and only if g and g' are σ-twistedly conjugate in G_1. For real variables $\theta, t>0$ and x set

(1) $$k_\theta=\begin{pmatrix}\cos\theta/2 & \sin\theta/2\\ -\sin\theta/2 & \cos\theta/2\end{pmatrix},\quad a_t=\begin{pmatrix}\sqrt{t} & \\ & \sqrt{t}^{-1}\end{pmatrix}\ \text{and}\ n(x)=\begin{pmatrix}1 & x\\ & 1\end{pmatrix}.$$

It follows from Lemma 3.4~Corollary 3.7 of [4] that a complete set of representatives for σ-twisted conjugate classes of G_1 is given by $\{k_\theta;\, 0\leqq\theta\leqq\pi\}\cup\{a_t,\, t>1\}\cup n(1)$. For each $g\in G_1$ set

$$Z_\sigma(g)=\{x\in G_1;\, x^{\sigma}gx^{-1}=g\}.$$

Then we have

(2) $$Z_\sigma(k_\theta)=\begin{cases} G_{1\boldsymbol{R}}\cup G_{1\boldsymbol{R}}\begin{pmatrix}i & \\ & -i\end{pmatrix} & \text{if } \theta=0, \text{ where we put } G_{1\boldsymbol{R}}=SL(2,\boldsymbol{R})/\pm1\\ K=SO(2)/\pm1 & \text{if } 0<\theta<\pi,\\ PU(2)=SU(2)/\pm1 & \text{if } \theta=\pi,\end{cases}$$

$Z_\sigma(a_t)=A\cup A\begin{pmatrix}i & \\ & -i\end{pmatrix}$ if $t\neq1$, where we put $A=\{a_t;\, t>0\}$, and $Z_\sigma(n(1))=N=\{n(x);\, x\in\boldsymbol{R}\}$.

We normalize an invariant measure on G_1 by setting $dg=dudz\frac{dt}{t}$ for $g=un(z)a_t$ ($u\in PU(2)$, $z\in\boldsymbol{C}$, $t>0$) where du is the normalized Haar measure of $SU(2)/\pm1$. Similarly, we normalize an invariant measure on $G_{1\boldsymbol{R}}$ by setting $dg=dkdx\frac{dt}{t}$ for $g=kn(x)a_t$ ($k\in K$, $x\in\boldsymbol{R}$, $t>0$), where dk is the normalized Haar measure of K. Further, we normalize invariant measures on A and on N by setting $da_t=\frac{dt}{t}$ and $dn(x)=dx$. For each compactly supported smooth function f on G_1, set

(3) $$F^1_{f,\sigma}(t)=|t-t^{-1}|\int_{G_1/A}f(\dot{x}^{\sigma}a_t\dot{x}^{-1})d\dot{x}\qquad(t\in\boldsymbol{R}_+,\, t\neq1),$$

(4) $$F^2_{f,\sigma}(\theta)=\sin\theta\int_{G_1/K}f(\dot{x}^{\sigma}k_\theta\dot{x}^{-1})d\dot{x}\qquad(\theta\in\boldsymbol{R},\, \sin\theta\neq0),$$

(5) $$\delta^1_\sigma(f)=\int_{G_1/G_{1\boldsymbol{R}}}f(\dot{x}^{\sigma}\dot{x}^{-1})d\dot{x},$$

(6) $$\delta^2_\sigma(f)=\int_{G_1/PU(2)}f\left(\dot{x}^{\sigma}\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\dot{x}^{-1}\right)d\dot{x},$$

(7) $$\delta^3_\sigma(f)=\int_{G_1/N}f\left(\dot{x}^{\sigma}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\dot{x}^{-1}\right)d\dot{x},$$

where invariant measures on quotient spaces G_1/A, G_1/K etc. are quotient measures of previously normalized Haar measures of respective groups.

PROPOSITION 1. *Notations being as above,*

(i) *the function* $F^1_{f,\sigma}$ $(f\in C_0^\infty(G_1))$ *on* $\boldsymbol{R}_+-\{1\}$ *is extended to a smooth function on* $\boldsymbol{R}_+$ *with the following properties:*

$$F^1_{f,\sigma}(t)=F^1_{f,\sigma}(t^{-1}), \quad F^1_{f,\sigma}(1)=4\delta^3_\sigma(f).$$

(ii) *The function* $F^2_{f,\sigma}$ $(f\in C_0^\infty(G_1))$ *on* $\boldsymbol{R}-\pi\boldsymbol{Z}$ *is extended to a smooth odd function on* $\boldsymbol{R}-2\pi\boldsymbol{Z}$ *with the following properties:*

$$\lim_{\theta\to\pm 0} F^2_{f,\sigma}(\theta)=\pm 2\pi\delta^3_\sigma(f),$$

$$\lim_{\theta\to 0}\frac{d}{d\theta}F^2_{f,\sigma}(\theta)=-2\pi\delta^1_\sigma(f),$$

$$\Big(\frac{d}{d\theta}F^2_{f,\sigma}\Big)(\pi)=-\delta^2_\sigma(f).$$

PROOF. (i) We have

$$F^1_{f,\sigma}(t)=|t-t^{-1}|\int_{PU(2)\times\boldsymbol{C}} f(u^\sigma n(z^\sigma)a_t n(-z)u^{-1})dzdu$$

$$=t^{-1}\int_{PU(2)\times\boldsymbol{C}} f(u^\sigma n(z)a_t u^{-1})dzdu.$$

Thus, $F^1_{f,\sigma}$ is extended to a smooth function on $\boldsymbol{R}_+$. Since a_t and $a_{t^{-1}}$ are σ-twistedly conjugate to each other, $F^1_{f,\sigma}(t^{-1})=F^1_{f,\sigma}(t)$. On the other hand,

$$\delta^3_\sigma(f)=\int_{PU(2)\times\boldsymbol{R}\times\boldsymbol{R}_+} f(u^\sigma n(-ix)a_t n(1)a_t^{-1}n(-ix)u^{-1})dudxdt$$

$$=4^{-1}\int_{PU(2)\times C} f(u^\sigma n(z)u^{-1})dudz.$$

Hence $F^1_{f,\sigma}(1)=4\delta^3_\sigma(f)$.

(ii) It is easy to see that $F^2_{f,\sigma}$ is a smooth odd function on $\boldsymbol{R}-\pi\boldsymbol{Z}$ with period 2π. For any compact subset I of the interval $(0, 2\pi)$, the mapping $(x,\theta)\mapsto\varphi(x,\theta)=x^\sigma k_\theta x^{-1}$ is a *proper* mapping from $G_1\times I$ into G_1. In fact, let M_1 be a compact subset in $\varphi(G_1\times I)$ and set $M_2=\varphi^{-1}(M_1)$. Let $\{(g_i,\theta_i)\}$ $(i=1, 2, \cdots)$ be an infinite sequence in M_2. Set $g_i=u_i n(z_i)a_{t_i}$ $(t_i>0,\ z_i\in\boldsymbol{C},\ u_i\in PU(2))$. Choosing a suitable subsequence if necessary, we may assume that $\theta_i\mapsto\theta\in I$, $u_i\mapsto u\in PU(2)$ and $\varphi(g_i,\theta_i)=g_i^\sigma k_{\theta_i}g_i^{-1}\mapsto g\in M_1$. Put $(u^\sigma)^{-1}gu=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $x_i=(n(z_i)a_{t_i})^\sigma k_{\theta_i}\times(n(z_i)a_{t_i})^{-1}=\begin{pmatrix}* & *\\ -t_i^{-1}\sin\frac{\theta_i}{2} & *\end{pmatrix}$. Then $\lim_{i\to\infty}x_i=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $\lim\Big(-t_i^{-1}\sin\frac{\theta_i}{2}\Big)\mapsto c$. Since $\sin\frac{\theta_i}{2}\mapsto\sin\frac{\theta}{2}\neq 0$ $(\theta\in I\subset(0,2\pi))$, $-\lim_{i\to\infty}t_i^{-1}=c/\sin\frac{\theta}{2}$.

If c were equal to zero, the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sigma}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ would be positive while the eigenvalues of $x_i^{\sigma}x_i$ are $\exp(\pm\sqrt{-1}\theta_i)$ $(\theta_i \in I)$. Thus, $c \neq 0$ and $\lim a_{t_i} \mapsto a_t$ $(t^{-1}=-c/\sin\theta/2>0)$. Hence $n(z_i^{\sigma})a_t k_\theta a_t^{-1} n(z_i)^{-1} = \begin{pmatrix} \cos\frac{\theta}{2}+z_i^{\sigma}c & * \\ c & * \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $i \mapsto \infty$. We have $z_i^{\sigma} \mapsto \frac{1}{c}\left(a-\cos\frac{\theta}{2}\right)$. We have proved that from any infinite sequence in M_2, we can choose a convergent subsequence. Hence M_2 is compact. Thus the integral defining $F_{f,\sigma}^2(\theta)$ is absolutely convergent for any $\theta \in (0, 2\pi)$ and $F_{f,\sigma}^2(\theta)$ is smooth on $(0, 2\pi)$. Since

$$F_{f,\sigma}^2(\theta) = \sin\theta \int_{R^+} \frac{dt}{t} \int_C dz \int_{PU(2)/K} f(a_t n(z^{\sigma})\dot{u}^{\sigma} k_\theta \dot{u}^{-1} n(z)^{-1} a_t^{-1}) d\dot{u}$$

($d\dot{u}$ is the quotient measure of the normalized Haar measures of $PU(2)$ by that of K), and

$$\delta_\sigma^2(f) = \int_{R^+} \frac{dt}{t} \int_C dz\, f\left(a_t n(z^{\sigma}) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} n(z)^{-1} a_t^{-1}\right),$$

$F_{f,\sigma}^2(\pi)=0$ and $\left(\frac{d}{d\theta}F_{f,\sigma}^2\right)(\pi) = -\delta_\sigma^2(f)$.

Next, we will show that, for a given compact subset C of G_1 there exists a compact subset C' of G_1 such that $x^{\sigma}k_\theta x^{-1} \in C$ for some $\theta \in (-\pi/4, \pi/4)$ implies $x^{\sigma}x^{-1} \in C'$. Denote by M the image of $G_1 \times (-\pi/4, \pi/4)$ under the mapping $(x, \theta) \mapsto x^{\sigma}k_\theta x^{-1}$. Set $C_2 = \{z^{\sigma}z\, ;\, z \in C \cap M\}$. Then C_2 is a relatively compact subset of G_1. The eigen values of any element of C_2 are given by $\{e^{i\varphi}, e^{-i\varphi}\}$ for a suitable $\varphi \in (-\pi/4, \pi/4)$. Hence, the binomial series $\sqrt{g} = \sum_{m=0}^{\infty} \binom{1/2}{m}(g-1)^m$ is absolutely convergent on C_2. Let C_3 be the image of C_2 under the continuous mapping $g \mapsto \sqrt{g}$. Then C_3 is also a relatively compact subset of G. For $g \in C_2$, $\sqrt{g}$ is an element of G characterized by the following two properties:

(i) $(\sqrt{g})^2 = g$, (ii) Arguments of eigen values of $\sqrt{g}$ are both in the interval $(-\pi/8, \pi/8)$. Hence, $y = x^{\sigma}k_\theta x^{-1} \in C$ for some $\theta \in (-\pi/4, \pi/4)$ implies $\sqrt{y^{\sigma}y} = xk_\theta x^{-1}$. Thus, $x^{\sigma}k_\theta x^{-1} \in C$ $(\theta \in (-\pi/4, \pi/4))$ implies $xk_\theta x^{-1} \in C_3$. Let C' be the closure of CC_3^{-1}. Then C' is compact and $x^{\sigma}k_\theta x^{-1} \in C$ for some $\theta \in (-\pi/4, \pi/4)$ implies $x^{\sigma}x^{-1} \in C'$. It is easy to see that the mapping $x \mapsto x^{\sigma}x^{-1}$ establishes a bicontinuous 1 to 1 correspondence between the homogeneous space $G_1/Z_\sigma(1)$ and the closed submanifold $\{z \in G_1\, ;\, z^{\sigma}z = 1\}$ of G_1. Since

$$F_{f,\sigma}^2(\theta) = \sin\theta \int_{G_1/G_{1R}} d\dot{x} \int_{G_{1R}/K} f(\dot{x}^{\sigma}\dot{y}k_\theta \dot{y}^{-1}\dot{x}^{-1}) d\dot{y},$$

for a given compactly supported smooth function f on G_1 there exists a compact subset C of G_1/G_{1R} such that

$$F^2_{f,\sigma}(\theta)=\sin\theta\int_C d\dot{x}\int_{G_{1R}/K} f(\dot{x}^\sigma \dot{y} k_\theta \dot{y}^{-1}\dot{x}^{-1})d\dot{y} \qquad (\theta\neq 0,\ |\theta|<\pi/4).$$

By the Lemma of [1],

$$\lim_{\theta\downarrow 0}\sin\theta\int_{G_{1R}/K} f(\dot{x}^\sigma \dot{y} k_\theta y^{-1}\dot{x}^{-1})d\dot{y}=2\pi\int_{G_{1R}/N} f\left(\dot{x}^\sigma \dot{y}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\dot{y}^{-1}\dot{x}^{-1}\right)d\dot{y},$$

$$\lim_{\theta\downarrow 0}\frac{d}{d\theta}\sin\theta\int_{G_{1R}/K} f(\dot{x}^\sigma \dot{y} k_\theta \dot{y}^{-1}\dot{x}^{-1})d\dot{y}=-2\pi f(\dot{x}^\sigma \dot{x}^{-1}).$$

Moreover, if x remains in a compact subset, both convergences are uniform with respect to $\dot{x}$. Thus,

$$\lim_{\theta\to\pm 0} F^2_{f,\sigma}(\theta)=\pm 2\pi\delta^{(3)}_\sigma(f),$$

$$\lim\frac{d}{d\theta}F^2_{f,\sigma}(\theta)=-2\pi\delta^{(1)}_\sigma(f).$$

We omit the proof of the following proposition which is an analogue of the Weyl integral formula.

PROPOSITION 2. *For each compactly supported smooth function f on G_1,*

$$\int_{G_1} f(g)dg=\frac{1}{2}\int_0^\infty F^1_{f,\sigma}(t)|t-t^{-1}|\frac{dt}{t}+$$
$$+\frac{1}{2\pi}\int_0^{2\pi} F^2_{f,\sigma}(\theta)4\sin\theta\, d\theta.$$

2 Let G be a connected complex semi-simple Lie group with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ and G_0 be the Lie subgroup of G corresponding to $\mathfrak{g}_0$. We assume that there exists an anti-holomorphic involutive automorphism σ of G such that G_0 is the subgroup of fixed points of σ. Denote by $\mathfrak{B}$ the universal enveloping algebra of $\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. We identify $\mathfrak{B}$ with the algebra of left invariant differential operators on G in a usual manner. For a smooth function f on G, we write $(bf)(x)=f(x, b)$ $(x\in G, b\in\mathfrak{B})$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and let A_0 be the Lie subgroup of G_0 corresponding to $\mathfrak{h}_0$. Denote by $\mathfrak{A}$ the universal enveloping algebra of $\mathfrak{h}_0$ and identify it with the algebra of left invariant operators on A_0. For $X\in\mathfrak{g}$, L_X (resp. R_X) is a linear mapping in $\mathfrak{B}$ given by left (resp. right) multiplication by X. The proof of the next lemma is quite similar to that of Lemma 15 of [2].

LEMMA 3. *For each $a\in A_0$, there exists a unique linear mapping $\Gamma_{a,\sigma}$ from $\mathfrak{B}\times\mathfrak{A}$ into $\mathfrak{B}$ such that $\Gamma_{a,\sigma}(1\times\nu)=\nu$ and $\Gamma_{a,\sigma}(X_1\cdots X_r\times\nu)=(L_{Ad(a^{-1})X_1^\sigma}-R_{X_1})\cdots(L_{Ad(a^{-1})X_r^\sigma}-R_{X_r})\nu$ for $X_1,\cdots,X_r\in\mathfrak{g}$, $\nu\in\mathfrak{A}$ ($Ad(a^{-1})$ is the adjoint transformation corresponding to a^{-1}).*

For a smooth function f on G, denote by F a smooth function on $G\times A_0$ given by $F(x, a)=f(x^\sigma ax^{-1})$. It is proved that, for $b\in\mathfrak{B}$ and $\nu\in\mathfrak{A}$,

$$F(x, b; a, \nu)=f(x^\sigma ax^{-1}, w^x),$$

where $w=\Gamma_{a,\sigma}(b\times\nu)$ and w^x is the image of w under the adjoint transformation corresponding to x (see p. 114 of [2]). Let $\mathfrak{q}$ be the real subspace of $\mathfrak{g}$ spanned by $\sqrt{-1}\ \mathfrak{h}_0$ and by root vectors of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}=\mathfrak{h}_0+\sqrt{-1}\ \mathfrak{h}_0$. Denote by λ the canonical mapping from $S(\mathfrak{g}\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C})$ (the symmetric algebra over $\mathfrak{g}\underset{\boldsymbol{R}}{\otimes}\boldsymbol{C}$) onto $\mathfrak{B}$ and set $\mathfrak{Q}=\lambda(S(\mathfrak{q}))$, where $S(\mathfrak{q})$ is the symmetric algebra over $\mathfrak{q}$. Put

$$A_0'=\{a\in A_0;\ \det(\mathrm{Ad}(a^{-1})\cdot\sigma-1)|_{\mathfrak{q}}\neq 0\}.$$

LEMMA 4 (cf. Lemma 15 of [2]). *Notations being as above, for $a\in A_0'$, the mapping $\Gamma_{a,\sigma}$ is a linear bijection from $\mathfrak{Q}\times\mathfrak{A}$ onto $\mathfrak{B}$.*

Set $\mathfrak{Q}'=\sum_{r=1}^{\infty}\lambda(S_r(\mathfrak{q}))$, where $S_r(\mathfrak{q})$ is the space of homogeneous elements of degree r in $S(\mathfrak{q})$. By lemma 4, for $b\in\mathfrak{B}$ and $a\in A_0'$, there exists a uniquely determined $\delta_{a,\sigma}(b)\in\mathfrak{A}$ such that

$$b-\delta_{a,\sigma}(b)\in\Gamma_{a,\sigma}(\mathfrak{Q}'\times\mathfrak{A}).$$

A smooth function f on G is said to be σ-twistedly G-invariant if $f(x^\sigma gx^{-1})=f(g)$ for any $x, g\in G$. The following lemma is an easy consequence of preceding results.

LEMMA 5. *Notations being as above, let z be in the center of $\mathfrak{B}$ and let f be a smooth σ-twistedly G-invariant function on G. Then, for $a\in A_0'$, $f(a, z)=f(a, \beta_{a,\sigma}(z))$, where $\beta_{a,\sigma}(z)$ is a differential operator on A_0' whose local expression (the definition of "local expression" is given at* p. 112 *of* [2]) *at a coincides with $\delta_{a,\sigma}(z)$.*

An example of Lemma 5. Set $G=SL(2, \boldsymbol{C})/\pm 1$, $G_0=SL(2, \boldsymbol{R})/\pm 1$, $g^\sigma=\bar{g}$ $(g\in G)$. Put $H=\begin{pmatrix}1 & \\ & -1\end{pmatrix}$, $X=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $Y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$, $H_1=\sqrt{-1}\ H$, $X_1=\sqrt{-1}\ X$, $Y_1=\sqrt{-1}\ Y$. Then $\{H, X, Y, H_1, X_1, Y_1\}$ is an $\boldsymbol{R}$-base of the Lie algebra $\mathfrak{g}$ of G and $\Omega_0=\frac{1}{2}H^2+XY+YX-\left(\frac{1}{2}H_1^2+X_1Y_1+Y_1X_1\right)$ is the Casimir operator on G.

1. Set $\mathfrak{h}_0=\boldsymbol{R}H$, $A_0=\left\{a_t=\begin{pmatrix}\sqrt{t} & \\ & \sqrt{t^{-1}}\end{pmatrix};\ t>0\right\}$. Then $A_0'=A_0-\{1\}$. For $a_t\in A_0'$, we have, by an elementary calculation,

$$\Omega_0=\Gamma_{a_t,\sigma}\left(1\times\left\{\frac{1}{2}H^2+\left(\frac{t+1}{t-1}+\frac{t-1}{t+1}\right)H\right\}\right)$$
$$+\Gamma_{a_t,\sigma}\left(\left\{-\frac{1}{8}H_1^2-\frac{t}{(t-1)^2}(XY+YX)-\frac{t}{(t+1)^2}(X_1Y_1+Y_1X_1)\right\}\times 1\right).$$

Thus, $\delta_{a,\sigma}(\Omega_0)=\frac{1}{2}H^2+2\frac{t+t^{-1}}{t-t^{-1}}H$. Hence, for a σ-twistedly G-invariant smooth function f on G,

$$f(a_t,\ \Omega_0)=\left\{2(t-t^{-1})^{-1}\left(t\frac{d}{dt}\right)^2(t-t^{-1})-2\right\}f(a_t)\,. \tag{8}$$

2. Set $\mathfrak{h}_0=\boldsymbol{R}(X-Y)$, $A_0=\left\{k_\theta=\begin{pmatrix}\cos\theta/2 & \sin\theta/2\\ -\sin\theta/2 & \cos\theta/2\end{pmatrix}\right\}$. Then $A_0'=\{k_\theta;$ $\sin\theta\neq 0\}$. For $k_\theta\in A_0'$, we have, by an elementary calculation

$$\Omega_0=\Gamma_{k\theta,\sigma}\left(1\times\left\{-\frac{1}{2}K^2-\frac{\sin\theta}{1-\cos\theta}K+\frac{\sin\theta}{1+\cos\theta}K\right\}\right)+$$

$$+\Gamma_{k\theta,\sigma}\left(\left\{\frac{1}{8}K_1^2+\frac{1}{4(1-\cos\theta)}(H^2+W^2)-\frac{1}{4(1+\cos\theta)}(H_1^2+W_1^2)\right\}\times 1\right),$$

where we put $K=X-Y$, $K_1=\sqrt{-1}\,K$, $W=X+Y$, $W_1=\sqrt{-1}\,W$. Thus,

$$\delta_{k\theta,\sigma}(\Omega_0)=-\frac{1}{2}K^2-2\frac{\cos\theta}{\sin\theta}K\,.$$

Hence, for a σ-twistedly G-invariant smooth function f on G,

$$f(k_\theta,\ \Omega_0)=\left\{-2\frac{1}{\sin\theta}\frac{d^2}{d\theta^2}\sin\theta-2\right\}f(k_\theta)\,. \tag{9}$$

Set

$$\Omega=2\Omega_0+4=H^2+2(XY+YX)-H_1^2-2(X_1Y_1+Y_1X_1)+4\,. \tag{10}$$

The following proposition follows from (8), (9), and Proposition 2

PROPOSITION 6. *For* $f\in C_0^\infty(G_1)$ $(G_1=SL(2,\boldsymbol{C})/\pm 1)$,

$$F^1_{\Omega f,\sigma}(t)=4\left(t\frac{d}{dt}\right)^2F^1_{f,\sigma}(t)\,,$$

$$F^2_{\Omega f,\sigma}(\theta)=-4\frac{d^2}{d\theta^2}F^2_{f,\sigma}(\theta)\qquad\text{(for notations, see (3) and (4)).}$$

3 Let M be an m-dimensional smooth manifold and let N be an n-dimensional submanifold of M. For each $p\in N$, there exists a relatively compact coordinate neighborhood U of p in M and a system of coordinate functions $\{x_1,\cdots,x_n,$ $y_1,\cdots,y_{m-n}\}$ such that

$U\cap N=\{q\in U;\, y_1(q)=\cdots=y_{m-n}(q)=0\}$. For each $(m-n)$-tuple of non-negative integers $I=(i_1,\cdots,i_{m-n})$, we put

$$\partial_y^I=\left(\frac{\partial}{\partial y_1}\right)^{i_1}\cdots\left(\frac{\partial}{\partial y_{m-n}}\right)^{i_{m-n}},\qquad y^I=y_1^{i_1}\cdots y_{m-n}^{i_{m-n}},$$

$$|I|=i_1+\cdots+i_{m-n},\qquad I!=i_1!\cdots i_{m-n}!\,.$$

Let T be a distribution on M *whose support is contained in* N. As is well-known, the localization of T to U has the following expression:

$$T(f)=\sum_{|I|\leqq r} T_I(\partial_y^I f|_N) \qquad (f\in C_0^\infty(U)), \tag{11}$$

where each T_I is a distribution on $U\cap N$. In (11), if there is an I such that $T_I\neq 0$ and $|I|=r$, the non-negative integer r is called the normal rank of T on $U\cap N$. Now *assume that a Lie group G operates smoothly on M and that N is a single G-orbit.*

LEMMA 7. *Notations and assumptions being as above, assume further that T is invariant under the action of G.*

(i) *The normal rank of T is constant on N.*

(ii) *In* (11), *each T_I is a smooth function on $N\cap U$.*

PROOF. (i) is obvious. We identify $U\cap N$ with its image under the mapping: $q\mapsto x(q)=(x_1(q),\cdots,x_n(q))$. Take a neighborhood $U_0\subset U$ of p such that $x(U_0)\subset x(U\cap N)$. For each $\varphi\in C_0^\infty(U\cap N)$, denote by φy^I a smooth function on U_0 given by

$$\varphi y^I(p)=\varphi(x(p))y^I(p).$$

Since the support of T is concentrated to N, $T(\varphi y^I)$ is defined in a natural manner. Moreover, $T(\varphi y^I)=I!\,T_I(\varphi)$. Let $\{X_1,\cdots,X_l\}$ be an $\boldsymbol{R}$ base of the Lie algebra $\mathfrak{g}$ of G. For each $X\in\mathfrak{g}$, we denote by the same symbol a differential operator on M given by $(Xf)(x)=\frac{d}{dt}f(\exp tX)\cdot x)|_{t=0}$. Set

$$X_i=\sum_{j=1}^{n} a_{ij}(x,y)\frac{\partial}{\partial x_j}+\sum_{k=1}^{m-n} b_{ik}(x,y)\frac{\partial}{\partial y_k}$$

on U. Since T is G-invariant, $T(X_i(\varphi y^I))=0$ $(1\leqq i\leqq l)$. Thus

$$\sum_{j=1}^{n}\sum_{|J|\leqq r-|I|}\frac{(I+J)!}{J!}T_{I+J}\Big(\frac{\partial\varphi}{\partial x_j}(\partial_y^J a_{ij})|_N\Big)+$$
$$+\sum_{k=1}^{m-n}\sum_{|J|\leqq r-|I|+1} i_k\frac{(I_k+J)!}{J!}T_{I_k+J}(\varphi\partial_y^J b_{jk}|_N)=0,$$

where, for $I=(i_1,\cdots,i_{m-n})$, we put $I_k=(i_1,\cdots,i_k-1,\cdots,i_{m-n})$. Since N is a G-orbit, $b_{jk}|_N=0$ and the rank of the matrix $(a_{jk}(x,0))$ $(1\leqq j\leqq l,\ 1\leqq k\leqq n)$, is n for each $x\in U\cap N$. Thus, a system of distributions $\{T_I;\ |I|\leqq r\}$ satisfies an elliptic system of differential equations with smooth coefficients. Hence, each T_I is a smooth function on $U\cap N$.

Let us further assume that there exists a non-zero G-invariant n-form ω on N. Then, in Lemma 7, there exists a smooth function $T_I(x)$ on $U\cap N$ such that $T_I(\varphi)=\int_N\varphi(x)T_I(x)\omega$ for any $\varphi\in C_0^\infty(U\cap N)$.

Denote by $S^r(T_N(M)/T(N))$ the symmetric tensor product of homogeneous degree r of the normal tangent bundle of N in M.

The next lemma is easily proved.

LEMMA 8. *Notations and assumptions being as above, the mapping:* $p \mapsto \sum_{|I|=r} T_I(p)(\partial_y^I)_p$ *gives a G-invariant section of* $S^r(T_N(M)/T(N))$.

In the following, for each G-invariant distribution T supported in N, we denote by $s(T)$ the G-invariant section of $S^r(T_N(M)/T(N))$ given by Lemma 8. We note that if T_1 and T_2 are G-invariant distributions supported on N of normal rank r, $s(T_1)=s(T_2)$ implies that the normal rank of T_1-T_2 is less than r.

From now on, we regard the group $G_1=SL(2, \boldsymbol{C})/\pm 1$ as a transformation space of G_1 under the action

$$(12) \qquad x \longrightarrow g\cdot x = g^{\sigma}xg^{-1} \qquad (g, x \in G_1).$$

A distribution T on G_1 is said to be *σ-twistedly G_1-invariant* if T is invariant under the transformations (12). Set $N_1=\{g\in G_1, g^{\sigma}g=1\}$. Then N_1 is a three dimensional closed submanifold of G_1 which is a single G_1-orbit. It is easy to see that there exists a G_1-invariant non-zero 3-form on N_1. For each non-zero integer k, $\Omega^k\delta_{\sigma}^1$ (see (5), (10)) is a σ-twistedly G_1-invariant distribution supported on N_1 with the normal rank $2k$. Moreover, any G_1-invariant section of $S^r(T_{N_1}(G_1)/T(N_1))$ is zero if r is odd and is a multiple of $s(\Omega^{r/2}\delta_{\sigma}^1)$ if r is even. The following lemma is now easily proved, by the induction with respect to the normal rank.

LEMMA 9. *Any σ-twistedly G_1-invariant distribution supported on N_1 is a finite linear combinations of* $\Omega^k\delta_{\sigma}^1$ $(k=0, 1, 2, \cdots)$.

Set $N_2=\{g\in G_1, g^{\sigma}g=-1\}$. Then N_2 is also a closed three dimensional submanifold of G_1 which is a single G_1-orbit. The next lemma is derived in quite a similar manner.

LEMMA 10. *Any σ-twistedly G_1-invariant distribution supported on N_2 is a finite linear combinations of* $\Omega^k\delta_{\sigma}^2$ $(k=0, 1, 2, \cdots)$ (*see* (6), (10)).

Set $S_{\sigma}=\{g\in G_1; \operatorname{tr} g^{\sigma}g=\pm 2\}$. Put $N_3=\{g\in G_1; \operatorname{tr} g^{\sigma}g=2, g^{\sigma}g\neq 1\}$, then $S_{\sigma}=N_1\cup N_2\cup N_3$ is a decomposition of S_{σ} into G_1-orbits under the action: $x\mapsto g\cdot x=g^{\sigma}xg^{-1}$. The orbit N_3 is open in S_{σ} and is of codimension 1 in G_1.

LEMMA 11. *Let T be a σ-twistedly G_1-invariant distribution on G_1 supported in S_{σ}. Then, on N_3, T is equal to a suitable finite linear combinations of* $\Omega^k\delta_{\sigma}^3$ $(k=0, 1, 2, \cdots)$ (*see* (7), (10)).

PROOF. Set $x(g)=\operatorname{tr} g^{\sigma}g-2$ $(g\in G_1)$. Then it is easy to see that $dx\neq 0$ on N_3. Hence, if the support of T has a nonempty intersection with N_3, there exists the largest non-negative integer r such that $x(g)^rT\neq 0$ on N_3. We call r the normal rank of T on N_3. It is easy to see that, for a suitable constant c, $x^rT=c\delta_{\sigma}^3$ on N_3. We note that the normal rank of $\Omega^k\delta_{\sigma}^3$ on N_1 is k $(k=0, 1, 2, \cdots)$. In fact, it follows from Proposition 1 and Proposition 6 that

$$
\begin{aligned}
(4x^r\Omega^k\delta_\sigma^3)(f) &= \lim_{t\to 1} F^1_{\Omega^k x^r f,\sigma}(t) \\
&= \lim_{t\to 1}\left(2t\frac{d}{dt}\right)^{2k}(t+t^{-1}-2)^r F^1_{f,\sigma}(t) \\
&= \begin{cases} 0 & \text{if } r>k \\ 2^{2k}(2k)!\,4\delta_\sigma^3(f) & \text{if } r=k\,. \end{cases}
\end{aligned}
$$

Thus, the normal rank of $T-\frac{c}{2^{2k}(2k)!}\Omega^r\delta_\sigma^3$ on N_3 is smaller than r. The lemma is now obtained easily, by applying the induction with respect to r.

Lemma 9, Lemma 10 and Lemma 11 imply the following:

LEMMA 12. *Notations being as above, any σ-twistedly G_1-invariant distribution on G_1 supported on S_σ is a finite linear combinations of $\Omega^k\delta_\sigma^1$, $\Omega^k\delta_\sigma^2$ and $\Omega^k\delta_\sigma^3$ ($k=0, 1, 2, \cdots$).*

COROLLARY. *If T is a σ-twistedly G_1-invariant distribution on G_1 supported on S_σ and is an eigen distribution of Ω, then $T=0$.*

§ 2.

1. Let T be an irreducible unitary representation of the group $G_1^\sim$ on a Hilbert space $\mathfrak{H}$ ($G_1^\sim$ is the semi-direct product of $G_1=SL(2, \boldsymbol{C})/\pm 1$ and $\langle\sigma\rangle$ introduced in § 1, 1). Let T_0 be the restriction of T to the group G_1, which is the connected component of 1 of $G_1^\sim$. Then T_0 is either irreducible or the direct sum of two mutually inequivalent irreducible unitary representations of G_1. The representation T is said to be *of the first kind* or *of the second kind* according as T_0 is irreducible or reducible. If T is of the first kind,

$$
(12) \qquad T_0(g)=JT_0(g^\sigma)J^{-1} \qquad (g\in G_1)\,,
$$

where $T_0(g)=T(g, 1)$, $J=T(1, \sigma)$. Hence, T_0 is equivalent to its "conjugate representation" given by $g\mapsto T_0(g^\sigma)$. We note that a unitary operator J on $\mathfrak{H}$ which satisfies (12) is either $T(1, \sigma)$ or $-T(1, \sigma)$.

2. We recall a description of irreducible unitary representations of $G_1=SL(2, \boldsymbol{C})/\pm 1$. For details, see [**3**]. For an integer m, denote by $\mathfrak{H}^{2m}$ the space of measurable functions f on $PU(2)\cong SU(2)/\pm 1$ which satisfy the following conditions (i) and (ii).

$$
\text{(i)} \qquad f\left(\begin{pmatrix} e^{+i\theta} & \\ & e^{-i\theta}\end{pmatrix}u\right)=e^{2im\theta}f(u) \qquad (\forall\theta\in\boldsymbol{R})\,,
$$

$$
\text{(ii)} \qquad \int_{PU(2)}|f(u)|^2du<\infty\,,
$$

where du is the normalized invariant measure on $PU(2)$. The sapce $\mathfrak{H}^{2m}$ is a Hilbert space with the inner product

$$(f_1, f_2)=\int_{PU(2)} f_1(u)\overline{f_2(u)}du\,.$$

For each $g\in G_1$, there exists a uniquely determined triple $(z(g), t(g), k(g))\in \boldsymbol{C}\times\boldsymbol{R}_+\times PU(2)$ such that $g=\begin{pmatrix}1 & z(g)\\ & 1\end{pmatrix}\begin{pmatrix}t(g)^{-1} & \\ & t(g)\end{pmatrix}k(g)$. Denote by $R^{(2m,\rho)}$ $(\rho\in\boldsymbol{R})$ a representation of G_1 on $\mathfrak{H}^{2m}$ given by the following formula;

$$(R^{(2m,\rho)}(g)f)(u)=t(ug)^{i\rho-2}f(k(ug))\,.$$

It is known that $R^{(2m,\rho)}$ is an irreducible unitary representation of G_1. Two representations $R^{(2m,\rho)}$ and $R^{(2m',\rho')}$ are equivalent if and only if $(2m, \rho)=\pm(2m', \rho')$.

For a positive τ $(0<\tau<2)$, denote by $\mathfrak{H}_\tau$ the space of measurable functions on $PU(2)$ which satisfy the following conditions:

(i) $$f\left(\begin{pmatrix}e^{-i\theta} & \\ & e^{i\theta}\end{pmatrix}u\right)=f(u) \qquad (\forall\theta\in\boldsymbol{R})\,.$$

(ii) $$\int_{PU(2)^2}\Phi_\tau(u_1u_2^{-1})f_1(u_1)\overline{f_2(u_2)}du_1du_2<\infty\,,$$

where we put $\Phi_\tau(u)=|u_{21}|^{-2+\tau}$, u_{21} being the (2, 1)-entry of u. Then $\mathfrak{H}_\tau$ is the Hilbert space with the inner product

$$(f_1, f_2)=\pi\int_{PU(2)^2}\Phi_\tau(u_1u_2^{-1})f_1(u_1)\overline{f_2(u_2)}du_1du_2\,.$$

Denote by R_τ a representation of G on $\mathfrak{H}_\tau$ given by the following:

$$(R_\tau(g)f)(u)=t(ug)^{-2-\tau}f(k(ug))\,.$$

Then R_τ is an irreducible unitary representation of G. Two representations R_τ and $R_{\tau'}$ $(0<\tau, \tau'<2)$ are equivalent if and only if $\tau=\tau'$. Representations R_τ and $R^{(2m,\rho)}$ are never equivalent. It is known that any non-trivial irreducible unitary representation of G_1 is equivalent either to $R^{(2m,\rho)}$ $(m\in\boldsymbol{Z}, \rho\in\boldsymbol{R})$ or to R_τ $(0<\tau<2)$. An irreducible unitary representation R of G_1 is said to be self-conjugate if R is equivalent to $R^\sigma: g\mapsto R(g^\sigma)$. It is easy to see that representations $R^{(0,\rho)}$ $(\rho\in\boldsymbol{R})$, $R^{(2m,0)}$ $(m\in\boldsymbol{Z})$ and R_τ $(0<\tau<2)$ are all self-conjugate and that any non-trivial self-conjugate irreducible representation of G_1 is equivalent to one of them.

Denote by I_σ a unitary operator of order 2 on $\mathfrak{H}^0$ (or $\mathfrak{H}_\tau$) given by

(13) $$(I_\sigma f)(u)=f(\bar{u})\,.$$

We extend representation $R^{(0,\rho)}$ (resp. R_τ) of G_1 to a representation $T_\pm^{(0,\rho)}$ (resp. $T_{\tau\pm}$) of $G_1^\sim$ by setting $T_\pm^{(0,\rho)}((g, \sigma))=\pm I_\sigma R^{(0,\rho)}(g)$ (resp. $T_{\tau\pm}((g, \sigma))=\pm I_\sigma R_\tau(g)$).

Let m, n and r be integers which satisfy inequalities: $n\geqq|m|$, $|r|$. Denote by $C_{m,r}^n$ the function on $SU(2)/\pm1$ given by

$$\sqrt{2n+1}\,C^n_{m,r}(u)=\sum_{\max(0,-m-r)\leqq k\leqq\min(n-r,n-m)}\binom{n-r}{k}\binom{n+r}{n-m-k}a^{k+m+r}\times$$
$$\times(-\bar{b})^{n-m-k}b^{n-r-k}\bar{a}^k$$

for $u=\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}$. Namely, $\sqrt{2n+1}\,C^n_{m,r}(u)$ is the coefficient of $X^{n+m}Y^{n-m}$ in $(aX-\bar{b}Y)^{n+r}(bX+\bar{a}Y)^{n-r}$. It is well known that $\{C^n_{m,r}(u);\ n=|m|, |m|+1, \cdots, r=-n, \cdots, n\}$ forms a complete orthonormal base of the Hilbert space $\mathfrak{H}^{2m}$.

LEMMA 13. *Set* $\Phi^{2m}_s(u)=|u_{21}|^{-2m-2+2s}u^{2m}_{21}$ *for* $u=\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix}$. *If* Re $s\geqq 1$,

$$\int_{PU(2)}\Phi^{2m}_s(vu^{-1})C^n_{m,r}(u)du$$
$$=(-1)^{n+m}\Big(\prod_{k=0}^{n+m-1}\frac{s-n+k}{s-m+k}\Big)\frac{1}{s+n}C^n_{-m,r}(v)\,.$$

PROOF. Making use of the change of variable: $u\mapsto uv$, we see that the above integral is equal to

$$\int_{PU(2)}u_{12}^{-2m}|u_{12}|^{2m+2s-2}C^n_{m,r}(uv)du$$
$$=\int_{PU(2)}u_{12}^{-2m}|u_{12}|^{2m+2s-2}\int_{|t|=1}C^n_{m,r}\Big(u\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}v\Big)t^{2m}dtdu\,,$$

(where dt is the normalized Haar measure on the unit circle)

$$=C^n_{-m,r}(v)\int_{PU(2)}u_{12}^{-2m}|u_{12}|^{2m+2s-2}C^n_{m,-m}(u)du\sqrt{2n+1}\,.$$

Set $u_{11}=\sqrt{1-t}\,e^{i\varphi_1}$ and $u_{12}=\sqrt{t}\,e^{i\varphi_2}$ $(0\leqq t\leqq 1,\ 0\leqq\varphi_1,\ \varphi_2\leqq 2\pi)$. Then $du=\frac{1}{4\pi^2}dtd\varphi_1d\varphi_2$ and

$$\sqrt{2n+1}\,C^n_{m,-m}(u)=t^m e^{2mi\varphi_2}\frac{d^{n+m}}{dt^{n+m}}(1-t)^{n+m}t^{n-m}\times(n+m)!^{-1}\,.$$

Thus the integral is equal to

$$C^n_{-m,r}(v)\{(n+m)!\}^{-1}\int_0^1 t^{m+s-1}\frac{d^{n+m}}{dt^{n+m}}(1-t)^{n+m}t^{n-m}dt$$
$$=C^n_{-m,r}(v)(-1)^{n+m}\Big(\prod_{k=0}^{n+m-1}\frac{s-n+k}{s-m+k}\Big)\frac{1}{s+n}\,.$$

Lemma 13 shows that if $m>0$, for a smooth function φ in $\mathfrak{H}^{2m}$ the integral $\int_{PU(2)}\Phi^{2m}_s(v^\sigma u^{-1})\varphi(u)du$, which is absolutely convergent for Re$s\geqq 1$, gives rise to a holomorphic function of s in the domain $\{s;\ \mathrm{Re}s>-|m|\}$. Set

(14) $$I^{(2m)}_\sigma(\varphi)(v)=(-1)^m m\int_{PU(2)}\Phi^{2m}_s(v^\sigma u^{-1})\varphi(u)du\Big|_{s=0}\,.$$

Since $C^n_{m,r}(u^\sigma)=(-1)^{m+r}C^n_{-m-r}(u)$, Lemma 13 shows that $I^{(2m)}_\sigma$ is a unitary operator of order 2 of $\mathfrak{H}^{2m}$. Furthermore, it is easy to see that $I^{(2m)}_\sigma R^{(2m,0)}(g)=R^{(2m,0)}(g^\sigma)I^{(2m)}_\sigma$ ($\forall g\in G_1$). We extend a representation $R^{(2m,0)}$ ($m>0$) of G_1 to a representation $T^{(2m,0)}_\pm$ of $G_1^\sim$ by setting

$$T^{(2m,0)}_\pm(g,\sigma)=\pm I^{(2m)}_\sigma R^{(2m,0)}(g)\,.$$

So far, we have proved the following proposition.

PROPOSITION 14. *Notations being as above, any non-trivial irreducible unitary representation of* $G_1^\sim$ *of the first kind is equivalent to* $T^{(0,\rho)}_\pm$ ($\rho\geqq 0$) *or* $T_{\tau\pm}$ ($0<\tau<2$) *or to* $T^{(2m,0)}_\pm$ ($m=1,2,\cdots$).

3. If T is an irreducible unitary representation of $G_1^\sim$ of the first kind, for any $f\in C_0^\infty(G_1)$, the linear operator $\int_{G_1}f(g)T(g,\sigma)dg$ is known to be of trace class. In the following we calculate the trace of this operator for each representation T. We use notations introduced in 2 without further comment.

PROPOSITION 15. (i) *Notations being as above, for* $f\in C_0^\infty(G_1)$, *the trace of the linear operator*

$$\int_G f(g)I_\sigma R^{(0,\rho)}(g)dg \tag{15}$$

on $\mathfrak{H}^{(0)}$ *is given by* $\int_{G_1}f(g)S^{(0,\rho)}(g)dg$, *where* $S^{(0,\rho)}$ *is a function on* G_1 *given as follows:*

$$S^{(0,\rho)}(g)=\begin{cases}\dfrac{t^{i\rho/2}+t^{-i\rho/2}}{|t-t^{-1}|}, & \textit{if } g \textit{ is } \sigma\textit{-twistedly conjugate to } a_t\ (t>0),\\ 0 & \textit{otherwise.}\end{cases}$$

(ii) *The trace of the linear operator* $\int_{G_1}f(g)I_\sigma R_\tau(g)dg$ *on* $\mathfrak{H}_\tau$ *is given by* $\int_{G_1}f(g)S_\tau(g)dg$, *where* $S_\tau(g)$ *is given by*

$$S_\tau(g)=\begin{cases}\dfrac{t^{\tau/2}+t^{-\tau/2}}{|t-t^{-1}|}, & \textit{if } g \textit{ is } \sigma\textit{-twistedly conjugate to } a_t\ (t>0).\\ 0 & \textit{otherwise.}\end{cases}$$

PROOF. (i) For $\varphi\in\mathfrak{H}^0$,

$$\begin{aligned}&\Big(\int_{G_1}f(g)I_\sigma R^{(0,\rho)}(g)dg\varphi\Big)(u)\\&=\int_{G_1}f(g)t(u^\sigma g)^{i\rho-2}\varphi(k(u^\sigma g))dg\\&=\int_{R_+\times C\times PU(2)}f((u^\sigma)^{-1}a_tn(z)v)t^{1-i\rho/2}\varphi(v)\,\frac{dt}{t}\,dzdv\end{aligned}$$

$$=\int_{PU(2)} K(u, v)\varphi(v)dv\,,$$

where $K(u, v)$ is a smooth function on $PU(2)\times PU(2)$ given by

$$K(u, v)=\int_{R_+\times C} f((u^\sigma)^{-1}a_t n(z)v)t^{-i\rho/2+1}\frac{dt}{t}dz\,.$$

Hence, the trace of the operator (15) is given by

$$\int_{PU(2)} K(u, u)du=\int_{R_+} t^{-i\rho/2}F^1_{f,\sigma}(a_t)\frac{dt}{t} \qquad \text{(see (3))}$$

$$=\int_G f(g)S^{(0,\rho)}(g)dg \qquad \text{(see Proposition 2).}$$

The second part of the proposition is proved in a similar manner.

For $z\in C$, put $\chi(z)=\exp 2\pi\sqrt{-1}\,\mathrm{Re}z$. For $f\in C_0^\infty(G_1)$ and $(x, y)\in C^2$, set

$$I_f(x, y)=\int_{G_1\times T} f(g)\chi(t\{(a\bar{x}+c\bar{y})y-(b\bar{x}+d\bar{y})x\})t^{-2m}dgdt$$

$\left(g=\begin{pmatrix} a & b \\ c & d\end{pmatrix},\ m=1, 2, \cdots\right)$, where T is the unit circle and dt is the normalized Haar measure on T.

PROPOSITION 16. *Notations being as above, the integral*

$$\int_{C\times C} I_f(x, y)dxdy$$

is absolutely convergent and is equal to the trace of the linear operator $\frac{1}{2}\int_{G_1} I_\sigma^{(2m)}R^{(2m,0)}(g)f(g)dg$ *on* $\mathfrak{H}^{2m}$.

PROOF. Set $c(s, m)=(-1)^m\frac{\pi^{-2s}}{2}\frac{\Gamma(s+m)}{\Gamma(1+m-s)}$. Then

$$\int_T \chi(tz)t^{-2m}dt=\frac{z^{2m}|z|^{-2m}}{\pi i}\int_{\mathrm{Re}s=\sigma_0} c(s, m)|z|^{-2s}ds\,,$$

$(-m<\sigma_0<0)$, the integral is absolutely convergent. Put $(x, y)=r(0, 1)u$ $(r\geqq 0, u\in SU(2))$. We denote by u' the image of u in $SU(2)/\pm 1$. We have

$$I_f(x, y)=\frac{1}{\pi i}\int_{\mathrm{Re}s=\sigma_0} c(s, m)\left\{\int_{PU(2)} F_f(v, u', s)r^{-4s}\times\Phi^{2m}_{1-s}(u'^\sigma v^{-1})dv\right\}ds\,,$$

where we put

$$F_f(v, u', s)=\int_{R_+\times C} f\left(v^{-1}\begin{pmatrix} 1 & z \\ & 1\end{pmatrix}a_t u'\right)t^{-s}\frac{dt}{t}dz$$

and

$$\Phi^{2m}_s(u)=u_{21}^{2m}|u_{21}|^{-2m-2+2s}\,.$$

Set

$$\Phi_f^{(2m)}(u', s)=\int_{PU(2)} F_f(v, u', s)\Phi_{1-s}^{2m}(u'^{\sigma}v^{-1})dv .$$

It follows from Lemma 13 that

$$\Phi_f^{(2m)}(u', s)=\sum_{\substack{n\geqq m\\ n\geqq |k|}}\int_{PU(2)} F_f(v, u', s)\overline{C_{m,k}^{n}(v)}dvC_{-m,k}^{n}(u'^{\sigma})a_{m,n}(1-s) ,$$

where $a_{m,n}(s)=(-1)^{n+m}\left(\prod_{k=0}^{n+m-1}\frac{s-n+k}{s-m+k}\right)\frac{1}{s+n}$.

It is now easy to see that for any polynomial $P(s)$ in s, $|\Phi_f^{(2m)}(u', s)P(s)|$ is bounded, uniformly with respect to u', when s remains in the strip $\left\{s\in \boldsymbol{C};\ -1<\mathrm{Re}\, s<\frac{3}{2}\right\}$. We have, for $(x, y)=r(0, 1)u$,

$$I_f(x, y)=\frac{1}{\pi i}\int_{\mathrm{Re}s=\sigma_0} c(s, m)\Phi_f^{(2m)}(u', s)r^{-4s}ds .$$

Thus, the function $I_f(x, y)$ is integrable on $\boldsymbol{C}\times\boldsymbol{C}$ and

$$\begin{aligned}&\int_{C\times C} I_f(x, y)dxdy\\ &=\frac{2\pi^2}{\pi i}\int_0^{\infty} r^{3-4s}dr\left[\int_{\mathrm{Re}s=\sigma_0} c(s, m)\left\{\int_{PU(2)}\Phi_f^{(2m)}(u', s)du'\right\}ds\right]\\ &=\frac{2\pi^2}{\pi i}\frac{2\pi i}{4}c(1, m)\int_{PU(2)}\Phi_f^{(2m)}(u', 1)du'\\ &=\frac{1}{2}\ \mathrm{trace}\int_{G_1} I_{\sigma}^{(2m)}R^{(2m,0)}(g)f(g)dg .\end{aligned}$$

PROPOSITION 17. *Assume that the support of $f\in C_0^{\infty}(G_1)$ is contained in the following set:*

$$G'_{\sigma}=\{g\in G_1\, ;\ \mathrm{tr}\, g^{\sigma}g\neq\pm 2\} . \tag{16}$$

Then

$$\int_{C\times C} I_f(x, y)dxdy=\frac{1}{2}\int_{G_1} f(g)S^{(2m,0)}(g)dg ,$$

where

$$S^{(2m,0)}(g)=\begin{cases} -\dfrac{e^{mi\theta}-e^{-mi\theta}}{e^{i\theta}-e^{-i\theta}} & \textit{if } g \textit{ is } \sigma\textit{-twistedly conjugate to } k_{\theta},\\[2ex] \dfrac{2t^{-m}}{t-t^{-1}} & \textit{if } g \textit{ is } \sigma\textit{-twistedly conjugate to } a_t\ (t>1). \end{cases}$$

PROOF. It follows from Proposition 16 that

$$\begin{aligned}&\int_{C\times C} I_f(x, y)dxdy\\ &=\lim_{\varepsilon\downarrow 0}\int_{C\times C}\exp\{-\pi\varepsilon(|x|^2+|y|^2)\}I_f(x, y)dxdy .\end{aligned}$$

We note that for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G_1$,

$$\varepsilon(|x|^2+|y|^2)-2i\mathrm{Re}t\{(a\bar{x}+c\bar{y})y-(b\bar{x}+d\bar{y})x\}$$
$$=(\varepsilon+ibt+i\bar{b}t^{-1})|x|^2-i(at-\bar{d}t^{-1})\bar{x}y-i(\bar{a}t^{-1}-dt)x\bar{y}+$$
$$+(\varepsilon-ict-i\bar{c}t^{-1})|y|^2.$$

Changing the orders of integrations, we have

$$\int_{C\times C} I_f(x, y)dxdy$$
$$=\lim_{\varepsilon\downarrow 0}\int_{G_1}\int_T \frac{f(g)t^{-2m}}{\{\varepsilon^2+i\varepsilon(bt+\bar{b}t^{-1}-ct-\bar{c}t^{-1})+trg^\sigma g-t^2-t^{-2}\}}dgdt.$$

We note that the set of roots of the following equation in t is invariant under the transformation: $t\mapsto-(\bar{t})^{-1}$.

(17) $$t^4-i\varepsilon(\bar{b}-\bar{c})t^3-(\varepsilon^2+\mathrm{tr}\, g^\sigma g)t^2-i\varepsilon(b-c)t+1=0.$$

For $\varepsilon>0$, the above equation has no root on the unit circle. Denote by $\{\lambda_\varepsilon(g),\ \mu_\varepsilon(g),\ -\bar{\lambda}_\varepsilon^{-1}(g),\ -\bar{\mu}_\varepsilon^{-1}(g)\}$ $(|\lambda_g(\varepsilon)|<1,\ |\mu_g(\varepsilon)|<1)$ the set of roots of the above equation (17). By the assumption, if g is in the support of f, the matrix $g^\sigma g$ has two different eigenvalues $\{e^{i\theta}, e^{-i\theta}\}$ $(\sin\theta\neq 0)$ or $\{\lambda, \lambda^{-1}\}$ $(\lambda>1)$. In the former case, we may assume

$$\lim_{\varepsilon\downarrow 0}(\lambda_\varepsilon(g),\ \mu_\varepsilon(g))=(e^{i\theta/2},\ e^{-i\theta/2})$$

or

$$\lim_{\varepsilon\downarrow 0}(\lambda_\varepsilon(g),\ \mu_\varepsilon(g))=(e^{i\theta/2},\ -e^{-i\theta/2}).$$

In the latter case, we may assume that $\lim_{\varepsilon\downarrow 0}(\lambda_\varepsilon(g),\ \mu_\varepsilon(g))=(\sqrt{\lambda}^{-1},\ -\sqrt{\lambda}^{-1})$. Since the support of f is a compact subset of (16), there exists a positive number η such that $\lambda_\varepsilon(g)\neq\mu_\varepsilon(g)$ if $0<\varepsilon\leqq\eta$ and $f(g)\neq 0$. Thus, for $0<\varepsilon\leqq\eta$, if $f(g)\neq 0$,

$$\int_T \frac{t^{-2m}dt}{\{\varepsilon^2+i\varepsilon(bt+\bar{b}t^{-1}-ct-\bar{c}t^{-1})+\mathrm{tr}\, g^\sigma g-(t^2+t^{-2})\}}$$
$$=-\frac{2\pi}{2\pi}\left\{\frac{\lambda_\varepsilon(g)^{2m+1}}{(\lambda_\varepsilon(g)+\bar{\lambda}_\varepsilon^{-1}(g))(\lambda_\varepsilon(g)-\mu_\varepsilon(g))(\lambda_\varepsilon(g)+\bar{\mu}_\varepsilon^{-1}(g))}\right.$$
$$\left.+\frac{\mu_\varepsilon(g)^{2m+1}}{(\mu_\varepsilon(g)-\lambda_\varepsilon(g))(\mu_\varepsilon(g)+\bar{\lambda}_\varepsilon^{-1}(g))(\mu_\varepsilon(g)+\bar{\mu}_\varepsilon^{-1}(g))}\right\}.$$

Thus,

$$I_f(x, y)=\frac{1}{2}\int_{G_1} f(g)S^{(2m,0)}(g)dg$$

under our assumption that the support of f is contained in the set (16).

PROPOSITION 18. *For $m=1, 2, \cdots$, the trace of linear operator*

$$\int_{G_1} f(g) I_\sigma^{2m} R^{(2m,0)}(g) dg$$

on $\mathfrak{H}^{2m}$ ($f \in C_0^\infty(G)$) is given by $\int_{G_1} f(g) S^{(2m,0)}(g) dg$.

PROOF. Set

$$T_1(f) = \operatorname{trace} \int_{G_1} f(g) I_\sigma^{2m} R^{(2m,0)}(g) dg ,$$

$$T_2(f) = \int_{G_1} f(g) S^{(2m,0)}(g) dg .$$

Then, both T_1 and T_2 are σ-twistedly invariant distributions on G_1 (see 3, § 1).

Moreover, by Proposition 16 and Proposition 17, the support of $T_1 - T_2$ is contained in the set $S_\sigma = \{g \in G ; \operatorname{tr} g^\sigma g = \pm 2\}$. Let Ω be the Casimir operator on G_1 given by (10). It is well-known that $\Omega T_1 = 4m^2 T_1$. We will show that $\Omega T_2 = 4m^2 T_2$. By Proposition 2, we have

$$T_2(\Omega f) = \int_1^\infty \frac{(t-t^{-1})}{t} \frac{2t^{-m}}{t-t^{-1}} F^1_{\Omega f,\sigma}(t) dt$$

$$+ \frac{4}{2\pi} \int_0^{2\pi} \sin\theta \Big(-\frac{\sin m\theta}{\sin\theta} \Big) F^2_{\Omega f,\sigma}(\theta) d\theta .$$

By Proposition 6,

$$F^1_{\Omega f,\sigma}(a_t) = 4\Big(t\frac{d}{dt} \Big)^2 F^1_{f,\sigma}(t)$$

and

$$F^2_{\Omega f,\sigma}(k_\theta) = -4 \frac{d^2}{d\theta^2} F^2_{f,\sigma}(\theta) .$$

Hence, in view of Proposition 1 (i),

$$2\int_1^\infty t^{-m} F^1_{\Omega f,\sigma}(t) \frac{dt}{t} = 8m \int_1^\infty t^{-m} \Big(t\frac{t}{dt} \Big) F^1_{f,\sigma}(t) \frac{dt}{t}$$

$$= 8(m^2) \int_1^\infty t^{-m} F^1_{f,\sigma}(t) - 8m \cdot 4\delta_\sigma^3(f) .$$

On the other hand, by (ii) of Proposition 1,

$$-\frac{2}{\pi} \int_0^{2\pi} \sin m\theta F^2_{\Omega f,\sigma}(\theta) d\theta = -\frac{8m}{\pi} \int_0^{2\pi} \cos m\theta \frac{d}{d\theta} F^2_{f,\sigma}(\theta) d\theta$$

$$= -\frac{8m}{\pi} (-2\pi - 2\pi) \delta_\sigma^3(f) - \frac{8m^2}{\pi} \int_0^{2\pi} \sin m\theta F^2_{f,\sigma}(\theta) d\theta .$$

Thus, again by Proposition 2,

$$T_2(\Omega f)=8m^2\int_1^\infty t^{-m}F^1_{f,\sigma}(t)\frac{dt}{t}-\frac{8m^2}{\pi}\int_0^{2\pi}\sin m\theta F^2_{f,\sigma}(\theta)d\theta$$

$$=4m^2T_2(f)\,.$$

Hence, T_1-T_2 is a σ-twistedly G-invariant distribution on G_1 with support concentrated to the set S_σ and satisfies the differential equation $\Omega(T_1-T_2)=4m^2(T_1-T_2)$. Thus, by Corollary to Lemma 12, $T_1=T_2$.

4. For an $f\in C_0^\infty(G_1)$, set

$$S_n(f)=\mathrm{trace}\int_{G_1}f(g)I_\sigma^{(2n)}R^{(2n,0)}(g)dg \qquad (n=1,2,\cdots)$$

and

$$T_\lambda(f)=\mathrm{trace}\int_{G_1}f(g)I_\sigma R^{(2\lambda i,0)}(g)dg\,.$$

The following formula is quite analogous to the Plancherel formula for $SL(2, \boldsymbol{R})$.

PROPOSITION 19. *Notations being as above, we have*

$$4\pi\delta^1_\sigma(f)=\sum_{n=1}^\infty nS_n(f)+\frac{1}{2}\int_{-\infty}^\infty\lambda\frac{ch\,\pi\lambda}{sh\,\pi\lambda}T_\lambda(f)d\lambda \qquad (\text{cf. (5)})\,.$$

PROOF. It follows from Proposition 18, Proposition 17 and Proposition 2 that

$$\sum_{n=1}^N nS_n(f)=-\frac{4}{2\pi}\int_0^{2\pi}\sum_{n=1}^N n\sin n\theta F^2_{f,\sigma}(\theta)d\theta+2\int_1^\infty\sum_{n=1}^N nt^{-n}F^1_{f,\sigma}(t)\frac{dt}{t}$$

$$=\frac{1}{\pi}\int_0^{2\pi}F^2_{f,\sigma}(\theta)\left\{\frac{d}{d\theta}\frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right\}d\theta$$

$$-\int_1^\infty F^1_{f,\sigma}(t)\frac{d}{dt}\frac{1+t^{-1}-2t^{-N-1}}{1-t^{-1}}dt\,.$$

Proposition 1 suggests that

$$\int_0^{2\pi}F^2_{f,\sigma}(\theta)\left\{\frac{d}{d\theta}\frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right\}d\theta$$

$$=-4\pi(2N+1)\delta^3_\sigma(f)-\int_0^{2\pi}\left(\frac{d}{d\theta}F^2_{f,\sigma}(\theta)\right)\frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}d\theta$$

and

$$\int_1^\infty F^1_{f,\sigma}(t)\frac{d}{dt}\frac{1+t^{-1}-2t^{-N-1}}{1-t^{-1}}dt$$

$$=-(2N+1)4\delta^3_\sigma(f)-\int_1^\infty\Big(\frac{d}{dt}F^1_{f,\sigma}(t)\Big)\frac{1+t^{-1}-2t^{-N-1}}{1-t^{-1}}dt\,.$$

Thus, we have

$$\sum_{n=1}^N nS_n(f)=-\frac{1}{\pi}\int_0^{2\pi}\frac{\sin\Big(N+\dfrac{1}{2}\Big)\theta}{\sin\dfrac{\theta}{2}}\frac{d}{d\theta}F^2_{f,\sigma}(\theta)d\theta$$

$$+\int_1^\infty\frac{1+t^{-1}-2t^{-N-1}}{1-t^{-1}}\frac{d}{dt}F^1_{f,\sigma}(t)dt\,.$$

It follows from Proposition 1 that

$$\sum_{n=1}^\infty nS_n(f)=4\pi\delta^1_\sigma(f)+\int_1^\infty\frac{1+t^{-1}}{1-t^{-1}}\frac{d}{dt}F^1_{f,\sigma}(t)dt\,.$$

In view of Proposition 15, we have

$$T_\lambda(f)=\int_0^\infty F^1_{f,\sigma}(t)t^{i\lambda}\frac{dt}{t}\,.$$

Hence,

$$\int_1^\infty\frac{1+t^{-1}}{1-t^{-1}}\frac{d}{dt}F^1_{f,\sigma}(t)dt=-\frac{1}{2}\int_{-\infty}^\infty\lambda\frac{ch\,\pi\lambda}{sh\,\pi\lambda}T_\lambda(f)d\lambda\,.$$

Thus we get the Proposition.

5. Set $G=GL(2,\boldsymbol{C})$ and let $\tilde{G}$ be the semi-direct product of G with $\langle\sigma\rangle$ (for details, see the introduction). We are going to construct (up to equivalence) all the irreducible unitary representations of the first kind of $\tilde{G}$. In the following we use notations in § 2, 2. without further comment. Let $L^2(\boldsymbol{C})$ be the Hilbert space of square-integrable functions on $\boldsymbol{C}$. We denote by $X(\boldsymbol{R}^\times)$ the character group of $\boldsymbol{R}^\times$. For $(\mu_1,\mu_2)\in X(\boldsymbol{R}^\times)\times X(\boldsymbol{R}^\times)$, let $\Pi_\pm^{(\mu_1,\mu_2)}$ be the representation of $\tilde{G}$ on $L^2(\boldsymbol{C})$ given by the following formula:

$$\Pi_\pm^{(\mu_1,\mu_2)}(g,1)f(z)$$

$$=f\Big(\frac{az+c}{bz+d}\Big)\mu_1\Big(\Big|\frac{ad-bc}{bz+d}\Big|^2\Big)\mu_2(|bz+d|^2)\frac{|ad-bc|}{|bz+d|^2}\quad\Big(g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G\Big),$$

$$\Pi_\pm^{(\mu_1,\mu_2)}(g,\sigma)=\pm J_\sigma\Pi_\pm^{(\mu_1,\mu_2)}(g,1)\,,$$

where J_σ is a unitary operator on $L^2(\boldsymbol{C})$ given by $(J_\sigma f)(z)=f(z^\sigma)$. Denote by $T_\pm^{(\mu_1,\mu_2)}$ the representation of $\tilde{G}$ given by

$$T_\pm^{(\mu_1,\mu_2)}(g,\lambda)=(\mu_1\mu_2)^{-1}(|\det g|)\Pi_\pm^{(\mu_1,\mu_2)}(g,\lambda)\qquad(g\in G,\ \lambda\in\langle\sigma\rangle)\,.$$

The representation $T_\pm^{(\mu_1,\mu_2)}$ is naturally regarded as a representation of $\tilde{G}_1$. Take a real number ρ such that $\mu_1^{-1}\mu_2(t)=(t)^{i\rho}$ $(t>0)$. Denote by $M_{i\rho}$ a linear mapping from $\mathfrak{H}^0$ into $L^2(\boldsymbol{C})$ given by the following formula:

$$M_{i\rho}(F)(z)=\frac{1}{\sqrt{\pi}}(1+|z|^2)^{i\rho-1}F(u(z)) \quad (F\in\mathfrak{H}^0),$$

where $u(z)=\begin{pmatrix}\bar{\alpha} & -\bar{\beta}\\ \beta & \alpha\end{pmatrix}$ $\left(\alpha=\frac{1}{\sqrt{1+|z|^2}},\ \beta=\frac{z}{\sqrt{1+|z|^2}}\right)$.

It is easy to see that $M_{i\rho}$ is an isometric linear mapping from $\mathfrak{H}^0$ onto $L^2(\boldsymbol{C})$ such that $M_{i\rho}(I_\sigma)=J_\sigma M_{i\rho}$ and

$$M_{i\rho}T_{\pm}^{(0,2\rho)}(g)=T_{\pm}^{(\mu_1,\mu_2)}(g)M_{i\rho} \quad (\forall g\in\tilde{G}_1).$$

For a positive integer m, denote by $M^{(m)}$ the linear mapping from $\mathfrak{H}^{2m}$ into $L^2(\boldsymbol{C})$ given by the following formula:

$$M^{(m)}F(z)=\frac{1}{\sqrt{\pi}}(1+|z|^2)^{-1}F(u(z)) \qquad (F\in\mathfrak{H}^{2m}).$$

Let $J_\sigma^{(2m)}$ be the linear operator on $L^2(\boldsymbol{C})$ given by the following:

$$J_\sigma^{(2m)}f(z)=\frac{(-1)^m m}{\pi}\lim_{s\to+0}\int_{\boldsymbol{C}}|z^\sigma-w|^{-2m-2+s}(z^\sigma-w)^{2m}f(w)dw.$$

It is easy to see that $M^{(m)}$ is an isometric linear mapping from $\mathfrak{H}^{2m}$ onto $L^2(\boldsymbol{C})$ and that $J_\sigma^{(2m)}M^{(m)}=M^{(2m)}I_\sigma^{(2m)}$ (cf. (14)).

Hence, $J_\sigma^{(2m)}$ is a unitary operator of order 2 on $L^2(\boldsymbol{C})$. For a $\mu\in X(\boldsymbol{R}^\times)$ and a positive integer m, let $\Pi_{m\pm}^\mu$ be the representation of $\tilde{G}$ on $L^2(\boldsymbol{C})$ given by the following formula:

$$\Pi_{m\pm}^\mu(g,1)f(z)$$
$$=\mu(|ad-bc|^2)\frac{(ad-bc)^m}{|ad-bc|^{m-1}}\frac{|bz+d|^{2m-2}}{(bz+d)^{2m}}f\left(\frac{az+c}{bz+d}\right) \quad \left(g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G\right),$$

$$\Pi_{m\pm}^\mu(g,\sigma)=\pm J_\sigma^{(2m)}\Pi_{m\pm}^\mu(g,1).$$

Denote by $T_{m\pm}^\mu$ the representation of $\tilde{G}$ given by

$$T_{m\pm}^\mu(g,\lambda)=\mu^{-1}(|\det g|^2)\Pi_{m\pm}^\mu(g,\lambda) \qquad (g\in G,\ \lambda\in\langle\sigma\rangle).$$

The representation $T_{m\pm}^\mu$ is naturally regarded as a representation of the group $\tilde{G}_1$. Moreover, it is easy to see that

$$T_{m\pm}^\mu(g)M^{(m)}=M^{(m)}T_{\pm}^{(2m,0)}(g) \qquad (\forall g\in\tilde{G}_1).$$

For each $\tau(0<\tau<1)$, denote by H_τ the space of measurable functions on $\boldsymbol{C}$ such that

$$\int_{\boldsymbol{C}^2}|z_1-z_2|^{2(\tau-1)}f(z_1)\overline{f(z_2)}dz_1dz_2<\infty.$$

Then the space H_τ is a Hilbert space with the inner product

$$(f_1, f_2)=\int_{C^2}|z_1-z_2|^{2(\tau-1)}f_1(z_1)\overline{f_2(z_2)}dz_1dz_2 .$$

For a $\mu\in X(\boldsymbol{R}^\times)$ and a τ $(0<\tau<1)$, denote by $\Pi^\mu_{\tau\pm}$ the representation of $\tilde{G}$ on H_τ given by the following formula:

$$\Pi^\mu_{\tau\pm}(g, 1)f(z)$$

$$=\mu(|ad-bc|^2)\left|\frac{ad-bc}{(bz+d)^2}\right|^{\tau+1}f\Big(\frac{az+c}{bz+d}\Big) \qquad \Big(g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in G\Big),$$

$$\Pi^\mu_{\tau\pm}(g, \sigma)=\pm J_\sigma \Pi^\mu_{\tau\pm}(g, 1) .$$

Set

$$\Pi^\mu_{\tau\pm}(g, \lambda)=\mu(|\det g|^2)T^\mu_{\tau\pm}(g, \lambda) \qquad (g\in G,\ \lambda\in\langle\sigma\rangle) .$$

Then $T^\mu_{\tau\pm}$ is naturally regarded as a representation of $\tilde{G}_1$. Let L_τ be a linear mapping from $\mathfrak{H}_{2\tau}$ into H_τ given by

$$L_\tau F(z)=\frac{1}{\sqrt{\pi}}(1+|z|^2)^{-1-\tau}F(u(z)) \qquad (F\in\mathfrak{H}_{2\tau}) .$$

Then it is easy to see that L_τ is an isometric linear mapping from $\mathfrak{H}_{2\tau}$ onto H_τ which satisfies

$$L_\tau T_{2\tau\pm}(g)=T^\mu_{\tau\pm}(g)L_\tau \qquad (\forall g\in\tilde{G}_1) .$$

The next proposition is now an easy consequence of Prop. 14, Prop. 15, Prop. 18 and the classification theory of irreducible unitary representations of $GL(2, \boldsymbol{C})$.

PROPOSITION 20. *Let notations be as above,*

(i) *Any infinite dimensional irreducible unitary representation T of $\tilde{G}$ of the first kind is equivalent to a suitable $\Pi^{(\mu_1,\mu_2)}_\pm$ $((\mu_1, \mu_2)\in X(\boldsymbol{R}^\times)^2)$ or to $\Pi^\mu_{m\pm}$ $(\mu\in X(\boldsymbol{R}^\times)$, $m=1, 2, \cdots)$ or to $\Pi^\mu_{\tau\pm}$ $(\mu\in X(\boldsymbol{R}^\times)$, $0<\tau<1)$.*

(ii) *For each $F\in C_0^\infty(G)$, the trace of the operator $\int_G F(g)T(g, \sigma)dg$ is equal to $\int_G F(g)$*trace*$T(g, \sigma)dg$, where* trace$T(.\,, \sigma)$ *is a locally integrable function on G given by the following formulas:*

a) *If $T=\Pi^{(\mu_1,\mu_2)}_\pm$,*

$$\operatorname{trace} T(g, \sigma)=\begin{cases} \pm|\det g|\dfrac{\mu_1(\lambda_1)\mu_2(\lambda_2)+\mu_1(\lambda_2)\mu_2(\lambda_1)}{|\lambda_1-\lambda_2|}, & \text{if } gg^\sigma \text{ has} \\ & \text{distinct positive eigenvalues } \lambda_1 \text{ and } \lambda_2 \\ 0, & \text{otherwise.} \end{cases}$$

b) *If $T=\Pi^\mu_{\tau\pm}$*

$$\operatorname{trace} T(g,\sigma)=\begin{cases}\pm\mu(|\det g|^2)|\det g|^{1-\tau}\dfrac{\lambda_1^\tau+\lambda_2^\tau}{|\lambda_1-\lambda_2|}, & \textit{if } gg^\sigma \textit{ has two distinct positive eigenvalues } \lambda_1 \textit{ and } \lambda_2,\\ 0, & \textit{otherwise.}\end{cases}$$

c) *If* $T=\Pi^\mu_{m,\pm}$,

$$\operatorname{trace} T(g,\sigma)=\begin{cases}\pm\mu(|\det g|^2)|\det g|^{1-m}\dfrac{2\lambda_1^m}{|\lambda_2-\lambda_1|}, & \textit{if } gg^\sigma \textit{ has two distinct positive eigenvalues } \lambda_1 \textit{ and } \lambda_2\ (\lambda_1<\lambda_2)\\ \pm\mu(|\det g|^2)\Big(-\dfrac{\sin m\theta}{\sin\theta}\Big), & \textit{if } gg^\sigma \textit{ has complex eigenvalues } re^{i\theta},\ re^{-i\theta}.\end{cases}$$

6. Let us recall a description of irreducible unitary representations of $G_{\boldsymbol{R}}=GL(2,\boldsymbol{R})$. For $(\mu_1,\mu_2)\in X(\boldsymbol{R}^\times)^2$, $r^{(\mu_1,\mu_2)}$ is a representation of $G_{\boldsymbol{R}}$ on $L^2(\boldsymbol{R})$ (=the Hilbert space of square integrable functions on $\boldsymbol{R}$) given by

$$r^{(\mu_1,\mu_2)}(g)f(x)$$
$$=\mu_1\Big(\frac{ad-bc}{bx+d}\Big)\mu_2(bx+d)\frac{|ad-bc|^{1/2}}{|bx+d|}f\Big(\frac{ax+c}{bx+d}\Big)\qquad\Big(g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G_{\boldsymbol{R}}\Big).$$

For a τ $(0<\tau<1)$, let h_τ be the space of measurable functions on $\boldsymbol{R}$ such that

$$\int_{\boldsymbol{R}\times\boldsymbol{R}}|x_1-x_2|^{\tau-1}f(x_1)\overline{f(x_2)}\,dx_1dx_2<\infty\,.$$

Then h_τ is a Hilbert space with the inner product

$$(f_1,f_2)=\int_{\boldsymbol{R}\times\boldsymbol{R}}|x_1-x_2|^{\tau-1}f_1(x_1)\overline{f_2(x_2)}\,dx_1dx_2\,.$$

For a $\mu\in X(\boldsymbol{R}^\times)$ and a τ $(0<\tau<1)$, the representation r^μ_τ of $G_{\boldsymbol{R}}$ on h_τ is given by

$$r^\mu_\tau(g)f(x)=\mu(ad-bc)\Big|\frac{ad-bc}{(bx+d)^2}\Big|^{(\tau+1)/2}f\Big(\frac{ax+c}{bx+d}\Big)\qquad\Big(g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in G_{\boldsymbol{R}}\Big).$$

For a positive integer m, denote by $\mathcal{H}_m$ the space of holomorphic functions on $\boldsymbol{C}-\boldsymbol{R}$ such that

$$\int_{\boldsymbol{C}}|f(z)|^2|\operatorname{Im} z|^{m-1}dz<\infty\,.$$

For $\mu\in X(\boldsymbol{R}^\times)$ and a positive integer m, the representation $r^{\mu,m}$ of $G_{\boldsymbol{R}}$ on $\mathcal{H}_m$ is given by

$$r^{\mu,m}(g)f(z)=\mu(ad-bc)|ad-bc|^{(m+1)/2}f\Big(\frac{az+c}{bz+d}\Big)(bz+d)^{-m-1}\,.$$

The next theorem is now an immediate consequence of Prop. 20 and the well-known character formulas for irreducible unitary representations of $GL(2, \boldsymbol{R})$.

We employ notations in the introduction, § 2, 4. and in § 2, 5.

THEOREM. *For each irreducible unitary representation T of $\tilde{G}$ of the first kind, there exists an irreducible unitary representation r of $G_{\boldsymbol{R}}$ such that*

$$\operatorname{trace} T(g, \sigma) = \varepsilon \operatorname{trace} r(gg^{\sigma}) \qquad (\forall g \in GL(2, \boldsymbol{C}))\,,$$

where $\varepsilon = \pm 1$ does not depend upon g.

More precisely, for $T = \Pi_{\pm}^{(\mu_1, \mu_2)}$, one may put $\varepsilon = \pm 1$, $r = r^{(\mu_1, \mu_2)}$, for $T = \Pi_{m\pm}^{\mu}$, one may put $\varepsilon = \pm 1$, $r = r^{\mu, m}$ and for $T = \Pi_{\tau\pm}^{(\mu)}$ one may put $\varepsilon = \pm 1$, $r = r_{\tau}^{\mu}$.

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