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# Riemannian submersions and critical Riemannian metrics

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**Introduction.**

In the present paper we consider only Riemannian submersions $\pi:(\tilde{M},\tilde{g})\rightarrow(B,{}^{B}g)$ such that fibers $F$ are complete and connected and imbedded in $(\tilde{M},\tilde{g})$ regularly as totally geodesic submanifolds.

There are many examples of Riemannian manifolds $(\tilde{M},\tilde{g})$ any of which admits such a submersion and at the same time $\tilde{g}$ is a critical Riemannian metric on $\tilde{M}$. But it is in general not true that ${}^{B}g$ is a critical Riemannian metric on $B$ although $\tilde{g}$ is a critical Riemannian metric on $\tilde{M}$. Nevertheless there exist some cases in which $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics on $\tilde{M}$ and $B$ simultaneously. A Sasakian manifold $(\tilde{M},\tilde{g},\tilde{\xi})$ admits a Riemannian submersion (Sasakian submersion) $\pi:(\tilde{M},\tilde{g},\tilde{\xi})\rightarrow(B,{}^{B}g)$ such that, if ${}^{B}g$ is an Einstein metric satisfying a subsidiary condition, $\tilde{g}$ and ${}^{B}g$ are simultaneously critical Riemannian metrics or not. If $N$ is a certain integer an $N$-dimensional sphere $(S^{N},g_{0})$ with the standard Riemannian metric $g_{0}$ admits one or more Riemannian submersions $\pi:(S^{N},g_{0})\rightarrow(B,{}^{B}g)$ [**2**] and ${}^{B}g$ is always a critical Riemannian metric on $B$.

On any $C^{\infty}$ complete manifold $M$ there can exist various Riemannian metrics $g$. Among them a critical Riemannian metric is defined as follows if $M$ is compact orientable. Let $\mathcal{M}(M)$ be the space of $C^{\infty}$ Riemannian metrics $g$ on $M$ satisfying

$$\int_{M}dV_{g}=1$$

where $dV_{g}$ is the volume element measured by $g$. Consider a point $g\in\mathcal{M}(M)$. Let $f(K)$ be a scalar field on $M$ determined by $g$ as the contraction of a tensor product of the curvature tensor. Then

$$F_{M}[g]=\int_{M}f(K)dV_{g}$$

defines a mapping $F:\mathcal{M}(M)\rightarrow\boldsymbol{R}$. A critical point of $F$ is denoted by $g_{F}$ and is called a critical Riemannian metric with respect to the field $f(K)$ or the integral $F_M[g]$. Thus, following M. Berger [1] we have four kinds of critical Riemannian metrics $g_A, g_B, g_C$ and $g_D$ as the most prominent ones. The corresponding integrals are

$$A_M[g]=\int_M K dV_g\,,\qquad B_M[g]=\int_M K^2 dV_g\,,$$

$$C_M[g]=\int_M K_{ji}K^{ji} dV_g\,,\qquad D_M[g]=\int_M K_{kjih}K^{kjih} dV_g$$

where $K$ is the scalar curvature, $K_{ji}$ is the Ricci tensor and $K_{kjih}$ is the curvature tensor expressed in components. The equations of the critical Riemannian metrics were obtained by M. Berger which we can write in the following form in tensor notations,

$$A_{ji}=c_A g_{ji}\,,\qquad B_{ji}=c_B g_{ji}\,,\qquad C_{ji}=c_C g_{ji}\,,\qquad D_{ji}=c_D g_{ji}\,, \tag{0.1}$$

where $c_A$, $c_B$, $c_C$, $c_D$ are undetermined constants and $A_{ji}$, $B_{ji}$, $C_{ji}$, $D_{ji}$ are given by

$$A_{ji}=-K_{ji}+\frac{1}{2}Kg_{ji}\,, \tag{0.2A}$$

$$B_{ji}=2\nabla_j\nabla_i K-2\nabla_t\nabla^t K g_{ji}-2KK_{ji}+\frac{1}{2}K^2 g_{ji}\,, \tag{0.2B}$$

$$C_{ji}=\nabla_j\nabla_i K-\nabla_t\nabla^t K_{ji}-\frac{1}{2}\nabla_t\nabla^t K g_{ji} -2K_{jtsi}K^{ts}+\frac{1}{2}K_{ts}K^{ts}g_{ji}\,, \tag{0.2C}$$

$$D_{ji}=2\nabla_j\nabla_i K-4\nabla_t\nabla^t K_{ji}+4K_{jt}K_i{}^t -4K_{jtsi}K^{ts}-2K_{jtsr}K_i{}^{tsr} +\frac{1}{2}K_{tsrq}K^{tsrq}g_{ji}\,. \tag{0.2D}$$

REMARK. $\nabla$ means covariant differentiation with respect to the Riemannian connection induced by $g$. When $T$ and $S$ are tensor fields, $\nabla TS$ means in the present paper $(\nabla T)\otimes S$.

Although critical Riemannian metrics were first defined on a compact manifold, it is easy to generalize the definition when $M$ is not compact. We only need to consider variations of the Riemannian metric with compact support. The resulting equations are the same as the foregoing ones and (0.1) and (0.2) are valid.

We get for example following results.

Let $\pi:(\tilde{M}, \tilde{g}, \tilde{\xi})\rightarrow(B, {}^{B}g)$ be a Sasakian submersion. If $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics $g_D$ on $\tilde{M}$ and $B$ respectively, then the scalar curvature $\tilde{K}$ is constant and $(B, {}^{B}g)$ is an Einstein manifold satisfying

$$ {}^{B}K_{dcba}{}^{B}K^{dcba}=6(\tilde{n}+2){}^{B}K-2(3\tilde{n}+2)(\tilde{n}^{2}-1) $$

where the first member is the square of the curvature tensor of $(B, {}^{B}g)$, ${}^{B}K$ the scalar curvature and $\tilde{n}=\dim\tilde{M}$. Conversely let us consider the case where $(B, {}^{B}g)$ is an Einstein manifold satisfying the above equation. If one of $\tilde{g}$ and ${}^{B}g$ is a critical Riemannian metric $g_D$, then the other is also a critical Riemannian metric $g_D$.

The standard Riemannian metrics on $CP(n)$ and $QP(n)$ are critical Riemannian metrics in the sense $A$, $B$, $C$, $D$.

§ 1 is devoted to a summary of known results concerning general properties of Riemannian submersions with totally geodesic fibers. In § 2 a Sasakian manifold $(M, g, \xi)$ is studied when $g$ turns out to be a critical Riemannian metric. In § 3 we study a Riemannian submersion $\pi:(\tilde{M}, \tilde{g}, \tilde{\xi})\rightarrow(B, {}^{B}g)$ where $(\tilde{M}, \tilde{g}, \tilde{\xi})$ is a Sasakian manifold for the purpose of finding the cases where $\tilde{g}$ and ${}^{B}g$ become critical Riemannian metrics simultaneously. § 4 is devoted to the study of Riemannian submersions $\pi:(S^{N}, g_{0})\rightarrow(B, {}^{B}g)$ for the purpose of proving that ${}^{B}g$ is a critical Riemannian metric.

## § 1. Riemannian submersions with totally geodesic fibers.

Riemannian submersions were extensively studied by the authors R. H. Escobales [**2**], S. Ishihara [**3**], S. Ishihara and M. Konishi [**4**], Y. Mutō [**5**], T. Nagano [**8**], B. O'Neill [**9**], K. Yano and S. Ishihara [**11**], [**12**] and others.

Riemannian submersions considered in the present paper are limited to those with totally geodesic fibers only, and this means that the tensor $T$ of B. O'Neill vanishes [**9**]. Tensors in the total manifold $\tilde{M}$, in the base manifold $B$ or in the fiber $F$ are written in such letters as $\tilde{S}$, ${}^{B}S$, or ${}^{F}S$. Thus the Riemannian metrics on $\tilde{M}$, $B$ and $F$ are denoted respectively by $\tilde{g}$, ${}^{B}g$ and ${}^{F}g$.

Let $\tilde{W}$ be any vector field on $\tilde{M}$, $\tilde{E}$ any horizontal vector field on $\tilde{M}$ and $\tilde{X}$ any vertical vector field on $\tilde{M}$. Then, for example, from any (1, 1)-tensor field $\tilde{S}$ on $\tilde{M}$, we get four (1, 1)-tensor fields $S_{H}{}^{H}$, $S_{H}{}^{V}$, $S_{V}{}^{H}$, $S_{V}{}^{V}$ such that

$$ \tilde{S}=S_{H}{}^{H}+S_{H}{}^{V}+S_{V}{}^{H}+S_{V}{}^{V}, $$

$$ S_{H}{}^{H}\tilde{X}=S_{H}{}^{V}\tilde{X}=S_{V}{}^{H}\tilde{E}=S_{V}{}^{V}\tilde{E}=0, $$

$$ \tilde{g}(S_{H}{}^{H}\tilde{W}, \tilde{X})=0, \qquad \tilde{g}(S_{V}{}^{H}\tilde{W}, \tilde{X})=0, $$

$$ \tilde{g}(S_{H}{}^{V}\tilde{W}, \tilde{E})=0, \qquad \tilde{g}(S_{V}{}^{V}\tilde{W}, \tilde{E})=0. $$

It is easy to see that such a decomposition of $\tilde{S}$ is unique. Similarly, if $\tilde{S}$ is a (1, 2)-tensor field, we have a unique decomposition

$$\tilde{S}=S_{HH}{}^{H}+S_{HH}{}^{V}+S_{HV}{}^{H}+S_{HV}{}^{V} \tag{1.1}$$
$$+S_{VH}{}^{H}+S_{VH}{}^{V}+S_{VV}{}^{H}+S_{VV}{}^{V}.$$

The (0, 2)-tensor field and the (2, 0)-tensor field associated with the Riemannian metric $\tilde{g}$ are decomposed into $g_{HH}+g_{VV}$ and $g^{HH}+g^{VV}$ respectively since $g_{HV}$ and $g^{HV}$ vanish.

We define a tensor field $\tilde{R}$ with the following property.

$\tilde{R}$ has only one non-vanishing part, namely,

$$\tilde{R}=R_{HH}{}^{V}. \tag{1.2}$$

Let $\tilde{A}$ be the tensor field $A$ in O'Neill's paper [**9**]. Let $\tilde{E}$ and $\tilde{F}$ be any horizontal vector fields and $\tilde{X}$ any vertical vector field. Then $\tilde{R}$ satisfies

$$\tilde{A}_{\tilde{E}}\tilde{F}=-\frac{1}{2}\tilde{R}_{\tilde{E}}\tilde{F}, \tag{1.3}$$
$$\tilde{g}(\tilde{A}_{\tilde{E}}\tilde{X},\ \tilde{F})=\frac{1}{2}\tilde{g}(\tilde{R}_{\tilde{E}}\tilde{F},\ \tilde{X}).$$

Such a vector field $\tilde{R}$ also appears in [**5**].

We can cover $\tilde{M}$ by a set $\{V\}$ of coordinate neighborhoods with the following property. $\pi V$ is a coordinate neighborhood of $B$ and for any point $P\in V$ we have local coordinates $P\Leftrightarrow(x^1,\cdots,x^n,\ y^1,\cdots,y^m)=(x^1,\cdots,x^n,\ x^{n+1},\cdots,x^{n+m})$ such that $\pi P\Leftrightarrow(x^1,\cdots,x^n)$. If we use the natural frame attached to such a coordinate neighborhood $V$, the components $(\tilde{X}^1,\cdots,\tilde{X}^n,\ \tilde{X}^{n+1},\cdots,\tilde{X}^{n+m})$ of a vertical vector $\tilde{X}$ satisfy $\tilde{X}^h=0$ where $h=1,\cdots,n$.

Now we use indices in the following ranges:

$$a,\ b,\ c,\cdots,h,\ i,\ j,\cdots=1,\cdots,n,$$
$$\alpha,\ \beta,\ \gamma,\cdots,\kappa,\ \lambda,\ \mu,\cdots=n+1,\cdots,n+m,$$
$$A,\ B,\ C,\cdots,R,\ S,\ T,\cdots=1,\cdots,n+m.$$

Then the covariant componets of the Riemannian metric $\tilde{g}$ are $\tilde{g}_{CB}$, or separately, $\tilde{g}_{ji}$, $\tilde{g}_{j\lambda}$, $\tilde{g}_{\mu i}$, $\tilde{g}_{\mu\lambda}$ where $\tilde{g}_{j\lambda}=\tilde{g}_{\lambda j}$. If ${}^{F}g_{\mu\lambda}$ are the covariant components of the Riemannian metric ${}^{F}g$ on the fiber, it is easy to see that ${}^{F}g_{\mu\lambda}=\tilde{g}_{\mu\lambda}$. The inverse matrix of $({}^{F}g_{\mu\lambda})$ is denoted by $({}^{F}g^{\mu\lambda})$ where ${}^{F}g^{\mu\lambda}$ are the contravariant components of ${}^{F}g$.

Now we define $\Gamma_i^\kappa$ by

$$\Gamma_i^\kappa={}^{F}g^{\kappa\tau}\tilde{g}_{i\tau}.$$

For any vector $\tilde{W}$ we have the decomposition formula $\tilde{W}=W^H+W^V$. If $\tilde{W}^A$, namely $\tilde{W}^h$ and $\tilde{W}^\kappa$, are the components of $\tilde{W}$ and the components of $W^H$ and $W^V$ are denoted by $(W^H)^A$ and $(W^V)^A$ respectively, then we have

$$(1.4)\qquad \begin{aligned} &(W^H)^h=\tilde{W}^h, \qquad (W^H)^\kappa=-\Gamma_i^\kappa\tilde{W}^i,\\ &(W^V)^h=0, \qquad (W^V)^\kappa=\tilde{W}^\kappa+\Gamma_i^\kappa\tilde{W}^i, \end{aligned}$$

as it is easy to see from $\tilde{g}(W^H, W^V)=0$.

For any covariant vector $\tilde{U}$ we have $\tilde{U}=U_H+U_V$. As we have $U_H(\tilde{X})=0$, $U_V(\tilde{E})=0$ for any vertical vector $\tilde{X}$ and any horizontal vector $\tilde{E}$, we get

$$(1.5)\qquad \begin{aligned} &(U_H)_h=\tilde{U}_h-\Gamma_h^\kappa\tilde{U}_\kappa, \qquad (U_H)_\kappa=0,\\ &(U_V)_h=\Gamma_h^\kappa\tilde{U}_\kappa, \qquad (U_V)_\kappa=\tilde{U}_\kappa. \end{aligned}$$

We observe from (1.4) and (1.5) that $(W^H)^h$, $(W^V)^\kappa$, $(U_H)_h$ and $(U_V)_\kappa$ are the leading components of $W^H$, $W^V$, $U_H$ and $U_V$ respectively.

Using such local coordinates and natural frames we can deduce that $\tilde{R}$ has components

$$\tilde{R}_{ji}{}^\kappa=(R_{HH}{}^V)_{ji}{}^\kappa=D_j\Gamma_i^\kappa-D_i\Gamma_j^\kappa$$

where

$$D_i=\partial_i-\Gamma_i^\alpha\partial_\alpha, \qquad \partial_i=\partial/\partial x^i, \qquad \partial_\alpha=\partial/\partial y^\alpha.$$

All other components of $\tilde{R}$ vanish and we shall write $R_{ji}{}^\kappa$ for the sake of convenience instead of $\tilde{R}_{ji}{}^\kappa$.

For the Riemannian metric ${}^Bg$ on the base manifold $B$ we have

$${}^Bg_{ji}=\tilde{g}_{ji}-\Gamma_j^\mu\Gamma_i^\lambda\tilde{g}_{\mu\lambda}, \qquad {}^Bg^{ji}=\tilde{g}^{ji}.$$

We have observed that a tensor of $\tilde{M}$ is decomposed into several parts as in (1.1). If we take any one part, indices can be raised and lowered with the use of ${}^Bg^{ji}$, ${}^Fg^{\mu\lambda}$ and ${}^Bg_{ji}$, ${}^Fg_{\mu\lambda}$. Thus we can define new tensors from $\tilde{R}$, especially $R_H{}^{HV}$ and $R_{HHV}$ whose leading components are $R_j{}^{h\kappa}=R_{jt}{}^\kappa\, {}^Bg^{th}$ and $R_{ji\kappa}=R_{ji}{}^\alpha\, {}^Fg_{\alpha\kappa}$. It is easy to see that $(g_{HH})_{ji}={}^Bg_{ji}$, $(g_{VV})_{\mu\lambda}={}^Fg_{\mu\lambda}$, $(g^{HH})^{ji}={}^Bg^{ji}=\tilde{g}^{ji}$, $(g^{VV})^{\mu\lambda}={}^Fg^{\mu\lambda}=\tilde{g}^{\mu\lambda}-\Gamma_t^\mu\Gamma_s^\lambda\tilde{g}^{ts}$.

Relations between the curvature tensor $\tilde{K}_{DCBA}$ of $(\tilde{M}, \tilde{g})$, the curvature tensor ${}^BK_{kjih}$ of $(B, {}^Bg)$ and the curvature tensor ${}^FK_{\nu\mu\lambda\kappa}$ of the fiber $(F, {}^Fg)$ have been obtained by B. O'Neill [**9**]. In our terminology they are

$$(1.6)\qquad \begin{aligned} (K_{HHHH})_{kjih}={}^BK_{kjih}&-\frac{1}{4}(R_{ji}{}^\alpha R_{kh\alpha}-R_{ki}{}^\alpha R_{jh\alpha})\\ &+\frac{1}{2}R_{kj}{}^\alpha R_{ih\alpha}, \end{aligned}$$

(1.7) $$(K_{HHHV})_{kji\kappa}=\frac{1}{2}R_{ikj\kappa},$$

(1.8) $$(K_{HVHV})_{k\mu i\kappa}=\frac{1}{2}R_{\mu ki\kappa}-\frac{1}{4}R_{k}{}^{t}{}_{\mu}R_{it\kappa},$$

(1.9) $$(K_{VVHV})_{\nu\mu i\kappa}=0,$$

(1.10) $$(K_{VVVV})_{\nu\mu\lambda\kappa}={}^{F}K_{\nu\mu\lambda\kappa}$$

where $R_{kji\kappa}=R_{kji}{}^{\alpha}{}^{F}g_{\alpha\kappa}$, $R_{\mu ji\kappa}=R_{\mu ji}{}^{\alpha}{}^{F}g_{\alpha\kappa}$ and

$$R_{kji}{}^{\kappa}=((\tilde{\nabla}\tilde{R})_{HHH}{}^{V})_{kji}{}^{\kappa},\qquad R_{\mu ji}{}^{\kappa}=((\tilde{\nabla}\tilde{R})_{VHH}{}^{V})_{\mu ji}{}^{\kappa}$$

$\tilde{\nabla}$ denoting covariant differentiation in $(\tilde{M}, \tilde{g})$ with the use of Christoffel's symbols. Then, as we have $(K_{HVHV})_{k\mu i\kappa}=(K_{HVHV})_{i\kappa k\mu}$, we get $R_{\mu ki\kappa}-R_{\kappa ik\mu}=0$, hence

(1.11) $$R_{\mu ji\lambda}+R_{\lambda ji\mu}=0.$$

REMARK. Though $\tilde{R}$ has only the part $R_{HH}{}^{V}$, $\tilde{\nabla}\tilde{R}$ has several parts, namely, $\tilde{\nabla}\tilde{R}=(\tilde{\nabla}\tilde{R})_{HHH}{}^{H}+(\tilde{\nabla}\tilde{R})_{HHH}{}^{V}+(\tilde{\nabla}\tilde{R})_{HHV}{}^{V}+(\tilde{\nabla}\tilde{R})_{HVH}{}^{V}+(\tilde{\nabla}\tilde{R})_{VHH}{}^{V}$. Hence the vanishing of $(\tilde{\nabla}\tilde{R})_{HHH}{}^{V}$ and $(\tilde{\nabla}\tilde{R})_{VHH}{}^{V}$ does not mean that $\tilde{R}$ is parallel in $(\tilde{M}, \tilde{g})$. But, if $(\tilde{\nabla}\tilde{R})_{HHH}{}^{H}$ vanishes, we get $R_{k}{}^{h}{}_{\alpha}R_{ji}{}^{\alpha}=0$ and moreover $\tilde{R}=0$ from $R_{ji}{}^{\alpha}R^{ji}{}_{\alpha}=0$. Similarly, if $(\tilde{\nabla}\tilde{R})_{HHV}{}^{V}$ vanishes, we get $R_{k}{}^{t}{}_{\lambda}R_{jt}{}^{\kappa}=0$ and moreover $\tilde{R}=0$ (Escobales [**2**]).

We have some other formulas such as

$$(K_{HHHH})_{kji\kappa}=0,$$

$$(K_{HHHV})_{kjih}=(K_{HHHV})_{kji\alpha}\Gamma^{\alpha}_{h},$$

$$(K_{HHVV})_{kj\lambda\kappa}=(K_{HVHV})_{k\lambda j\kappa}-(K_{HVHV})_{k\kappa j\lambda}$$

and so on. These are direct consequences of the decomposition $\tilde{K}=K_{HHHH}+K_{HHHV}+\cdots+K_{VVVV}$ and the formulas similar to (1.5), or the well-known Bianchi identity.

For the Ricci tensors $\tilde{Ric}$, ${}^{B}Ric$ and ${}^{F}Ric$ of $(\tilde{M}, \tilde{g})$, $(B, {}^{B}g)$ and $(F, {}^{F}g)$ we have

(1.12) $$(K_{HH})_{ji}={}^{B}K_{ji}-\frac{1}{2}R_{j}{}^{t\alpha}R_{it\alpha},$$

(1.13) $$(K_{HV})_{j\lambda}=\frac{1}{2}g^{ts}R_{tjs\lambda}$$

(1.14) $$(K_{VV})_{\mu\lambda}={}^{F}K_{\mu\lambda}+\frac{1}{4}R^{ts}{}_{\mu}R_{ts\lambda}$$

where $\tilde{R}ic = K_{HH}+K_{HV}+K_{VH}+K_{VV}$ and ${}^BK_{ji}$ and ${}^FK_{\mu\lambda}$ are respectively components of ${}^BRic$ and ${}^FRic$. For the scalar curvatures $\tilde{K}$, ${}^BK$ and ${}^FK$ we have

$$\tilde{K} = {}^BK + {}^FK - \frac{1}{4} R^{ts\alpha} R_{ts\alpha}. \tag{1.15}$$

Fundamental formulas of covariant differentiation have been obtained by B. O'Neill [**9**]. They are also given in [**4**]. The following is only a translation into our terminology, where $\tilde{W}$ is a vector field and $\tilde{U}$ a 1-form.

$$((\tilde{\nabla}\tilde{W})_H{}^H)_j{}^h = D_j \tilde{W}^h + {}^B\left\{ {h \atop jt} \right\} \tilde{W}^t + \frac{1}{2} R_j{}^h{}_\alpha (\tilde{W}^\alpha + \Gamma^\alpha_t \tilde{W}^t),$$

$$((\tilde{\nabla}\tilde{U})_{HH})_{ji} = D_j(\tilde{U}_i - \Gamma^\alpha_i \tilde{U}_\alpha) - {}^B\left\{ {t \atop ji} \right\} (\tilde{U}_t - \Gamma^\alpha_t \tilde{U}_\alpha) + \frac{1}{2} R_{ji}{}^\alpha \tilde{U}_\alpha,$$

$$((\tilde{\nabla}\tilde{W})_H{}^V)_j{}^\kappa = D_j(\tilde{W}^\kappa + \Gamma^\kappa_h \tilde{W}^h) + \partial_\alpha \Gamma^\kappa_j (\tilde{W}^\alpha + \Gamma^\alpha_t \tilde{W}^t) - \frac{1}{2} R_{jt}{}^\kappa \tilde{W}^t,$$

$$((\tilde{\nabla}\tilde{U})_{HV})_{j\lambda} = D_j \tilde{U}_\lambda - \partial_\lambda \Gamma^\alpha_j \tilde{U}_\alpha - \frac{1}{2} R_j{}^t{}_\lambda (\tilde{U}_t - \Gamma^\alpha_t \tilde{U}_\alpha),$$

$$((\tilde{\nabla}\tilde{W})_V{}^H)_\mu{}^h = \partial_\mu \tilde{W}^h + \frac{1}{2} R_t{}^h{}_\mu \tilde{W}^t,$$

$$((\tilde{\nabla}\tilde{U})_{VH})_{\mu i} = \partial_\mu(\tilde{U}_i - \Gamma^\alpha_i \tilde{U}_\alpha) - \frac{1}{2} R_i{}^t{}_\mu (\tilde{U}_t - \Gamma^\alpha_t \tilde{U}_\alpha),$$

$$((\tilde{\nabla}\tilde{W})_V{}^V)_\mu{}^\kappa = \partial_\mu(\tilde{W}^\kappa + \Gamma^\kappa_h \tilde{W}^h) + {}^F\left\{ {\kappa \atop \mu\tau} \right\} (\tilde{W}^\tau + \Gamma^\tau_t \tilde{W}^t),$$

$$((\tilde{\nabla}\tilde{U})_{VV})_{\mu\lambda} = \partial_\mu \tilde{U}_\lambda - {}^F\left\{ {\alpha \atop \mu\lambda} \right\} \tilde{U}_\alpha.$$

Formulas for covariant differentiation of tensor fields are easily deduced from the above formulas.

## § 2. A Sasakian manifold where the Riemannian metric is a critical Riemannian metric.

We can consider a Sasakian manifold $(M, g, \xi)$ as a Riemannian manifold $(M, g)$ with a unit Killing vector field $\xi$ satisfying

$$K_{tji}{}^h \xi^t = g_{ji}\xi^h - \xi_i \delta^h_j.$$

Hence in a Sasakian manifold $(M, g, \xi)$ we have

$$K_t{}^h \xi^t = (n-1)\xi^h,$$

$$\nabla_j \nabla_i \xi_h = g_{jh}\xi_i - g_{ji}\xi_h,$$

$$\nabla_s \nabla^s \xi^h = -(n-1)\xi^h ,$$

$$K_{jt}\nabla_i \xi^t + K_{it}\nabla_j \xi^t = 0 ,$$

$$\nabla_t \xi_j \nabla^t \xi_i = g_{ji} - \xi_j \xi_i .$$

We show that, if the Riemannian metric $g$ is a critical Riemannian metric $g_D$, then the scalar curvature $K$ is constant.

First we get from (0.1) and (0.2D)

$$2\xi^t \nabla_t \nabla_i K - 4\xi^t \nabla_s \nabla^s K_{ti} + 4\xi^t K_{ts} K_i{}^s - 4\xi^t K_{tsri} K^{sr} - 2\xi^t K_{tsrq} K_i{}^{srq} + \frac{1}{2} K_{tsrq} K^{tsrq} \xi_i = c_D \xi_i . \tag{2.1}$$

But we have

$$\xi^t \nabla_t \nabla_i K = \xi^t \nabla_i \nabla_t K = \nabla_i (\xi^t \nabla_t K) - \nabla_i \xi^t \nabla_t K = \nabla_t K \nabla^t \xi_i ,$$

$$\begin{aligned} \xi^t \nabla_s \nabla^s K_{ti} &= \nabla_s \nabla^s (\xi^t K_{ti}) - (\nabla_s \nabla^s \xi^t) K_{ti} - 2\nabla^s \xi^t \nabla_s K_{ti} \\ &= (n-1)\nabla_s \nabla^s \xi_i + \nabla_s \nabla^s \xi^t K_{ti} - 2\nabla_s (\nabla^s \xi^t K_{ti}) \\ &= -2(n-1)^2 \xi_i - 2\nabla_s (K^{st} \nabla_t \xi_i) \\ &= -2(n-1)^2 \xi_i - \nabla^t K \nabla_t \xi_i - 2K^{st} \nabla_s \nabla_t \xi_i \\ &= -2(n-1)^2 \xi_i - \nabla^t K \nabla_t \xi_i - 2K^{st} (g_{si} \xi_t - g_{st} \xi_i) \\ &= 2\{K - n(n-1)\} \xi_i - \nabla^t K \nabla_t \xi_i , \end{aligned}$$

$$\xi^t K_{ts} K_i{}^s = (n-1)^2 \xi^i ,$$

$$\xi^t K_{tsri} K^{sr} = \{K - (n-1)\} \xi_i ,$$

$$\xi^t K_{tsrq} K_i{}^{srq} = 2(n-1)\xi_i .$$

Hence we get

$$-12K + 4(n-1)(3n-1) + \frac{1}{2} K_{tsrq} K^{tsrq} = c_D \tag{2.2}$$

and

$$\nabla_t K \nabla^t \xi_i = 0 . \tag{2.3}$$

From (2.3) and $\xi^t \nabla_t K = 0$ we get $K = \text{const}$ as we have $\nabla_t \xi_j \nabla^t \xi_i = g_{ji} - \xi_j \xi_i$. From (2.2) we get $K_{tsrq} K^{tsrq} = \text{const}$.

By a similar argument we get

$$K=\text{const}$$

and

$$-4K+\frac{1}{2}K_{ts}K^{ts}+2(n-1)(n+1)=c_C$$

if $g$ is a critical Riemannian metric $g_C$, and similarly

$$K=\text{const}$$

and

$$-2(n-1)K+\frac{1}{2}K^2=c_B$$

if $g$ is a critical Riemannian metric $g_B$. On the other hand it is well-known that in an Einstein manifold $K$=const.

Thus we have proved the following theorem.

THEOREM 2.1. *If $g$ is a critical Riemannian metric $g_A$, $g_B$, $g_C$ or $g_D$ in a Sasakian manifold $(M, g, \xi)$, then the scalar curvature is constant.*

## § 3. Sasakian submersions and critical Riemannian metrics.

Let $(\tilde{M}, \tilde{g}, \tilde{\xi})$ be a Sasakian manifold where $\dim\tilde{M}=\tilde{n}$. Let the indices $A, B, C, \cdots, R, S, T, \cdots$ run the range $\{1, \cdots, \tilde{n}\}$ and the indices $a, b, c, \cdots, h, i, j, \cdots, r, s, t, \cdots$ the range $\{1, \cdots, n\}$ where $n=\tilde{n}-1$. Then we have

$$\tilde{K}_{TCB}{}^{A}\tilde{\xi}^{T}=\tilde{g}_{CB}\tilde{\xi}^{A}-\tilde{\xi}_{B}\delta_{C}^{A} \tag{3.1}$$

and

$$\begin{gathered}\tilde{K}_{T}{}^{A}\tilde{\xi}^{T}=(\tilde{n}-1)\tilde{\xi}^{A}\,,\\ \tilde{\nabla}_{C}\tilde{\nabla}_{B}\tilde{\xi}_{A}=\tilde{g}_{CA}\tilde{\xi}_{B}-\tilde{g}_{CB}\tilde{\xi}_{A}\,,\\ \tilde{K}_{CT}\tilde{\nabla}_{B}\tilde{\xi}^{T}+\tilde{K}_{BT}\tilde{\nabla}_{C}\tilde{\xi}^{T}=0\,,\\ \tilde{\xi}^{T}\tilde{\nabla}_{T}\tilde{K}=0\,.\end{gathered} \tag{3.2}$$

A Sasakian manifold $(\tilde{M}, \tilde{g}, \tilde{\xi})$ admits a Riemannian submersion where the unit Killing vector $\tilde{\xi}$ is a vertical vector and the fibers are geodesics tangent to $\tilde{\xi}$. Let us call such a Riemannian submersion a *Sasakian submersion*. For the existence of such a submersion see K. Yano and S. Ishihara [**11**], [**12**].

One of our purposes is to find the condition for a Sasakian submersion $(\tilde{M}, \tilde{g}, \tilde{\xi})\rightarrow(B, {}^{B}g)$ that $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics simultaneously in $\tilde{M}$ and $B$ respectively. We also study some related problems.

To that end we take local coordinates introduced in § 1. But, as dim $F=1$ in the present case, Greek indices can take only one number $\tilde{n}$. Hence we can replace all Greek indices by 0. Thus, for example, the components of the unit Killing vector $\tilde{\xi}$ are $\tilde{\xi}^h$ and $\tilde{\xi}^0$. As $\tilde{\xi}$ is a vertical vector, we have

$$\tilde{\xi}^h=0, \qquad \tilde{\xi}^0=1, \tag{3.3}$$

for we take $\tilde{g}_{00}=1$ as the Riemannian metric on the one dimensional fiber.

The index 0 can be dropped if there is no possibility of confusion. For example, we can write $\Gamma_j$ for $\Gamma_j^0$ which was written $\Gamma_j^\kappa$ in § 1. Thus we get

$$\Gamma_j=\tilde{g}_{j0}=\tilde{g}_{j0}\tilde{\xi}^0=\tilde{g}_{jT}\tilde{\xi}^T=\tilde{\xi}_j,$$

and

$$\tilde{g}^{00}=1+\Gamma_j\Gamma_i\tilde{g}^{ji}.$$

The non-vanishing components of the tensor $\tilde{R}$ are $R_{ji}{}^0$ which we write $R_{ji}$.

From (1.11) we get $R_{0ji}{}^0=0$. As $\tilde{\xi}$ is a Killing vector field, we have

$$\tilde{\xi}^C\partial_C\tilde{g}_{BA}+\tilde{g}_{CB}\partial_A\tilde{\xi}^C+\tilde{g}_{CA}\partial_B\tilde{\xi}^C=0.$$

From this and (3.3) we get $\partial_0\tilde{g}_{BA}=0$ and especially $\partial_0\tilde{g}_{i0}=0$, $\partial_0\tilde{\xi}_i=0$, namely,

$$\partial_0\Gamma_i=0. \tag{3.4}$$

We also get

$$R_{ji}=\partial_j\tilde{\xi}_i-\partial_i\tilde{\xi}_j. \tag{3.5}$$

(3.1) is equivalent to

$$\tilde{K}_{TCBA}\tilde{\xi}^T=\tilde{g}_{CB}\tilde{\xi}_A-\tilde{g}_{CA}\tilde{\xi}_B. \tag{3.6}$$

As the pure horizontal part of the right hand member vanishes, we get $K_{VHHH}=0$. Hence we get $R_{ikj}{}^0=0$ from (1.7). On the other hand we have (3.4). Applying the fundamental formulas of covariant differentiation in § 1 to the tensor field $\tilde{R}$ we get

$${}^B\nabla_i R_{kj}=0,$$

which means that $R_{ji}$ considered as a tensor field of the base manifold $(B, {}^Bg)$ is covariantly constant.

From (3.6) we get $K_{VHHV}=g_{HH}$, that is,

$$(K_{VHHV})_{0ji0}={}^Bg_{ji}.$$

On the other hand, as we have $R_{0ji0}=0$, we get from (1.8)

$$(K_{VHHV})_{0ji0}=\frac{1}{4}R_j{}^tR_{it}.$$

Hence we have

$$\frac{1}{4}R_j{}^tR_{it}={}^Bg_{ji}.$$

As $R_{ji}$ is a skew tensor and satisfies ${}^B\nabla_k R_{ji}=0$, we can conclude that $F_j{}^i$ defined by

$$F_j{}^i=\frac{1}{2}R_j{}^i=\frac{1}{2}R_{jt}\,{}^Bg^{ti}$$

represents a complex structure $J$ such that $(J, {}^Bg)$ is a Kähler structure on $B$.

Let us decompose the Ricci tensor of $(\tilde{M}, \tilde{g}, \tilde{\xi})$ into the sum $K_{HH}+K_{HV}+K_{VH}+K_{VV}$. Then we get from the above results

$$(3.7)\qquad \begin{aligned}(K_{HH})_{ji}&={}^BK_{ji}-2\,{}^Bg_{ji}\,,\\ (K_{HV})_{j0}&=0\,,\\ (K_{VV})_{00}&=\tilde{n}-1\,,\end{aligned}$$

for the one dimensional fiber is flat. For the scalar curvatures we have

$$(3.8)\qquad \tilde{K}={}^BK-(\tilde{n}-1)\,.$$

Thus, the scalar curvatures are constant if any one of them is constant. The Ricci tensor of $(\tilde{M}, \tilde{g}, \tilde{\xi})$ has the following components,

$$(3.9)\qquad \begin{aligned}\tilde{K}_{ji}&={}^BK_{ji}-2\,{}^Bg_{ji}+(\tilde{n}-1)\Gamma_j\Gamma_i\,,\\ \tilde{K}_{j0}&=(\tilde{n}-1)\Gamma_j\,,\\ \tilde{K}_{00}&=\tilde{n}-1\,.\end{aligned}$$

We have defined tensors $A$, $B$, $C$, $D$ by (0.2) in a Riemannian manifold $(M, g)$. Corresponding tensors of a Sasakian manifold $(\tilde{M}, \tilde{g}, \tilde{\xi})$ will be denoted by $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and their components by $\tilde{A}_{BA}$, $\tilde{B}_{BA}$, $\tilde{C}_{BA}$, $\tilde{D}_{BA}$, while those of the base manifold $(B, {}^Bg)$ by ${}^BA$, ${}^BB$, ${}^BC$, ${}^BD$ and their components by ${}^BA_{ji}$, ${}^BB_{ji}$, ${}^BC_{ji}$, ${}^BD_{ji}$. In order to find the relation between, for example, $\tilde{D}$ and ${}^BD$, we need the following identities,

$$\begin{aligned}\tilde{\nabla}_T\tilde{\nabla}^T\tilde{K}_{ji}&={}^B\nabla_t\,{}^B\nabla^t\,{}^BK_{ji}-2\,{}^BK_{ji}+2(\tilde{n}+1)\,{}^Bg_{ji}\\ &\quad-2\Gamma_jF^{ts}\,{}^B\nabla_t\,{}^BK_{is}-2\Gamma_iF^{ts}\,{}^B\nabla_t\,{}^BK_{js}\\ &\quad+2\{{}^BK-(\tilde{n}-1)(\tilde{n}+1)\}\Gamma_j\Gamma_i\,,\\ \tilde{\nabla}_T\tilde{\nabla}^T\tilde{K}_{0i}&=-2F^{ts}\,{}^B\nabla_t\,{}^BK_{is}+2\{{}^BK-(\tilde{n}-1)(\tilde{n}+1)\}\Gamma_i\,,\\ \tilde{\nabla}_T\tilde{\nabla}^T\tilde{K}_{00}&=2\{{}^BK-(\tilde{n}-1)(\tilde{n}+1)\}\,,\\ \tilde{K}_{jT}\tilde{K}_i{}^T&={}^BK_{jt}\,{}^BK_i{}^t-4\,{}^BK_{ji}+4\,{}^Bg_{ji}+(\tilde{n}-1)^2\Gamma_j\Gamma_i\,,\\ \tilde{K}_{0T}\tilde{K}_i{}^T&=(\tilde{n}-1)^2\Gamma_i\,,\end{aligned}$$

$$\tilde{K}_{0T}\tilde{K}_0{}^T=(\tilde{n}-1)^2\,,$$

$$\tilde{K}_{jTSi}\tilde{K}^{TS}={}^BK_{jtsi}\,{}^BK^{ts}-5\,{}^BK_{ji}+(\tilde{n}+5)\,{}^Bg_{ji}$$
$$+\{{}^BK-2(\tilde{n}-1)\}\Gamma_j\Gamma_i\,,$$

$$\tilde{K}_{0TSi}\tilde{K}^{TS}=\{{}^BK-2(\tilde{n}-1)\}\Gamma_i\,,$$

$$\tilde{K}_{0TS0}\tilde{K}^{TS}={}^BK-2(\tilde{n}-1)\,,$$

$$\tilde{K}_{jTSR}\tilde{K}_i{}^{TSR}={}^BK_{jtsr}\,{}^BK_i{}^{tsr}-12\,{}^BK_{ji}+(6\tilde{n}+2)\,{}^Bg_{ji}$$
$$+2(\tilde{n}-1)\Gamma_j\Gamma_i\,,$$

$$\tilde{K}_{0TSR}\tilde{K}_i{}^{TSR}=2(\tilde{n}-1)\Gamma_i\,,$$

$$\tilde{K}_{0TSR}\tilde{K}_0{}^{TSR}=2(\tilde{n}-1)\,,$$

$$\tilde{K}_{DCBA}\tilde{K}^{DCBA}={}^BK_{dcba}\,{}^BK^{dcba}-12\,{}^BK+2(3\tilde{n}+2)(\tilde{n}-1)\,,$$

$$\tilde{K}_{TS}\tilde{K}^{TS}={}^BK_{ts}\,{}^BK^{ts}-4\,{}^BK+(\tilde{n}-1)(\tilde{n}+3)\,,$$

where ${}^B\nabla$ denotes covariant differentiation with respect to the metric ${}^Bg$. These identities are obtained from the fundamental formulas by straightforward calculation.

If the scalar curvature $\tilde{K}$ is constant, we have $F^{ts}\,{}^B\nabla_t\,{}^BK_{is}=0$ because of

$$F^{ts}\,{}^B\nabla_t\,{}^BK_{is}={}^B\nabla_t(F^{ts}\,{}^BK_{si})={}^B\nabla_t({}^BK^{ts}F_{si})$$
$$=-\frac{1}{2}F_i{}^s\,{}^B\nabla_s\,{}^BK\,.$$

Then we get from the above identities

$$\tilde{D}_{ji}={}^BD_{ji}+36\,{}^BK_{ji}+[-6\,{}^BK+(3\tilde{n}+2)(\tilde{n}-9)]\,{}^Bg_{ji}$$
$$+\Big[\frac{1}{2}\,{}^BK_{dcba}\,{}^BK^{dcba}-18\,{}^BK+5(3\tilde{n}+2)(\tilde{n}-1)\Big]\Gamma_j\Gamma_i\,, \tag{3.10}$$

$$\tilde{D}_{0i}=\Big[\frac{1}{2}\,{}^BK_{dcba}\,{}^BK^{dcba}-18\,{}^BK+5(3\tilde{n}+2)(\tilde{n}-1)\Big]\Gamma_i\,, \tag{3.11}$$

$$\tilde{D}_{00}=\frac{1}{2}\,{}^BK_{dcba}\,{}^BK^{dcba}-18\,{}^BK+5(3\tilde{n}+2)(\tilde{n}-1)\,. \tag{3.12}$$

First, let us assume that $\tilde{g}$ is a critical Riemannian metric $g_D$ on $\tilde{M}$ and at the same time ${}^Bg$ is a critical Riemannian metric $g_D$ on $B$. Then $\tilde{K}$ is constant and (3.10), (3.11), (3.12) are valid. Substituting $\tilde{D}_{ji}=\tilde{c}_D\tilde{g}_{ji}$ and ${}^BD_{ji}={}^Bc_D\,{}^Bg_{ji}$ into (3.10) we get

$$\tilde{c}_D({}^B g_{ji}+\Gamma_j\Gamma_i)={}^B c_D\,{}^B g_{ji}+36\,{}^B K_{ji}$$
$$+[-6\,{}^B K+(3\tilde{n}+2)(\tilde{n}-9)]\,{}^B g_{ji}$$
$$+\Big[\frac{1}{2}{}^B K_{dcba}\,{}^B K^{dcba}-18\,{}^B K+5(3\tilde{n}+2)(\tilde{n}-1)\Big]\Gamma_j\Gamma_i\,,$$

and substituting $\tilde{D}_{00}=\tilde{c}_D\tilde{g}_{00}=\tilde{c}_D$ into (3.12) we get

(3.13) $$\tilde{c}_D=\frac{1}{2}{}^B K_{dcba}\,{}^B K^{dcba}-18\,{}^B K+5(3\tilde{n}+2)(\tilde{n}-1)\,.$$

Hence we have

$$[{}^B c_D-\tilde{c}_D-6\,{}^B K+(3\tilde{n}+2)(\tilde{n}-9)]\,{}^B g_{ji}+36\,{}^B K_{ji}=0$$

and, consequently,

(3.14) $${}^B K_{ji}=\frac{{}^B K}{\tilde{n}-1}\,{}^B g_{ji}\,,$$

(3.15) $${}^B c_D-\tilde{c}_D+(3\tilde{n}+2)(\tilde{n}-9)-\frac{6(\tilde{n}-7)}{\tilde{n}-1}\,{}^B K=0\,.$$

Substituting (3.13) and

$${}^B c_D=\frac{1}{\tilde{n}-1}\,{}^B D_{ji}\,{}^B g^{ji}=\Big(\frac{1}{2}-\frac{2}{\tilde{n}-1}\Big)\,{}^B K_{dcba}\,{}^B K^{dcba}$$

into (3.15) we get

(3.16) $${}^B K_{dcba}\,{}^B K^{dcba}=6(\tilde{n}+2)\,{}^B K-2(3\tilde{n}+2)(\tilde{n}^2-1)\,.$$

Secondly, let us assume that $\tilde{K}$ is constant and (3.14) and (3.16) are satisfied. If moreover $\tilde{g}$ is a critical Riemannian metric $g_D$ on $\tilde{M}$, we have $\tilde{D}_{BA}=\tilde{c}_D\tilde{g}_{BA}$, hence

(3.17) $$\tilde{D}_{ji}=\tilde{c}_D({}^B g_{ji}+\Gamma_j\Gamma_i)\,,\qquad \tilde{D}_{00}=\tilde{c}_D\,.$$

Then we get from (3.10), (3.12) and (3.14)

$${}^B D_{ji}=\Big[\tilde{c}_D-\frac{36}{\tilde{n}-1}\,{}^B K+6\,{}^B K-(3\tilde{n}+2)(\tilde{n}-9)\Big]{}^B g_{ji}$$

which proves that ${}^B g$ is a critical Riemannian metric $g_D$ on $B$.

Thirdly, let us assume that $\tilde{K}$ is constant and (3.14) and (3.16) are satisfied. If moreover ${}^B g$ is a critical Riemannian metric $g_D$ on $B$, we have

(3.18) $${}^B D_{ji}={}^B c_D\,{}^B g_{ji}$$

and

(3.19)
$$^{B}c_D=\frac{1}{\tilde{n}-1}\,{}^{B}D_{ji}\,{}^{B}g^{ji}$$
$$=\frac{1}{\tilde{n}-1}\Big(\frac{\tilde{n}-1}{2}-2\Big)\,{}^{B}K_{dcba}\,{}^{B}K^{dcba}.$$

Substituting (3.16) into (3.19) we get

(3.20)
$$^{B}c_D=\frac{3(\tilde{n}-5)(\tilde{n}+2)}{\tilde{n}-1}\,{}^{B}K-(\tilde{n}-5)(3\tilde{n}+2)(\tilde{n}+1).$$

From (3.10), (3.14), (3.18) and (3.20) we get

$$\tilde{D}_{ji}=(\tilde{n}-4)[3\,{}^{B}K-(3\tilde{n}+2)(\tilde{n}-1)]\,{}^{B}g_{ji}$$
$$+\Big[\frac{1}{2}\,{}^{B}K_{dcba}\,{}^{B}K^{dcba}-18\,{}^{B}K+5(3\tilde{n}+2)(\tilde{n}-1)\Big]\Gamma_j\Gamma_i.$$

But, as we have

$$\frac{1}{2}\,{}^{B}K_{dcba}\,{}^{B}K^{dcba}-18\,{}^{B}K+5(3\tilde{n}+2)(\tilde{n}-1)$$
$$=(\tilde{n}-4)[3\,{}^{B}K-(3\tilde{n}+2)(\tilde{n}-1)]$$

from (3.16), we get

$$\tilde{D}_{ji}=\nu\tilde{g}_{ji}$$

where

$$\nu=(\tilde{n}-4)[3\,{}^{B}K-(3\tilde{n}+2)(\tilde{n}-1)].$$

At the same time we get, from (3.11) and (3.12), $\tilde{D}_{0i}=\nu\Gamma_i$ and $\tilde{D}_{00}=\nu$, which proves that $\tilde{g}$ is a critical Riemannian metric $g_D$ on $\tilde{M}$.

Thus we have proved the following theorem.

THEOREM 3.1. *Let $\pi$: $(\tilde{M}, \tilde{g}, \xi)\to(B, {}^{B}g)$ be a Sasakian submersion. If $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics $g_D$ on $\tilde{M}$ and $B$ respectively, then the scalar curvature $\tilde{K}$ is constant and $(B, {}^{B}g)$ is an Einstein manifold satisfying* (3.16). *Conversely, let us consider the case where $(B, {}^{B}g)$ is an Einstein manifold satisfying* (3.16). *If one of $\tilde{g}$ and ${}^{B}g$ is a critical Riemannian metric $g_D$, then the other is also a critical Riemannian metric $g_D$.*

For $\tilde{C}_{BA}$ and ${}^{B}C_{ji}$ we get following identities if $\tilde{K}$ is constant,

(3.21)
$$\tilde{C}_{ji}={}^{B}C_{ji}+12\,{}^{B}K_{ji}+\Big[-2\,{}^{B}K+\frac{1}{2}(\tilde{n}+3)(\tilde{n}-9)\Big]\,{}^{B}g_{ji}$$
$$+\Big[\frac{1}{2}\,{}^{B}K_{ts}\,{}^{B}K^{ts}-6\,{}^{B}K+\frac{5}{2}(\tilde{n}-1)(\tilde{n}+3)\Big]\Gamma_j\Gamma_i,$$

(3.22)
$$\tilde{C}_{0i}=\Big[\frac{1}{2}\,{}^{R}K_{ts}\,{}^{B}K^{ts}-6\,{}^{B}K+\frac{5}{2}(\tilde{n}-1)(\tilde{n}+3)\Big]\Gamma_i,$$

(3.23)
$$\tilde{C}_{00}=\frac{1}{2}\,{}^{B}K_{ts}\,{}^{B}K^{ts}-6\,{}^{B}K+\frac{5}{2}(\tilde{n}-1)(\tilde{n}+3).$$

If $\tilde{g}$ and ${}^B g$ are critical Riemannian metrics $g_C$, we get

$$\tilde{c}_C({}^B g_{ji}+\Gamma_j\Gamma_i)={}^B c_C\,{}^B g_{ji}+12\,{}^B K_{ji}+\Big[-2\,{}^B K+\frac{1}{2}(\tilde{n}+3)(\tilde{n}-9)\Big]\,{}^B g_{ji}$$

$$+\Big[\frac{1}{2}\,{}^B K_{ts}\,{}^B K^{ts}-6\,{}^B K+\frac{5}{2}(\tilde{n}-1)(\tilde{n}+3)\Big]\Gamma_j\Gamma_i\,,$$

(3.24) $$\tilde{c}_C=\frac{1}{2}\,{}^B K_{ts}\,{}^B K^{ts}-6\,{}^B K+\frac{5}{2}(\tilde{n}-1)(\tilde{n}+3)$$

from (3.21) and (3.23). Hence ${}^B g$ is an Einstein metric and

(3.25) $$\tilde{c}_C={}^B c_C-2\,\frac{\tilde{n}-7}{\tilde{n}-1}\,{}^B K+\frac{1}{2}(\tilde{n}+3)(\tilde{n}-9)\,.$$

On the other hand we get from (0.2C)

$${}^B c_C=\frac{1}{\tilde{n}-1}\,{}^B C_{ji}\,{}^B g^{ji}=\frac{\tilde{n}-5}{2(\tilde{n}-1)}\,{}^B K_{ts}\,{}^B K^{ts}$$

which becomes

(3.26) $${}^B c_C=\frac{\tilde{n}-5}{2(\tilde{n}-1)^2}({}^B K)^2\,.$$

From (3.24), (3.25) and (3.26) we get

(3.27) $$({}^B K)^2-2(\tilde{n}-1)(\tilde{n}+2)\,{}^B K+(\tilde{n}+1)(\tilde{n}+3)(\tilde{n}-1)^2=0$$

whose solution is

$${}^B K=\tilde{n}^2-1 \quad \text{or} \quad (\tilde{n}-1)(\tilde{n}+3)\,.$$

It is also easy to prove, following almost the same process as in the case of $g_D$, that, in a Sasakian submersion with Einsteinian $(B, {}^B g)$ satisfying (3.27), if one of $\tilde{g}$ and ${}^B g$ is a critical Riemannian metric $g_C$, then the other is also a critical Riemannian metric $g_C$.

Thus we have proved the following theorem.

THEOREM 3.2. *Let* $\pi: (\tilde{M}, \tilde{g}, \tilde{\xi})\to(B, {}^B g)$ *be a Sasakian submersion. If* $\tilde{g}$ *and* ${}^B g$ *are critical Riemannian metrics* $g_C$ *on* $\tilde{M}$ *and* $B$ *respectively, then the scalar curvature* $\tilde{K}$ *is constant and* $(B, {}^B g)$ *is an Einstein manifold satisfying* (3.27). *Conversely, consider the case where* $(B, {}^B g)$ *is an Einstein manifold satisfying* (3.27). *If one of* $\tilde{g}$ *and* ${}^B g$ *is a critical Riemannian metric* $g_C$, *then the other is also a critical Riemannian metric* $g_C$.

For $\tilde{B}_{BA}$ and ${}^B B_{ji}$ we get the following identities if $\tilde{K}$ is constant.

$$\tilde{B}_{ji}={}^B B_{ji}+2(\tilde{n}-1)\,{}^B K_{ji}+\Big[-(\tilde{n}-5)\,{}^B K-4(\tilde{n}-1)+\frac{1}{2}(\tilde{n}-1)^2\Big]\,{}^B g_{ji}$$

$$+\Big[\frac{1}{2}({}^B K)^2-3(\tilde{n}-1)\,{}^B K+\frac{5}{2}(\tilde{n}-1)^2\Big]\Gamma_j\Gamma_i\,,$$

$$\tilde{B}_{0i}=\left[\frac{1}{2}({}^BK)^2-3(\tilde{n}-1)\,{}^BK+\frac{5}{2}(\tilde{n}-1)^2\right]\Gamma_i\,,$$

$$\tilde{B}_{00}=\frac{1}{2}({}^BK)^2-3(\tilde{n}-1)\,{}^BK+\frac{5}{2}(\tilde{n}-1)^2\,.$$

By an argument similar to the foregoing one we get the following theorem.

THEOREM 3.3. *Let* $\pi:(\tilde{M},\tilde{g},\tilde{\xi})\to(B,{}^Bg)$ *be a Sasakian submersion. If* $\tilde{g}$ *and* ${}^Bg$ *are critical Riemannian metrics* $g_B$ *on* $\tilde{M}$ *and* $B$ *respectively, then the scalar curvature* $\tilde{K}$ *is constant and* $(B,{}^Bg)$ *is an Einstein manifold satisfying*

$$({}^BK)^2-(\tilde{n}-1)(\tilde{n}+2)\,{}^BK+(\tilde{n}-1)^2(\tilde{n}+1)=0\,. \tag{3.28}$$

*Conversely consider the case where* $(B,{}^Bg)$ *is an Einstein manifold where* ${}^BK=\tilde{n}^2-1$ *or* $\tilde{n}-1$. *If one of* $\tilde{g}$ *and* ${}^Bg$ *is a critical Riemannian metric* $g_B$, *then the other is also a critical Riemannian metric* $g_B$.

For $\tilde{A}_{BA}$ and ${}^BA_{ji}$ we have the following identities

$$\tilde{A}_{ji}={}^BA_{ji}-\frac{1}{2}(\tilde{n}-5)\,{}^Bg_{ji}+\frac{1}{2}\{{}^BK-3(\tilde{n}-1)\}\Gamma_j\Gamma_i\,,$$

$$\tilde{A}_{0i}=\frac{1}{2}\{{}^BK-3(\tilde{n}-1)\}\Gamma_i\,,$$

$$\tilde{A}_{00}=\frac{1}{2}\{{}^BK-3(\tilde{n}-1)\}\,.$$

If $\tilde{g}$ and ${}^Bg$ are critical Riemannian metrics $g_A$, namely, Einstein metrics, we get from the above identities

$$\tilde{K}=\tilde{n}^2-\tilde{n}\,,\qquad {}^BK=\tilde{n}^2-1\,. \tag{3.29}$$

Conversely, if $\tilde{K}$ and ${}^BK$ are such, then from (3.7) we can conclude that $\tilde{g}$ and ${}^Bg$ are simultaneously Einstein metrics or not.

REMARK. A $(2n+1)$-dimensional sphere with standard Riemannian metric $g_0$ admits a Sasakian submersion $\pi:(S^{2n+1},g_0)\to(B,{}^Bg)$ where $(B,{}^Bg)$ is a Kaehler manifold of constant holomorphic sectional curvature $k$. Here $g_0$ and ${}^Bg$ are critical Riemannian metrics on $S^{2n+1}$ and $B$ respectively in the sense $A$, $B$, $C$, $D$ simultaneously (see also § 4). As we have

$$K_{ji}=\frac{1}{2}(n+1)kg_{ji}\,,\qquad K_{dcba}K^{dcba}=2n(n+1)k^2\,,$$

$$K=n(n+1)k$$

for a Kaehler manifold of constant holomorphic sectional curvature and $k=4$ in the present case, (3.16), (3.27), (3.28), (3.29) are all satisfied.

## § 4. Riemannian submersions with totally geodesic fibers on spheres.

Let $g_0$ be the standard Riemannian metric on an $(n+m)$-dimensional sphere. All possible Riemannian submersions $\pi:(S^{n+m}, g_0)\to(B, {}^Bg)$ with totally geodesic fibers were obtained by R. H. Escobales [2]. In the present paper the following theorem is proved independent of Escobales' result.

THEOREM 4.1. *If* $(S^{n+m}, g_0)$ *admits a Riemannian submersion* $\pi:(S^{n+m}, g_0)\to(B, {}^Bg)$, $\dim B=n$, *with totally geodesic fibers, then* ${}^Bg$ *is a critical Riemannian metric in the sense A, B, C and D.*

The remaining part of the present paper is devoted to the proof of this theorem. As the curvature tensor of $(S^{n+m}, g_0)$ satisfies

$$\tilde{K}_{DCBA}=\tilde{g}_{CB}\tilde{g}_{DA}-\tilde{g}_{DB}\tilde{g}_{CA}, \tag{4.1}$$

we get

$$\begin{aligned}(K_{HHHH})_{kjih}&={}^Bg_{ji}\,{}^Bg_{kh}-{}^Bg_{ki}\,{}^Bg_{jh},\\ (K_{HHHV})_{kji\kappa}&=0,\\ (K_{HVHV})_{k\mu i\kappa}&=-{}^Bg_{ki}\,{}^Fg_{\mu\kappa}.\end{aligned} \tag{4.2}$$

Substituting this into (1.6), (1.7) and (1.8) we get

$$\begin{aligned}{}^BK_{kjih}={}&{}^Bg_{ji}\,{}^Bg_{kh}-{}^Bg_{ki}\,{}^Bg_{jh}\\ &+\frac{1}{4}(R_{ji}{}^{\alpha}R_{kh\alpha}-R_{ki}{}^{\alpha}R_{jh\alpha})-\frac{1}{2}R_{kj}{}^{\alpha}R_{ih\alpha},\end{aligned} \tag{4.3}$$

$$R_{kji}{}^{\kappa}=0, \tag{4.4}$$

$$\frac{1}{2}R_{\mu ji\lambda}-\frac{1}{4}R_j{}^t{}_{\mu}R_{it\lambda}=-{}^Bg_{ji}\,{}^Fg_{\mu\lambda}. \tag{4.5}$$

From (4.5) we deduce

$$R_{\mu ji}{}^{\kappa}=\frac{1}{4}(R_j{}^t{}_{\mu}R_{it}{}^{\kappa}-R_i{}^t{}_{\mu}R_{jt}{}^{\kappa}), \tag{4.6}$$

$${}^Fg_{\mu\lambda}\,{}^Bg_{ji}=\frac{1}{8}(R_j{}^t{}_{\mu}R_{it\lambda}+R_i{}^t{}_{\mu}R_{jt\lambda}), \tag{4.7}$$

$$R^{ts}{}_{\mu}R_{ts\lambda}=4n\,{}^Fg_{\mu\lambda}, \tag{4.8}$$

$$R_j{}^{t\alpha}R_{it\alpha}=4m\,{}^Bg_{ji}. \tag{4.9}$$

For the Ricci tensor of $(S^{n+m}, g_0)$ we have

$$\tilde{K}_{CB}=(n+m-1)\tilde{g}_{CB}.$$

Substituting this into (1.12) we get

$$ {}^B K_{ji} = \frac{1}{2} R_j{}^{t\alpha} R_{it\alpha} + (n+m-1)\, {}^B g_{ji} $$

and from (4.9)

$$ (4.10) \qquad {}^B K_{ji} = (n+3m-1)\, {}^B g_{ji} \,, $$

$$ (4.11) \qquad {}^B K = n(n+3m-1) \,. $$

For these formulas see also Ishihara and Konishi [4].

(4.10) shows that the base manifold is an Einstein manifold. Hence we get

$$ {}^B\nabla_j {}^B\nabla_i {}^B K = 0 \,, \qquad {}^B\nabla_t {}^B\nabla^t {}^B K_{ji} = 0 \,, $$

$$ {}^B K_{jt} {}^B K_i{}^t = (n+3m-1)^2\, {}^B g_{ji} \,, $$

$$ {}^B K_{jtsi} {}^B K^{ts} = (n+3m-1)^2\, {}^B g_{ji} \,, $$

$$ {}^B K_{ts} {}^B K^{ts} = n(n+3m-1)^2 = \frac{({}^B K)^2}{n} \,. $$

By straightforward calculation we get

$$ \begin{aligned} {}^B K_{jtsr} {}^B K_i{}^{tsr} = {} & 2(n-1)\, {}^B g_{ji} + 3 R_{js}{}^\alpha R_i{}^s{}_\alpha \\ & + \frac{1}{4} R_{ts}{}^\alpha R^{ts\beta} R_{rj\alpha} R^r{}_{i\beta} - \frac{1}{4} R^{ts\beta} R^r{}_{i\beta} R_{tj}{}^\alpha R_{sr\alpha} \\ & - \frac{1}{4} R_{ts}{}^\alpha R_{rj\alpha} R^t{}_i{}^\beta R^{sr}{}_\beta + \frac{1}{8} R_{tj}{}^\alpha R_{sr\alpha} R^t{}_i{}^\beta R^{sr}{}_\beta \\ & - \frac{1}{8} R_{sj}{}^\alpha R_{tr\alpha} R^t{}_i{}^\beta R^{sr}{}_\beta \,. \end{aligned} $$

Applying (4.8) and (4.9) to this formula we get

$$ \begin{aligned} {}^B K_{jtsr} {}^B K_i{}^{tsr} = {} & \{2(n-1+6m)+6nm\}\, {}^B g_{ji} \\ & - \frac{1}{4} R^{ts\beta} R^r{}_{i\beta} R_{tj}{}^\alpha R_{sr\alpha} - \frac{1}{4} R_{ts}{}^\alpha R_{rj\alpha} R^t{}_i{}^\beta R^{sr}{}_\beta \\ & - \frac{1}{8} R_{sj}{}^\alpha R_{tr\alpha} R^t{}_i{}^\beta R^{sr}{}_\beta \,, \end{aligned} $$

and finally applying (4.7)

$$ {}^B K_{jtsr} {}^B K_i{}^{tsr} = 2(n-1+12m+3nm-3m^2)\, {}^B g_{ji} \,. $$

This proves that ${}^B g$ is a critical Riemannian metric $g_D$. That ${}^B g$ is a critical Riemannian metric $g_C$ and $g_B$ simultaneously is immediately proved.

From Escobales' result we obtain the following corollary.

COROLLARY 4.2. *The standard Riemannian metrics on $CP(n)$ and $QP(n)$ are critical Riemannian metrics in the sense A, B, C, D.*

In this corollary a standard Riemannian metric means a Riemannian metric induced by the projection $\pi$.

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