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Concerning the bounded case of the Bernstein-Nachbin approximation problem

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Summary. Recently we gave a solution to the Bernstein-Nachbin Approximation Problem in the general complex case. As a corollary, we obtained the quasi-analytic, the analytic, and the bounded criteria for localizability in the general complex case. This generalizes the known results of the real or self-adjoint complex cases, in the same way that Bishop's Theorem generalizes the Weierstrass-Stone Theorem. In this paper, we present a direct proof of the bounded criterion for localizability, and show how it can be used to get a new proof of our solution of the Bernstein-Nachbin Approximation Problem. The proof in the real case is based in the idea of the proof of the Weierstrass-Stone Theorem discovered by one of us; the general complex case follows by Zorn's Lemma.

§1. Introduction.

Throughout this paper X denotes a Hausdorff topological space, and $A \subset C(X; K)$, where $K=R$ or C, denotes a subalgebra. A *vector fibration over* X is a pair $(X, (F_x)_{x\in X})$, where each F_x is a vector space over the field K. A *cross-section* is then any element f of the vector space Cartesian product of the vector spaces F_x, i.e., $f=(f(x))_{x\in X}$. A *weight on* X is a function v on X such that $v(x)$ is a seminorm over F_x for each $x\in X$. A *Nachbin space* LV_∞ is a vector space of cross-sections f such that the mapping $x\in X\mapsto v(x)[f(x)]$ is upper semicontinuous and null at infinity on X for each weight $v\in V$, equipped with the topology defined by the family of seminorms of the form

$$\|f\|_v=\sup\{v(x)[f(x)];\ x\in X\}.$$

For simplicity, and without loss of generality, the set V is assumed to be *directed*, i.e., given $u, v\in V$ there is $w\in W$ and $t>0$ such that $u(x)\leq t\cdot w(x)$ and $v(x)\leq t\cdot w(x)$, for all $x\in X$.

Throughout this paper $W\subset LV_\infty$ denotes a vector subspace which is an A-module, i.e., if $a\in A$ and $g\in W$ then the cross-section $ag=(a(x)g(x))_{x\in X}$ be-

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longs to W. In this context, the *Bernstein-Nachbin approximation problem* consists in asking for a description of the closure of W in LW_∞. Let P be a closed, pairwise disjoint covering of X. We say that W is *P-localizable in* LV_∞ if its closure consists of those $f \in LV_\infty$ such that, given any $S \in P$, any $v \in V$ and any $\varepsilon > 0$, there is some $g \in W$ such that $v(x)[f(x)-g(x)] < \varepsilon$ for all $x \in S$. The *strict Bernstein-Nachbin approximation problem* consists in asking for necessary and sufficient conditions for an A-module W to be P-localizable, when P is the set P_A of all equivalence classes modulo X/A. We recall that the equivalence relation X/A is defined as follows. For any pair $x, y \in X$, x is equivalent to y modulo X/A if, and only if $a(x)=a(y)$ for all $a \in A$. The *bounded case* of the Bernstein-Nachbin approximation problem is the case in which every $a \in A$ is bounded in the support of every weight $v \in V$.

DEFINITION 1. Let P be a closed, pairwise disjoint covering of X. We say that W is *sharply P-localizable in* LV_∞ if, for any $f \in LV_\infty$ and any $v \in V$, there is $S \in P$ such that

$$\inf\{\|f-g\|_v \,;\, g \in W\} = \inf\{\|f|S - g|S\|_v \,;\, g \in W\}\,.$$

When W is sharply P_A-localizable in LV_∞ we say that W is *sharply localizable under* A *in* LV_∞.

The following partition of the unity result had a fundamental role in the proofs of the results of [**5**]. It is going to be again a basic piece in the proofs that follow.

LEMMA 1. *Let* $A \subset C_b(X; R)$ *be a subalgebra containing the constants. For each equivalence class* $Y \subset X$ *modulo* X/A, *let there be given a compact set* $K_Y \subset X$, *disjoint from* Y. *Then, there exist equivalence classes* $Y_1, \cdots, Y_m \subset X$ *modulo* X/A *such that to each* $\delta > 0$, *there correspond* $a_1, \cdots, a_m \in A$ *with* $0 \leqq a_i \leqq 1$; $0 \leqq a_i(t) < \delta$ *for all* $t \in K_{Y_i}$, $i=1, \cdots, m$; *and* $a_1 + \cdots + a_m = 1$.

PROOF. See Lemma 8, [**5**].

§ 2. The real bounded case.

In this section we shall prove the *bounded case* of the Bernstein-Nachbin approximation problem for the case of modules of cross-sections over algebras of *real-valued* functions. As we said in the Introduction, the bounded case is the one in which every $a \in A$ is bounded on the support of every $v \in V$. According to Theorem 2 below, there is always sharp localizability in the real bounded case. Since sharp localizability implies localizability, Theorem 2 below extends Theorem 2, [**7**], in the real case.

THEOREM 1. *Let* $A \subset C(X; R)$ *be a subalgebra such that every* $a \in A$ *is bounded on the support of every* $v \in V$. *Let* W *be an* A-*module. For each* $f \in LV_\infty$ *and* $v \in V$ *we have*

$$\inf\{\|f-g\|_v;\ g\in W\}=\sup\{\inf\{\|f|Y-g|Y\|_v;\ g\in W\};\ Y\in P_A\}.$$

PROOF. The above formula was inspired by the so-called "strong" Stone-Weierstrass theorems proved by Buck [1] for vector-valued functions, and by Cunningham and Roy [2] for vector-fibrations of a special kind. Earlier, Glicksberg [3] had established such a formula for Bishop's theorem, i.e., for the partition of X into maximal anti-symmetric sets for A.

Let $d=\inf\{\|f-g\|_v;\ g\in W\}$, and let $c=\sup\{\inf\{\|f|Y-g|Y\|_v;\ g\in W\};\ Y\in P_A\}$. Clearly, $c\leqq d$.

To prove the reverse inequality, let $0<\varepsilon$. We may assume without loss of generality that A contains the constants and that $A\subset C_b(X;\ R)$. For each $Y\in P_A$ there exists $g_Y\in W$ such that $v(x)[f(x)-g_Y(x)]<c+\varepsilon/2$ for all $x\in Y$. Let $U_Y=\{t\in X;\ v(t)[f(t)-g_Y(t)]<c+\varepsilon/2\}$. Then U_Y is an open subset containing Y and such that its complement K_Y in X is a compact set. By Lemma 1, there exist equivalence classes $Y_1,\cdots,Y_n\in P_A$ such that to each $\delta>0$, there correspond functions $a_1,\cdots,a_n$ in A with $0\leqq a_i\leqq 1$; $0\leqq a_i(x)<\delta$ for $x\in K_i$, for $i=1,\cdots,n$, where $K_i=K_Y$ with $Y=Y_i$. Moreover, $a_1+\cdots+a_n=1$ on X. Let us choose $\delta>0$ such that $nM\delta<\varepsilon/2$, where $M=\max\{\|f-g_i\|_v;\ i=1,\cdots,n\}$ and $g_i=g_Y$ with $Y=Y_i$, and consider the corresponding functions $a_1,\cdots,a_n$ in A. Let $g=a_1g_1+a_2g_2+\cdots+a_ng_n$, which belongs to W, since W is an A-module. We claim that

$$v(x)[f(x)-g(x)]<c+\varepsilon$$

for all $x\in X$. Indeed, given $x\in X$, we have

$$v(x)[f(x)-g(x)]=v(x)[a_1(x)(f(x)-g_1(x))+\cdots+a_n(x)(f(x)-g_n(x))]$$

$$\leqq\sum_{i=1}^{n}a_i(x)v(x)[f(x)-g_i(x)].$$

Now, if $x\in K_i$ then $a_i(x)<\delta$, and therefore

$$a_i(x)v(x)[f(x)-g_i(x)]<\delta\|f-g_i\|_v<\delta M.$$

On the other hand, if $x\notin K_i$, then the following estimate is true

$$a_i(x)v(x)[f(x)-g_i(x)]\leqq a_i(x)(c+\varepsilon/2).$$

Combining both estimates, we get

$$v(x)[f(x)-g(x)]<nM\delta+(c+\varepsilon/2)(a_1(x)+\cdots+a_n(x))<c+\varepsilon.$$

This shows that $d<c+\varepsilon$. Since $\varepsilon>0$ was arbitrary, $d\leqq c$.

THEOREM 2. *Assume the hypothesis of Theorem* 1. *Then, every A-module $W\subset LV_\infty$ is sharply localizable under A in LV_∞.*

PROOF. Let $f\in LV_\infty$ and $v\in V$ be given. Let Z be the quotient space of

X by the equivalence relation X/A and let $\pi: X \to Z$ be the quotient map. By Lemma 1, [7], the map

$$z \in Z \longmapsto \|f|\pi^{-1}(z) - g|\pi^{-1}(z)\|_v$$

is upper semicontinuous and null at infinity on Z, for each $g \in W$. Hence the map

$$z \in Z \longmapsto \inf \{\|f|\pi^{-1}(z) - g|\pi^{-1}(z)\|_v ; g \in W\}$$

is upper semicontinuous and null at infinity on Z too. Therefore, it attains its supremum over Z for some point z. Let $Y = \pi^{-1}(z) \in P_A$. By Theorem 1, the above supremum is just $\inf \{\|f-g\|_v ; g \in W\}$. Hence, $\inf \{\|f-g\|_v ; g \in W\} = \inf \{\|f|Y - g|Y\|_v ; g \in W\}$, i. e., W is sharply lecalizable under A in LV_∞.

§ 3. The complex bounded case.

In this section we shall obtain the general, i. e., the not necessarily self-adjoint complex case of Theorem 2 above. This will be done by an application of Zorn's Lemma. Since the algebra A is not self-adjoint in general, we shall prove sharp localizability not with respect with the partition P_A of X into equivalence classes modulo X/A, but with respect to the partition $\mathcal{K}_A$ of X into maximal anti-symmetric sets for A. Of course, when A is self-adjoint both partitions agree. We recall that a subset $K \subset X$ is said to be *anti-symmetric for* A if, for any $a \in A$, the restriction $a|K$ being real-valued implies that $a|K$ is constant. We will follow the reasoning of [4], where a sharpened form of Bishop's theorem was proved.

THEOREM 3. *Let $A \subset C(X; C)$ be a complex subalgebra such that every $a \in A$ is bounded on the support of every $v \in V$, and let $W \subset LV_\infty$ be an A-module. Then W is sharply $\mathcal{K}_A$-localizable in LV_∞.*

PROOF. Let $f \in LV_\infty$ and $v \in V$ be given. Let

$$d = \inf \{\|f-g\|_v ; g \in W\} .$$

The case $d=0$ is trivial, since then

$$0 \leqq \inf \{\|f|K - g|K\|_v ; g \in W\} \leqq d = 0, \qquad \text{for any } K \in \mathcal{K}_A .$$

Assume $d>0$. Let $\mathcal{D}$ be the collection of all pairs (P, S) such that:

(a) P is a partition of X into non-empty closed pairwise disjoint subsets of X;

(b) S is an element of P such that $d = \inf \{\|f|S - g|S\|_v ; g \in W\}$.

The collection $\mathcal{D}$ is non-empty, since the pair $(\{X\}, X)$ satisfies properties (a) and (b). We define a partial order in $\mathcal{D}$ by setting $(P, S) \leqq (Q, T)$ if the partition Q is finer than P and $T\subset S$. Let $\mathcal{C}$ be a chain in $\mathcal{D}$. We define an upper bound of $\mathcal{C}$ in $\mathcal{D}$ as follows. For any $(P, S)\in\mathcal{C}$ and any $x\in X$, let $P(x)$ be the element of P which contains the point x. Define $Q(x)=\cap\{P(x)\,;\,(P, S)\in\mathcal{C}\}$. It is clear that $Q(x)$ is a non-empty closed snbset of X. Moreover, for any pair $x, y\in X$, $Q(x)\cap Q(y)\neq\emptyset$ implies that $Q(x)=Q(y)$. Hence, the collection of all distinct $Q(x)$, $x\in X$, is a partition Q of X into non-empty closed subsets. Define now $T=\cap\{S\,;\,(P, S)\in\mathcal{C}\}$. We will prove that $T\neq\emptyset$ and $d=\inf\{\|f|T-g|T\|_v\,;\,g\in W\}$ at the same time. Once this is proved, it is clear that (Q, T) will be an upper bound for the chain $\mathcal{C}$ in $\mathcal{D}$.

Since we have assumed $d>0$, let $\varepsilon>0$ be such that $d-\varepsilon>0$. For each $g\in W$, let $K(g)=\{x\in T\,;\,v(x)[f(x)-g(x)]\geqq d-\varepsilon\}$. For each $(P, S)\in\mathcal{C}$, define $K(g,(P, S))=\{x\in S\,;\,v(x)[f(x)-g(x)]\geqq d-\varepsilon\}$. Then $\{K(g,(P, S))\,;\,(P, S)\in\mathcal{C}\}$ is a family of compact subsets whose intersection is $K(g)$. Assume that $K(g)=\emptyset$. By the finite-intersection property, there are $(P_1, S_1), \cdots, (P_n, S_n)$ in $\mathcal{C}$, which we may assume to be in increasing order, because $\mathcal{C}$ is a chain, such that

$$K(g,(P_n, S_n))=\bigcap_{i=1}^{n} K(g,(P_i, S_i))=\emptyset\,.$$

This is impossible, since $d=\inf\{\|f|S_n-g|S_n\|_v\,;\,g\in W\}$. This contradiction shows that $K(g)\neq\emptyset$. Since $K(g)\subset T$, $T\neq\emptyset$ too. Moreover, $\|f|T-g|T\|_v\geqq d-\varepsilon$. Since $\varepsilon>0$ was arbitrary, $\|f|T-g|T\|_v\geqq d$. Therefore, $\inf\{\|f|T-g|T\|_v\,;\,g\in W\}\geqq d$. The reverse inequality being trivial, this ends the proof that (Q, T) is an upper bound for $\mathcal{C}$ in $\mathcal{D}$.

By Zorn's Lemma, there is a maximal element $(P, S)\in\mathcal{D}$. We claim that S is anti-symetric for A. Indeed, assume by contradiction that the set S is not anti-symmetric for A. Let A_S be the subalgebra of A consisting of the elements $a\in A$ which are real-valued on S. Then the subalgebra $A_S|S$ of $C(S\,;\,R)$ contains non-constant elements. By Theorem 2 above, applied to the $(A_S|S)$-module $W|S$ of the Nachbin space $L(V|S)_\infty$, there is a partition P_S of S, distinct from $\{S\}$, into equivalence classes modulo $S/(A_S|S)$, such that for some $T\in P_S$ we have

$$\inf\{\|f|S-g|S\|_v\,;\,g\in W\}=\inf\{\|f|T-g|T\|_v\,;\,g\in W\}\,.$$

The partition Q obtained by the elements of P distinct from S, and the elements of the above partition P_S is then strictly finer than P; and the element (Q, T) belongs to $\mathcal{D}$ and it is such that $(P, S)<(Q, T)$, contradicting the maximality of (P, S). So, S is anti-symmetric for A. Finally, let $K\in\mathcal{K}_A$ be the maximal anti-symmetric set for A containing the set S. It is the thing we are looking for.

REMARK 1. *Notice that a module W is sharply $\mathcal{K}_A$-localizable if, and only* *if, W is sharply localizable under A in LV_∞ in the sense of Definition* 5 *of* [**5**].

§ 4. Solution of the Bernstein-Nachbin approximation problem.

In this section we are concerned with the reduction of the search for sufficient conditions of sharp localizability to the problem of finding fundamental weights in the sense of Serge Bernstein. This reduction was accomplished in our previous paper [**5**]. Here, we want to show how Theorem 2 above can be used to do the mentioned reduction, thus presenting a different proof from [**5**]. Hence, our objective in this section is to prove Theorem 9 of [**5**] using Theorem 2 of the present paper. Namely we want to prove the following.

THEOREM 4. *Let $W \subset LV_\infty$ be an A-module. Suppose that there exist sets of generators $G(A)$ and $G(W)$, for A and W respectively, such that:*

(1) *$G(A)$ consists only of real-valued functions;*

(2) *given any $v \in V$, $a_1, \cdots, a_n \in G(A)$ and $g \in G(W)$, there exist $a_{n+1}, \cdots, a_N \in G(A)$, with $N \geqq n$, and $\omega \in \Omega_N$ such that $v(x)[g(x)] \leqq \omega(a_1(x), \cdots, a_n(x), \cdots, a_N(x))$ for all $x \in X$.*

Then W is sharply localizable under A in LV_∞.

We first explain the notation used above. Given a subalgebra $A \subset C(X; K)$, we denote by $G(A)$ a subset of A which topologically generates the algebra A as an algebra over K, i.e. the K-subalgebra of A generated by $G(A)$ is dense in A for the topology of $C(X; K)$. Similarly, $G(W)$ denotes a subset of W which topologically generates W as an A-module, i.e., the A-submodule of W generated by $G(W)$ is dense in W for the topology of the space LV_∞.

We next present some lemmas needed in the proof of Theorem 4. (See Nachbin [**6**]).

LEMMA 2. *Suppose that the hyphothesis of Theorem 4 are satisfied. Given $v \in V$, $a_1, \cdots, a_n \in G(A)$, $g \in G(W)$, $\alpha \in C_b(R^n; C)$ and $\delta > 0$, there is $h \in W$ such that $\|h - \alpha(a_1, \cdots, a_n)g\|_v < \delta$.*

PROOF. Given $v \in V$, $a_1, \cdots, a_n \in G(A)$ and $g \in G(W)$ there are $a_{n+1}, \cdots, a_N \in G(A)$, with $N \geqq n$, and $\omega \in \Omega_N$ such that $v(x)[g(x)] \leqq \omega(a_1(x), \cdots, a_n(x), \cdots, a_N(x))$ for all $x \in X$. Define the function $\beta \in C_b(R^N; C)$ by $\beta(t) = \alpha(t_1, \cdots, t_n)$ for all $t = (t_1, \cdots, t_n, \cdots, t_N) \in R^N$. By hypothesis, $\omega \in \Omega_N$; hence $C_b(R^N; C) \subset C\omega_\infty(R^N; C)$ and the polynomials are densely contained in the space $C\omega_\infty(R^N; C)$. Therefore a polynomial p in N variables can be found such that $\|p - \beta\|_\omega < \delta$. From this it follows that $\|h - \alpha(a_1, \cdots, a_n)g\|_v < \delta$, where $h = p(a_1, \cdots, a_n, \cdots, a_N)g$.

For the next two lemmas let us introduce some notations. If the algebra A has a set of generators consisting only of real-valued functions, say $G(A)$, we define B as the subalgebra of $C_b(X; C)$ of all functions of the form $\alpha(a_1, \cdots, a_n)$, where $n \geqq 1$, $a_1, \cdots, a_n \in G(A)$ and $\alpha \in C_b(R^n; C)$ are arbitrary. Notice that the equivalence relations X/A and X/B are the same. Notice too that B is self-adjoint. Next define $W(B)$ to be the B-submodule of LV_∞ generated by $G(W)$. (Recall that LV_∞ is a $C_b(X; C)$-module.) Finally, we remark that the vector subspace of W generated by $G(W)$ is contained in $W(B)$.

LEMMA 3. *Suppose that the hypothesis of Theorem 4 are satisfied. Given $f \in LV_\infty$ and $v \in V$, then*

$$d = \inf \{\|f-g\|_v ;\ g \in W\} \leqq \inf \{\|f-g\|_v ;\ g \in W(B)\} = d(B) .$$

PROOF. Let $\varepsilon > 0$ be given. There exists $g_B \in W(B)$ such that $\|f-g_B\|_v < d(B) + \varepsilon/2$. The cross-section g_B is of the form

$$g_B = \sum_{i=1}^{m} \alpha_i(a_1, \cdots, a_n) g_i$$

where $a_1, \cdots, a_n \in G(A)$, $g_i \in G(W)$, and $\alpha_i \in C_b(R^n; C)$, $i=1, \cdots, m$. By Lemma 2 applied with $\delta = \varepsilon/2m$, there are $h_1, \cdots, h_m \in W$ such that $\|h_i - \alpha_i(a_1, \cdots, a_n)g_i\|_v < \varepsilon/2m$. Let $g = h_1 + \cdots + h_m \in W$. Then $\|f-g\|_v \leqq \|f-g_B\|_v + \|g_B - g\|_v < d(B) + \varepsilon$. Hence, $d < d(B) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $d \leqq d(B)$.

LEMMA 4. *Suppose that $G(A)$ consists only of real-valued functions. For any $f \in LV_\infty$, $v \in V$, and $Y \subset X$ an equivalence class modulo X/A, we have*

$$c(Y, B) = \inf \{\|f|Y - g|Y\|_v ;\ g \in W(B)\} \leqq \inf \{\|f|Y - g|Y\|_v ;\ g \in W\} = c(Y) .$$

PROOF. Let $\varepsilon > 0$ be given. There exists $g \in W$ such that $\|f|Y - g|Y\|_v < c(Y) + \varepsilon$. In fact, we may assume that g belongs to the vector subspace generated by $G(W)$, because the A-submodule generated by $G(W)$ is dense in W for the topology of LV_∞, and the elements of A are constant on Y. Since the vector subspace generated by $G(W)$ is contained in $W(B)$, $g \in W(B)$. That is, $c(Y, B) < c(Y) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $c(Y, B) \leqq c(Y)$.

PROOF OF THEOREM 4. Let $f \in LV_\infty$ and $v \in V$ be given. We will prove that

$$d = \inf \{\|f-g\|_v ;\ g \in W\} = \sup \{ \inf \{\|f|Y - g|Y\|_v ;\ g \in W\} ;\ Y \in P_A\} = c .$$

Once this is done, the rest of the proof is exactly the same as the proof of Theorem 2 from Theorem 1.

Clearly, $c \leqq d$. On the other hand, by Lemma 3, $d \leqq d(B)$. By Theorem 3 applied to the self-adjoint algebra B and the B-module $W(B)$, we have $d(B) = \sup \{c(Y, B) ;\ Y \in P_B\}$. Since X/A and X/B are the same $\sup \{c(Y, B) ;\ Y \in P_B\} = \sup \{c(Y, B) ;\ Y \in P_A\}$. By Lemma 4, $\sup \{c(Y, B) ;\ Y \in P_A\} \leqq c$, whence $d(B) \leqq c$, and therefore $d \leqq c$.

REMARK 2. We used Theorem 3 in the proof of Theorem 4. However, it was the self-adjoint case that was used. This case follows easily from Theorem 2, since any self-adjoint subalgebra has a set of generators consisting only of real-valued functions.

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