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Fundamental solutions for operators of regularly hyperbolic type

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§ 0. Introduction.

In a recent paper [6] we have studied a global calculus of Fourier integral operators on R^n, and, as an application, constructed the fundamental solution $E_\phi(t, s)$ for the single equation of hyperbolic type:

$$(0.1)\qquad \begin{cases} Lu=D_t u-\lambda(t, X, D_x)u=f & \text{in } (0, T)\ (T>0), \\ u|_{t=0}=u_0, \end{cases}$$

where $E_\phi(t, s)$ is a Fourier integral operator with phase function $\phi(t, s; x, \xi)$ and symbol $e(t, s; x, \xi)$ of class $\mathcal{B}_t^1(S^0)$ $(0\leqq s\leqq t\leqq T)$.

In the present paper we shall construct the fundamental solutions for a regularly hyperbolic system of first order operators and for a regularly hyperbolic operator of higher order in the exact form on R^n. We shall first consider a system with the diagonal principal part, and reduce this system to a diagonal system $(\mathrm{mod}\, S^{-\infty})$ by the perfect diagonalizer (see Definition 2.2). Then, the fundamental solutions for general operators will be constructed by using the approximate fundamental solution (see Definition 2.3) for the equation (0.1). We note that our method is applicable to the diagonalizable hyperbolic system of first order with constant multiplicity.

§ 1. Fourier integral operators.

For a point $x=(x_1, \cdots, x_n)\in R^n$ and a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ of non-negative integers α_j we use notations: $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\alpha!=\alpha_1!\cdots\alpha_n!$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$, $\partial_{x_j}=\partial/\partial x_j$, $D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$, $D_{x_j}=-i\partial/\partial x_j$, $\langle x\rangle=\sqrt{1+|x|^2}$.

Let $\mathcal{S}$ denote the Schwartz space of rapidly decreasing functions, and let $\mathcal{B}$ denote the space of C^∞-functions in R^n whose derivatives of any order are all bounded. For $u\in\mathcal{S}$ the Fourier transform $\hat{u}(\xi)$ is defined by $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$ $(x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n)$. Let S^m denote the space of C^∞-symbols in R^{2n}

(1.1) $$\{p(x,\xi)\,;\,\forall\alpha,\beta,\ |\partial_\xi^\alpha D_x^\beta p(x,\xi)|\leqq \exists C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}\quad \text{on}\quad R^{2n}\}\,,$$

and define semi-norms $|p|_l^{(m)}$ $(l=0,1,\cdots)$ by

$$|p|_l^{(m)}=\underset{|\alpha+\beta|\leqq l}{\mathrm{Max}}\ \inf\{C_{\alpha,\beta}\ \text{of (1.1)}\}\,.$$

Then, the pseudo-differential operator $P=p(X,D_x)$ (denoted by $P\in S^m$) with the symbol $\sigma(P)(x,\xi)=p(x,\xi)$ is defined by

(1.2) $$Pu(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)d\xi\,,\qquad u\in\mathcal{S}\ \ (d\xi=(2\pi)^{-n}d\xi)\,.$$

As in [6] the real valued C^∞-function $\phi(x,\xi)$ is called a phase function, when $\phi(x,\xi)$ satisfies following conditions:

(1.3) $$\begin{cases}\text{(i)}\ \ \phi(x,\xi)-x\cdot\xi\in S^1\,,\\ \text{(ii)}\ \ |\nabla_x\phi(x,\xi)-\xi|\leqq(1-\varepsilon_0)|\xi|+C_0\quad(0<\varepsilon_0\leqq 1,\ 0<C_0)\,,\\ \text{(iii)}\ \ \|\nabla_x\nabla_\xi\phi(x,\xi)-I\|\leqq(1-\varepsilon_0')\quad(0<\varepsilon_0'\leqq 1)\end{cases}$$

(compare with Hörmander's in [3]).

Then, the Fourier integral operator $P=p_\phi(X,D_x)$ (denoted by $P\in S_\phi^m$) with phase function $\phi(x,\xi)$ and symbol $p(x,\xi)$ of class S^m is defined by

(1.4) $$P_\phi u(x)=\int e^{i\phi(x,\xi)}p(x,\xi)\hat{u}(\xi)d\xi\,,\qquad u\in\mathcal{S}\,.$$

We note that if we write

$$\begin{cases}P_\phi u(x)=\int e^{ix\cdot\xi}(e^{iJ(x,\xi)}p(x,\xi))\hat{u}(\xi)d\xi\,,\\ Pu(x)=\int e^{i\phi(x,\xi)}(e^{-iJ(x,\xi)}p(x,\xi))\hat{u}(\xi)d\xi\end{cases}$$

for $p(x,\xi)\in S^{-\infty}=\bigcap_m S^m$ and $J(x,\xi)=\phi(x,\xi)-x\cdot\xi$, then we can easily see that

(1.5) $$S_\phi^{-\infty}\ (=\bigcap_m S_\phi^m)=S^{-\infty}\ (=\bigcap_m S^m)\,.$$

§ 2. Construction of fundamental solutions.

Consider the Cauchy problem of the form

(2.1) $$\begin{cases}LU=D_tU+A^{(1)}(t)U+A^{(0)}(t)U=F\quad \text{in } (0,T)\ (T>0)\,,\\ U|_{t=0}=U_0\,,\end{cases}$$

where $A^{(j)}(t)\in\mathcal{B}_t(S^j)$ on $[0,T]$ $(j=0,1)$. which means that $A^{(j)}(t)$ are $l\times l$ matrices of operators of class S^j, and the symbols $\sigma(A^{(j)}(t))$ are infinitely differentiable with respect to t in the topology of S^j. For simplicity we also assume that the symbol $\sigma(A^{(1)}(t))(x, \xi)$ is real valued.

DEFINITION 2.1. The operator $\boldsymbol{L}$ of (2.1) is said to be regularly hyperbolic, when all the eigenvalues $\lambda_j^{(1)}(t, x, \xi)$ $(j=1, \cdots, l)$ of $\sigma(A^{(1)}(t))$ are real valued, belong to $\mathscr{B}_t(S^1)$ on $[0, T]$, and satisfy for a constant $c_0>0$

$$|\lambda_j^{(1)}(t, x, \xi)-\lambda_{j'}^{(1)}(t, x, \xi)| \geqq c_0\langle\xi\rangle \quad \text{on } [0, T]\times R^{2n} \ (j \neq j'). \tag{2.2}$$

DEFINITION 2.2. i) The operator $\boldsymbol{L}$ of (2.1) is said to be diagonalizable, when there exists $N_0(t) \in \mathscr{B}_t(S^0)$ on $[0, T]$ such that

$$|\det(\sigma(N_0(t)))| \geqq c_1 \qquad \text{on } [0, T]\times R^{2n} \tag{2.3}$$

for a constant $c_1>0$, and we can write

$$\sigma(N_0(t))^{-1}\sigma(A^{(1)}(t))\sigma(N_0(t))=\sigma(\mathscr{D}^{(1)}) \tag{2.4}$$

for a diagonal matrix

$$\sigma(\mathscr{D}^{(1)}(t))=\begin{pmatrix} \lambda_1^{(1)}(t) & & 0 \\ & \ddots & \\ 0 & & \lambda_l^{(1)}(t) \end{pmatrix} \in \mathscr{B}_t(S^1) \qquad \text{on } [0, T]. \tag{2.5}$$

ii) The operator $\boldsymbol{L}$ of (2.1) is said to be perfectly diagonalizable, when there exists $N(t) \in \mathscr{B}_t(S^0)$ on $[0, T]$ such that $\sigma(N(t))$ satisfies (2.3) and we can write

$$\boldsymbol{L}N(t) \equiv N(t)(D_t+\mathscr{D}(t)) \qquad \text{on } [0, T] \ (\bmod \mathscr{B}_t(S^{-\infty})) \tag{2.6}$$

for a diagonal $\mathscr{D}(t) \in \mathscr{B}_t(S^1)$ ($N_0(t)$, $N(t)$ are called the diagonalizer, the perfect diagonalizer for $\boldsymbol{L}$, respectively).

DEFINITION 2.3. $\tilde{E}_\phi(t, s)$ $(\in \mathscr{B}_t(S_\phi^0))$ is said to be the approximate fundamental solution of (2.1), when $\tilde{E}_\phi(t, s)$ satisfies

$$\begin{cases} \boldsymbol{L}_t\tilde{E}_\phi(t, s)=R(t, s) \in \mathscr{B}_t(S^{-\infty}) & \text{on } [0, T], \\ \tilde{E}_\phi(s, s)=I \text{ (the identity matrix operator)}. \end{cases} \tag{2.7}$$

LEMMA 2.4. *Let the operator* $\boldsymbol{L}$ *of* (2.1) *have the form*

$$\boldsymbol{L}U=D_tU+\mathscr{D}^{(1)}(t)U+B^{(0)}(t)U, \tag{2.8}$$

where $\mathscr{D}^{(1)}(t) \in \mathscr{B}_t(S^1)$ *and* $B^{(0)}(t) \in \mathscr{B}_t(S^0)$ *on* $[0, T]$, *and* $\sigma(\mathscr{D}^{(1)})$ *has the form* (2.5) *and satisfies* (2.2). *Then,* $\boldsymbol{L}$ *is perfectly diagonalizable.*

COROLLARY. *The regularly hyperbolic operator* $\boldsymbol{L}$ *is diagonalizable, and, moreover, perfectly diagonalizable.*

PROOF OF LEMMA 2.4. Assume that the perfect diagonalizer of $\boldsymbol{L}$ has the form

$$\sigma(N(t)) \sim I + \sum_{\nu=1}^{\infty} \sigma(N^{(-\nu)}(t)), \tag{2.9}$$

where

$$\sigma(N^{(-\nu)}(t)) = (n_{jk}^{(-\nu)}(t)\,;\, n_{jj}^{(-\nu)}(t)=0) \in \mathscr{B}_t(S^{-\nu}) \quad \text{on } [0, T]. \tag{2.10}$$

Then, the parametrix $Q(t)$ of $N(t)$ has the form

$$\begin{aligned}\sigma(\mathbf{Q}(t)) &\sim I + \sum_{k=1}^{\infty} (-1)^k \sigma\Big(\Big(\sum_{\nu=1}^{\infty} N^{(-\nu)}(t)\Big)^k\Big) \\ &\sim I + \sum_{\nu=1}^{\infty} \sigma(M^{(-\nu)}(t)),\end{aligned} \tag{2.11}$$

where $M^{(-\nu)}(t)$ belong to $\mathscr{B}_t(S^{-\nu})$ and are determined by $N^{(-1)}(t), \cdots, N^{(-\nu)}(t)$. Then, we can write

$$\boldsymbol{L}N(t) \equiv N(t)(D_t + \mathscr{D}^{(1)}(t) + F(t)) \quad (\text{mod } \mathscr{B}_t(S^{-\infty})), \tag{2.12}$$

where $F(t) \in \mathscr{B}_t(S^0)$ and has the form

$$F(t) = Q(t)[\mathscr{D}^{(1)}(t), N(t)] + Q(t)B^{(0)}(t)N(t) + Q(t)D_tN(t).$$

Then, noting (2.9) and (2.11) we can write

$$\begin{aligned}\sigma(F) \sim & \sum_{\nu=1}^{\infty} \sigma([\mathscr{D}^{(1)}, N^{(-\nu)}]) \\ & + \{\sigma(B^{(0)}) + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \sigma(M^{(-\nu)}[\mathscr{D}^{(1)}, N^{(-\nu)}]) + \sum_{\nu=1}^{\infty} \sigma(B^{(0)}N^{(-\nu)}) \\ & + \sum_{\mu=1}^{\infty} \sigma(M^{(-\nu)}B^{(0)}) + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \sigma(M^{(-\mu)}B^{(0)}N^{(-\nu)}) \\ & + \sum_{\nu=1}^{\infty} \sigma(D_tN^{(-\nu)}) + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} \sigma(M^{(-\mu)}D_tN^{(-\nu)})\}.\end{aligned}$$

Hence, we can rewrite $\sigma(F)$ in the form

$$\sigma(F(t)) \sim \sum_{\nu=0}^{\infty} (\sigma(J^{(-\nu)}) - \sigma(\Gamma^{(-\nu)})), \tag{2.13}$$

where

$$\begin{cases} \sigma(J^{(-\nu)}) = \sigma(\mathscr{D}^{(1)})\sigma(N^{(-\nu-1)}) - \sigma(N^{(-\nu-1)})\sigma(\mathscr{D}^{(1)}) \in \mathscr{B}_t(S^{-\nu}), \\ \sigma(\Gamma^{(-\nu)}) = (\gamma_{jk}^{(-\nu)}) \in \mathscr{B}_t(S^{-\nu}) \quad (\nu = 0, 1, \cdots), \end{cases}$$

and $\sigma(\Gamma^{(-\nu)})$ are determined by $\sigma(\mathscr{D}^{(1)})$, $\sigma(B^{(0)})$, $\sigma(N^{(-1)}), \cdots, \sigma(N^{(-\nu)})$. Now we set

$$n_{jk}^{(-\nu)} = (\lambda_j^{(1)} - \lambda_k^{(1)})^{-1} \cdot \gamma_{jk}^{(-\nu+1)} \ (j \neq k), \quad = 0 \ (j = k), \quad \nu = 1, 2, \cdots,$$

then, noting $\sigma(J^{(-\nu)}) = ((\lambda_j^{(1)} - \lambda_k^{(1)}) \cdot n_{jk}^{(-\nu-1)})$, we see that $\sigma(F(t))$ is equal to a diagonal matrix asymptotically. Hence the perfect diagonalizer $N(t)$ can be found by (2.9). Q. E. D.

PROOF OF COROLLARY. Let $N_0(t)$ be the diagonalizer for $\boldsymbol{L}$ and let $Q_0(t)$ be the parametrix of $N_0(t)$. Then we can write

$$\boldsymbol{L}N_0(t) \equiv N_0(t)\tilde{\boldsymbol{L}}_0 \qquad (\text{mod } \mathcal{B}_t(S^{-\infty})),$$

where

$$\tilde{\boldsymbol{L}}_0 = D_t + Q_0 A^{(1)} N_0 + (Q_0 A^{(0)} N_0 + Q_0 D_t N_0),$$

and using (2.4) we see that $\tilde{\boldsymbol{L}}_0$ has the form (2.8). Hence, by Lemma 2.4 we can find the perfect diagonalizer $\tilde{N}(t)$ of $\tilde{\boldsymbol{L}}_0$, and setting $N(t)=N_0(t)\tilde{N}(t)$ we see that $N(t)$ is the perfect diagonalizer for $\boldsymbol{L}$. Q. E. D.

THEOREM 2.5. *Let the operator* $\boldsymbol{L}$ *of* (2.1) *is regularly hyperbolic. Then, for a small* T_0 $(0<T_0\leqq T)$ *the approximate fundamental solution* $\tilde{E}_\phi(t, s)$ *can be found in the form*

$$\tilde{E}_\phi(t, s) = N(t)E_{0,\phi}(t, s)Q(s) + (I - N(s)Q(s)), \tag{2.14}$$

where $N(t)$ *is the perfect diagonalizer of* $\boldsymbol{L}$ *and* $Q(t)$ *is the parametrix of* $N(t)$ *such that* $(I-N(t)Q(t)) \in \mathcal{B}_t(S^{-\infty})$ *on* $[0, T]$, *and* $E_{0,\phi}(t, s)$ *is the approximate fundamental solution for* $\boldsymbol{L}_0 = D_t + \mathcal{D}(t)$ *of* (2.6). *Moreover, adding some* $e^{-\infty}(t, s) \in \mathcal{B}_t(S^{-\infty})$ $(0\leqq s\leqq t\leqq T_0)$, *the fundamental solution* $E_\phi(t, s)$ *can be found in the form*

$$E_\phi(t, s) = \tilde{E}_\phi(t, s) + e^{-\infty}(t, s). \tag{2.15}$$

REMARK. When $\boldsymbol{L}$ is diagonalizable and $\mathcal{D}^{(1)}(t)$ has the form

$$\mathcal{D}^{(1)} = \begin{pmatrix} \mathcal{D}_1^{(1)} & & 0 \\ & \ddots & \\ 0 & & \mathcal{D}_k^{(1)} \end{pmatrix}, \quad \text{where } \sigma(\mathcal{D}_j^{(1)}) = \begin{pmatrix} \lambda_j^{(1)}(t) & & 0 \\ & \ddots & \\ 0 & & \lambda_j^{(1)}(t) \end{pmatrix}$$

and $\{\lambda_j^{(1)}(t)\}_{j=1}^k$ satisfies (2.2), then, by the same procedure with the proof of Lemma 2.4, we can find $N(t)$ such that $\mathcal{D}(t)$ of (2.6) has the form: $\mathcal{D}(t) = \begin{pmatrix} \mathcal{D}_1(t) & & 0 \\ & \ddots & \\ 0 & & \mathcal{D}_k(t) \end{pmatrix}$, where $\mathcal{D}_j(t) = \mathcal{D}_j^{(1)}(t) + B_j^{(0)}(t)$ for some $B_j^{(0)}(t) \in \mathcal{B}_t(S^0)$, and construct the fundamental solution for such $\boldsymbol{L}$. In Lax [7] and Ludwig [8] it seems that the approximate fundamental solutions are constructed for our non-analytic case. We also note that in Èskin [2] the exact fundamental solutions are constructed when R^n is replaced by a compact C^∞ manifold M.

PROOF. Since $\sigma(\mathcal{D}) - \sigma(\mathcal{D}^{(1)}) \in \mathcal{B}_t(S^0)$ on $[0, T]$, by means of Theorem 3.2° in [6] we can construct the approximate fundamental solution $E_{0,\phi}(t, s)$ for $\boldsymbol{L}_0$ of the form

(2.16) $$E_{0,\phi}(t,s)=\begin{pmatrix} E_{0,\phi_1}(t,s) & & 0 \\ & \ddots & \\ 0 & & E_{0,\phi_l}(t,s)\end{pmatrix} \quad (0\leqq s\leqq t\leqq T_0)$$

for a small T_0 $(0<T_0\leqq T)$ which belongs to $\mathcal{B}_t^1(S^0)$ and moreover to $\mathcal{B}_t(S^0)$ since $\mathcal{D}(t)\in\mathcal{B}_t(S^1)$, where ϕ_j $(j=1,\cdots,l)$ are the solution of

(2.17) $$\partial_t\phi+\lambda_j^{(1)}(t,x,\nabla_x\phi)=0 \qquad (0\leqq s\leqq t\leqq T_0),\ \phi|_{t=s}=x\cdot\xi,$$

which satisfy the conditions of (1.3). Then, using (2.6) we get the approximate fundamental solution $\tilde{E}_\phi(t,s)$ by (2.14). Next as in [10] we set

(2.18) $$\begin{cases} W_1(t,s)=-iR(t,s) \qquad (=-iL_t\tilde{E}_\phi), \\ W_k(t,s)=\int_s^t W_1(t,\theta)W_{k-1}(\theta,s)d\theta \qquad (k=2,3,\cdots) \end{cases}$$

$$\Big(=\int_s^t\int_s^{s_1}\cdots\int_s^{s_{k-2}} W_1(t,s_1)W_1(s_1,s_2)\cdots W_1(s_{k-1},s) \cdot ds_1ds_2\cdots ds_{k-1} \qquad (s_0=t)\Big),$$

and regard for any real τ

$$W_1(t,s_1)\in S^\tau,\quad W_1(s_j,s_{j+1})\in S^0 \quad (j=1,\cdots,k-1 \text{ and } s_k=s).$$

Then, from a theorem on the theory of pseudo-differential operators of multiple symbol (see [4] or [5]) we have for

$$F(t,s;s_1,\cdots,s_{k-1})=W_1(t,s_1)W_1(s_1,s_2)\cdots W_1(s_{k-1},s)$$

an inequality

$$|\sigma(F)(t,s)|_{l_0}^{(\tau)}\leqq C^k\,(\max_{0\leqq s\leqq t\leqq T}\{|\sigma(R(t,s))|_l^{(\tau)}\})^k \qquad (k=2,3,\cdots)$$

for any τ and l_0, where C and l are constants depending on τ and l_0 (but independent of k). Hence, from (2.18) we have

$$|\sigma(W_k(t,s))|_{l_0}^{(\tau)}\leqq C_1^k\frac{(t-s)^{k-1}}{(k-1)!}\leqq C_1^k\frac{T_0^{k-1}}{(k-1)!} \quad (0\leqq s\leqq t\leqq T_0),$$

and consequently we see that $W(t,s)=\sum_{k=1}^\infty W_k(t,s)$ converges in $\mathcal{B}_t(S^\tau)$ for any τ. Then, setting $E_\phi(t,s)=\tilde{E}_\phi(t,s)+\int_s^t\tilde{E}_\phi(t,\theta)W(\theta,s)d\theta$ and noting (1.6) we get the desired fundamental solution $E_\phi(t,s)$ of $\boldsymbol{L}$. Q.E.D.

Now consider the single higher order operator of regularly hyperbolic type

(2.19) $$L=D_t^l+\sum_{j=1}^{l-1}a_j(t,X,D_x)D_t^j+\sum_{j=1}^{l-1}b_j(t,X,D_x)D_t^j \quad \text{on } [0,T],$$

where $a_j(t)\in\mathscr{B}_t(S^{l-j})$ and $b_j(t)\in\mathscr{B}_t(S^{l-j-1})$, and the roots $\lambda_j^{(1)}(t, x, \xi)$ $(j=1, \cdots, l)$ of the equation

$$\lambda^l+\sum_{j=1}^{l-1} a_j(t, x, \xi)\lambda^j=0 \tag{2.20}$$

are real and distinct, and satisfy (2.2). Set for $\Lambda=\langle D_x\rangle$

$$A^{(1)}(t)=\begin{pmatrix} 0 & -\Lambda & \cdots\cdots & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 0 \\ & & \ddots & \\ 0 & \cdots\cdots & 0 & -\Lambda \\ a_0\Lambda^{-(l-1)} & \cdots & & a_{l-1} \end{pmatrix}, \quad A^{(0)}(t)=\begin{pmatrix} 0 & \cdots\cdots\cdots & 0 \\ & \cdots\cdots\cdots & \\ 0 & \cdots\cdots\cdots & 0 \\ b_0\Lambda^{-(l-1)} & \cdots & b_{l-1} \end{pmatrix} \tag{2.21}$$

and let V be one of the Fréchet spaces $\mathcal{S}$, $\mathscr{B}$, $\mathcal{S}'$. Then, we have

THEOREM 2.6. i) *The solution $U(t)\in\mathscr{B}_t^1(V)$ of the Cauchy problem* (2.1) *for $U_0\in V$ and $F\in\mathscr{B}_t^0(V)$ exists uniquely in $(0, T_0)$ and can be solved by*

$$U(t)=E_\phi(t, 0)U_0+\int_0^t E_\phi(t, s)F(s)ds\,, \tag{2.22}$$

where $E_\phi(t, s)$ is the fundamental solution of $\boldsymbol{L}$.

ii) *The solution $u(t)\in\mathscr{B}_t^l(V)$ of the Cauchy problem*

$$Lu=f \text{ in } (0, T_0), \quad D_t^{j-1}u|_{t=0}=\varphi_j \quad (j=1, \cdots, l) \tag{2.23}$$

for $\varphi_j\in V$ and $f\in\mathscr{B}_t^0(V)$ exists uniquely and can be solved by

$$\begin{aligned} u(t)=&\sum_{j=1}^{l} \Lambda^{-(l-1)}\Big(\sum_{k=1}^{l} N_{1,k}(t)E_{\phi_k}(t, 0)Q_{kj}(0)+e_j^{-\infty}(t, 0)\Big)\Lambda^{l-j}\varphi_j \\ &+\int_0^t \Lambda^{-(l-1)}\Big(\sum_{k=1}^{l} N_{1,k}(t)E_{\phi_k}(t, s)Q_{kl}(s)+e_l^{-\infty}(t, s)\Big)f(s)ds\,, \end{aligned} \tag{2.24}$$

where $N(t)=(N_{jk}(t))$ is the perfect diagonalizer for $\boldsymbol{L}$ of (2.1) *corresponding to $A^{(j)}(t)$ $(j=1, 0)$ of* (2.21) *and $F(t)=(0, \cdots, f(t))^t$, $Q(t)=(Q_{jk}(t))$ is the parametrix for $N(t)$, and $e_j^{-\infty}(t, s)$ $(j=1, \cdots, l)$ are appropriate operators of class $\mathscr{B}_t(S^{-\infty})$.*

REMARK 1°. Let H_s be the Sobolev space $\{u\in\mathcal{S}'\,;\ \Lambda^s u\in L^2(R^n)\}$. Then, by means of the H_s-theory of Fourier integral operators in [**6**] (which states that $P_\phi(\in S_\phi^m)$ maps H_{s+m} continuously into H_s for any H_s) we see that the solution $U(t)$ belongs to $\mathscr{B}_t^0(H_s)\cap\mathscr{B}_t^1(H_{s-1})$ for $U_0\in H_s$ and $F\in\mathscr{B}_t^0(H_s)$, and that the solution $u(t)$ belongs to $\mathscr{B}_t^0(H_{s+l-1})\cap\mathscr{B}_t^1(H_{s+l-2})\cap\cdots\cap\mathscr{B}_1^l(H_{s-1})$ for $\varphi_j\in H_{s+l-j}$ $(j=1, \cdots, l)$ and $f\in\mathscr{B}_t^0(H_s)$.

2°. In Calderón [**1**] and Mizohata [**9**] the problems (2.1) and (2.23) are solved by means of energy inequalities.

3°. The principal symbol of $N(t)$ is equal to

$$\Big((\lambda_k^{(1)}/\langle\xi\rangle)^j\,;\ \begin{matrix} j\downarrow 0, \cdots, l-1 \\ k\rightarrow 1, \cdots, l \end{matrix}\Big).$$

PROOF. i) is clear. ii) Set $U=(u_1, \cdots, u_l)^t$ for $u_j=\Lambda^{l-j}D_t^{j-1}u$ $(j=1, \cdots, l)$, $U_0=(\varphi_1, \cdots, \varphi_l)^t$ and $F(t)=(0, \cdots, 0, f(t))^t$. Then, we get the regularly hyperbolic system $\boldsymbol{L}$ of the form (2.1), and the diagonalizer $N_0(t)$ of $\boldsymbol{L}$ is given by

$$(N_0(t))=\begin{pmatrix} 1 & \cdots\cdots\cdots & 1 \\ \lambda_1 & \cdots\cdots\cdots & \lambda_l \\ & \cdots\cdots\cdots & \\ \lambda_1^{l-1} & \cdots\cdots & \lambda_1^{l-1} \end{pmatrix}.$$

Hence, from the corollary of Lemma 2.4 $\boldsymbol{L}$ has the perfect diagonalizer $N(t)$. Then the solution $u(t)$ of the problem (2.23) is given by $u=\Lambda^{-(l-1)}u_1$, which is represented by (2.24). Q. E. D.

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