J. Math. Soc. Japan
Vol. 29, No. 4, 1977

Real parts of Banach function algebras

By Jun-ichi TANAKA

(Received Sept. 28, 1976)
(Revised Dec. 18, 1976)

§ 1. Introduction.

In this paper we study real parts of Banach function algebras. In § 2, we shall show that uniformly closed ideals in algebras of continuous functions are determined by their real parts. In § 3, we shall give a slight generalization of a theorem of Arenson [**1**] on real parts of function algebras to the case of Banach function algebras, and use it to derive results on complex conjugation.

Let X be a compact Hausdorff space and $C(X)$ (resp. $C_R(X)$) be the space of all complex (resp. real) valued continuous functions on X. For any subset B of $C(X)$, we write as follows:

$$\mathrm{Re}\, B=\{\mathrm{Re}\, f;\ f\in B\}\ ,$$

$$B_R=\{f\in B;\ f \text{ is real-valued}\}\ ,$$

$$[B]=\text{the uniform closure of } B \text{ in } C(X)\ ,$$

$$\bar{B}=\{\bar{f};\ f\in B\}\ (\bar{f} \text{ is the complex conjugate of } f)\ ,$$

$$Z(B)=\{x\in X;\ f(x)=0 \text{ for any } f\in B\}\ .$$

We shall call A a *function algebra* on X if A is a uniformly closed subalgebra of $C(X)$ which separates points in X and contains constant functions. A subalgebra A in $C(X)$ is called a *Banach function algebra* if A is a Banach algebra in its own norm. Let A be a linear subspace of $C(X)$. Then we denote by $\mathcal{K}_0(A)$ (resp. $\mathcal{K}_1(A)$) the Shilov (resp. Bishop) decomposition for A, that is the collection of all subsets of X which are maximal with respect to inclusion and on which every function from A_R is constant (resp. the collection of maximal antisymmetric subsets).

In our discussions in the forthcoming sections, we need the following lemma, which is a somewhat strengthened version of Bishop's theorem (cf. [**4**], [**6**; 3.3]).

LEMMA 1.1. *Let B be a uniformly closed subspace of $C(X)$ and let A be a linear subspace of $C(X)$ such that $B\cdot A\subset B$, i.e., $fg\in B$ whenever $f\in A$, $g\in B$. If $f\in C(X)$ and $f|K\in B|K$ for any $K\in\mathcal{K}_1(A)$, then $f\in B$.*

This lemma can be proved in the same way as the proof of Theorem 1.1 in [4].

§ 2. Real parts of uniformly closed ideals.

In this section, we shall show that uniformly closed ideals in algebras of continuous functions are determined by their real parts. We begin with the following

THEOREM 2.1. *Let I be a uniformly closed subspace of $C(X)$, and let B be a linear subspace of $C(X)$. Suppose that*

$$\text{(a)}\quad I\subset B, \qquad \text{(b)}\quad \operatorname{Re} I=\operatorname{Re} B, \qquad \text{(c)}\quad I\cdot B_R\subset I,$$

then $I=B$.

PROOF. It is not hard to see that these conditions imply $Z(I_R)=Z(B_R)$. In fact, if $x\notin Z(B_R)$, then there exists a $f\in I$ such that $\operatorname{Re} f(x)\neq 0$, and $\operatorname{Re} f$, $\operatorname{Im} f\in B_R$ by the conditions (a) and (b). Since $|f|^2\in I_R$ by the condition (c), $x\notin Z(I_R)$. The other inclusion follows from the condition (a). Hence, we obtain $(B\cap\bar{B})|K=(I\cap\bar{I})|K$ for any $K\in\mathcal{K}_1(B\cap\bar{B})$. On the other hand, $I\cap\bar{I}$ is a uniformly closed subalgebra of $C(X)$ and $fg\in I\cap\bar{I}$ whenever $f\in I\cap I$, $g\in B\cap\bar{B}$. It follows from Lemma 1.1 that $B\cap\bar{B}\subset I\cap\bar{I}$ and thus $B\cap\bar{B}=I\cap\bar{I}$. If $f\in B$, then there exists a $g\in I$ such that $\operatorname{Re} f=\operatorname{Re} g$ by the condition (b). Since $f-g\in B\cap\bar{B}$, we have $f=g+(f-g)\in I$. This completes the proof.

COROLLARY 2.2. *Let A be a subalgebra of $C(X)$ and let I be a uniformly closed ideal of A. Then we obtain the following:*

(i) *If $f\in A$, $\operatorname{Re} f\in\operatorname{Re} I$, $\operatorname{Im} f\in\operatorname{Re} I$, then $f\in I$.*

(ii) *If J is an ideal in A and $\operatorname{Re} I=\operatorname{Re} J$, then $I=[J]$.*

PROOF. (i) Let B be the linear subspace of $C(X)$ generated by f and I. Then we can easily see that B and I satisfy the hypothesis of Theorem 2.1, and it concludes that $B=I$.

(ii) We set $I+J=\{f+g;\ f\in I,\ g\in J\}$. Then $I+J$ is an ideal in A, $I+J\supset I$ and $\operatorname{Re}(I+J)=\operatorname{Re} I$. By Theorem 2.1, we have that $I+J=I$, and so $J\subset I$. It follows that $[J]\subset I$ and $\operatorname{Re}[J]=\operatorname{Re} I$. By using Theorem 2.1 again, we have $[J]=I$.

REMARK. In Theorem 2.1, the condition (c) cannot be removed from the hypothesis (see [2; 2.3]).

§ 3. A theorem of Arenson.

We shall extend a theorem of Arenson [1] to the case of Banach function algebras. For any subset E of X and any function f defined on a set containing E, we set

$$\|f\|_E = \sup\{|f(x)|;\ x \in E\}.$$

Let A be a subalgebra of $C(X)$. We define

$$d(x, y) = \sup\{|f(x) - f(y)|;\ f \in A,\ \|f\|_X \leqq 1\}$$

for any $x, y \in X$, and

$$D_A(F) = \sup\{d(x, y);\ x, y \in F\}$$

for any subset F of X.

THEOREM 3.1. *Let A_2 be a Banach function algebra on X which contains constant functions, and let A_1 be a subalgebra of A_2. If* $[\operatorname{Re} A_1] \subset \operatorname{Re} A_2$, *then there exists a non-negative number $C < 2$ such that*

$$(*) \qquad D_{A_1+C}(K) \leqq C \qquad \text{for any } K \in \mathcal{K}_0(A_2).$$

Conversely, if (*) *holds for some nonnegative $C < 2$, then* $[\operatorname{Re} A_1] \subset \operatorname{Re}[A_2]$.

PROOF. If $\|\cdot\|_2$ denotes the norm of A_2, then this is finer than the uniform norm on X, automatically. We note first that $\operatorname{Re} A_2$ can be considered as a real Banach space with respect to the following norm:

$$N(u) = \inf\{\|f\|_2;\ f \in A_2,\ \operatorname{Re} f = u\}$$

for $u \in \operatorname{Re} A_2$. Now assume that $[\operatorname{Re} A_1] \subset \operatorname{Re} A_2$. Since $\|f\|_X \leqq \|f\|_2$ for any $f \in A_2$, we have that $\|u\|_X \leqq N(u)$ for any $u \in \operatorname{Re} A_2$. The closed graph theorem asserts the existence of a constant $\mathfrak{K}$ such tnat

$$\|u\|_X \leqq N(u) \leqq \mathfrak{K} \cdot \|u\|_X \tag{1}$$

for any $u \in \operatorname{Re} A_1$. We claim that (*) is true for some non-negative constant $C < 2$. Suppose not. Then for arbitrary $\delta > 0$, there exists a $K_\delta \in \mathcal{K}_0(A_2)$ such that

$$D_{A_1+C}(K_\delta) > 2(1-\delta). \tag{2}$$

We can here choose a polynomial $P(z)$ which satisfies the following: there exists an $\varepsilon > 0$ such that $P(z)$ map the open disc $D = \{z \in \boldsymbol{C};\ |z| < 1+\varepsilon\}$ into $\{z \in \boldsymbol{C}:\ |\operatorname{Re} z| < 1\}$, $P(0) = 0$ and $\operatorname{Im} P(1) > 2\mathfrak{K} + 3$ (cf. [**3**; Lemma 3.4]). Now let δ be a positive number with $1 - (1+\varepsilon/2)^{-1} > \delta$. Using this δ in (2), we have x_1, $x_2 \in K_\delta$ and $f \in A_1 + \boldsymbol{C}$ such that $\|f\|_X < 1$ and $|f(x_1) - f(x_2)| > 2(1-\delta)$. If we get $g = (f - f(x_1))(f(x_2) - f(x_1))^{-1}$, we see easily $g \in A_1 + \boldsymbol{C}$, $\|g\|_X < 1 + \varepsilon/2$, $g(x_1) = 0$ and $g(x_2) = 1$. Next, we get $h = P \circ g$. Then $h \in A_1 + \boldsymbol{C}$, $\|\operatorname{Re} h\|_X \leqq 1$, $\operatorname{Im} h(x_2) > 2\mathfrak{K} + 3$. Now, from (1) we have $v \in C_R(X)$ such that $(\operatorname{Re} h) + iv \in A_2$ and

$$\|\operatorname{Re} h + iv\|_2 \leqq (\mathfrak{K}+1)\|\operatorname{Re} h\|_X \leqq \mathfrak{K} + 1.$$

With this v, we set $k = -i(h - (\operatorname{Re} h + iv))$. Then $k \in A_2$ and real-valued on X.

Hence k is constant on K_δ and so $k(x_1)=k(x_2)$. But

$$|k(x_1)|=|v(x_1)|\leqq\|\mathrm{Re}\,h+iv\|_2\leqq\mathfrak{K}+1\,.$$

$$|k(x_2)|\geqq|\mathrm{Im}\,h(x_2)|-|v(x_2)|\geqq\mathfrak{K}+2\,.$$

This contradiction shows that (*) is true.

Conversly, suppose that (*) holds for some non-negative constant $C<2$. Since $\mathcal{K}_0([A_2])$ is finer than $\mathcal{K}_0(A_2)$ and $D_{[A_1+\boldsymbol{C}]}(E)=D_{A_1+\boldsymbol{C}}(E)$ for all subset E of X, we have

$$D_{[A_1+\boldsymbol{C}]}(K)\leqq C \qquad \text{for any } K\in\mathcal{K}_0([A_2])\,.$$

By Arenson's theorem in [1], we have $[\mathrm{Re}\,[A_1+\boldsymbol{C}]]\subset\mathrm{Re}\,[A_2]$. Thus we get Theorem 3.1.

COROLLARY 3.2. *Under the assumptions of the first half of Theorem* 3.1, *if* $\mathcal{K}_0(A_2)$ *is a finite class, then* $A|K$ *for any* $K\in\mathcal{K}_0(A_2)$ *contains no non-constant functions.*

PROOF. We first note that, since $\mathcal{K}_0(A_2)$ is a finite class, every $f\in A_2$ assumes infinitely many distinct values on any given $K\in\mathcal{K}_0(A_2)$ unless f is constant on K. In fact, suppose that f assumes finite many distinct values on K. Since A_2 contains characteristic function of K, we easily find a real-valued function in A_2 which is not constant on K. Next, take any non-zero $f\in A_2$. We can choose a $K_1\in\mathcal{K}_0(A_2)$ such that $\|f\|_X=\|f\|_{K_1}$. By Theorem 3.1, there exists a nonnegative number $C<2$ such that $D_{A_1+\boldsymbol{C}}(K_1)\leqq C$. This implies that f is equal to a constant, α_1, on K_1. For, if f is not constant on K_1, then there are $x_1, x_2\in K_1$ and $g\in A_2$ such that $|g(x_1)|<\delta$, $g(x_2)=\|g\|_X$, and $1-\delta<\|g\|_X<1$ for any given $\delta>0$. On the other hand, we know that for any $\varepsilon>0$ there exist a polynomial $Q(z)$ and $\delta>0$ such that $Q(z)$ maps open unit disc into itself and satisfies that

$$|Q(z)+1|<\varepsilon/2 \qquad \text{for any } z\in\{z\in\boldsymbol{C};\ |z|<\delta\}\,,$$

$$|Q(z)-1|<\varepsilon/2 \qquad \text{for any } z\in\{z\in\boldsymbol{C};\ |z-1|<\delta\}\,.$$

We put $G=Q\circ g$, then $G\in A_1+\boldsymbol{C}$, $\|G\|_X\leqq1$, $|G(x_1)+1|<\varepsilon/2$, and $|G(x_2)-1|<\varepsilon/2$. It follows that $D_{A_1+\boldsymbol{C}}(K)=2$, thus we have a contradiction. Assume by induction that we have found $K_1,\cdots,K_n\in\mathcal{K}_0(A_2)$ and $\alpha_1,\cdots\alpha_n\in\boldsymbol{C}$ such that f is identically equal to α_i on K_i $(1\leqq i\leqq n)$. We put

$$F=f(f-\alpha_1)(f-\alpha_2)\cdots(f-\alpha_n)\,.$$

If F vanishes identically on X, then f can take only a finite number of values, $0, \alpha_1,\cdots,\alpha_n$. So the above remark shows that f is constant on any $K\in\mathcal{K}_0(A_2)$, which completes the proof. Otherwise, there exists a $K_{n+1}\in\mathcal{K}_0(A_2)$ on which

F is a non-zero constant. The above remark again shows that f assumes a constant value, α_{n+1}, on K_{n+1}. Hence, the induction finishes the proof.

From Theorem 3.1, we can prove the following theorem due to S. Saeki [7].

THEOREM 3.3 ([7]). *Let A be a Banach function algebra on X, and let I be a subalgebra of A. If*

$$I\cdot A_R \subset I \quad \text{and} \quad [\mathrm{Re}\, I] \subset A+\bar{A},$$

then $[I]$ is closed under complex conjugation.

PROOF. We have $[\mathrm{Re}\, I] \subset \mathrm{Re}\, A$. So there exists a non-negative number $C<2$ such that $D_{I+C}(K) \leqq C$ for any $K \in \mathcal{K}_0(A)$. On the other hand, as we shall see below, the assumption $I\cdot A_R \subset I$ implies that $D_{I+C}(K)=0$ or 2 for any $K \in \mathcal{K}_0(A)$. Thus we have $D_{I+C}(K)=0$ for any $K \in \mathcal{K}_0(A)$, and so have the conclusion by Lemma 1.1. Hence it suffices to show only $D_{I+C}(K)=0$ or 2 for any $K \in \mathcal{K}_0(A)$. If $D_{I+C}(K)>0$ for some $K \in \mathcal{K}_0(A)$, then there are $x_1, x_2 \in K$ and $f \in I$ such that

$$f(x_1)=0 \quad \text{and} \quad f(x_2)=\|f\|_K>0.$$

We fix an arbitrary $\varepsilon>0$, and let $\delta>0$ and polynomial $Q(z)$ be as in the proof of Corollary 3.2. We may assume that $1-\delta<f(x_2)<1$. Next we set $U=\{x \in X;\ |f(x)|<1\}$, then U is open and $U \supset K$. Since K is an intersection of peak sets of A_R, there exists $h \in A_R$ such that $h=1$ on K,

$$\|h\|_X=1 \quad \text{and} \quad |h|<(2\cdot\|f\|_X)^{-1} \quad \text{on } X \backslash U.$$

If $g=fh \in I$, we get $\|g\|_X<1$, $g(x_1)=0$ and $g(x_2)>1-\delta$. We put $F=Q\circ g$, then $F \in I+\boldsymbol{C}$, $\|F\|_X \leqq 1$, $|F(x_1)+1|<\varepsilon/2$ and $|F(x_2)-1|<\varepsilon/2$. It follows that $D_{I+C}(K)=2$. Thus the proof is complete.

Saeki's theorem implies the following corollary (cf. [7]).

COROLLARY 3.4 ([5], [8]). *Let A be a function algebra on X, and let I be a closed ideal in A. If $A+\bar{I}$ is uniformly closed in $C(X)$, then $I=\bar{I}$.*

§ 4. Ring conditions.

Let A be a function algebra on X, and let I be a uniformly closed ideal in A. Since K is an intersection peak sets for any $K \in \mathcal{K}_1(A)$, $(I+\boldsymbol{C})|K$ is uniformly closed in $C(K)$. By this fact and a results of Wermer ([3; Corollary 3.6]), we can prove the following two propositions (compare [8]).

PROPOSITION 4.1. *Let A be a function algebra on X, and let I be a closed ideal in A. If N is a subset of A such that $N \supset I$ and $N+\bar{I}$ is a ring, then $I=\bar{I}$.*

PROOF. We first note that $(N+\bar{I})|K\cap C_R(K)$ is a ring for any $K\in\mathcal{K}_1(A_2)$. If $h\in(N+\bar{I})|K\cap C_R(K)$, then there exist $f\in N$ and $g\in I$ such that $h=(f+\bar{g})|K$ and $(f-g)|K$ is a real valued function in $A|K$. So $f|K=g|K+\alpha$ where α is real constant. Therefore $h\in\mathrm{Re}\{(I+\boldsymbol{C})|K\}$. In particular, if $u, v\in\mathrm{Re}(I|K)$, then $u\cdot v\in(N+\bar{I})|K\cap C_R(K)$, so $u\cdot v\in\mathrm{Re}\{(I+\boldsymbol{C})|K\}$. Hence, $\mathrm{Re}\{(I+\boldsymbol{C})|K\}$ is a ring. Since $(I+\boldsymbol{C})|K$ is uniformly closed, it follows that from Wermer's theorem that $(I+\boldsymbol{C})|K=\overline{(I+\boldsymbol{C})|K}$. Thus we have $I=\bar{I}$ by Lemma 1.1.

PROPOSITION 4.2. *Suppose A is a function algebra on X, and I, J are two closed ideals in A. Then $I+\bar{J}$ is a ring if and only if $I\cap J=\overline{I\cap J}$.*

PROOF. Observe that

$$(I+\bar{J}+\boldsymbol{C})|K\cap C_R(K)=\mathrm{Re}\{(I+\boldsymbol{C})|K\cap(J+\boldsymbol{C})|K\}$$

for any $K\in\mathcal{K}_1(A)$. Since $(I+\bar{J}+\boldsymbol{C})|K\cap C_R(K)$ is a ring, it follows from Wermer's theorem that $(I+\boldsymbol{C})|K\cap(J+\boldsymbol{C})|K$ contains no non-constant functions. Since

$$(I\cap J)|K\subset(I+\boldsymbol{C})|K\cap(J+\boldsymbol{C})|K$$

and $I\cap J$ is a closed ideal in A, we have $I\cap J=\overline{I\cap J}$ by Lemma 1.1. Conversely, suppose that $I\cap J=\overline{I\cap J}$. We easily see that

$$I\cap J=\{f\in C(X);\ f=0 \text{ on } Z(I)\cup Z(J)\}$$

by Stone-Weierstrass theorem. So $f\cdot\bar{g}\in I\cap J$ for any $f\in I$ and $g\in J$. Thus $I+\bar{J}$ is a ring.

The author wishes to express his hearty thanks to Professor J. Wada for his valuable advices and constant encouragements.

References

[1] E.L. Arenson, Gleason parts and the Wermer-Hoffman theorem on characterization of $C(X)$, Siberian Math. J., **13** (1973) 831-838.
[2] A. Bernard, Espace des parties réelles des éléments dune algèbre de Banach de founctions, J. Fuctional Anal., **10** (1972), 387-409.
[3] R.B. Burckel, Characterization of $C(X)$ among its subalgebras, Marcel Dekker, 1972.
[4] I. Glicksberg, Measures orthogonal to algebras and sets of antisymmetry, Trans. Amer. Math. Soc., **105** (1962), 415-435.
[5] I. Glicksberg, On two consequences of a theorem of Hoffman and Wermer, Math. Scand., **23** (1968), 188-192.
[6] G.M. Leibowitz, Lectures on Complex Function Algebras, Scott Foresman, 1970.

[7] S. Saeki, On Banach algebras of continuous functions, (1972) un-published.
[8] J. Wada, On a theorem of I. Glicksberg, Proc. Japan Acad., **48** (1972), 227-230.

Jun-ichi TANAKA
Department of Mathematics
Waseda University

Present address:
Department of Mathematics
Tsuru University
Tsuru, Yamanashi
Japan