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Trotter's product formula for nonlinear semigroups generated by the subdifferentials of convex functionals

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1. Introduction.

It was proved in [1, 2] that

$$\underset{n\to\infty}{s\text{-}\lim}\,[e^{-(t/n)A_2}e^{-(t/n)A_1}]^n=e^{-tA'}P', \qquad t>0, \tag{1.1}$$

whenever A_1, A_2 are nonnegative selfadjoint operators in a Hilbert space H (with no restriction on their domains). Here P' is the orthogonal projection of H onto the subspace H' spanned by $D'=D(A_1^{1/2})\cap D(A_2^{1/2})$ and A' is the *form sum* of A_1, A_2 (i.e. the selfadjoint operator in H' associated with the densely-defined, closed quadratic form $\|A_1^{1/2}u\|^2+\|A_2^{1/2}u\|^2$).

The purpose of the present paper is to prove a nonlinear analogue of (1.1). As a natural generalization of a nonnegative selfadjoint operator, A_j will be replaced by the subdifferential $\partial\varphi_j$ of a lower semicontinuous, convex function $\varphi_j\not\equiv+\infty$ on H to $]-\infty, +\infty]$; $-\partial\varphi_j$ generates a semigroup $\{e^{-t\partial\varphi_j}\}$ of nonlinear nonexpansive operators on $E_j=cl.D(\varphi_j)$. (For these notions see section 2.) Moreover, we shall admit any finite number N of such semigroups. Thus our result will take the form

$$\lim_{n\to\infty}[e^{-(t/n)\partial\varphi_N}P_N\cdots e^{-(t/n)\partial\varphi_1}P_1]^n x=e^{-t\partial\varphi}x,$$

$$\varphi=\varphi_1+\cdots+\varphi_N, \qquad t\geqq 0,\ x\in cl.D(\varphi), \tag{1.2}$$

where P_j is the nonlinear projection of H onto the closed convex set E_j, and it is assumed that $\varphi\not\equiv+\infty$. Note that $\partial\varphi$ is the analogue of the form sum of the $\partial\varphi_j$. The factors P_j are necessary to ensure that the product on the left of (1.2) makes sense, since $e^{-t\partial\varphi_j}$ is defined only on E_j.

REMARK 1.1. The condition $x\in cl.D(\varphi)$ in (1.2) is a new restriction which was not needed in the linear case (1.1). A straightforward generalization of the latter would be to admit every $x\in H$ and replace x by Px on the right-

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hand side of (1.2), where P is the projection onto $cl.D(\varphi)$. But this is in general impossible, as is seen from the following example. Let $N=2$ and let φ_j be the indicator function of a closed convex set $E_j \subset H$ (i. e. $\varphi_j(u)=0$ for $u \in E_j$ and $=+\infty$ otherwise) so that $D(\varphi_j)=E_j=P_jH$, $e^{-t\partial\varphi_j}=1$ on E_j and $e^{-t\partial\varphi}=1$ on $E=E_1 \cap E_2$. Then the suggested generalization would give $\lim (P_2P_1)^n x=Px$ for all $x \in H$. But this formula (a familiar one for linear projections) is in general false for nonlinear projections. This is seen from the special case in which $H=R^2$, E_1 is the unit disk, and E_2 is a straight line through the origin.

Except for the restriction on x, (1.2) includes (1.1) as a special case. It is interesting to note that we are able to include the case $N>2$ without essential complication of the proof, whereas the proof given in [**1**, **2**] does not seem to generalize to $N>2$ easily.

Actually we shall prove (1.2) in a more general case in which the semigroup $e^{-t\partial\varphi_j}P_j$ is replaced by a *φ_j-family* $U_j(t)$ to be introduced in section 2 (which includes some useful approximations to the semigroup, for example the *resolvent* $(1+t\partial\varphi_j)^{-1}$). For the precise statement of the result, see the Theorem below.) (A similar generalization was considered in the linear case [**1**, **2**]. For these generalizations, comparison of the results for linear and nonlinear cases is not easy, since the conditions for the approximating families are different.)

For nonlinear product formulas similar to (1.2) but under different assumptions, see [**3**; Propositions 4.3, 4.4].

2. Definitions. The main theorem.

(For basic notions and results regarding maximal monotone operators and the subdifferentials of convex functions, we refer to the book [3] by Brezis.) Let Φ denote the set of all lower semicontinuous, convex functions φ on a real Hilbert space H to $]-\infty, +\infty]$ such that $\varphi \not\equiv +\infty$. For $\varphi \in \Phi$, the *effective domain* $D(\varphi)$ is the (nonempty) set of all $u \in H$ with $\varphi(u)<+\infty$. Because of convexity, φ is lower semicontinuous in the weak topology of H. Let $\partial\varphi$ be the *subdifferential* of φ; $\partial\varphi$ is a multiple-valued, *maximal monotone* operator in H, and the relation $f \in (\partial\varphi)u$ is characterized by

$$\varphi(v) \geqq \varphi(u)+(v-u, f) \qquad \text{for all} \quad v \in H. \tag{2.1}$$

$-\partial\varphi$ generates a strongly continuous semigroup $\{e^{-t\partial\varphi}; t>0\}$ of nonlinear nonexpansive operators on $E=cl.D(\varphi)=cl.D(\partial\varphi)$, where $cl.$ denotes the closure and $D(\partial\varphi)$ is the set of all $u \in H$ such that $(\partial\varphi)u$ is not empty.

DEFINITION 2.1. Let $\varphi \in \Phi$. A family $\{U(t); t>0\}$ of nonexpansive operators on H to H will be called a φ-family if

$$\varphi(v)\geqq\varphi(U(t)y)+t^{-1}(v-y,\ y-U(t)y)+(2t)^{-1}\gamma\|y-U(t)y\|^{2} \tag{2.2}$$

for every $v,\ y\in H$ and $t>0$, where γ is a positive constant. The largest number γ with this property will be called the φ-index of $\{U(t)\}$ and will be denoted by γ again.

REMARK 2.2. (a) The definition of a φ-family is rather implicit; we shall give below several examples of φ-families.

(b) It is convenient to note that (2.2) is equivalent to:

$$\varphi(v)\geqq\varphi(U(t)y)+t^{-1}(v-w,\ y-U(t)y)-(2t)^{-1}\|y-w\|^{2}$$
$$+(2t)^{-1}\|w-U(t)y\|^{2}+(2t)^{-1}(\gamma-1)\|y-U(t)y\|^{2} \tag{2.3}$$

for every $v,\ y,\ w\in H$.

(c) If $\{U(t)\}$ is a φ-family, $U(t)$ maps H into $D(\varphi)$, as is seen from (2.2) by choosing $v\in D(\varphi)$. Also $U(t)y\rightarrow y,\ t\downarrow 0$, if $y\in E$. Moreover, $t^{-1}(1-U(t))\rightarrow\partial\varphi$ in the sense of *resolvent convergence*. This is a special case of the Theorem for $N=1$.

(d) The value of γ is important; the family is "nice" if $\gamma\geqq 1$ (see the theorem). On the other hand, $\gamma>2$ is impossible except when $\varphi=$const and $U(t)=1$. (To see this set $v=U(t)y$ in (2.2).)

EXAMPLE 2.3. Let $\varphi\in\Phi$, $J(t)=(1+t\partial\varphi)^{-1}$, $t>0$ (the resolvent of $\partial\varphi$). As is well known [**3**; Proposition 2.2], $J(t)$ is a nonexpansive operator on H to $D(\partial\varphi)\subset D(\varphi)$. $\{J(t)\}$ is a "nice" φ-family with $\gamma\geqq 2$.

To see this, we note that $t^{-1}(y-J(t)y)\in(\partial\varphi)J(t)y$ [**3**; after Theorem 2.2]. Hence $\varphi(v)\geqq\varphi(J(t)y)+(v-J(t)y,\ t^{-1}(y-J(t)y))$ by (2.1), from which (2.2) follows with $\gamma\geqq 2$.

EXAMPLE 2.4. For any fixed positive integer m, $\{J(t/m)^{m}\}$ is a "nice" φ-family with $\gamma\geqq 1+m^{-1}$.

To see this, write $y_{k}=J(t/m)^{k}y,\ y_{0}=y$. Then $mt^{-1}(y_{k-1}-y_{k})=mt^{-1}(1-J(t/m))y_{k-1}\in(\partial\varphi)J(t/m)y_{k-1}=(\partial\varphi)y_{k}$ as above, so that by (2.1)

$$\varphi(v)\geqq\varphi(y_{k})+(v-y_{k},\ mt^{-1}(y_{k-1}-y_{k})),\qquad 1\leqq k\leqq m\,. \tag{2.4}$$

On setting $v=y_{k-1}$, we see that $\varphi(y_{k})$ is decreasing in k. Hence we may replace $\varphi(y_{k})$ by $\varphi(y_{m})$ in (2.4) and take the average for $k=1,\cdots,m$, obtaining

$$\varphi(v)\geqq\varphi(y_{m})+t^{-1}(v-y,\ y-y_{m})+t^{-1}\sum_{k=1}^{m}(y-y_{k},\ y_{k-1}-y_{k})\,. \tag{2.5}$$

The last term in (2.5) becomes, with $a_{k}=y_{k-1}-y_{k}$,

$$t^{-1}\sum_{k=1}^{m}(a_{1}+\cdots+a_{k},\ a_{k})=(2t)^{-1}\Big[\sum_{k=1}^{m}\|a_{k}\|^{2}+\|\sum_{k=1}^{m}a_{k}\|^{2}\Big]$$
$$\geqq(2t)^{-1}(1+m^{-1})\|\sum_{k=1}^{m}a_{k}\|^{2}=(2t)^{-1}(1+m^{-1})\|y-y_{m}\|^{2},$$

which proves the assertion. (A more precise estimate shows that $\gamma \geqq 1 + O((\log m)^{-1})$ for large m.)

EXAMPLE 2.5. Let $U(t)=e^{-t\partial\varphi}P$, where P is the nonlinear projection of H onto $E=cl.D(\varphi)$, which is a closed convex set [3; Theorem 2.2]. $\{U(t)\}$ is a "nice" φ-family with $\gamma \geqq 1$.

This follows from Example 2.4, since $U(t)y=\lim_{m\to\infty} J(t/m)^m y$ for every $y \in H$ [3; Corollary 4.4].

REMARK 2.6. All the examples given above apply to the linear case in which $\varphi(u)=(1/2)\|A^{1/2}u\|^2$ with A nonnegative selfadjoint, so that $\partial\varphi=A$, $E=H$ and $P=1$. In particular, $U(t)=e^{-tA}$ forms a φ-family with $\gamma \geqq 1$. In this case we have actually $\gamma \geqq \gamma_0 > 1$, where γ_0 is a universal constant, as one can prove using the spectral formula.

We are now in a position to state the main theorem. Let $\varphi_j \in \Phi$, $j=1, \cdots, N$, with $\varphi=\varphi_1+\cdots+\varphi_N \not\equiv +\infty$, so that $\varphi \in \Phi$ too. Let $D_j=D(\varphi_j)$, $D=D(\varphi)$, $E_j=cl.D_j$, $E=cl.D$, and let P_j, P be projections of H onto E_j, E, respectively.

THEOREM. *Let $\{U_j(t)\,;\, t>0\}$ be a φ_j-family with φ_j-index $\gamma_j>0$, $j=1, \cdots, N$. Assume that one of the following conditions is satisfied.*

(i) *$\gamma_j \geqq 1$ for all $j=1, \cdots, N$, and for k with $\gamma_k=1$*

$$U_k(t)u=U_k(t)P_k u\,, \qquad t>0,\ u \in H\,. \tag{2.6}$$

(ii) *There is a k such that $\gamma_j>1$ for all $j \neq k$ and*

$$(\gamma_k-1)\sum_{j\neq k}(\gamma_j-1)^{-1}>-1\,. \tag{2.7}$$

Then we have

$$\lim_{t\downarrow 0}\,[1+\lambda t^{-1}(1-U_N(t)\cdots U_1(t))]^{-1}x=(1+\lambda\partial\varphi)^{-1}x\,, \qquad \lambda>0,\ x\in H\,, \tag{2.8}$$

$$\lim_{n\to\infty}\,[U_N(t/n)\cdots U_1(t/n)]^n x=e^{-t\partial\varphi}x\,, \qquad t\geqq 0,\ x\in E\,, \tag{2.9}$$

the convergence in (2.9) *being uniform in $t\in[0, T]$ for any $T>0$.*

COROLLARY. (1.2) *is true.*

REMARK 2.7. If $N=1$, assume simply $\gamma_1>0$ instead of (i), (ii). If $N=2$, (2.7) becomes $\gamma_1+\gamma_2>2$, and (2.6) is required only when $\gamma_1=\gamma_2=1$.

The proof of the theorem will be given in the following section. Here we give two lemmas required in the proof, under the assumptions of the theorem.

LEMMA 2.8. *There is* $M<+\infty$ *such that for any $a_j \in H$, $j=1, \cdots, N$,*

$$\sum_{j=1}^{N}(\gamma_j-1)\|a_j\|^2+M\|\sum_{j=1}^{N}a_j\|^2\geqq 0\,. \tag{2.10}$$

PROOF. This is trivial in case (i), and also in case (ii) if $\gamma_k \geqq 1$. Thus we may assume (ii) with $\gamma_k<1$. Then (2.7) implies that there is $s>0$ such that

$$(1+s)\Sigma'(\gamma_j-1)^{-1}<(1-\gamma_k)^{-1},$$

where Σ' means $\sum_{j\neq k}$. Writing $a=a_1+\cdots+a_N$, we then have

$$\begin{aligned}\|a_k\|^2=\|a-\Sigma' a_j\|^2&\leqq(1+s^{-1})\|a\|^2+(1+s)\|\Sigma' a_j\|^2\\ &\leqq(1+s^{-1})\|a\|^2+(1+s)[\Sigma'(\gamma_j-1)^{-1}][\Sigma'(\gamma_j-1)\|a_j\|^2]\\ &\leqq(1+s^{-1})\|a\|^2+(1-\gamma_k)^{-1}\Sigma'(\gamma_j-1)\|a_j\|^2,\end{aligned}$$

which implies (2.10) with $M=(1-\gamma_k)(1+s^{-1})$.

LEMMA 2.9. *Let* $z\in D$ *and* $z_0(t)=z$, $z_j(t)=U_j(t)\cdots U_1(t)z$, $t>0$, $j=1,\cdots,N$. *Then* $z_j(t)-z=O(t^{1/2})$ *as* $t\downarrow 0$.

PROOF. (2.2) gives, with $v=y=z$,

$$\begin{aligned}+\infty>\varphi_j(z)&\geqq\varphi_j(U_j(t)z)+(2t)^{-1}\gamma_j\|z-U_j(t)z\|^2\\ &\geqq -M\|U_j(t)z-z\|-M+(2t)^{-1}\gamma_j\|U_j(t)z-z\|^2,\end{aligned}$$

where M is a constant; note that any $\varphi\in\Phi$ is bounded from below by a (inhomogeneous) linear functional as is seen from (2.1). Since $\gamma_j>0$, it follows that $U_j(t)z-z=O(t^{1/2})$. In particular $z_1(t)-z=O(t^{1/2})$. The same result for $z_j(t)$ can be proved by induction, since

$$\begin{aligned}\|z_j(t)-z\|&\leqq\|z_j(t)-U_j(t)z\|+\|U_j(t)z-z\|\\ &\leqq\|z_{j-1}(t)-z\|+O(t^{1/2});\end{aligned}$$

note that $z_j(t)=U_j(t)z_{j-1}(t)$ and that $U_j(t)$ is nonexpansive.

3. Proof of Theorem.

According to a lemma due to Chernoff and Brezis-Pazy (see [**3**; Theorem 4.3]), (2.8) implies (2.9). Thus it suffices to prove (2.8). To this end, let

$$y_0(t)=[1+\lambda t^{-1}(1-U_N(t)\cdots U_1(t))]^{-1}x, \qquad t>0, \tag{3.1}$$

$$y_j(t)=U_j(t)\cdots U_1(t)y_0(t), \qquad j=1,\cdots,N, \tag{3.2}$$

$$a_j(t)=y_{j-1}(t)-y_j(t), \qquad j=1,\cdots,N, \tag{3.3}$$

$$a(t)=a_1(t)+\cdots+a_N(t)=y_0(t)-y_N(t). \tag{3.4}$$

(3.1) and (3.4) imply that

$$y_0(t)+\lambda t^{-1}a(t)=x. \tag{3.5}$$

PROPOSITION 3.1. $y_0(t)$, $y_j(t)$ *and* $\varphi_j(y_j(t))$ *are bounded as* $t\downarrow 0$, $j=1,\cdots,N$.

PROOF. Let $z\in D$ be fixed and construct $z_j(t)$, $j=0, 1, \cdots, N$ as in Lemma 2.9. Since $U_j(t)$ is nonexpansive, we have

$$\|y_j(t)-z_j(t)\|\leqq\|y_{j-1}(t)-z_{j-1}(t)\| . \tag{3.6}$$

Since $z_j(t)\to z$ as $t\downarrow 0$ by Lemma 2.9, it follows that

$$\limsup_{t\downarrow 0}\,[\|y_j(t)-z\|-\|y_0(t)-z\|]\leqq 0, \qquad j=1, \cdots, N. \tag{3.7}$$

To prove that the $y_j(t)$ are bounded, therefore, it suffices to show that $y_0(t)$ is bounded.

Let $v\in D=\cap D_j$. Since $\{U_j(t)\}$ is a φ_j-family, we can apply (2.2) to $\varphi=\varphi_j$ with $y=y_{j-1}(t)$, obtaining

$$\varphi_j(v)\geqq\varphi_j(y_j)+t^{-1}(v-y_{j-1}, a_j)+(2t)^{-1}\gamma_j\|a_j\|^2 ; \tag{3.8}$$

here and in what follows we write simply y_j, a_j, etc. for $y_j(t)$, $a_j(t)$, etc. Summing (3.8) for $j=1, \cdots, N$ and using (3.4), (3.5), we obtain for any $v\in H$

$$\varphi(v)\geqq \sum_{j=1}^{N}\varphi_j(y_j)+\lambda^{-1}(v-y_0, x-y_0)+(2t)^{-1}b , \tag{3.9}$$

with

$$\begin{aligned} b&=\Sigma[\gamma_j\|a_j\|^2+2(y_0-y_{j-1}, a_j)] \\ &=\Sigma[\gamma_j\|a_j\|^2+2(a_1+\cdots+a_{j-1}, a_j)] \\ &=\Sigma(\gamma_j-1)\|a_j\|^2+\|\Sigma a_j\|^2 \\ &\geqq -M\|\Sigma a_j\|^2=-M\|a\|^2\geqq -Mt^2\lambda^{-2}\|x-y_0\|^2 , \end{aligned} \tag{3.10}$$

where we have used Lemma 2.8 and (3.5). Thus (3.9) gives

$$\varphi(v)\geqq \sum_{j=1}^{N}\varphi_j(y_j)+\lambda^{-1}(v-y_0, x-y_0)-Mt(2\lambda^2)^{-1}\|x-y_0\|^2 . \tag{3.11}$$

We note again that $\varphi_j(y_j)$ is bounded from below by a (inhomogeneous) linear functional in y_j. In view of (3.7), therefore, we have

$$\varphi_j(y_j)\geqq -M-M\|x-y_0\| . \tag{3.12}$$

Now it is easy, using (3.11) with $v\in D$, to show that $\|x-y_0(t)\|$ is bounded as $t\downarrow 0$. Hence all the $y_j(t)$ are bounded by (3.7).

To show that the $\varphi_j(y_j)$ are bounded, it suffices to note that they are bounded from below because of (3.12) and, consequently, also from above by (3.11).

PROPOSITION 3.2. *For each $z\in D$ we have*

$$\|y_j(t)-z\|^2-\|y_0(t)-z\|^2 \longrightarrow 0, \qquad t\downarrow 0, \quad j=1, \cdots, N. \tag{3.13}$$

PROOF. Since $y_0(t)$ is bounded, it follows from (3.4) and (3.5) that $y_0-y_N=a\to 0$. In view of (3.6) and $z_j\to z$, we have then $\|y_j-z\|-\|y_0-z\|\to 0$. (3.13) follows from this since y_j and y_0 are bounded.

PROPOSITION 3.3. *There exists a sequence $t_n\downarrow 0$ such that $y_j(t_n)\rightharpoonup y_j^*\in H$, $j=1, \cdots, N$, where $\rightharpoonup$ denotes weak convergence in H. Furthermore, we have*

$$y_0^*=y_N^*, \qquad y_j^*\in D_j, \qquad j=1, \cdots, N, \tag{3.14}$$

$$\varphi(v)\geqq \sum_{j=1}^{N}\varphi_j(y_j^*)+\lambda^{-1}(v-y_0^*, x-y_0^*), \qquad v\in H. \tag{3.15}$$

PROOF. This follows directly from the boundedness of the $y_j(t)$ and $\varphi_j(y_j(t))$, and inequality (3.11). Recall that $y_0(t)-y_N(t)\to 0$ and that φ_j are lower semicontinuous in the weak topology.

PROPOSITION 3.4. *Set $a_j^*=y_{j-1}^*-y_j^*$, $j=1, \cdots, N$. Then*

$$a_1^*+\cdots+a_N^*=0, \tag{3.16}$$

and for k with $\gamma_k\geqq 1$

$$2(u_k-z, a_k^*)+(\gamma_k-1)\|a_k^*\|^2\leqq 0, \qquad z\in D,\ u_k\in E_k. \tag{3.17}$$

PROOF. (3.16) follows from $y_0^*=y_N^*$ (see (3.14)). To prove (3.17), we may assume that $u_k\in D_k$. Then (2.3) gives, with $\varphi=\varphi_k$, $U=U_k$, $v=u_k$, $y=y_{k-1}(t)$, $w=z$, and $\gamma=\gamma_k$,

$$2t\varphi_k(u_k)\geqq 2t\varphi_k(y_k)+2(u_k-z, a_k)-\|y_{k-1}-z\|^2$$
$$+\|y_k-z\|^2+(\gamma_k-1)\|a_k\|^2.$$

Letting $t=t_n\downarrow 0$ and using (3.13), we obtain (3.17); note that $\|a_k^*\|^2\leqq \liminf\|a_k(t_n)\|^2$.

PROPOSITION 3.5. *We have*

$$(y_{k-1}^*-z, a_k^*)\leqq 0, \qquad z\in D,\ k=1, \cdots, N. \tag{3.18}$$

PROOF. First we note that $a_k^*=0$ whenever $\gamma_k>1$, as is seen from (3.17) with $u_k=z$.

It follows that in case (ii), $a_k^*=0$ for all k so that (3.18) is true a fortiori. Indeed, we then have $\gamma_k>1$, hence $a_k^*=0$, except possibly for one k. But (3.16) then shows that all a_k^* must be zero.

It only remains to consider the case (i) with $\gamma_k=1$. Then we have

$$y_k(t)=U_k(t)y_{k-1}(t)=U_k(t)P_ky_{k-1}(t) \tag{3.19}$$

by (2.6). Now apply (2.3) to $\varphi=\varphi_k$, $v=w=z$ and $y=P_ky_{k-1}(t)$. Then we obtain by (3.19)

$$2t\varphi_k(z) \geqq 2t\varphi_k(y_k) - \|P_k y_{k-1} - z\|^2 + \|y_k - z\|^2 .$$

In view of Propositions 3.1 and 3.2, we thus obtain

$$\limsup_{t\downarrow 0} (\|y_{k-1} - z\|^2 - \|P_k y_{k-1} - z\|^2) \leqq 0 . \tag{3.20}$$

On the other hand, a simple geometric consideration shows that $\|y-z\|^2 - \|P_k y - z\|^2 \geqq \|y - P_k y\|^2$ for any $y \in H$, since $z \in D \subset D_k \subset E_k$ and P_k is the projection onto E_k. Thus (3.20) gives $y_{k-1} - P_k y_{k-1} \to 0$ as $t \downarrow 0$. Hence $y_{k-1}^* \in E_k$ because E_k is weakly closed. Then we can apply (3.17) with $u_k = y_{k-1}^*$ and $\gamma_k = 1$, obtaining (3.18).

PROPOSITION 3.6. $a_k^* = 0$, $y_k^* = y_0^*$, $k = 1, \cdots, N$.

PROOF. (This was already proved in case (ii) in the proof of Proposition 3.5. The following proof is necessary only for case (i).) We may assume $0 \in D$ without loss of generality. Then we can set $z=0$ in (3.18), obtaining $(y_{k-1}^*, a_k^*) \leqq 0$ for all $k=1, \cdots, N$. Then $\|y_k^*\|^2 = \|y_{k-1}^* - a_k^*\|^2 \geqq \|y_{k-1}^*\|^2 + \|a_k^*\|^2$ and so $\|a_1^*\|^2 + \cdots + \|a_N^*\|^2 \leqq \|y_N^*\|^2 - \|y_0^*\|^2 = 0$ by (3.14), hence $a_k^* = 0$ for all k.

PROPOSITION 3.7. $y_0^* = (1 + \lambda\partial\varphi)^{-1} x \in D(\partial\varphi) \subset D$.

PROOF. In view of Proposition 3.6, (3.15) becomes

$$\varphi(v) \geqq \varphi(y_0^*) + \lambda^{-1}(v - y_0^*, x - y_0^*) .$$

Since this is true for any $v \in H$, we have $y_0^* \in D(\partial\varphi)$ and $\lambda^{-1}(x - y_0^*) \in (\partial\varphi) y_0^*$ (see (2.1)). Hence $x \in (1 + \lambda(\partial\varphi)) y_0^*$, which is equivalent to the required result.

PROPOSITION 3.8. *$y_0(t) \to y_0^*$, so that (2.8) is true.*

PROOF. First we note that $y_j(t) \rightharpoonup y_j^* = y_0^*$ as $t \downarrow 0$ for $j = 0, 1, \cdots, N$. This follows from the standard argument since y_0^* as given by Proposition 3.7 is independent of the sequence $t_n \downarrow 0$ used above.

To prove that we have strong rather than weak convergence, it suffices to show that $\sup \|y_0(t)\| \leqq \|y_0^*\|$. To this end, we return to (3.11) and set $v = y_0^* \in D$. Then

$$\begin{aligned} &\lambda^{-1} \limsup\, (y_0^* - y_0(t), x - y_0(t)) \\ &\qquad \leqq \varphi(y_0^*) - \liminf \sum_{j=1}^{N} \varphi_j(y_j(t)) \leqq \varphi(y_0^*) - \varphi(y_0^*) = 0 . \end{aligned}$$

Since $y_0(t) \rightharpoonup y_0^*$, this implies $\limsup \|y_0(t)\|^2 \leqq \|y_0^*\|^2$ as required.

4. Remarks and examples.

The limit $u(t) = e^{-t\partial\varphi} x$ in (1.2) or (2.9) is characterized as the (strongly continuous) solution of the Cauchy problem

$$-du/dt\in(\partial\varphi)u(t), \quad \text{a.e. } t>0, \quad u(0)=x; \tag{4.1}$$

see [**3**; Theorems 3.1, 3.2]. In view of (2.1), (4.1) is equivalent to

$$\varphi(v)\geqq\varphi(u(t))-(v-u(t), du/dt) \quad \text{for all} \quad v\in H. \tag{4.2}$$

In this sense (4.1) is a *variational inequality.*

In general $\partial\varphi$ or (4.1) is difficult to describe more explicity. But it often happens that each φ_j, $\partial\varphi_j$ and $\varphi=\varphi_1+\cdots+\varphi_N$ are known in concrete form. In such a case, (1.2) or (2.9) is useful because it gives a constructive method for computing $u(t)$.

EXAMPLE 4.1. Let $N=2$ and $\varphi_2=\text{ind}_{E_2}$ (the indicator function of a convex closed set $E_2\subset H$). Then (4.2) becomes

$$\varphi_1(v)\geqq\varphi_1(u(t))-(v-u(t), du/dt) \quad \text{for all} \quad v\in E_2, \tag{4.3}$$

with the additional condition $u(t)\in E_2$. This is still rather implicit, but (1.2) or (2.9) gives the following formula, which is computable if one can compute the semigroup $e^{-t\partial\varphi_1}$ or the resolvent $(1+t\partial\varphi_1)^{-1}$ for $\partial\varphi_1$:

$$\begin{aligned} u(t)&=\lim\,[P_2e^{-(t/n)\partial\varphi_1}P_1]^n x=\lim\,[e^{-(t/n)\partial\varphi_1}P_1P_2]^n x \\ &=\lim\,[P_2(1+(t/n)\partial\varphi_1)^{-1}]^n x=\lim\,[(1+(t/n)\partial\varphi_1)^{-1}P_2]^n x; \end{aligned} \tag{4.4}$$

note that $e^{-t\partial\varphi_2}P_2=(1+t\partial\varphi_2)^{-1}=P_2$, $t>0$, in this case [**3**; Example 2.8.2].

EXAMPLE 4.2. In the above example, suppose that $H=L^2(\Omega)$, $\Omega\subset R^m$ (an open set), and $\varphi_1(u)=(1/2)\|\text{grad}\, u\|^2-(u,f)$, where $D(\varphi_1)=H_0^1(\Omega)$ (the Sobolev space) and $f\in H$. Furthermore, let $\varphi_2=\text{ind}_{E_2}$ with $E_2\subset H$ consisting of all $u\in H$ such that $u(x)\geqq g(x)$ a.e., where g is a given function on Ω such that E_2 is not empty. In this case $P_2u(x)=\sup\{u(x), g(x)\}$. Since $P_1=1$ and $(\partial\varphi_1)u=-\Delta u-f$ with $D(\partial\varphi_1)\subseteqq H_{loc}^2(\Omega)\cap H_0^1(\Omega)$, it is in principle possible to compute (4.4). For example, $[P_2e^{-(t/n)\partial\varphi_1}P_1]v$ is obtained by solving the (inhomogeneous) heat equation $dw/dt=\Delta w+f$ for the time interval t/n, starting with the initial value v, and then replacing w by $\sup\{w, g\}$. The variational inequality (4.2) is not so easy to handle directly. (The stationary variant of this example is a classical variational inequality studied by Lewy, Lions, Stampacchia, and others; see [**4, 5**].)

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