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Weakly closed cyclic 2-groups in finite groups

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1. Introduction.

In this paper we shall prove the following results.

THEOREM 1. *Let G be a finite group. Let S_0 be a subgroup of a Sylow 2-subgroup S of G such that $|S:S_0|\leqq 2$. If there exists an element x such that $\langle x\rangle \lhd S$, $x^2=z$, $|z|=2$, and $z^G \cap S_0=\{z\}$, then $Z^*(G)\neq 1$ or a Sylow 2-subgroup of $\langle z^G\rangle$ is dihedral or semidihedral.*

COROLLARY 1. *Let X be a cyclic subgroup of a Sylow 2-subgroup S of G. If X is weakly closed in S with respect to G, then $Z^*(G)\neq 1$ or a Sylow 2-subgroup of $\langle \Omega_1(X)^G\rangle$ is dihedral or semidihedral.*

D. M. Goldschmidt determined the structure of the finite groups with weakly closed four-groups. Jonathan I. Hall determined the structure of the finite groups with weakly closed cyclic group of order 4. This corollary is a generalization of Jonathan I. Hall's result.

COROLLARY 2. *Let S be a Sylow 2-subgroup of a finite group G. Suppose $S=R_1*\cdots *R_n$ and the following conditions hold;*

(1) R_1 has a cyclic subgroup of index 2 for $i=1, 2, \cdots, n$.

(2) $Z(S)$ is cyclic.

Then $\Omega_1(Z(S))\subseteqq Z^*(G)$ or a Sylow 2-subgroup of $\langle \Omega_1(Z(S))^G\rangle$ is dihedral or semidihedral.

We shall write $A*B$ for a central product of A and B.

THEOREM 2. *Let G be a finite group. Let S_0 be a subgroup of a Sylow 2-subgroup S of G, such that $|S:S_0|\leqq 2$. If every involution of S_0 is isolated each other, then $Z^*(G)\neq 1$ or there exists an involution z of S_0 such that a Sylow 2-subgroup of $\langle z^G\rangle$ is dihedral or semidihedral.*

In fact we find an example in symmetric group of degree 6 which has an involution z such that a Sylow 2-subgroup of $\langle z^G\rangle$ is neither dihedral nor semidihedral.

We shall say elements x, y of G are isolated if x any y are not conjugate in G.

2. Preliminaries.

LEMMA 2.1. *If A and B are conjugate subsets of a Sylow p-subgroup P of G, then there exist Sylow p-subgroups Q_i with $H_i=P\cap Q_i$ a time intersection, $1\leqq i\leqq n$ such that*

(1) $C_P(H_i)\subseteqq H_i$

(2) H_i ; *Sylow p-subgroup of* $O_{p,p'}(N(H_i))$

(3) $H_i=P$ *or* $N(H_i)/H_i$ *is p-isolated*

(4) $A\subseteqq H_1$, $A^{x_1\cdots x_i}\subseteqq H_{i+1}$ *for some* $x_i\in N(H_i)$ *if* $H_i=C_P(\Omega(Z(H_i)))$ *and for some* $x_i\in N(H_i)\cap C_G(\Omega(Z(H_i)))$ *if* $H_i\neq C_P(\Omega(Z(H_i)))$, *and* $A^{x_1\cdots x_{n-1}y}=B$ *for some* $y\in N_G(P)$.

This fusion lemma may be found in Goldschmidt [2].

LEMMA 2.2. *If element t, z of a Sylow p-subgroup P of G are conjugate and $z\in Z(P)$, then there exists an element g of G such that $t^g=z$ and $C_S(t)^g\subseteqq S$.*

PROOF. Since t and z are conjugate in G, there exists an element k such that $t^k=z$. Since $C_S(t)^k\subseteqq C_G(t^k)=C_G(z)$ and $S\subseteqq C_G(z)$, by Sylow's theorem there exists an element h of $C_G(z)$ such that $C_S(t)^{kh}\subseteqq S$. we set $g=kh$, then $t^g=t^{kh}=z^h=z$. So the lemma is proved.

We say that, for a subgroup K of a Sylow 2-subgroup S of G, K is strongly involution closed if $k\in I(K)$ and $k^g\in S$ for some $g\in G$ implies that $k^g\in K$.

In [3], Goldschmidt proved the following result.

LEMMA 2.3. *Suppose D is a strongly involution closed dihedral 2-subgroup of G. Then a Sylow 2-subgroup of $\langle D^G\rangle$ is dihedral or semidihedral.*

3. Proof of Theorem 1.

Let G be a finite group which satisfies the assumption of Theorem 1.

We may assume that $Z^*(G)=1$.

LEMMA 3.1. *There exists an involution t such that t, tz and z are conjugate each other in G.*

PROOF. By Z^*-theorem there exists an involution t of S which is conjugate to z and distinct from z. Let x be as in Theorem 1. Suppose t centralizes x. By Lemma 2.2 there exists an element g such that $t^g=z$, $C_S(t)^g\subseteqq S$. Since $z^g=(x^g)^2\in S_0$, $z^g=z$, this implies $t=z$, which contradicts the choice of t. By hypothesis $\langle x\rangle\lhd S$, so $x^t=x^{-1}$. Thus $t^x=x^{-1}txtt=tz$, which proves Lemma 3.1.

LEMMA 3.2. *Let D be weakly closed in $N_S(D)$ with respect to G, then we have $S\rhd D$.*

PROOF. Let g be an element of $N_S(N_S(D))$, then we have $D^g\subseteqq N_S(D)$. Since D is weakly closed in $N_S(D)$, we have $D^g=D$. Thus $g\in N_S(D)$, this implies $N_S(N_S(D))=N_S(D)$. Hence we have $S=N_S(D)$, which proves Lemma 3.2.

LEMMA 3.3. *G has a strongly involution closed dihedral* 2-*subgroup.*

PROOF. Let $D_0=\langle t\rangle\times\langle z\rangle$, where t and z are as in Lemma 3.1. If $z^G\cap N_S(D_0)\subseteqq D_0$, then D_0 is weakly closed in $N_S(D_0)$ since $D_0=\langle z^G\cap N_S(D_0)\rangle$. By Lemma 3.2 we have $D_0\lhd S$, hence $z^G\cap S\subseteqq D_0$, this implies that D_0 is strongly involution closed. Then the Lemma is proved. Therefore we may assume that $z^G\cap N_S(D_0)\nsubseteqq D_0$. Thus there exists an involution u such that $u\in z^G\cap N_S(D_0)-D_0$. Assume $C_{D_0}(u)=D_0$, then u centralizes t. Since u is conjugate to z, there exists an element g such that $u^g=z$ and $C_S(u)^g\subseteqq S$ by Lemma 3.2.

Assume $t^g\in S_0$, so that $t^g=z$ by hypothesis of Theorem 1. This implies $u=t$, which contradicts the choice of u. Similarly we have $z^g\notin S_0$. Therefore $(tz)^g\in S_0$, and hence $(tz)^g=z$ since tz is conjugate to z. This implies $tz=u$, which contradicts the choice of u. Thus $\langle u\rangle D_0$ is a dihedral group of order 8, and all involutions of $\langle u\rangle D_0$ are conjugate. Let D_1 be $\langle u\rangle D_0$. Assume $z^G\cap N_S(D_1)\subseteqq D_1$, then it is easy that D_1 is strongly involution closed. Thus we may assume $z^G\cap N_S(D_1)\nsubseteqq D_1$. We shall repeat this method. Assume that D_n is a dihedral subgroup of S, all involutions are conjugate to z, and that $z^G\cap N_S(D_n)\nsubseteqq D_n$. Let $v\in z^G\cap N_S(D_n)-D_n$. By previous method it is easy proved that $C_{D_n}(v)$ is cyclic group. Next we shall prove that $C_{D_n}(v)=\langle z\rangle$. Suppose false. Then there exists an element y of D_n such that $|y|=4$ and $[v,y]=1$. Clearly $y^2=z$. Since v is conjugate to z and $z\in Z(S)$, there exists an element g such that $v^g=z$ and $C_S(v)^g\subseteqq S$ by Lemma 2.2. In particular we have $y^g\in S$, hence $z^g=(x^g)^2\in S_0$. By hypothesis of Theorem 1 we have $z^g=z$. This implies $v=z$, which contradicts the choice of v. Therefore we have $C_{D_n}(v)=\langle z\rangle$. Let $D_{n+1}=\langle v\rangle D_n$, then D_{n+1} is dihedral. If we repeat this method, we have a dihedral subgroup D such that $z^G\cap N_S(D)\subseteqq D$ and $I(D)\subseteqq z^G$. This implies that D is a strongly involution closed dihedral subgroup. Hence Lemma 3.3 is proved.

Since all involutions of D are conjugate, $\langle D^G\rangle=\langle z^G\rangle$. By Lemma 2.3 a Sylow 2-subgroup of $\langle D^G\rangle$ is dihedral or semidihedral. This completes the proof of Theorem 1.

4. Proof of Theorem 2.

Let G be a finite group which satisfies the assumption of Theorem 2.

LEMMA 4.1. *There exists an involution z of S_0 which is conjugate to an involution t of S, moreover conjugate to tz.*

PROOF. Let z_0 be an involution of S_0. By Z^*-theorem we have an involution t_0 of S which is conjugate to z_0 and distinct from z_0. Since $S\rhd S_0$, we have $\Omega_1(S_0)\subseteqq Z(S)$ by hypothesis of Theorem 2. In particular $z_0\in Z(S)$. By Lemma 2.1 there exist an element g and 2-subgroup H such that $t_0{}^g=z_0$, $g\in N_G(H)$ and $H=C_S(\Omega_1(Z(H)))$. Since $z_0\in Z(S)$, $t_0\in\Omega_1(Z(H))$. Set $K=\Omega_1(Z(H))$, then $g\in N(K)$. Since H is a tame intersection, we may assume that g is an odd order element. Let $K_0=[K, g]$, then $|K_0 : K_0 \cap S_0|=2$. Since every involution of S_0 is isolated each other, $|K_0^{\#}| \geqq |(K_0 \cap S_0)^{\#}| \times 3$. This implies that K_0 is four-group and $g^3 \in C_G(K_0)$. Let z be an involution of $K_0 \cap S_0$ and t be an involution of $K_0 - S_0$, then Lemma 4.1 is proved.

Then it is easy that Theorem 2 can be proved by using of Lemma 3.2 and Lemma 3.3. Thus Theorem 2 is proved.

5. Proof of Corollary 1.

If $|X|=2$, then $Z^*(G) \neq 1$ by Z^*-theorem. Assume $|X|=4$. Let $S_0=C_S(X)$, then $|S : S_0| \leqq 2$. Let $\Omega_1(X)=\langle z \rangle$. If $t \in z^G \cap S_0$, then $[t, X]=1$. Since $z \in Z(S)$, we have an element g such that $t^g=z$ and $C_S(t)^g \subseteqq S$ by Lemma 2.2. Then $X^g \subseteqq S$ since $X \subseteqq C_S(t)$. Since X is weakly closed in S, we have $X^g=X$, this implies that $z^g=z$. Hence $t=z$, thus we have $z^G \cap S_0=\{z\}$. Since $X \lhd S$, the assumption of Theorem 1 is satisfied, which implies a conclusion of Corollary 1.

Assume $|X| \geqq 8$. Let $|X|=2^n$, $n \geqq 3$. We set $X=\langle x \rangle$, $y=x^2$, $y_0 \in \langle x \rangle$ such that $|y_0|=4$. Let $S_0=C_S(y_0)$, then $|S : S_0| \leqq 2$. Let $t \in z^G \cap S_0$. Since $|t|=2$ and $\langle x \rangle \lhd S$, $x^t=x$ or x^{-1}, $x^{-1}z$, xz. Since t centralizes y_0, $x^t=x$ or xz. Thus $y^t=y$ in each cases. By Lemma 2.2 there exists an element g such that $t^g=z$ and $C_S(t)^g \subseteqq S$. Since $y \in C_S(t)$, $y^g \in S$, hence y^g acts on X. Since $|X| \geqq 8$, automorphism of X is type of $(2^{n-2}, 2)$. Hence $(y^g)^{2^{n-2}}$ centralizes X. Since $|y|=2^{n-1}$, $(y^g)^{2^{n-1}}=z^g$. Let $t_0=z^g$, t_0 centralizes X. Since t_0 is conjugate to z, there exists an element k such that $t_0^k=z$ and $C_S(t)^k \subseteqq S$. Since $X \subseteqq C_S(t_0)$, $X^k=X$. Hence $z^k=z$. This implies $t_0=z$, hence $t=z$. Thus $z^G \cap S_0=\{z\}$. Since $\langle y_0 \rangle \lhd S$, the assumption of Theorem 1 is satisfied. This completes the proof of Corollary 1.

6. Proof of Corollary 2.

We set $\langle z \rangle=\Omega_1(Z(S))$. We may assume that exponent of $R_1 \geqq$ exponent of R_i for $i=1, \cdots, n$. R_1 has a maximal cyclic subgroup $\langle x \rangle$ such that $|R_1 : \langle x \rangle| \leqq 2$. We set $|x|=2^m$ and $S_0=\langle x \rangle * R_2 * \cdots * R_n$, then $|S : S_0| \leqq 2$. Assume $t \in z^G \cap S_0$, then there exists an element g such that $t^g=z$ and $C_S(t)^g \subseteqq S$ by Lemma 2.2. Since $t \in S_0$, $[x, t]=1$. Therefore $x^g \subseteqq S$. Then $z^g=(x^g)^{2^{m-1}} \in Z(S)$ by the assumption (1) of Corollary 2. By the assumption (2) of Corollary 2 we have $z^g=z$. This implies $t=z$, hence $z^G \cap S_0=\{z\}$. By Theorem 1 Corollary 2 is proved.

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