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On the values of Hecke's L-functions at non-positive integers

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§ 0. Introduction.

Let F be a totally real algebraic number field of finite degree g, and $\mathfrak{m}$ be an integral ideal of F. Let χ be a character of finite order of the narrow ray class group modulo $\mathfrak{m}$. Then we say that χ is totally real (resp. totally imaginary) if the field corresponding to the kernel of χ is totally real (resp. totally imaginary). Consider the L-function with character χ which is defined as usual by $L(s, \chi)=\sum_{(\mathfrak{a},\mathfrak{m})=1}\chi(\mathfrak{a})(N(\mathfrak{a}))^{-s}$, where the summation runs over all integral ideals $\mathfrak{a}$ of F prime to $\mathfrak{m}$, and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$ from F to $\boldsymbol{Q}$. In his paper [**12**], C. L. Siegel considered the ζ-function $\zeta(\mathfrak{b}, \mathfrak{m}, s)=\sum_{\mathfrak{g}}(N(\mathfrak{g}))^{-s}$, where the summation runs over all integral ideals $\mathfrak{g}$ in the same narrow ray class mod. $\mathfrak{m}$ as a fixed ideal $\mathfrak{b}$ prime to $\mathfrak{m}$, and showed that there is an algorithm to compute $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ for positive integers m, and that the value $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ is contained in the rational number field $\boldsymbol{Q}$. Especially, when $\mathfrak{m}=\mathfrak{O}_F$ (the maximal order of F), he obtained an arithmetical expression of $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ (see Siegel [**11**]). Recently, using a method essentially different from Siegel's, T. Shintani [**10**] has given a formula for the value $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ (hence that of $L(1-m, \chi)$ without our assumptions on χ (see below)) in terms of (a generalization of) Bernoulli polynomials in several variables.

In this paper, we shall introduce a certain trick which enables us to apply Siegel's Theorem (see Siegel [**11**] Satz 1, and see text Lemma 2.1) in the case of a non-trivial character χ which satisfies the following conditions:

(i) *χ is totally real or totally imaginary,*

(ii) *$\mathfrak{m}\neq 1$, and $\mathfrak{m}\cap\boldsymbol{Z}$ is a prime ideal $p\boldsymbol{Z}$.*

When the restriction of χ to $\boldsymbol{Z}$ is not trivial, we shall further assume a certain additional not too restrictive condition on χ (for details, see Theorem 2.1 and 2.2). Under the above assumptions on χ, if $k>1$, we shall derive a new formula for $L(1-k, \chi)$ (Theorem 2.2) which is similar to the Siegel formula for $\zeta(1-k)$. In particular, we can prove that

(*) *$L(1-k, \chi)$ is contained in $\boldsymbol{Q}(\chi)$ $(=\boldsymbol{Q}(\chi(a)\,;\, a\in\mathfrak{O}_F))$.*

If $k=1$, for a simple technical reason, our method gives the value of $L(0, \chi)$ only when χ is a real valued character. In other words, we can give a formula for the relative class number of the totally imaginary quadratic extension over F corresponding to χ (Theorem 2.1). So, our method and the final expression of our formula have quite different natures from those of Shintani's.

Now, let us explain our method in more detail. Let $\mathfrak{O}_F$ be the maximal order of F, and we denote by $GL_2^+(\mathfrak{O}_F)$ the subgroup consisting of all elements in $GL_2(\mathfrak{O}_F)$ with totally positive determinant, and put $\Gamma_0(\mathfrak{m})=\{\alpha\in GL_2^+(\mathfrak{O}_F);$ $\alpha=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, c\equiv 0 \pmod{\mathfrak{m}}\}$. We regard $GL_2^+(\mathfrak{O}_F)$ and $\Gamma_0(\mathfrak{m})$ as subgroups of $GL_2^+(\boldsymbol{R})^g$, the product of g-copies of $GL_2^+(\boldsymbol{R})=\{\alpha\in GL_2(\boldsymbol{R}); \det(\alpha)>0\}$, by the following injection:

$$GL_2^+(\mathfrak{O}_F)\ni\alpha\mapsto(\alpha^{(1)}, \cdots, \alpha^{(g)})\in GL_2^+(\boldsymbol{R})^g,$$

where we denote by $\alpha^{(i)}$ the i-th conjugate of α over $\boldsymbol{Q}$. Then $GL_2^+(\mathfrak{O}_F)$ acts on $\mathfrak{H}^g$, the product of g-copies of the complex upper half plane $\mathfrak{H}=\{z\in\boldsymbol{C};$ $\mathrm{Im}(z)>0\}$, as follows:

$$\alpha(z_1, \cdots, z_g)=(\alpha^{(1)}(z_1), \cdots, \alpha^{(g)}(z_g)),$$
$$\alpha^{(i)}(z_i)=(a^{(i)}z_i+b^{(i)})/(c^{(i)}z_i+d^{(i)}),$$

where $\alpha\in GL_2^+(\mathfrak{O}_F)$, $\alpha=(\alpha^{(1)}, \cdots, \alpha^{(g)})$, $\alpha^{(i)}=\begin{bmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{bmatrix}$. For every positive integer κ, and a function f on $\mathfrak{H}^g$, we write

$$(0.1)\qquad (f|_\kappa[\alpha])(z)=f(\alpha(z))J(\alpha, z)^{-\kappa},$$

where $J(\alpha, z)=\prod_{i=1}^{g}(c^{(i)}z_i+d^{(i)})$.

Let χ_0 be the character of $(\mathfrak{O}_F/\mathfrak{m})^\times$ which is naturally obtained from χ. Let $\mathcal{M}_\kappa(\Gamma_0(\mathfrak{m}), \chi_0)$ denote the vector space of all Hilbert modular forms of weight κ with respect to χ_0. We simply write it $M_\kappa(\Gamma_0(m))$, when χ_0 is the identity character. Namely $f\in\mathcal{M}_\kappa(\Gamma_0(\mathfrak{m}), \chi_0)$ if and only if

$$(0.2)\qquad f|_\kappa[\alpha]=\chi_0(d)f(z) \quad \text{for all} \quad \alpha=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in\Gamma^0(\mathfrak{m}),$$

and $f(z)$ is holomorphic on $\mathfrak{H}^g$ and finite at every cusp of $\Gamma_0(\mathfrak{m})$. Then f has the following Fourier expansion at infinity:

$$(0.3)\qquad f(z)=a_0+\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\\ \xi\gg 0}} a(\xi\mathfrak{d}_F)\exp(2\pi i\ \mathrm{Tr}(\xi z)),$$

where ξ runs over all totally positive ($\gg 0$) elements in $\mathfrak{d}_F^{-1}$, and $\mathfrak{d}_F$ is the different of F and $\mathrm{Tr}(\xi z)=\xi^{(1)}z_1+\cdots+\xi^{(g)}z_g$. The coefficient $a(\xi\mathfrak{d}_F)$ depends only on the ideal $\xi\mathfrak{d}_F$ by the property (0.2). Now we define the Dirichlet series corresponding to $f(z)\in\mathcal{M}_\kappa(\Gamma_0(\mathfrak{m}), \chi_0)$ by using (0.3) as follows:

(0.4) $$D_f(s)=\sum_{\mathfrak{a}} a(\mathfrak{a})(N(\mathfrak{a}))^{-s},$$

where $\mathfrak{a}$ runs over all integral ideals of F.

In § 1, without assuming the condition (ii) for χ, we shall introduce a Hilbert modular form $E_{F,k,\chi}\in\mathcal{M}_k(\Gamma_0(\mathfrak{m}),\chi_0)$ (k; a positive integer), whose constant term of the Fourier expansion at infinity is $2^{-g}L(1-k,\chi)$. This form $E_{F,k,\chi}$ is an Eisenstein series of "Neben"-type, and it corresponds to the Dirichlet series $\zeta_F(s)L(s-k+1,\chi)$, where $\zeta_F(s)$ is the Dedekind ζ-function of F. These forms have been constructed by Hecke [3], [5] in the case of one variable ($F=\boldsymbol{Q}$).

Now, for a subgroup $\Gamma_0(N)$ of $SL_2(\boldsymbol{Z})$ for a positive integer N, let $\mathrm{Tr}_{\Gamma_0(N)/SL_2(\boldsymbol{Z})}$ denote the trace operator from $\mathcal{M}_k(\Gamma_0(N))$ to $\mathcal{M}_k(SL_2(\boldsymbol{Z}))$ (cf. Serre [8] § 3), namely

(0.5) $$(\mathrm{Tr}_{\Gamma_0(N)/SL_2(\boldsymbol{Z})}f)(z)=\sum_{i=1}^{d} f|_k[\alpha_i], \qquad \text{for} \quad f\in\mathcal{M}_k(\Gamma_0(N)),$$

where $SL_2(\boldsymbol{Z})=\bigcup_{i=1}^{d}\Gamma_0(N)\alpha_i$, $d=(SL_2(\boldsymbol{Z}):\Gamma_0(N))$. Further we define the embedding $\boldsymbol{P}$ of $\mathfrak{H}$ into $\mathfrak{H}^g$ by $\boldsymbol{P}(z)=(z,\cdots,z)\in\mathfrak{H}^g$. Let $\tilde{\chi}$ denote the restriction of χ to $\boldsymbol{Q}$. In § 2, under the above assumptions (i), (ii), we shall calculate the Fourier expansion of $\mathrm{Tr}_{\Gamma_0(p)/SL_2(\boldsymbol{Z})}((E_{F,k,\chi}\circ\boldsymbol{P})\times E_{\boldsymbol{Q},\lambda,\bar{\tilde{\chi}}})$ (for every positive integer satisfying $\lambda\equiv kg \pmod{2}$) when $\tilde{\chi}\neq1$, and $\mathrm{Tr}_{\Gamma_0(p)/SL_2(\boldsymbol{Z})}(E_{F,k,\chi}\circ\boldsymbol{P})$ when $\tilde{\chi}=1$. Then, to our $\mathrm{Tr}_{\Gamma_0(p)/SL_2(\boldsymbol{Z})}((E_{F,k,\chi}\circ\boldsymbol{P})\times E_{\boldsymbol{Q},\lambda,\bar{\tilde{\chi}}})$ and $\mathrm{Tr}_{\Gamma_0(p)/SL_2(\boldsymbol{Z})}(E_{F,k,\chi}\circ\boldsymbol{P})$, we can apply Siegel's Theorem (Siegel [11] Satz 1, see text, Lemma 2.1.) that determines the constant term of a modular form with respect to $SL_2(\boldsymbol{Z})$. In the last section (§ 3), we shall discuss some numerical examples of Theorem 2.1 and Theorem 2.2.

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Notation and terminology. As usual we denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ respectively the ring of rational integers, the rational number field, the real number field, and the complex number field. For a ring X ($\ni1$), we denote by $X^\times$ the group of all invertible elements of X, and $M_2(X)$ denotes the ring of 2×2 matrices with entries in X, and we put $GL_2(X)=M_2(X)^\times$. If F is a totally real algebraic number field of finite degree g, we denote by $\mathfrak{O}_F$, $\mathfrak{d}_F$, F_A respectively the maximal order of F, the different of F, and the adele ring of F. Further $F_\infty^\times$ denotes the infinity part of $F_A^\times$, and $F_{\infty+}^\times$ denotes the identity component of $F_\infty^\times$. Let $G=GL_2$ the general linear group of 2 variables, and G_F denotes the group of all F-rational points of G, and G_A denotes its adelization. For a place v of F, we denote by F_v, and G_v respectively the completion of F and G_F at v. For a finite place v of F, we denote by $\mathfrak{O}_v$, $\mathfrak{p}_v$ respectively the valuation ring of F_v, and the maximal ideal of $\mathfrak{O}_v$. For an idele $x=(x_v)$ of $F_A^\times$, we define $\mathrm{div}(x)=\prod_v \mathfrak{p}_v^{\mathrm{ord}(x_v)}$, where the product is taken over all finite places v of F, and $|x|_A$ denotes the module of x on $F_A^\times$. For a finite algebraic extension K/F, we denote by $\mathfrak{d}_{K/F}$ and $D_{K/F}$ respectively the relative different, and the relative discriminant of K/F. We follow the notation and terminology of adelic language of F and G in Weil [**13**], [**14**].

§ 1. Eisenstein series of "Neben"-type with χ of F.

In this section, first we shall define the $\boldsymbol{C}^{2^g}$-valued function $\mathfrak{F}$, $\mathfrak{F}'$ on $B_A=\left\{\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix};\ x\in F_A^\times,\ y\in F_A\right\}$, and prove that $(\mathfrak{F}, C\cdot\mathfrak{F}')$ with a suitable constant C is an automorphic pair in Weil's sense (cf. [**14**], and see below). Then, we shall interpret our result into the classical language, and we shall obtain $E_{F,k,\chi}$.

To explain our result, we shall follow Weil [**14**] the terminology and notation. Let $\mathfrak{m}$ be an integral ideal of F, and $m=(m_v)$ be an idele satisfying $\mathrm{div}(m)=\mathfrak{m}$, and $m_v=1$ for any place v outside $\mathfrak{m}$. We always identify the character χ of $F_A^\times/F^\times$ of conductor $\mathfrak{m}$ and the character corresponding to χ of the narrow ray class group modulo $\mathfrak{m}$. For every finite place v of F, we define an open compact subgroup of G_v by

$\mathfrak{K}_v=\left\{\begin{pmatrix} u & z \\ m_v w & t\end{pmatrix};\ \begin{matrix} u,\ z,\ w,\ t\in\mathfrak{O}_v \\ ut-m_v wz\in\mathfrak{O}_v^\times\end{matrix}\right\}$. We write $\mathcal{Z}$ for the center of G, and $\mathcal{Z}_A$ for its adelization.

We shall deal with a $\boldsymbol{C}^{2^g}$-valued function Φ on G_A satisfying the following conditions (see Weil [**14**] § 11):

(A) *For all* $\gamma\in G_F$ *and* $g\in G_A$, $\Phi(\gamma g)=\Phi(g)$.

(B) *For all* $g\in G_A$ *and* $z\in\mathcal{Z}_A$, $\Phi(gz)=\Phi(g)\chi(z)$.

(C) *If v is any finite place outside* $\mathfrak{m}$, *then for all* $g\in G_A$ *and* $\mathfrak{p}\in\mathfrak{K}_v$, $\Phi(g\mathfrak{p})=\Phi(g)$.

(D) *If v is a place occurring in* $\mathfrak{m}$, *and* $\mathfrak{p}=\begin{pmatrix} u & z \\ m_v w & t\end{pmatrix}$ *is any element of* $\mathfrak{K}_v$ *then for all* $g\in G_A$, $\Phi(g\mathfrak{p})=\Phi(g)\chi_v(t)$.

(E) Φ *is admissible type of* $\mathcal{H}_{k,\chi}$ *for every infinite place of F* (see Weil [**14**] § 53).

We denote by $\mathcal{M}_k^0(\mathfrak{m}, \chi)$, the vector space over $\boldsymbol{C}$ consisting of all the functions satisfying the above conditions. A pair of functions $\Phi\in\mathcal{M}_k^0(\mathfrak{m}, \chi)$, $\Phi'\in\mathcal{M}_k^0(\mathfrak{m}, \bar{\chi})$ is called an automorphic pair, when they satisfy the following condition:

$$\Phi'(g)=\Phi(gm^*)\chi(\det(g))^{-1}, \tag{1.1}$$

where $m^*=\begin{pmatrix} 0 & 1 \\ -m & 0\end{pmatrix}\in G_A$.

Now we consider the Fourier expansion of such Φ following Weil [**14**] § 13, § 60. We shall define the Whittaker function $W(x)$ $(x\in F_\infty^\times)$, which is of $\boldsymbol{C}^{2^g}$-valued, corresponding to admissible type of $\mathcal{H}_{k,\chi}$ (for a positive integer k) at every infinite place w of F. To explain this explicitly, we denote by P_∞ the set of all archimedean places of F, and by I the set of $\{1, 2\}$-valued multiindices on P_∞, i. e. $I=\{(\alpha_w)_{w\in P_\infty} | \alpha_w \in \{1, 2\}\}$. For $w\in P_\infty$, we define a $\boldsymbol{C}^2$-valued function $W_w(x)=(W_{w,1}(x), W_{w,2}(x))$ on F_w ($\cong \boldsymbol{R}$) and a $\boldsymbol{C}$-valued function $W_\alpha(x)$ for $\alpha\in I$ by

(1.2)
$$W_{w,1}(x)=\begin{cases} 0, & \text{if } x>0, \\ |x|^{k/2}\cdot\exp(-2\pi|x|), & \text{if } x<0, \end{cases}$$
$$W_{w,2}(x)=W_{w,1}(-x)$$
$$W_\alpha(x)=\prod_{w\in P_\infty} W_{w,\alpha_w}(x_w), \qquad \text{for each } \alpha=(\alpha_w)_{w\in P_\infty}\in I.$$

Then the Whittaker function W corresponding to type $\mathcal{H}_{k,\chi}$ at every $w\in P_\infty$ is defined by $W(x)=\bigotimes_{w\in P_\infty} W_w(x_w)$, for $x=(x_w)_{w\in P_\infty}$, where the tensor product is taken over $\boldsymbol{C}$ for all $w\in P_\infty$. Namely

(1.3) $W(x)=(W_\alpha(x))_{\alpha\in I}$ (here we consider $W(x)$ as a vector valued function defined by a row vector $(W_\alpha(x))_{\alpha\in I}$).

To describe the Whittaker function corresponding to the other function Φ' in the automorphic pair (Φ, Φ'), we define a representation M_k of $O(2, \boldsymbol{R})$ by

$$M_k(r(\theta))=\begin{pmatrix} e^{ik\theta} & 0 \\ 0 & e^{-ik\theta}\end{pmatrix} \text{ for } r(\theta)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} \text{ and } M_k\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}.$$

We define a representation of $\prod_{w\in P_\infty} O(2, F_w)$ by

$$M_{k,\infty}=\bigotimes_{w\in P_\infty} M_{k,w}, \quad \text{where } M_{k,w}=M_k \text{ on } O(2, F_w)\cong O(2, \boldsymbol{R}).$$

Then the Whittaker function W' corresponding to Φ' is given by

(1.4)
$$W'(x)=W(x)M_{k,\infty}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\chi_\infty(x)^{-1}.$$

Assume k is even if χ is totally real and odd if χ is totally imaginary. Then by (1.2) and (1.4), we can easily prove

(1.5)
$$W(x)=i^{gk}W'(x) \quad (i=\sqrt{-1}).$$

We define a non trivial additive character ψ of F_A/F by the following condition:

(1.6)
$$\psi_v(x)=\exp(2\pi i\langle \mathrm{Tr}(x)\rangle) \quad \text{for a finite place } v \text{ of } F,$$

where $\langle \mathrm{Tr}(x)\rangle$ denotes the rational part of the trace of x from F_v to $\boldsymbol{Q}_p$ ($\mathfrak{p}_v | p$: a prime of $\boldsymbol{Z}$),

$$\psi_w(x)=\exp(-2\pi i x) \qquad \text{for } w\in P_\infty.$$

We denote by d the differental idele attached to ψ.

The function Φ satisfying the conditions (A)-(E) is uniquely determined by its restriction $\mathcal{F}$ to B_A, and $\mathcal{F}$ has the following Fourier expansion by the properties (A)-(E) (see [**14**] § 13).

$$\mathcal{F}(x, y)=c_0(x)+\sum_{\xi\in F^\times} c(\mathrm{div}(\xi dx))W(\xi x_\infty)\phi(\xi y)\,, \tag{1.7}$$

where $c_0(x)$ is a linear combination of $\phi(x)\cdot|x|_A^{k/2}$, and $\phi(x)$ is the character of finite order and the conductor of ϕ is 1, and c is a function on the divisor group of F, satisfying the condition that $c(\mathfrak{a})$ equals to 0 unless $\mathfrak{a}$ is integral (see [14] § 13, § 65).

We denote by Ω_F the group of all quasi-characters of $F_A^\times/F^\times$ with the natural complex structure (see [14] § 9). Then we define the Dirichlet series corresponding to F by $Z(\omega)=\sum_{\mathfrak{a}} c(\mathfrak{a})\omega(\mathfrak{a})$ for $\omega\in\Omega_F$, where the summation runs over all integral ideals $\mathfrak{a}$ of F.

Now we consider the Dirichlet series which is obtained as the product of the Dedekind ζ-function of F and Hecke's L-function with χ. We put for a positive integer k

$$\begin{aligned}\zeta_F(s)\cdot L(s-k+1, \chi)&=\sum_{\mathfrak{a}} a_{k,\chi}(\mathfrak{a})(N(\mathfrak{a}))^{-s}\\ &=\prod_{\mathfrak{p}}\{1-(N(\mathfrak{p}))^{-s}\}^{-1}\cdot\prod_{\mathfrak{p}}\{1-(N(\mathfrak{p}))^{k-1}\chi(\mathfrak{p})\cdot(N(\mathfrak{p}))^{-s}\}^{-1}\\ &=\prod_{\mathfrak{p}}\{1-(1+(N(\mathfrak{p}))^{k-1}\chi(\mathfrak{p}))\cdot(N(\mathfrak{p}))^{-s}+\chi(\mathfrak{p})\cdot(N(\mathfrak{p}))^{k-1-2s}\}^{-1}\,,\end{aligned} \tag{1.8}$$

where the summation runs over all integral ideals $\mathfrak{a}$ and the product is taken over all prime ideals $\mathfrak{p}$ of F. Therefore

$$a_{k,\chi}(\mathfrak{a})=\sum_{\mathfrak{b}|\mathfrak{a}}(N(\mathfrak{b}))^{k-1}\chi(\mathfrak{b})\,. \tag{1.9}$$

In the same way, if we put

$$\begin{aligned}\zeta_F(s-k+1)L(s, \chi)&=\sum_{\mathfrak{a}} a'_{k,\chi}(\mathfrak{a})(N(\mathfrak{a}))^{-s}\\ &=\prod_{\mathfrak{p}}\{1-((N(\mathfrak{p}))^{k-1}+\chi(\mathfrak{p}))(N(\mathfrak{p}))^{-s}+\chi(\mathfrak{p})(N(\mathfrak{p}))^{k-1-2s}\}^{-1}\,,\end{aligned} \tag{1.10}$$

then

$$a'_{k,\chi}(\mathfrak{a})=\sum_{\mathfrak{b}|\mathfrak{a}}(N(\mathfrak{b}))^{k-1}\chi(\mathfrak{a}/\mathfrak{b})\,. \tag{1.11}$$

Now we define the functions $c_{k,\chi}$ and $c'_{k,\chi}$ on the divisor group of F by

$$c_{k,\chi}(\mathfrak{a})=\begin{cases}(N(\mathfrak{a}))^{-k/2}a_{k,\chi}(\mathfrak{a})\,, & \text{if } \mathfrak{a} \text{ is integral (see (1.10)),}\\ 0\,, & \text{otherwise.}\end{cases} \tag{1.12}$$

$$c'_{k,\chi}(\mathfrak{a})=\begin{cases}(N(\mathfrak{a}))^{-k/2}a'_{k,\chi}(\mathfrak{a})\,, & \text{if } \mathfrak{a} \text{ is integral (see (1.12)),}\\ 0\,, & \text{otherwise.}\end{cases}$$

Further we put $c_{0,k,\chi}(x)=2^{-g}L(1-k, \chi)\cdot|dx|_A^{k/2}$, for $x\in F_A^\times$.

Then we have the following result:

PROPOSITION 1.1. *Under the assumption* (i) (see § 0), *we assume that* $\mathfrak{m}\neq 1$ *and* χ *is a primitive charactor* mod. $\mathfrak{m}$. *Further we assume* $k\equiv\delta_\chi$ (mod. 2), *where*

$$\delta_\chi=\begin{cases}0, & \text{if } \chi \text{ is totally real,}\\ 1, & \text{if } \chi \text{ is totally imaginary,}\end{cases}$$ *and we put*

(1.13) $$\mathcal{F}(x,\ y)=c_{0,k,\chi}(x)+\sum_{\xi\in F^\times}c_{k,\chi}(\mathrm{div}(\xi dx))W(\xi x_\infty)\psi(\xi y)\,,$$

$$\mathcal{F}'(x,\ y)=\delta_k c_{0,1,\bar{\chi}}(x)+\sum_{\xi\in F^\times}c'_{k,\bar{\chi}}(\mathrm{div}(\xi dx))W(\xi x_\infty)\psi(\xi y)\,,$$

where $\delta_k=\begin{cases}0, & \text{if } k>1,\\ 1, & \text{if } k=1.\end{cases}$

Then $(\mathcal{F},\ (-1)^{[k/2]g}(N(\mathfrak{m}))^{(k-1)/2}W(\chi)\mathcal{F}')$ *can be extended to an automorphic pair of admissible type* $\mathcal{H}_{k,\chi}$ *in* $\mathcal{M}_k^0(\mathfrak{m},\ \chi)$, *where* $W(\chi)$ $(=\kappa(\chi)\chi(dm)$ *in the notation of Weil* [**13**]) *is the constant factor of the functional equation of* $L(s,\ \chi)$.

PROOF. The Dirichlet series corresponding to F is given by

(1.14) $$Z(\omega_s)=\sum_{\mathfrak{a}}c_{k,\chi}(\mathfrak{a})\omega_s(\mathfrak{a})$$
$$=\zeta_F(k/2+s)L(1-k/2+s,\ \chi)\,,$$

where we define $\omega_s\in\Omega_F$ by $\omega_s(x)=|x|_A^s$. The Fourier expansion of $\mathcal{F}'$ with respect to $W'(x)$ (see (1.4), (1.5)) is as follows:

$$\mathcal{F}'(x,\ y)=\delta_k c_{0,1,\bar{\chi}}(x)+i^{gk}\sum_{\xi\in F^\times}c'_{k,\bar{\chi}}(\mathrm{div}(\xi dx))W'(\xi x_\infty)\psi(\xi y)\,.$$

So the Dirichlet series corresponding to $C\mathcal{F}'$ with $C=(-1)^{[k/2]g}(N(\mathfrak{m}))^{(k-1)/2}W(\chi)$ is given by

(1.15) $$Z'(\omega_s)=i^{gk}C\sum_{\mathfrak{a}}c'_{k,\bar{\chi}}(\mathfrak{a})\omega_s(\mathfrak{a})$$
$$=i^{gk}C\zeta_F(1-k/2+s)L(k/2+s,\ \bar{\chi})\,.$$

Now we consider the poles of $Z(\omega)$ and $Z'(\omega)$ on Ω_F (here the poles of $Z(\omega)$ and $Z'(\omega)$ mean the poles of the analytic continuation of the product of $Z(\omega)$ or $Z'(\omega)$ and the suitable Γ-factor as in (A), see below). $L(k/2+s,\ \omega)$ for $\omega\in\Omega_F$ has simple poles at $\omega_{-k/2}$, $\omega_{1-k/2}$. On the other hand, if $k>1$, $L(1-k/2+s,\ \chi\omega)$ has zero at $\omega_{1-k/2}$ by the assumption $k\equiv\delta_\chi$ (mod. 2). Consequently, by the assumption $\mathfrak{m}\neq 1$, $Z(\omega_s)$ (resp. $Z'(\omega_{-s})$) has a simple pole at $\omega_{-k/2}$ if $k>1$, and at $\omega_{1/2}$, $\omega_{-1/2}$ if $k=1$. By Weil [**14**] § 63 Proposition 14, § 59 Lemma 9 (see (1.7)) and the same discussion of [**14**] § 65, the form of the constant term of the Fourier expansion of $\mathcal{F}$, $\mathcal{F}'$ is $c\omega_{k/2}(x)$ for a suitable constant c. Consequently, the Fourier coefficients of $\mathcal{F}$, $\mathcal{F}'$ satisfy the conditions (a), (b) of Proposition 6 of [**14**] § 24 because of the Euler factorization property of $Z(\omega)$ and $Z'(\omega)$ as in (1.8), (1.10). Therefore $\mathcal{F}$ and $\mathcal{F}'$ are eigenfunctions of all the Hecke operator T_v of any place of F prime to $\mathfrak{m}$. In [**13**], we have studied only a B-cuspidal function on G_A, but after a slight modification, Corollary of Theorem 7 in [**13**] is also valid for other than B-cuspidal functions. Namely, $(\mathcal{F},\ C\mathcal{F}')$ is an automorphic pair of admissible type $\mathcal{H}_{k,\chi}$, if $Z(\omega)$ and

$Z'(\omega)$ satisfy the following conditions:

(I) *$Z(\omega)$ and $Z'(\omega)$ can be continued meromorphically to functions on Ω_F with finite many simple poles at $\phi\cdot\omega_{\pm k/2}$, where ϕ is a torsion element in Ω_F of conductor* 1, *bounded in every vertical strip of Ω_F outside circles around these poles.*

(II) *$Z(\omega)$ and $Z'(\omega)$ have the functional equation of the following type for $\omega\in\Omega_F$, whose conductor is prime to* $\mathfrak{m}$.

(A)
$$\prod_{w\in P_\infty} G_2(s+s_w+k/2-A_w)Z(\omega\omega_s)$$
$$=\kappa(\omega)^2\omega_\infty(-1)\cdot i^{-gk}\chi(f)\omega(m)\omega(df)^2\cdot|m|_A^s\cdot|df|_A^{2s}$$
$$\times\prod_{w\in P_\infty} G_2(-s-s_w+k/2+A_w)Z'(\omega^{-1}\omega_{-s}).$$

Here we denote by ω_∞ and ω_w $(w\in P_\infty)$, the infinite part of ω and the w-component of ω, respectively. Further, A_w and s_w are defined by $\omega_w(x)=x^{-A_w}\cdot|x|^{s_w}$, $A_w=0$ or 1, $s_w\in\boldsymbol{C}$, and f is an idele of F such that div(f) coincides with the conductor of ω. We put $G_2(s)=(2\pi)^{1-s}\Gamma(s)$, and $\kappa(\omega)$ is the normalized Gaussian sum with respect to f and d (see Weil [**13**]).

The condition (I) has already proved by (1.14), (1.15) and the above argument on the poles of $Z(\omega)$ and $Z'(\omega)$. Now we consider the condition (II). First we put

$$Z_1(\omega\omega_s)=L(k/2+s,\omega),\quad Z_2(\omega\omega_s)=L(1-k/2+s,\chi\omega),$$
$$Z_1'(\omega\omega_s)=L(1-k/2+s,\omega),\quad Z_2'(\omega\omega_s)=L(k/2+s,\bar{\chi}\omega).$$

The functional equation of $Z_1(\omega)$ is (see Weil [**13**]) given by

(B)
$$\prod_{w\in P_\infty} G_1(1/2+s+(k-1)/2+s_w)\cdot Z_1(\omega\omega_s)$$
$$=\kappa(\omega)\cdot\omega(df)\cdot|df|_A^{(k-1)/2+s}\prod_{w\in P_\infty} G_1(1/2-s-(k-1)/2-s_w+2A_w)\cdot Z_2'(\omega^{-1}\omega_{-s}),$$

where
$$G_1(s)=\pi^{-s/2}\Gamma(s/2).$$

The functional equation of $Z_2(\omega)$ is given by

(C)
$$\prod_{w\in P_\infty} G_1(1/2+s-(k-1)/2+s_w+1-2A_w)\cdot Z_2(\omega\omega_s)$$
$$=\kappa(\omega\chi)\cdot\omega\chi(dfm)\cdot|dfm|_A^s\cdot|dfm|_A^{-(k-1)/2}$$
$$\times\prod_{w\in P_\infty} G_1(1/2-s+(k-1)/2-s_w+1)\cdot Z_2'(\omega^{-1}\omega_{-s}),$$

if χ is totally imaginary.

(C′)
$$\prod_{w\in P_\infty} G_1(1/2+s-(k-1)/2+s_w)\cdot Z_2(\omega\omega_s)$$
$$=\kappa(\omega\chi)\cdot\omega\chi(dfm)\cdot|dfm|_A^s\cdot|dfm|_A^{-(k-1)/2}$$
$$\times\prod_{w\in P_\infty} G_1(1/2-s+(k-1)/2-s_w+2A_w)\cdot Z_2'(\omega^{-1}\omega_{-s}),$$

if χ is totally real. By (1.14), (1.15), we have

$$Z(\omega\omega_s)=Z_1(\omega\omega_s)\cdot Z_2(\omega\omega_s), \quad Z'(\omega\omega_s)=i^{gk}\cdot CZ_1'(\omega\omega_s)\cdot Z_2'(\omega\omega_s),$$

where $C=(-1)^{[k/2]g}\cdot|m|_A^{-(k-1)/2}\cdot W(\chi)$.

Thus we obtain the following functional equation of $Z(\omega)$ and $Z'(\omega)$ by (B), (C), (C$'$);

(i) if χ is totally imaginary,

(D) $$\prod_{w\in P_\infty} G_1(1/2+s+(k-1)/2+s_w)\cdot G_1(1/2+s-(k-1)/2+s_w+1-2A_w)\cdot Z(\omega\omega_s)$$
$$=C^{-1}(-i)^{gk}\cdot\kappa(\omega)\kappa(\omega\chi)\chi(dm)\chi(f)\omega(df)^2\omega(m)\cdot|df|_A^{2s}\cdot|m|_A^s\cdot|m|_A^{-(k-1)/2}$$
$$\times\prod_{w\in P_\infty} G_1(1/2-s-(k-1)/2-s_w+2A_w)$$
$$\cdot G_1(1/2-s+(k-1)/2-s_w+1)\cdot Z'(\omega^{-1}\omega_{-s}),$$

where $C=(-1)^{[k/2]g}\cdot|m|_A^{-(k-1)/2}\cdot W(\chi)$,

(ii) if χ is totally real,

(D$'$) $$\prod_{w\in P_\infty} G_1(1/2+s+(k-1)/2+s_w)\cdot G_1(1/2+s-(k-1)/2+s_w)\cdot Z(\omega\omega_s)$$
$$=C^{-1}(-i)^{gk}\cdot\kappa(\omega)\kappa(\omega\chi)\chi(dm)\chi(f)\omega(df)^2\omega(m)\cdot|df|_A^{2s}\cdot|m|_A^s\cdot|m|_A^{-(k-1)/2}$$
$$\times\prod_{w\in P_\infty} G_1(1/2-s-(k-1)/2-s_w+2A_w)$$
$$\cdot G_1(1/2-s+(k-1)/2-s_w+2A_w)\cdot Z'(\omega^{-1}\omega_{-s}),$$

where C is as above.

Now we shall prove that (D) (resp. (D$'$)) is equivalent to (A) if χ is totally imaginary (resp. totally real). First we consider the Γ-factor (G-part) of (D) or (D$'$), i. e. we shall prove the following relation between the G-part of (A) and the G-part of (D) or (D$'$).

Case (i)

(1.16) $$G_1(1/2+s+(k-1)/2+s_w)\cdot G_1(1/2+s-(k-1)/2+s_w+1-2A_w)$$
$$\times G_1(1/2-s-(k-1)/2-s_w+2A_w)^{-1}\cdot G_1(1/2-s+(k-1)/2-s_w+1)^{-1}$$
$$=(-1)^{[k/2]}\cdot G_2(s+s_w+k/2-A_w)\cdot G_2(-s-s_w+k/2+A_w)^{-1},$$

Case (ii)

(1.17) $$G_1(1/2+s+(k-1)/2+s_w)\cdot G_1(1/2+s-(k-1)/2+s_w)$$
$$\times G_1(1/2-s-(k-1)/2-s_w+2A_w)^{-1}\cdot G_1(1/2-s+(k-1)/2-s_w+2A_w)^{-1}$$
$$=(-1)^{A_w}\cdot(-1)^{[k/2]}\cdot G_2(s+s_w+k/2-A_w)\cdot G_2(-s-s_w+k/2+A_w)^{-1}.$$

To prove (1.16) and (1.17), we use the following formulae of G_1 and G_2:

(1.18) $$G_1(s-2n)=\prod_{j=1}^{n}(s/2-j)^{-1}\pi^n G_1(s), \qquad \text{for any positive integer } n.$$

(1.19) $$G_1(s)\cdot G_1(s+1)=\pi^{-1}G_2(s).$$

(1.20) $$G_2(s+1)=\pi^{-1}sG_2(s).$$

The formula (1.18) and (1.20) are easy consequences of the formula $\Gamma(s+1) = s\Gamma(s)$, and (1.19) is derived from $\Gamma(s/2)\Gamma((s+1)/2)=\pi^{1/2}2^{1-s}\Gamma(s)$.

We first consider Case (i) (In Case (i), $k\equiv 1 \pmod{2}$ by the assumption). As to the G-part of the left hand side of (D), we obtain by (1.18),

$$(1.21)\qquad G_1(1/2+s+s_w-(k-1)/2+1-2A_w)$$
$$=\prod_{i=1}^{(k-1)/2}\{1/2\cdot(1/2+s+s_w+(k-1)/2+1-2A_w)-i\}^{-1}\cdot\pi^{(k-1)/2}$$
$$\times G_1(1/2+s+s_w+(k-1)/2+1-2A_w),$$

and by (1.21) and (1.19) we obtain

$$(1.22)\qquad G_1(1/2+s-(k-1)/2+s_w+1-2A_w)\cdot G_1(1/2+s+s_w+(k-1)/2)$$
$$=\prod_{i=1}^{(k-1)/2}\{1/2\cdot(1/2+s+s_w+(k-1)/2+1-2A_w)-i\}^{-1}\cdot\pi^{(k-3)/2}$$
$$\times G_2(1/2+s+s_w+(k-1)/2-A_w).$$

In the same way, as to the G-part of the right hand side of (D), we obtain

$$(1.23)\qquad G_1(1/2-s-(k-1)/2-s_w+2A_w)\cdot G_1(1/2-s+(k-1)/2-s_w+1)$$
$$=\prod_{i=1}^{(k-1)/2}\{1/2\cdot(1/2-s-s_w+2A_w+(k-1)/2)-i\}^{-1}\cdot\pi^{(k-3)/2}$$
$$\times G_2(1/2-s-s_w+(k-1)/2+A_w).$$

Then we see easily

$$(1.24)\qquad \prod_{i=1}^{(k-1)/2}\{(1/2)\cdot(1/2+s+s_w+(k-1)/2+1-2A_w)-i\}^{-1}$$
$$=(-1)^{(k-1)/2}\prod_{i=1}^{(k-1)/2}\{(1/2)\cdot(-1/2-s-s_w-(k-1)/2-1+2A_w)+i\}^{-1}$$
$$=(-1)^{(k-1)/2}\prod_{i=1}^{(k-1)/2}\{(1/2)\cdot(1/2-s-s_w+2A_w+(k-1)/2)-1-(k-1)/2+i\}^{-1}$$
$$=(-1)^{(k-1)/2}\prod_{i=1}^{(k-1)/2}\{(1/2)\cdot(1/2-s-s_w+2A_w+(k-1)/2)-i\}^{-1}.$$

Therefore we obtain (1.16). As to case (ii) ($k\equiv 0 \pmod{2}$), we obtain exactly like as (1.21) and (1.22), for the G-part of the left hand side of (D′)

$$(1.25)\qquad G_1(1/2+s+(k-1)/2+s_w)\cdot G_1(1/2+s-(k-1)/2+s_w)$$
$$=\prod_{i=1}^{k/2}\{1/2\cdot(1/2+s+(k-1)/2+s_w+1)-i\}^{-1}\cdot\pi^{(k/2-1)}$$
$$\times G_2(1/2+s+(k-1)/2+s_w).$$

As for the right hand side of (D′), we have

$$(1.26)\qquad G_1(1/2-s-(k-1)/2-s_w+2A_w)\cdot G_1(1/2-s+(k-1)/2-s_w+2A_w)$$

$$=\prod_{i=1}^{k/2}\{1/2\cdot(1/2-s+(k-1)/2-s_w+2A_w+1)-i\}^{-1}\cdot\pi^{(k/2-1)}$$
$$\times G_2(1/2-s+(k-1)/2-s_w+2A_w).$$

If $A_w=0$, by the same argument as (1.24), we obtain

$$(1.27)\qquad \prod_{i=1}^{k/2}\{1/2\cdot(1/2+s+s_w+(k-1)/2+1)-i\}^{-1}$$
$$=(-1)^{k/2}\prod_{i=1}^{k/2}\{1/2\cdot(1/2-s+(k-1)/2-s_w+2A_w+1)-i\}^{-1}.$$

If $A_w=1$, we have by (1.20),

$$(1.28)\qquad G_2(1/2+s+(k-1)/2+s_w)=(1/2+s+(k-1)/2+s_w-A_w)\cdot\pi^{-1}$$
$$\times G_2(1/2+s+(k-1)/2+s_w-A_w)$$

and

$$(1.29)\qquad G_2(1/2-s+(k-1)/2-s_w+2A_w)=(1/2-s+(k-1)/2-s_w+A_w)\cdot\pi^{-1}$$
$$\times G_2(1/2-s+(k-1)/2-s_w+A_w).$$

Also like as (1.24), we obtain

$$(1.30)\qquad \prod_{i=1}^{k/2}\{1/2\cdot(1/2+s+(k-1)/2+s_w+1)-i\}^{-1}\cdot(1/2+s+(k-1)/2+s_w-A_w)$$
$$=(-1)^{k/2+1}\prod_{i=1}^{k/2}\{1/2\cdot(1/2-s+(k-1)/2-s_w+2A_w+1)-i\}^{-1}$$
$$\times(1/2-s+(k-1)/2-s_w+A_w).$$

Therefore, by (1.25~30), we obtain (1.17).

Comparing the equation (A) with (D) (resp. (D')), by (1.16) (resp. (1.17)), our task is to prove the following:

$$\kappa(\omega)^2\cdot\omega_\infty(-1)\cdot i^{-gk}\chi(f)\omega(m)\omega(df)^2\cdot|m|_A^s\cdot|df|_A^{2s}$$
$$\times\{C^{-1}\cdot(-i)^{gk}\kappa(\omega)\kappa(\omega\chi)\chi(dm)\chi(f)\omega(df)^2\omega(m)\cdot|df|_A^{2s}\cdot|m|_A^s\cdot|m|_A^{-(k-1)/2}\}^{-1}$$
$$=\begin{cases}(-1)^{[k/2]g} & :\ \text{Case (i)},\\ (-1)^{[k/2]g}\omega_\infty(-1)\quad(\omega_\infty(-1)=(-1)^{\Sigma A_w}) & :\ \text{Case (ii)},\end{cases}$$

where $C=(-1)^{[k/2]g}|m|_A^{-(k-1)/2}\cdot W(\chi)$. Then the equation is reduced to

$$(1.31)\qquad \omega_\infty(-1)\kappa(\omega)\times\{C^{-1}\kappa(\omega\chi)\chi(dm)\cdot|m|_A^{-(k-1)/2}\}^{-1}$$
$$=\begin{cases}(-1)^{[k/2]g} & :\ \text{Case (i)},\\ (-1)^{[k/2]g}\omega_\infty(-1) & :\ \text{Case (ii)}.\end{cases}$$

It is easy to see by the definition of $\kappa(\omega)$,

$$(1.32)\qquad \kappa(\omega\chi)=\begin{cases}\omega_\infty(-1)\kappa(\omega)\kappa(\chi) & :\ \text{Case (i)},\\ \kappa(\omega)\kappa(\chi) & :\ \text{Case (ii)}.\end{cases}$$

By the definition of C, and by $W(\chi)=\kappa(\chi)\chi(dm)$ (see Weil [13] Chap. VII), we can see (1.31) immediately from (1.32).

Finally, we calculate the constant term of $\mathcal{F}$, $\mathcal{F}'$. To formulate our result, we first explain some notations for the Haar measure on $F_A^\times$. We put

$$F_A^1=\{x\in F_A^\times;\ |x|_A=1\},\quad N=\{x\in F_\infty^\times;\ x=(\nu,\cdots,\nu)\ \nu\in R_+^\times\},$$

then we have $F_A^1\supset F^\times$ and $F_A^\times\cong F_A^1\times N$. We also put $G_1=F_A^1/F^\times$. For each $\nu\in R_+^\times$, we define u_ν as the element $(\nu,\cdots,\nu)\in N$. For each finite place v, we normalize the Haar measure $d^\times u_v$ on $F_v^\times$ by $\int_{O_v^\times} d^\times u_v=1$, and for an infinite place w we define the Haar measure $d^\times u_w$ by $d^\times u_w=|u_w|^{-1}\cdot du_w$, where du_w is the Lebesgue measure on F_w. We denote by $d^\times u$ the product measure on $F_A^\times$ of $d^\times u_v$ for every place v. On N, we take the Haar measure $d^\times\nu$ by $d^\times\nu=\nu^{-1}d\nu$, and on $F^\times$ we take the canonical Haar measure $d\delta$ defined by $\int_{F^\times} f\,d\delta=\sum_{\xi\in F^\times} f(\xi)$, and on G_1 we define the Haar measure d_1u by $d\delta\cdot d_1u\cdot d^\times\nu=d^\times u$. By the discussion of the beginning of this proof, the constant term of $\mathcal{F}$, $\mathcal{F}'$ is given by the following:

$$(1.33)\qquad c_{0,k,\chi}(x)=b\omega_{k/2}(x),$$
$$c'_{0,k,\bar\chi}(x)=b'\omega_{k/2}(x),\qquad \text{for suitable constants } b,\ b'.$$

Now we define $J(\omega_s)$ following to Weil [14] § 63,

$$J(\omega_s)=\int_0^\infty\int_{G_1}\{\mathcal{F}(tuu_\nu,0)-c_{0,k,\chi}(tuu_\nu)\}\cdot\omega_s(tuu_\nu)\cdot d_1u\,d^\times\nu\,.$$

Since, $(\mathcal{F}, C\mathcal{F}')$ is an automorphic pair (where $C=(-1)^{[k/2]g}\cdot W(\chi)\cdot|m|_A^{-(k-1)/2}$), $(\mathcal{F}, C\mathcal{F}')$ satisfies $\mathcal{F}(tuu_\nu,0)=C\mathcal{F}'(mt^{-1}u^{-1}u_\nu^{-1},0)$ (see Weil [14] § 64 (45)). Therefore we have

$$\begin{aligned}(1.34)\qquad J(\omega_s)=&\int_1^\infty\int_{G_1}\{\mathcal{F}(tuu_\nu,0)-b\cdot\omega_{k/2}(tuu_v)\}\cdot\omega_s(tuu_\nu)d_1u\,d^\times\nu\\&+C\cdot\int_0^1\int_{G_1}\{\mathcal{F}'(mt^{-1}u^{-1}u_\nu^{-1},0)-b'\cdot\omega_{k/2}(mt^{-1}u^{-1}u_\nu^{-1})\}\cdot\omega_s(tuu_\nu)d_1u\,d^\times\nu\\&+C\cdot b'\cdot\int_0^1\int_{G_1}\omega_{k/2}(mt^{-1}u^{-1}u_\nu^{-1})\cdot\omega_s(tuu_\nu)d_1u\,d^\times\nu\\&-b\cdot\int_0^1\int_{G_1}\omega_{k/2}(tuu_\nu)\cdot\omega_s(tuu_\nu)d_1u\,d^\times\nu\,.\end{aligned}$$

By Lemma 10 of Weil [14] § 61 and the argument of § 63, we see easily that the first and the second integral in (1.34) are entire functions in s. Then we calculate another integral by using the fact: $\int_{G_1} d_1u=2^{g-1}\cdot h_F\cdot R_F$, where h_F is the class number of F and R_F is the regulator of F. We obtain

$$(1.35)\quad J(\omega_s)=\text{“an entire function in } s\text{”}+C\cdot b'\cdot 2^{g-1}\cdot h_F\cdot R_F\cdot|m|_A^{k/2}\cdot|t|_A^{s-k/2}/(s-k/2)$$

$$-b\cdot 2^{g-1}\cdot h_F\cdot R_F\cdot |t|_A^{s+k/2}/(s+k/2)\,.$$

On the other hand, by Proposition 14 in Weil [**14**] § 63, and

$$\int_{F_\infty^\times} W_\alpha(u)\cdot\omega_{s,\infty}(u)\cdot \bigotimes_{w\in P_\infty} d^\times u_w=(2\pi)^{-g}\cdot G_2(s+k/2)^g\,,\qquad \text{for}\quad \alpha\in I\,,$$

we have

$$(1.36)\qquad J(\omega_s)=|d|_A^{-s}\cdot(2\pi)^{-g}G_2(s+k/2)^g\cdot Z(\omega_s)$$
$$=|d|_A^{-s}\cdot 2^{-g}\cdot G_1(s+k/2)^g G_1(s+k/2+1)^g Z_1(\omega_s)\cdot Z_2(\omega_s)\quad \text{(by (1.19))}.$$

Since the residue at $s=0, 1$ of $G_1(s)^g\cdot\zeta_F(s)$ is well known (see Weil [**13**] Chap. VII § 6 Theorem 3), the residue of $G_1(s+k/2)^g\cdot Z_1(\omega_s)$ at $s=-k/2$ is $-2^{g-1}\cdot h_F\cdot R_F$, and also $Z_2(\omega_{-k/2})=L(1-k, \chi)$. Consequently, the residue at $s=-k/2$ of $J(\omega_s)$ is $-2^{g-1}\cdot h_F\cdot R_F\cdot 2^{-g}L(1-k, \chi)\cdot|d|_A^{k/2}$ by (1.14). Therefore by (1.35) we obtain

$$b=2^{-g}L(1-k, \chi)\cdot|d|_A^{k/2}\,.$$

Repeating the same calculation, we obtain

$$b'=\begin{cases} 0 & \text{if}\quad k>1,\\ 2^{-g}L(0, \bar\chi)\cdot|d|_A^{1/2} & \text{if}\quad k=1.\end{cases}$$

Then all the assertion of Proposition 1.1 were already proved. q. e. d.

REMARK 1.1. We explain the same result in the case of $\mathfrak{m}=1$ without proof. In this case $Z(\omega)$ and $Z'(\omega)$ have poles at $\omega_{-k/2}$ and $\bar\chi\cdot\omega_{k/2}$ if $k>1$ and $\omega_{\pm 1/2}$, $\bar\chi\cdot\omega_{\pm 1/2}$ if $k=1$, then we can prove the same result as Proposition 1.1 for the following constant term of $\mathcal{F}$, $\mathcal{F}'$

if $k>1$ $\quad c_{0,k,\chi}(x)=2^{-g}L(1-k, \chi)\cdot|d|_A^{k/2}\cdot\omega_{k/2}(x)\,,$

$\qquad\qquad c'_{0,k,\bar\chi}(x)=\bar\chi(d)\cdot 2^{-g}L(1-k, \chi)\cdot|d|_A^{k/2}\cdot\bar\chi\omega_{k/2}(x)\,,$

if $k=1$ $\quad c_{0,k,\chi}(x)=2^{-g}L(0, \chi)\cdot|d|_A^{1/2}\cdot\omega_{1/2}(x)+\chi(d)\cdot 2^{-g}L(0, \bar\chi)\cdot|d|_A^{1/2}\cdot\chi\omega_{1/2}(x)\,,$

$\qquad\qquad c'_{0,k,\bar\chi}(x)=2^{-g}L(0, \bar\chi)\cdot|d|_A^{1/2}\cdot\omega_{1/2}(x)+\bar\chi(d)\cdot 2^{-g}L(0, \chi)\cdot|d|_A^{1/2}\cdot\bar\chi\omega_{1/2}(x)\,.$

Now, we shall interpret our result in the classical terminology. For details, the correspondence of classical modular forms and adelic ones are explained in Yoshida [**15**]. In this paper, we discuss only the restriction of $\mathcal{F}$, $\mathcal{F}'$ to $F_{\infty+}^\times\times F_\infty$. To formulate our result, we define some notations:

For $\alpha=(1, \cdots, 1)\in I=\{1, 2\}^g$, we denote by W_α the α component of the Whittaker function (see (1.3)). We put ϕ (resp. ϕ', ϕ'') for the component of $\mathcal{F}$ (resp. $\mathcal{F}'$, $\mathcal{F}'(mx, my)$) corresponding to α. For $(x, y)\in F_{\infty+}\times F_\infty$, we define $z\in\mathfrak{H}^g$ (the product of g copies of the upper half complex plane) by $z=ix+y$ $=(ix_w+y_w)$ $w\in P_\infty$. We define the function $E_{F,k,\chi}$, $E'_{F,k,\chi}$, $E''_{F,k,\chi}$ on $\mathfrak{H}^g$ by

$$E_{F,k,\chi}(z)=|d|_A^{-k/2}\cdot\prod_{w\in P_\infty}|x_w|^{-k/2}\cdot\phi(x, y)\,,$$

$$E'_{F,k,\bar{\chi}}(z)=|d|_A^{-k/2}\cdot\prod_{w\in P_\infty}|x_w|^{-k/2}\cdot\phi'(x, y),$$

$$E''_{F,k,\bar{\chi}}(z)=|md|_A^{-k/2}\cdot\prod_{w\in P_\infty}|x_w|^{-k/2}\cdot\phi''(x, y).$$

Then the condition (A)~(E) and Proposition 1.1 assert $E_{F,k,\chi}$, $E'_{F,k,\chi}\in\mathcal{M}_k(\Gamma_0(\mathfrak{m}), \chi_0)$. By (1.13), we have

(1.37) $$\phi(x, y)=c_{0,k,\chi}(x)+\sum_{\xi\in F^\times}c_{k,\chi}(\xi\mathfrak{d}_F)W_\alpha(\xi x)\exp(-2\pi i\mathrm{Tr}(\xi y)),$$

$$\phi'(x, y)=c_{0,k,\bar{\chi}}(x)+\sum_{\xi\in F^\times}c'_{k,\bar{\chi}}(\xi\mathfrak{d}_F)W_\alpha(\xi x)\exp(-2\pi i\mathrm{Tr}(\xi y)),$$

$$\phi''(x, y)=c_{0,k,\bar{\chi}}(mx)+\sum_{\xi\in F^\times}c'_{k,\bar{\chi}}(\xi\mathfrak{d}_F\mathfrak{m})W_\alpha(\xi x)\exp(-2\pi i\mathrm{Tr}(\xi y)).$$

By (1.2) and (1.3), we have

$$W_\alpha(x)=\begin{cases}\prod_{w\in P_\infty}|x_w|^{k/2}\cdot\exp(2\pi\mathrm{Tr}(x)), & \text{if } x_w<0 \text{ for all } w\in P_\infty,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, by the definition of $E_{F,k,\chi}$, $E'_{F,k,\chi}$, $E''_{F,k,\chi}$, and Proposition 1.1, we obtain

(1.38) $$E_{F,k,\chi}(z)=2^{-g}L(1-k, \chi)+\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\\ \xi\gg0}}a_{k,\chi}(\xi\mathfrak{d}_F)\exp(2\pi i\mathrm{Tr}(\xi z)),$$

$$E'_{F,k,\chi}(z)=\delta_k 2^{-g}L(0, \chi)+\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\\ \xi\gg0}}a'_{k,\chi}(\xi\mathfrak{d}_F)\exp(2\pi i\mathrm{Tr}(\xi z)),$$

$$E''_{F,k,\chi}(z)=\delta_k 2^{-g}L(0, \chi)+\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\mathfrak{m}^{-1}\\ \xi\gg0}}a'_{k,\chi}(\xi\mathfrak{d}_F\mathfrak{m})\exp(2\pi i\mathrm{Tr}(\xi z)).$$

Now we put $t_w=(x_w^2+y_w^2)^{1/2}$, $f_w=t_w^{-1}\cdot x_w$, $e_w=t_w^{-1}\cdot y_w$ and $e'_w=-e_w$ for $(x, y)\in F_{\infty+}^\times\times F$ and for $w\in P_\infty$, and $t_v=1$, $f_v=1$, $e_v=e'_v=0$ for any finite places v. Then by Proposition 4 and the condition (II′) of [14] § 17, $(\mathcal{F}, C\mathcal{F}')$ satisfies:

(1.39) $$\mathcal{F}(tf, te)=C\mathcal{F}'(mt^{-1}f, mt^{-1}e')\cdot\bigotimes_{w\in P_\infty}M_k(r(\pi/2-\arg(z_w))),$$

where $z=ix+y$ and $\arg(z_w)$ denotes the argument of z_w and M_k is defined in (1.3) below. So we restrict both hands side of (1.39) to $F_{\infty+}^\times\times F$, then by an easy calculation, we obtain:

(1.40) $$E_{F,k,\chi}(-1/z)\cdot\prod_{w\in P_\infty}z_w^{-k}=(-i)^{gk}(-1)^{[k/2]g}(N(\mathfrak{m}))^{-1/2}\cdot W(\chi)\cdot E''_{F,k,\bar{\chi}}(z),$$

where $-1/z=(-1/z_1, \cdots, -1/z_g)$. Summing up our results, we obtain the following Corollary:

COROLLARY TO PROPOSITION 1.1. *We assume* $\mathfrak{m}\neq1$ *and* $k\equiv\delta_\chi$ (mod. 2) *and we put for an integral ideal* $\mathfrak{a}$ *in* F,

$$a_{k,\chi}(\mathfrak{a})=\sum_{\mathfrak{b}|\mathfrak{a}}(N(\mathfrak{b}))^{k-1}\cdot\chi(\mathfrak{b}),\quad a'_{k,\chi}(\mathfrak{a})=\sum_{\mathfrak{b}|\mathfrak{a}}(N(\mathfrak{b}))^{k-1}\cdot\chi(\mathfrak{a}/\mathfrak{b}).$$

Then we obtain

(1.38) $$E_{F,k,\chi}(z)=2^{-g}\cdot L(1-k,\chi)+\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\\ \xi\gg 0}} a_{k,\chi}(\xi\mathfrak{d}_F)\cdot\exp(2\pi i\mathrm{Tr}(\xi z)),$$

$$E'_{F,k,\chi}(z)=\delta_k\cdot 2^{-g}\cdot L(0,\chi)+\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\\ \xi\gg 0}} a'_{k,\chi}(\xi\mathfrak{d}_F)\cdot\exp(2\pi i\mathrm{Tr}(\xi z)),$$

$$E''_{F,k,\chi}(z)=\delta_k\cdot 2^{-g}\cdot L(0,\chi)+\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\mathfrak{m}^{-1}\\ \xi\gg 0}} a'_{k,\chi}(\xi\mathfrak{d}_F\mathfrak{m})\cdot\exp(2\pi i\mathrm{Tr}(\xi z)),$$

where $\delta_k=\begin{cases}1, & \text{if } k=1\\ 0, & \text{if } k\neq 1.\end{cases}$

Furthermore $E_{F,k,\chi}$, $E'_{F,k,\chi}\in\mathcal{M}_k(\Gamma_0(\mathfrak{m}),\chi_0)$ *and*

(1.40) $$E_{F,k,\chi}(-1/z)\prod_{w\in P_\infty} z_w^{-k}=(-i)^{gk}(-1)^{[k/2]g}(N(\mathfrak{m}))^{-1/2}\cdot W(\chi)\cdot E''_{F,k,\bar{\chi}}(z).$$

REMARK 1.2. Corollary to Proposition 1.1 can be also proved by the classical result of Hecke (cf. Hecke [2], [4] Kloosterman [6]). One can also refer this fact to Serre [8] appendice.

REMARK 1.3. For the case $\mathfrak{m}=1$, we also obtain the same result as above. But if $F=\boldsymbol{Q}(\sqrt{D})$ with a positive integer D, has no signature distribution and χ has order 2, then it is easy to see that $E_{F,1,\chi}$ is reduced to 0.

§ 2. The values of Hecke's L-functions at non-positive integers.

For a positive even integer $k>2$, let G_k be the Eisenstein series of weight k with respect to $SL_2(\boldsymbol{Z})$. The Fourier expansion of G_k $(k>2)$ is given by

(2.1) $$G_k(z)=1-(2k/B_k)\cdot\sum_{n=1}^{\infty}\sigma_{k-1}(n)\cdot\exp(2\pi inz),$$

where B_k is the k-th Bernoulli number and σ_g is defined by $\sigma_g(n)=\sum_{\substack{t\mid n\\ t>0}} t^g$. We put also $G_0=1$. We define $\Delta(z)$ by $\Delta(z)=\exp(2\pi iz)\cdot\prod_{n=1}^{\infty}(1-\exp(2\pi inz))^{24}$. Further, for a positive even integer h, we put $T_h=G_{12r(h)-h+2}\Delta^{-r(h)}$ for

$$r(h)=\dim\mathcal{M}_h(SL_2(\boldsymbol{Z}))=\begin{cases}[h/12]+1, & \text{if } h\not\equiv 2 \pmod{12},\\ [h/12], & \text{if } h\equiv 2 \pmod{12}.\end{cases}$$

We define the rational integer $C_{h,j}$ $j=1,\cdots,r=r(h)$ by

$$T_h=C_{h,r}\cdot\exp(-2\pi irz)+\cdots+C_{h,1}\cdot\exp(-2\pi iz)+C_{h,0}+\cdots.$$

It is easy to see by the definition of T_h, $C_{h,r}=1$, if $r(h)\neq 0$.

Here we restate Siegel's result (Siegel [11] Satz 1, Satz 2).

LEMMA 2.1 (C. L. Siegel). *Let* $M=a_0+\sum_{n=1}^{\infty}a_n\exp(2\pi inz)$ *be the Fourier expansion of a modular form of weight* h *with respect to* $SL_2(\boldsymbol{Z})$. *Then we have*

$$C_{h,0}\cdot a_0+C_{h,1}\cdot a_1+\cdots+C_{h,r}\cdot a_r=0, \quad for \quad r=r(h).$$

Furthermore $C_{h,0}\neq 0$ *if* $r(h)\neq 0$.

REMARK 2.1. We recall here Siegel's table (Siegel [11] p. 91) of $C_{h,j}$, $j=1, \cdots, r=r(h)$.

h	$r(h)$	$C_{h,0}$		$C_{h,1}$		$C_{h,2}$
4	1	-240	$(-2^4\cdot3\cdot5)$	1		
6	1	504	$(2^3\cdot3^2\cdot7)$	1		
8	1	-480	$(-2^5\cdot3\cdot5)$	1		
10	1	264	$(2^3\cdot3\cdot11)$	1		
12	2	-196560	$(-2^4\cdot3^3\cdot5\cdot7\cdot13)$	24	$(2^3\cdot3)$	1
14	1	24	$(2^3\cdot3)$	1		
16	2	-146880	$(-2^6\cdot3^3\cdot5\cdot17)$	-216	$(-2^3\cdot3^3)$	1
18	2	86184	$(2^3\cdot3^4\cdot7\cdot19)$	528	$(2^4\cdot3\cdot11)$	1
20	2	-39600	$(-2^4\cdot3^2\cdot5^2\cdot11)$	-456	$(-2^3\cdot3\cdot19)$	1
22	2	14904	$(2^3\cdot3^4\cdot23)$	288	$(2^5\cdot3^2)$	1
24	3	-52416000	$(-2^9\cdot3^2\cdot5^3\cdot7\cdot13)$	-195660	$(-2^2\cdot3^2\cdot5\cdot1087)$	48 $(2^4\cdot3)$
26	2	1224	$(2^3\cdot3^2\cdot17)$	48	$(2^4\cdot3)$	1

From now on, we assume the condition (ii) of § 0, i. e.

(ii) $\mathfrak{m}\neq 1$ *and* $\mathfrak{m}\cap \boldsymbol{Z}$ *is a prime ideal* $p\boldsymbol{Z}$.

We put $N(\mathfrak{m})=p^f$, $f>0$. We consider the embedding $\boldsymbol{P}$ from $\mathfrak{H}$ into $\mathfrak{H}^g$ defined by $\boldsymbol{P}(z)=(z, \cdots, z)$, then $\boldsymbol{P}$ satisfies the following properties:

(2.2) (i) *For all* $\alpha\in GL_2^+(\boldsymbol{Q})$, $\alpha\circ\boldsymbol{P}=\boldsymbol{P}\circ\alpha$,

(ii) *For all* $f\in\mathcal{M}_k(\Gamma_0(\mathfrak{m}), \chi_0)$, $f\circ\boldsymbol{P}\in\mathcal{M}_{kg}(\Gamma_0(p), \tilde{\chi}_0)$,

where we denote by $\tilde{\chi}_0$ the restriction of χ_0 to $\boldsymbol{Z}$.

For a positive integer N, we have defined in Introduction (see (0.5)) the linear map $\mathrm{Tr}_{\Gamma_0(N)/SL_2(\boldsymbol{Z})}$ (if there is no confusion, we write it simply by Tr) from $\mathcal{M}_k(\Gamma_0(N))$ to $\mathcal{M}_k(SL_2(\boldsymbol{Z}))$ by

$$(2.3) \qquad (\mathrm{Tr}_{\Gamma_0(N)/SL_2(\boldsymbol{Z})}f)(z)=\sum_{j=1}^{d} f|_k[\alpha_j](z), \quad \text{for every} \quad f\in\mathcal{M}_k(\Gamma_0(N)),$$

where α_j, $j=1, \cdots, d$ is one of the left representatives of $\Gamma_0(N)$ in $SL_2(\boldsymbol{Z})$, i. e.

$SL_2(\boldsymbol{Z})=\bigcup_{j=1}^{d}\Gamma_0(N)\alpha_j$, (disjoint union and $d=(SL_2(\boldsymbol{Z}):\Gamma_0(N)))$.

To calculate the Fourier expansion of $\mathrm{Tr}(f)$ for a given $f\in\mathcal{M}_k(\Gamma_0(N))$, we prepare next Lemmas.

LEMMA 2.2. *We assume that N is a prime number, then*

$$SL_2(\boldsymbol{Z})=\bigcup_{0\leq d<N}\Gamma_0(N)\begin{pmatrix}0 & -1\\ 1 & d\end{pmatrix}\cup\Gamma_0(N)\,,\ (\textit{disjoint union}).$$

PROOF. It is easy to see by the definition of $\Gamma_0(N)$ that $\begin{pmatrix}0 & -1\\ 1 & d\end{pmatrix}$, $0\leqq d<N$ is not left equivalent to each other under $\Gamma_0(N)$ and using the fact $(SL_2(\boldsymbol{Z}):\Gamma^0(N))=N+1$ we obtain Lemma 2.2. q. e. d.

LEMMA 2.3. *Let N be a prime number and $f(z)$ be an element of $\mathcal{M}_k(\Gamma_0(N))$. Put*

(i) $f(z)=a_0+\sum_{n=1}^{\infty}a_n\cdot\exp(2\pi inz)$, *and*

(ii) $f(-1/z)\cdot z^{-k}=b_0+\sum_{n=1}^{\infty}b_n\cdot\exp(2\pi inz/N)$.

Then we have

$$(\mathrm{Tr}(f))(z)=(a_0+Nb_0)+\sum_{n=1}^{\infty}(a_n+Nb_{nN})\exp(2\pi inz)\,. \tag{2.4}$$

PROOF. We put $\alpha_d=\begin{pmatrix}0 & -1\\ 1 & d\end{pmatrix}$, $0\leqq d<N$, then by Lemma 2.2

(iii) $\mathrm{Tr}(f)=f+\sum_{d \bmod N}f|_k[\alpha_d]$.

By the assumption (ii) of this Lemma, we have

$$\begin{aligned}\sum_{0\leq d<N}f|_k[\alpha_d]&=\sum_{0\leq d<N}f(-1/(z+d))\cdot(z+d)^{-k}\\ &=Nb_0+\sum_{n=1}^{\infty}\Big(\sum_{0\leq d<N}\exp(2\pi ind/N)\Big)\cdot b_n\cdot\exp(2\pi inz/N)\ \ (\text{by (ii)})\\ &=Nb_0+N\sum_{n=1}^{\infty}b_{nN}\cdot\exp(2\pi inz),\quad \text{where we use}\end{aligned}$$

$$\sum_{0\leq d<N}\exp(2\pi ind/N)=\begin{cases}0\,, & \text{if } n\not\equiv 0 \ (\text{mod.}\,N)\,,\\ N\,, & \text{if } n\equiv 0 \ (\text{mod.}\,N)\,.\end{cases}$$

Therefore by (iii), we obtain Lemma 2.3. q. e. d.

By Corollary to Proposition 1.1, for a primitive character ψ of $(\boldsymbol{Z}/N\boldsymbol{Z})^{\times}$, we put the Eisenstein series associated with the character ψ of weight λ by

$$E_{\boldsymbol{Q},\lambda,\psi}(z)=-B_{\lambda,\psi}/2\lambda+\sum_{n=1}^{\infty}b_{\lambda,\psi}(n)\cdot\exp(2\pi inz)\,, \tag{2.5}$$

$$E'_{Q,\lambda,\psi}(z)=-\delta_\lambda B_{1,\psi}/2\lambda+\sum_{n=1}^{\infty} b'_{\lambda,\psi}(n)\cdot\exp(2\pi inz),$$

where $B_{\lambda,\psi}$ is the λ-th generalized Bernoulli number with ψ and,

$$\delta_\lambda=\begin{cases}0, & \text{if } \lambda\neq 1,\\ 1, & \text{if } \lambda=1, \text{ and}\end{cases}$$

(2.6)
$$b_{\lambda,\psi}(n)=\sum_{\substack{d\mid n\\ d>0}} d^{\lambda-1}\cdot\psi(d),$$
$$b'_{\lambda,\psi}(n)=\sum_{\substack{d\mid n\\ d>0}} d^{\lambda-1}\cdot\psi(n/d).$$

By the property (2.2) (ii), if we put $\phi_{k,\chi}=E_{F,k,\chi}\circ\boldsymbol{P}$, $\phi'_{k,\chi}=E'_{F,k,\chi}\circ\boldsymbol{P}$, and $\phi''_{k,\chi}=E''_{F,k,\chi}\circ\boldsymbol{P}$, then $\phi_{k,\chi}$, $\phi'_{k,\chi}\in\mathcal{M}_{kg}(\Gamma_0(p), \tilde{\chi}_0)$. Put

(2.7)
$$\phi_{k,\chi}(z)=2^{-g}L(1-k, \chi)+\sum_{n=1}^{\infty} A_{k,\chi}(n)\cdot\exp(2\pi inz),$$

$$\phi''_{k,\chi}(z)=\delta_k 2^{-g}L(0, \chi)+\sum_{n=1}^{\infty} A'_{k,\chi}(n)\cdot\exp(2\pi inz/p),$$

where $\delta_k=\begin{cases}0, & \text{if } k\neq 1,\\ 1, & \text{if } k=1,\end{cases}$ then by (1.38) of Corollary to Proposition 1.1, we have

(2.8)
$$A_{k,\chi}(n)=\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\\ \mathrm{Tr}(\xi)=n\\ \xi\gg 0}} a_{k,\chi}(\xi\mathfrak{d}_F),$$
$$A'_{k,\chi}(n)=\sum_{\substack{\xi\in\mathfrak{d}_F^{-1}\mathfrak{m}^{-1}\\ \mathrm{Tr}(\xi)=n/p\\ \xi\gg 0}} a'_{k,\chi}(\xi\mathfrak{d}_F\mathfrak{m}).$$

By (1.40) of Corollary to Proposition 1.1, we obtain

(2.9)
$$E_{Q,\lambda,\psi}(-1/z)\cdot z^{-\lambda}=(-i)^\lambda\cdot(-1)^{[\lambda/2]}\cdot N^{-1/2}\cdot W(\psi)\cdot E'_{Q,\lambda,\bar{\psi}}(z/N).$$
$$\phi_{k,\chi}(-1/z)\cdot z^{-kg}=(-i)^{gk}\cdot(-1)^{[k/2]g}\cdot(N(\mathfrak{m}))^{-1/2}\cdot W(\chi)\cdot\phi''_{k,\bar{\chi}}(z).$$

Now, we calculate the Fourier expansion of $\mathrm{Tr}(E_{Q,\lambda,\bar{\tilde{\chi}}}\cdot\phi_{k,\chi})$ ($\lambda\equiv kg \pmod 2$) if $\tilde{\chi}\neq 1$, and $\mathrm{Tr}(\phi_{k,\chi})$ if $\tilde{\chi}=1$ by using Lemma 2.3. After an easy calculation, we obtain the following Fourier coefficients:

the n-th Fourier coefficient of $E_{Q,\lambda,\bar{\tilde{\chi}}}\cdot\phi_{k,\chi}$

$$=2^{-g}L(1-k, \chi)b_{\lambda,\bar{\tilde{\chi}}}(n)-(B_{\lambda,\bar{\tilde{\chi}}}/2\lambda)\cdot A_{k,\chi}(n)+\sum_{j=1}^{n-1} b_{\lambda,\bar{\tilde{\chi}}}(j)\cdot A_{k,\chi}(n-j),$$

the constant term of $E_{Q,\lambda,\bar{\tilde{\chi}}}\cdot\phi_{k,\chi}=-2^{-g}L(1-k, \chi)\cdot(B_{\lambda,\bar{\tilde{\chi}}}/2\lambda)$,

the Fourier coefficient of $\exp(2\pi inz/p)$ of $E'_{Q,\lambda,\tilde{\chi}}(z/p)\cdot\phi''_{k,\bar{\chi}}(z)$

$$=\delta_k 2^{-g}L(0, \bar{\chi})\cdot b'_{\lambda,\tilde{\chi}}(n)-\delta_\lambda\cdot(B_{1,\tilde{\chi}}/2)\cdot A'_{k,\bar{\chi}}(n)+\sum_{j=1}^{n-1} b'_{\lambda,\tilde{\chi}}(j)\cdot A'_{k,\bar{\chi}}(n-j),$$

the constant term of $E'_{Q,\lambda,\tilde{\chi}}(z/p)\cdot\phi''_{k,\bar{\chi}}(z)=-\delta_k\cdot\delta_\lambda\cdot 2^{-g}L(0, \bar{\chi})\cdot(B_{1,\tilde{\chi}}/2)$.

By (2.9), $E_{Q,\lambda,\bar{\tilde{\chi}}}\cdot\phi_{k,\chi}$ and $E'_{Q,\lambda,\tilde{\chi}}(z/p)\cdot\phi''_{k,\bar{\chi}}(z)$ satisfy the assumption of

Lemma 2.3, therefore we can apply Lemma 2.3 to these forms, then we obtain the following Fourier expansion of $\mathrm{Tr}(E_{Q,\lambda,\tilde{\bar{\chi}}}\cdot\phi_{k,\chi})$, if $\tilde{\chi}\neq 1$.

(2.10) $$\mathrm{Tr}(E_{Q,\lambda,\tilde{\bar{\chi}}}\cdot\phi_{k,\chi})=-2^{-g}L(1-k,\ \chi)\cdot(B_{\lambda,\tilde{\bar{\chi}}}/2\lambda)-C_1\delta_k\cdot\delta_\lambda\cdot 2^{-g}L(0,\ \tilde{\chi})\cdot(B_{1,\tilde{\chi}}/2)$$
$$+\sum_{n=1}^{\infty}[2^{-g}L(1-k,\ \chi)\cdot b_{\lambda,\tilde{\bar{\chi}}}(n)-(B_{\lambda,\tilde{\bar{\chi}}}/2\lambda)\cdot A_{k,\chi}(n)+\sum_{j=1}^{n-1}b_{\lambda,\tilde{\bar{\chi}}}(j)\cdot A_{k,\chi}(n-j)$$
$$+C_1\{\delta_k\cdot 2^{-g}L(0,\ \tilde{\chi})\cdot b'_{\lambda,\tilde{\chi}}(np)-\delta_\lambda\cdot(B_{1,\tilde{\chi}}/2)\cdot A'_{k,\tilde{\chi}}(np)$$
$$+\sum_{j=1}^{np-1}b'_{\lambda,\tilde{\chi}}(j)\cdot A'_{k,\tilde{\chi}}(np-j)\}]\cdot\exp(2\pi)inz\,,$$

where we put $N(\mathfrak{m})=p^f$ for $f>0$ and
$$C_1=(-i)^{kg+\lambda}(-1)^{[k/2]g+[\lambda/2]}p^{(1-f)/2}W(\chi)\cdot W(\tilde{\tilde{\chi}})\,.$$
Exactly like as above, we obtain the following Fourier expansion of $\mathrm{Tr}(\phi_{k,\chi})$ if $\tilde{\chi}=1$.

(2.11) $$\mathrm{Tr}(\phi_{k,\chi})=2^{-g}L(1-k,\ \chi)+C_2\delta_k 2^{-g}L(0,\ \tilde{\chi})$$
$$+\sum_{n=1}^{\infty}(A_{k,\chi}(n)+C_2A'_{k,\tilde{\chi}}(np))\cdot\exp(2\pi)inz\,,$$

where we put $N(\mathfrak{m})=p^f$ for $f>0$ and $C_2=(-i)^{kg}\cdot(-1)^{[k/2]g}\cdot p^{1-f/2}\cdot W(\chi)$.

Before applying Lemma 2.1 to (2.10) and (2.11), we prove some Lemmas.

LEMMA 2.4. *Assume the conditions* (i), (ii) *in* § 0 *and that* χ *is a primitive character modulo* $\mathfrak{m}$ *and has order* 2. *Put* $N(\mathfrak{m})=p^f$. *Then* $\tilde{\chi}=1\Leftrightarrow f\equiv 0$ (mod. 2) *and* $\mathfrak{m}$ *is prime to* 2, *or* $\mathfrak{m}|2$.

PROOF. First we assume that $\mathfrak{m}$ is a prime ideal and prime to 2. We choose a primitive root x (mod. $\mathfrak{m}$). If we put $d=(p^f-1)/(p-1)=1+p+\cdots+p^{f-1}$, then x^d is a primitive root (mod. p), and $\tilde{\chi}=1\Leftrightarrow\chi(x^d)=1$. It is easy to see $f\equiv 0$ (mod. 2)$\Leftrightarrow d\equiv 0$ (mod. 2)$\Leftrightarrow\chi(x^d)=1$. In this case, we obtain Lemma 2.4.

Next we assume $\mathfrak{m}$ is prime to 2, therefore $\mathfrak{m}$ is squarefree. We put the prime ideal decomposition of $\mathfrak{m}$ as follows:

$\mathfrak{m}=\prod_{i=1}^{d}\mathfrak{p}_i$, and $N(\mathfrak{p}_i)=p^{f_i}$. We denote by χ_i the $\mathfrak{p}_i$-part of χ. Since χ is primitive, we have $\chi_i\neq 1$. By the above argument, $\tilde{\chi}_i=1\Leftrightarrow f_i\equiv 0$ (mod. 2). Now we may assume for a suitable positive integer $e\leqq d$ that $\tilde{\chi}_i=1\Leftrightarrow i>e$. Therefore we obtain

$$i\leqq e\Leftrightarrow f_i:\ \text{odd, then we have }\tilde{\chi}=\prod_{i=1}^{e}\tilde{\chi}_i=\left(\frac{-}{p}\right)^e,$$

where $\left(\frac{-}{p}\right)$ is the Legendre symbol. Consequently we have $\tilde{\chi}=1\Leftrightarrow e$: even$\Leftrightarrow$ $f=\sum_{i=1}^{d}f_i$: even. As to $\mathfrak{m}|2$, clearly we have $\tilde{\chi}=1$. q. e. d.

LEMMA 2.5. *Under the assumption* (i) *of* § 0,

(i) *if* $\tilde{\chi}\neq 1$, $W(\chi)\cdot W(\tilde{\tilde{\chi}})\cdot p^{(1-f)/2}\in\boldsymbol{Q}(\chi)$, *if* $\tilde{\chi}=1$, $W(\chi)\cdot p^{(1-f)/2}\in \boldsymbol{Q}(\chi)$,

(ii) *if* χ *has order* 2, *then* $W(\chi)=1$.

PROOF. To prove (i) we consider the Gaussian sum $G(\chi)$ of χ, i.e.

$$G(\chi)=\chi(\rho\mathfrak{d}_F\mathfrak{m})\sum_{\substack{\beta \bmod .\mathfrak{m}\\ \beta\gg 0\\ \beta\in\mathfrak{d}_F^{-1}}}\chi_0(\beta)\cdot\exp(2\pi i\mathrm{Tr}(\rho\beta)),$$

where ρ is an element of F satisfying $(\rho\mathfrak{d}_F\mathfrak{m}, \mathfrak{m})=\mathfrak{O}_F$. Then we have

$$W(\chi)=\begin{cases}(-i)^g G(\chi)p^{-f/2} & \text{if } \chi \text{ is totally imaginary,}\\ G(\chi)p^{-f/2} & \text{if } \chi \text{ is totally real.}\end{cases} \tag{2.12}$$

For a positive integer b prime to p, we denote by σ_b the element of $\mathrm{Gal}(\boldsymbol{Q}(\chi, \zeta)/\boldsymbol{Q}(\chi))$ ($\zeta=\exp(2\pi i/p^f)$) satisfying $\zeta^{\sigma_b}=\zeta^b$ and $\chi^{\sigma_b}=\chi$. From the definition of $G(\chi)$, we obtain easily

(*) $G(\chi)^{\sigma_b}=\tilde{\chi}(b)G(\chi)=\tilde{\tilde{\chi}}(b)G(\chi)$, $G(\tilde{\tilde{\chi}})^{\sigma_b}=\tilde{\chi}(b)G(\tilde{\tilde{\chi}})$,

so that we have $(G(\chi)G(\tilde{\tilde{\chi}}))^{\sigma_b}=G(\chi)G(\tilde{\tilde{\chi}})$. Therefore we obtain

$$G(\chi)G(\tilde{\tilde{\chi}})\in\boldsymbol{Q}(\chi).$$

On the other hand, if χ is totally imaginary and g is odd, then $\tilde{\chi}$ is imaginary, and if χ is as above and g is even, then $\tilde{\chi}$ is real. So, for a non-trivial $\tilde{\chi}$, we obtain the followings:

$$W(\chi)W(\tilde{\tilde{\chi}})=\begin{cases}(-i)^{g+1}G(\chi)G(\tilde{\tilde{\chi}})p^{-(f+1)/2}, & \text{if } g \text{ is odd and } \chi \text{ is totally imaginary,}\\ (-i)^g G(\chi)G(\tilde{\tilde{\chi}})p^{-(f+1)/2} & \text{if } g \text{ is even and } \chi \text{ is totally real,}\\ G(\chi)G(\tilde{\tilde{\chi}})p^{-(f+1)/2} & \text{otherwise.}\end{cases} \tag{2.13}$$

Then by (2.13), we have $W(\chi)W(\tilde{\tilde{\chi}})p^{(1-f)/2}=\pm G(\chi)G(\tilde{\tilde{\chi}})p^{-f}\in\boldsymbol{Q}(\chi)$. If χ is the identity character, then we obtain by (2.12) and (*),

$$G(\chi)\in\boldsymbol{Q}(\chi) \quad\text{and}\quad W(\chi)p^{1-f/2}=\pm G(\chi)p^{1-f}\in\boldsymbol{Q}(\chi).$$

This proves (i). The assertion (ii) is obvious from the fact that $W(\chi)$ is also the constant factor of the functional equation of the Dedekind ζ-function of the field corresponding to χ. q.e.d.

REMARK 2.2. In terms of $G(\chi)$, the constant C_1, C_2 in (2.10) and (2.11) is expressed more simply in

$$C_1=G(\chi)G(\tilde{\tilde{\chi}})p^{-f},\quad C_2=G(\chi)p^{1-f}.$$

This is an easy consequences from (2.12), (2.13) and the expression of C_1 and C_2 in (2.10) and (2.11).

Now we apply Lemma 2.1 (Siegel's Lemma) to (2.10) and (2.11). When $k=1$, we assume that χ is a real valued character (in this case, χ is totally imaginary). To formulate our result, we use the same notations as in (2.5), (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11). In the case that $\tilde{\chi}$ is not trivial, put for a positive integer $\lambda \equiv kg \pmod 2$ as in (2.9), (2.10) and (2.11),

$$(2.14)\quad R_k(\lambda,\chi)=\{-B_{\lambda,\tilde{\chi}}/2\lambda-\delta_\lambda(-i)^{g+1}p^{(1-f)/2}B_{1,\tilde{\chi}}/2\}\cdot C_{g+\lambda,0} + \sum_{j=1}^{r(g+\lambda)} C_{g+\lambda,j}\{b_{\lambda,\tilde{\chi}}(j)+(-1)^{[g/2]+\lambda}p^{(1-f)/2}b'_{\lambda,\tilde{\chi}}(jp)\} \quad \text{if } k=1,$$

$$R_k(\lambda,\chi)=-(B_{\lambda,\tilde{\tilde{\chi}}}/2\lambda)C_{kg+\lambda,0}+\sum_{j=1}^{r(kg+\lambda)} C_{kg+\lambda,j}b_{\lambda,\tilde{\tilde{\chi}}}(j) \quad \text{if } k>1,$$

where $r(h)=\begin{cases}[h/12]+1 & \text{if } h\not\equiv 2 \pmod{12}\\ [h/12] & \text{if } h\equiv 2 \pmod{12}\end{cases}$

for a positive even integer h.

Further put for a positive integer $j\leqq r(kg+\lambda)$

$$(2.15)\quad S_k(\lambda,\chi,j)=-(B_{\lambda,\tilde{\tilde{\chi}}}/2\lambda)A_{k,\chi}(j)+\sum_{m=1}^{j-1} b_{\lambda,\tilde{\tilde{\chi}}}(m)A_{k,\chi}(j-m) + C_1\Big\{\sum_{m=1}^{jp-1} b'_{\lambda,\tilde{\chi}}(m)A'_{k,\tilde{\chi}}(jp-m)-\delta_\lambda(B_{1,\tilde{\chi}}/2)A'_{k,\tilde{\chi}}(jp)\Big\},$$

where $C_1=\begin{cases}(-1)^{[g/2]+\lambda}p^{(1-f)/2} & \text{if } k=1,\\ G(\chi)G(\tilde{\tilde{\chi}})p^{-f}\ (=(-1)^{[kg/2]+[k/2]g+\lambda}p^{(1-f)/2} \text{ if } \chi=\bar{\chi}) & \text{if } k>1.\end{cases}$

Then, by (2.10), Lemma 2.1, Lemma 2.5 and Remark 2.2, we obtain

$$(2.16)\quad 2^{-g}L(1-k,\chi)(-R_k(\lambda,\chi))=\sum_{j=1}^{r(kg+\lambda)} C_{kg+\lambda,j}S_k(\lambda,\chi,j).$$

In the case that $\tilde{\chi}$ is trivial, by (2.11) and Lemma 2.1, we obtain

$$(2.17)\quad C\cdot 2^{-g}L(1-k,\chi)=\sum_{j=1}^{r(kg)} C_{kg,j}(A_{k,\chi}(j)+C_2A'_{k,\tilde{\chi}}(jp)),$$

where $C=\begin{cases}-(1+(-i)^g p^{(1-f)/2})C_{g,0} & \text{if } k=1,\\ -C_{kg,0} & \text{if } k>1, \text{ and}\end{cases}$

$C_2=\begin{cases}(-i)^g p^{1-f/2} & \text{if } k=1,\\ G(\chi)p^{1-f}\ (=(-1)^{[kg/2]+[k/2]g}p^{1-f} \text{ if } \chi=\bar{\chi}) & \text{if } k>1,\end{cases}$

(in this case, kg is always an even integer).

When $k=1$, put

K: the totally imaginary quadratic extension over F corresponding to χ,

h_K (resp. h_F): the class number of K (resp. F),

W_K: the number of roots of unity in K,

E_K° (resp. E_F°): the unit group generated only by the fixed system of fundamental units of K (resp. F).

Then it is well known $2^{-g}L(0,\chi)=W_K^{-1}(E_K^\circ : E_F^\circ)^{-1}h_K/h_F$. In this case, the conductor of χ is the relative discriminant $D_{K/F}$ of K/F by the discriminant and conductor theorem. Then we obtain, by (2.16) and (2.17), the following two theorems:

THEOREM 2.1. *Let the notation be as above. Let K be totally imaginary quadratic extension of F with relative discriminant $D_{K/F}$. Assume the condition* (i), (ii) *in* § 0 *for the character χ of F corresponding to K, and put $N(D_{K/F})=p^f$ for a positive integer f. $\tilde{\chi}$ denotes the restriction of χ to $\boldsymbol{Z}$.*

(i) *If $\tilde{\chi}$ is the identity character, further assume $f\neq 2$ or $g\equiv 0$* (mod. 4), *then*

$$W_K^{-1}(E_K^\circ : E_F^\circ)^{-1}h_K/h_F = -(1+(-i)^g p^{1-f/2})^{-1}C_{g,0}^{-1}\cdot \sum_{j=1}^{r(g)} C_{g,j}(A_{1,\chi}(j)+(-i)^g p^{1-f/2}A'_{1,\chi}(jp)) \tag{2.18}$$

(for details in the case of $f=2$ and $g\equiv 2$ (mod. 4), see Remark 2.3).

(ii) *If $\tilde{\chi}$ is not trivial, further assume $R_1(\lambda, \chi)\neq 0$ for a positive integer $\lambda\equiv g$* (mod. 2) (for details of this condition, see Remark 2.4), *then*

$$W_K^{-1}(E_K^\circ : E_F^\circ)^{-1}h_K/h_F = -R_1(\lambda, \chi)^{-1}\sum_{j=1}^{r} C_{g+\lambda,j}S_1(\lambda, \chi, j) \qquad \text{for} \quad r=r(g+\lambda) \tag{2.19}$$

(for the definition of $R_1(\lambda, \chi)$ and $S_1(\lambda, \chi, j)$, see (2.14) and (2.15)).

THEOREM 2.2. *Let the notation be as in* (2.14), (2.15), (2.16) *and* (2.17). *Let χ be a ray class character modulo $\mathfrak{m}$ satisfying the condition* (i), (ii) *in* § 0 *and put $N(\mathfrak{m})=p^f$ for a positive integer f. $\tilde{\chi}$ denotes the restriction of χ to $\boldsymbol{Z}$. Let k be a positive integer larger than* 1.

(i) *If $\tilde{\chi}$ is the identity character, then*

$$2^{-g}L(1-k, \chi) = -C_{kg,0}^{-1}\cdot \sum_{j=1}^{r} C_{kg,j}\{A_{k,\chi}(j)+G(\chi)p^{1-f}A'_{k,\bar{\chi}}(jp)\}, \tag{2.20}$$

where $r=r(kg)$ and $G(\chi)$ is the Gaussian sum for χ.

(ii) *If $\tilde{\chi}$ is not trivial, further assume $R_k(\lambda, \chi)\neq 0$ for a positive integer $\lambda\equiv kg$* (mod. 2) (such λ exists for any k and χ, see Lemma 2.6), *then*

$$2^{-g}L(1-k, \chi) = -R_k(\lambda, \chi)^{-1}\sum_{j=1}^{r} C_{kg+\lambda,j}S_k(\lambda, \chi, j) \qquad \text{for} \quad r=r(kg+\lambda). \tag{2.21}$$

COROLLARY TO THEOREM 2.2. *Let χ be a ray class character of F satisfying the conditions* (i), (ii) *in* § 0, *and k be a positive integer larger than* 1. *Then $L(1-k, \chi)\in \boldsymbol{Q}(\chi)$.*

PROOF. Assume that there exist some λ such that $R_k(\lambda, \chi)\neq 0$, when $\tilde{\chi}$ is not trivial. Then this corollary is an easy consequence from Theorem 2.2 and Lemma 2.5. The existence of such λ is proved in the following

LEMMA 2.6. *Let χ be a ray class character of F satisfying the conditions* (i), (ii) *of* § 0. *Assume that $\tilde{\chi}$ is not trivial. Then, for a positive integer $k>1$, there are infinitely many λ such that $R_k(\lambda, \chi)\neq 0$.*

PROOF. Put, for an arbitrary non-trivial Dirichlet character ψ with prime conductor, a positive even integer h and a positive integer λ,

$$Q_h(\lambda, \psi)=(-B_{\lambda,\bar{\psi}}/2\lambda)C_{h,0}+\sum_{j=1}^{r(h)} C_{h,j}b_{\lambda,\bar{\psi}}(j),$$

where $b_{\lambda,\psi}(j)$ is as in (2.6). Fix a prime q and an integer $s=4, 6, 8, 10, 14$. Further put for a positive integer n, $h_n=12(q^n-1)+s$ and $l(n)=12r(n)-n+2$. Then $r(h_n)=q^n$ and $l(h_n)=l(s)=14-s$. Now we assume $G_l\equiv 1$ (mod. q) for $l=l(s)$, where G_l is the Eisenstein series of weight l defined in (2.1) (in the case of $l=0$, we put $G_0=1$). Here the congruence means that all Fourier coefficients of G_l-1 are divisible by q. Then, by the definition (see the beginning of this section),

$$T_{h_n}=G_{l(h_n)}\Delta^{-r(h_n)}=G_l\Delta^{-q^n},$$

so that

$$T_{h_n}(z)\equiv\Delta^{-q^n}(z)\equiv G_l(q^nz)\Delta^{-1}(q^nz)\equiv T_s(q^nz) \pmod{q},$$

since $G_l(z)\equiv G_l(q^nz)\equiv 1$ (mod. q) and Δ^{-1} has integral Fourier coefficients. Therefore we obtain for $s=4, 6, 8, 10, 14$,

$$C_{h_n,q^nj}\equiv C_{s,j} \pmod{q} \text{ for } j=0, 1, \text{ and}$$

$$C_{h_n,m}\equiv 0 \pmod{q} \text{ for } 0<m<q^n.$$

At first, we take 3 as q. Applying Lemma 2.1 to G_l, we obtain $2s/B_l=C_{l,0}$ for $l=4, 6, 8, 10, 14$. Hence, by the table in Remark 2.1 and (2.1), $G_l\equiv 1$ (mod. 3) for all $l=0, 4, 6, 8, 10, 14$. So the assumption of the above consideration is satisfied. We have for $s=4, 6, 8, 10, 14$,

$$C_{h_n,3^nj}\equiv C_{s,j} \pmod{3} \text{ for } j=0, 1, \text{ and}$$

$$C_{h_n,m}\equiv 0 \pmod{3} \text{ for } 0<m<3^n.$$

By the table in Remark 2.1, $C_{s,0}\equiv 0$ (mod. 3) for $s=4, 6, 8, 10, 14$, so that $C_{h_n,0}\equiv 0$ (mod. 3).

Now we assume that $B_{\lambda,\bar{\psi}}/2\lambda$ is a 3-adic integer. This condition is satisfied for a character ψ whose conductor is prime to 3 (see Carlitz [**1**] and Leopoldt [**7**]). Since $C_{h_n,m}\equiv 0$ (mod. 3) for $0\leqq m<3^n$ and $C_{h_n,3^n}=1$, we obtain

$$Q_{h_n}(\lambda, \psi)\equiv b_{\lambda,\bar{\psi}}(3^n)\equiv 1 \pmod{3} \text{ when } \lambda\geqq 2,$$

here we consider the congruence in the 3-adic completion of $\boldsymbol{Q}(\psi)$. Hence $Q_{h_n}(\lambda, \psi)\neq 0$ for $\lambda>1$.

When the conductor of ψ is 3, then $\psi=\left(\frac{\ }{3}\right)$ (the Legendre symbol). By [**1**] Theorem 6, for $\lambda>2$,

$$B_{\lambda,\psi}/\lambda\equiv\frac{1}{3}(1-\psi(2))\sum_{m=1}^{3} m\psi(m) \pmod{2},$$

so that $B_{\lambda,\psi}/\lambda\equiv 0$ (mod. 2) for $\lambda>2$, since $\psi(2)\equiv 1$ (mod. 2). Therefore $B_{\lambda,\psi}/2\lambda$ is a 2-adic integer for $\lambda>2$. Now, we take 2 as q. Then, in the same manner as above, we have $Q_{h_n}(\lambda, \psi)\neq 0$ for $\psi=\left(\frac{}{3}\right)$ for $\lambda>2$.

Finally, let χ be an arbitrary ray class character of F satisfying the assumptions of the Lemma. Then we see $R_k(\lambda, \chi)=Q_{kg+\lambda}(\lambda, \tilde{\chi})$. Take enough large n such that $h_n-kg>2$, and put $\lambda=h_n-kg$, then $\lambda\equiv kg \pmod{2}$ and $R_k(\lambda, \chi)=Q_{h_n}(\lambda, \tilde{\chi})\neq 0$. This completes our proof. q. e. d.

REMARK 2.3. When $\tilde{\chi}$ is trivial $g\equiv 2 \pmod{4}$ and $f=2$, Theorem 2.1 gives no formula. So we apply the same method to $E^2_{F,1,\chi}\circ \boldsymbol{P}$, then we obtain the following class number formula which is a quadratic equation whose root is the relative class number of K/F:

$$C_{2g,0}(1+(-i)^{2g}p^{1-f})(h_K/h_F)^2+2\sum_{j=1}^{r} C_{2g,j}R(j)(h_K/h_F) + \sum_{j=1}^{r} C_{2g,j}S(j)=0\,, \tag{2.22}$$

where $r=r(2g)$ and

$$R(j)=(E_K^\circ : E_F^\circ)W_K\cdot\{\sum_{\substack{\mathrm{Tr}(\xi)=j\\ \xi\in\mathfrak{d}_F^{-1}D_{K/F}^{-1}\\ \xi\gg 0}}((-i)^{2g}p^{1-f}a_{1,\chi}(\xi\mathfrak{d}_F D_{K/F})+a_{1,\chi}(\xi\mathfrak{d}_F)\}\,,$$

$$S(j)=(E_K^\circ : E_F^\circ)^2W_K^2\cdot\sum_{\substack{\mathrm{Tr}(\xi)=j\\ \xi\in\mathfrak{d}_F^{-1}D_{K/F}^{-1}\\ \xi\gg 0}}\{(-i)^{2g}p^{1-f}\cdot\sum_{\substack{\mu+\nu=\xi\\ \mu,\nu\gg 0\\ \mu,\nu\in\mathfrak{d}_F^{-1}D_{K/F}^{-1}}} a_{1,\chi}(\mu\mathfrak{d}_F D_{K/F})\cdot a_{1,\chi}(\nu\mathfrak{d}_F D_{K/F}) + B(\xi)\}$$

for

$$B(\xi)=\begin{cases}\displaystyle\sum_{\substack{\mu+\nu=\xi\\ \mu,\nu\gg 0\\ \mu,\nu\in\mathfrak{d}_F^{-1}}} a_{1,\chi}(\mu\mathfrak{d}_F)\cdot a_{1,\chi}(\nu\mathfrak{d}_F) & \text{if } \xi\in\mathfrak{d}_F^{-1}\,,\\ 0 & \text{otherwise}\end{cases}$$

(for the definition of $a_{1,\chi}$, see Corollary to Proposition 1.1). It is easy to see that (2.22) degenerates if g is odd and $f=1$. For an actual calculation of h_K/h_F by hand, the formula (2.18) and (2.19) are less easy than (2.22).

REMARK 2.4. In the case of $k=1$, we cannot prove, in general, the existence of λ such that $R_1(\lambda, \chi)\neq 0$ for a fixed χ. But in a special case, if $g\equiv 1 \pmod{4}$ and $f\neq 1$, or $g\equiv 3 \pmod{4}$ and $g\leqq 23$, we have $R_1(1, \chi)<0$. To prove this, we see

$$R_1(1, \chi)=(1+(-i)^{g+1}p^{(1-f)/2})\{(-B_{1,\tilde{\chi}}/2)C_{g+1,0}+\sum_{j=1}^{r} C_{g+1,j}b_{1,\tilde{\chi}}(j)\}$$

for $r=r(g+1)$. Hence if $g\equiv 1 \pmod{4}$ and $f=1$, we have $R_1(1, \chi)=0$. Since $C_{g+1,j}>0$ if $g\equiv 1 \pmod{4}$ (see [11]), $-B_{1,\tilde{\chi}}/2>0$ and $b_{1,\chi}(j)\geqq 0$ for all j, we obtain

$$(-B_{1,\tilde{\chi}}/2)C_{g+1,0}+\sum_{j=1}^{r} C_{g+1,j}b_{1,\tilde{\chi}}(j)>0\,.$$

Therefore $R_1(1, \chi)<0$ if $g\equiv 1$ (mod. 4) and $f\neq 1$. In the case of $g\equiv 3$ (mod. 4) and $g\leqq 23$, we see easily, by the table of Remark 2.1 and the fact that $b_{1,\tilde{\chi}}(j)\leqq 2$ for $j=2, 3$,

$$(-B_{1,\tilde{\chi}}/2)C_{g+1,0}+\sum_{j=1}^{r} C_{g+1,j}b_{1,\tilde{\chi}}(j)<0\,.$$

This proves our assertion. In a similar manner as in Lemma 2.6, we can prove the existence of λ such that $R_1(\lambda, \chi)\neq 0$, under the assumption of $\chi(3)=-1$, for an arbitrary g. But the other case, i. e. $\chi(3)=1$ or 0, this method gives no general proof.

§ 3. Examples.

Here, we calculate some numerical examples for Theorem 2.1 and Theorem 2.2.

EXAMPLE 1. Let us take $g=2$, $\lambda=2$, $f=1$, then $r=1$, and by Theorem 2.1,

$$W_K^{-1}\cdot(E_K^\circ : E_F^\circ)^{-1}\cdot h_K/h_F=-R_1(2, \chi)^{-1}\cdot S_1(2, \chi, 1)\,, \tag{3.1}$$

where $R_1(2, \chi)=-(B_{2,\tilde{\chi}}/4)\cdot C_{4,0}+1-p$,

$$S_1(2, \chi, 1)=-(B_{2,\tilde{\chi}}/4)\cdot A_{1,\chi}(1)-\sum_{j=1}^{p-1} b'_{2,\tilde{\chi}}(j)\cdot A'_{1,\chi}(p-j)\,.$$

Now we take $F=\boldsymbol{Q}(\sqrt{2})$, $K=\boldsymbol{Q}(\sqrt{2}, \sqrt{-5-2\sqrt{2}})$, then we have $h_F=1$, $(E_K^\circ : E_F^\circ)=1$ (because F has an arbitrary signature distribution of the unit group), $W_K=2$, $\mathfrak{d}_F=(2\sqrt{2})$ and $D_{K/F}=(5+2\sqrt{2})$. So by (3.1), we have

$$h_K/2=\{-(B_{2,\tilde{\chi}}/4)\cdot C_{4,0}+1-p\}^{-1}\cdot\{(B_{2,\tilde{\chi}}/4)\cdot A_{1,\chi}(1)+\sum_{j=1}^{p-1} b'_{2,\tilde{\chi}}(j)\cdot A'_{1,\chi}(p-j)\}\,, \tag{3.2}$$

and $p=17$, $\tilde{\chi}=\left(\frac{}{17}\right)$. $B_{2,\tilde{\chi}}$ can be calculated easily (see Leopoldt [7]), we obtain $B_{2,\left(\frac{}{17}\right)}=8$. By the table of Remark 2.1, we have $C_{4,0}=-240$. Therefore we have

$$R_1(2, \chi)=464\,. \tag{3.3}$$

For the value of $\chi(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of F, we can easily calculate by using the equation $x^2=-5-2\sqrt{2}$, see the following table:

Table A. (The value of χ at a prime ideal)

$\mathfrak{p}$	$N(\mathfrak{p})$	$\chi(\mathfrak{p})$	$\mathfrak{p}$	$N(\mathfrak{p})$	$\chi(\mathfrak{p})$
$(\sqrt{2})$	2	1	$(17+11\sqrt{2})$	47	1
(3)	3^2	-1	$(17-11\sqrt{2})$	47	1
(5)	5^2	-1	$(13+7\sqrt{2})$	71	1

Table A. (continued)

$\mathfrak{p}$	$N(\mathfrak{p})$	$\chi(\mathfrak{p})$	$\mathfrak{p}$	$N(\mathfrak{p})$	$\chi(\mathfrak{p})$
$(3+\sqrt{2})$	7	1	$(13-7\sqrt{2})$	71	-1
$(3-\sqrt{2})$	7	-1	$(19+12\sqrt{2})$	73	-1
(11)	11^2	-1	$(19-12\sqrt{2})$	73	1
(13)	13^2	1	$(23+15\sqrt{2})$	79	-1
$(5+2\sqrt{2})$	17	0	$(23-15\sqrt{2})$	79	1
$(5-2\sqrt{2})$	17	-1	$(11+4\sqrt{2})$	89	1
(19)	19^2	1	$(11-4\sqrt{2})$	89	1
$(11+7\sqrt{2})$	23	1	$(21+13\sqrt{2})$	103	-1
$(11-7\sqrt{2})$	23	-1	$(21-13\sqrt{2})$	103	-1
(29)	29^2	-1	$(25+16\sqrt{2})$	113	1
$(7+3\sqrt{2})$	31	1	$(25-16\sqrt{2})$	113	-1
$(7-3\sqrt{2})$	31	-1			
$(13+8\sqrt{2})$	41	-1			
$(13-8\sqrt{2})$	41	1			

To calculate $S_1(2, \chi, 1)$, we shall give in Table C the ideal $\xi D_{K/F}\mathfrak{d}_F$ for a totally positive number ξ in $D_{K/F}^{-1}\mathfrak{d}_F^{-1}$ whose trace to $\boldsymbol{Q}$ is $j/17$ ($j=1, \cdots, 16$), and the value of $a_{1,\chi}(\xi D_{K/F}\mathfrak{d}_F)$, $b'_{2,(\frac{}{17})}(17-j)$, and $b'_{2,(\frac{}{17})}(17-j)\cdot a_{1,\chi}(\xi D_{K/F}\mathfrak{d}_F)$. By Table C, we obtain,

(3.4) $$\sum_{j=1}^{16} b'_{2,\bar{\chi}}(17-j)\cdot A'_{1,\chi}(j)=224\,.$$

A totally positive element ξ in $\mathfrak{d}_F^{-1}$ whose trace to $\boldsymbol{Q}$ equals to 1 is as follows;

(3.5) $$\xi=(2\pm\sqrt{2})/4,\ 1/2\,.$$

Therefore by Table A, we obtain $A_{1,\chi}(1)=4$. Then by (3.1), (3.4) and $B_{2,(\frac{}{17})}=8$, we obtain $S_1(2, \chi, 1)=-232$. Therefore $h_K/2=-(-232)/464=1/2$. So we have $h_K=1$.

By Theorem 2.1 and the formula in Remark 2.3, we obtain the following class numbers of CM-field of degree 4 and 8.

Table B. (Class number of *CM*-field)

F	K	h_F	h_K	$D_{K/F}$	$N(D_{K/F})$
$\boldsymbol{Q}(\sqrt{2})$	$*F(\sqrt{-3})$	1	1	(3)	3^2
〃	$*F(\sqrt{-11})$	1	1	(11)	11^2
〃	$F(\sqrt{-19-12\sqrt{2}})$	1	1	$(19+12\sqrt{2})$	73
〃	$F(\sqrt{-11-4\sqrt{2}})$	1	1	$(11+4\sqrt{2})$	89
〃	$F(\sqrt{-13-6\sqrt{2}})$	1	3	$(13+6\sqrt{2})$	97
〃	$F(\sqrt{-15-4\sqrt{2}})$	1	5	$(15+4\sqrt{2})$	193
$**\boldsymbol{Q}(\sqrt{3})$	$F(\sqrt{-11-4\sqrt{3}})$	1	4	$(11+4\sqrt{3})$	73
$\boldsymbol{Q}(\sqrt{10})$	$F(\sqrt{-9+2\sqrt{10}})$	2	4	$(9-2\sqrt{10})$	41
$\boldsymbol{Q}(\sqrt{2},\sqrt{5})$	$*F(\sqrt{-7-4\sqrt{2}})$	1	1	$(7+4\sqrt{2})$	17^2
〃	$*F(\sqrt{-8-3\sqrt{2}-\sqrt{5}-\sqrt{10}})$	1	3	$(8+3\sqrt{2}+\sqrt{5}+\sqrt{10})$	1601

* These class numbers are calculated by the formula in Remark 2.2. We use the electronic computor for the field of degree 8.

** $\boldsymbol{Q}(\sqrt{3})$ has no signature distribution, but by Shimura [9], if the Galois group of the Galois closure of K is a dihedral group of order 8, then $(E_K^\circ : E_F^\circ)=1$. So we can calculate h_K/h_F by Theorem 2.1.

EXAMPLE 2. We take the same character χ as above. Here, we shall calculate $L(-2, \chi)$. For $g=2$, $k=3$, $f=1$, $\lambda=2$, by Theorem 2.2, we obtain

$$(3.6)\qquad 2^{-2}L(-2, \chi)=-R_3(2, \chi)^{-1}\cdot S_3(2, \chi, 1),$$

$$R_3(2, \chi)=-(B_{2,\tilde{\chi}}/4)\cdot C_{8,0}+1,$$

$$S_3(2, \chi, 1)=-(B_{2,\tilde{\chi}}/4)\cdot A_{3,\chi}(1)-\sum_{j=1}^{p-1} b'_{2,\tilde{\chi}}(j)\cdot A'_{3,\chi}(p-j).$$

By Table A and (3.5), $A_{3,\chi}(1)=7$, and by $B_{2,(\frac{}{17})}=8$ and $C_{8,0}=-480$ (see Remark 2.1), $R_3(2, \chi)=961$. To calculate $S_3(2, \chi, 1)$, we give the Table of ξ in $D_{K/F}^{-1}\mathfrak{d}_F^{-1}$ satisfying the following conditions:

(i) ξ is a totally positive element in $D_{K/F}^{-1}\mathfrak{d}_F^{-1}$,

(ii) $\mathrm{Tr}_{F/Q}(\xi)=j/17$, for $j=1, \cdots, 16$.

Now, we put $\mathfrak{d}_F=(\Delta)$, $\Delta=2(2+\sqrt{2})$, $D_{K/F}=(d)$, $d=5+2\sqrt{2}$

Table C.

$d\Delta\xi$	j	$N(d\Delta\xi)$	$a'_{1,\chi}((d\Delta\xi))$	$a'_{3,\chi}((d\Delta\xi))$	$b'_{2,\chi}(17-j)$	$a'_{1,\chi}((d\Delta\xi)) \times b'_{2,\tilde{\chi}}(17-j)$	$a'_{3,\chi}((d\Delta\xi)) \times b'_{2,\tilde{\chi}}(17-j)$
$3+2\sqrt{2}$	3	1	1	1	18	18	18
$2+\sqrt{2}$	5	2	2	5	14	28	70

Table C. (continued)

$d\Delta\xi$	j	$N(d\Delta\xi)$	$a'_{1,\chi}((d\Delta\xi))$	$a'_{3,\chi}((d\Delta\xi))$	$b'_{2,\chi}(17-j)$	$a'_{1,\chi}((d\Delta\xi)) \times b'_{2,\bar{\chi}}(17-j)$	$a'_{3,\chi}((d\Delta\xi)) \times b'_{2,\bar{\chi}}(17-j)$
$6+4\sqrt{2}$	6	2^2	3	21	10	30	210
1	7	1	1	1	12	12	12
$10+7\sqrt{2}$	7	2	2	5	12	24	60
$5+3\sqrt{2}$	8	7	0	48	7	0	336
$9+6\sqrt{2}$	9	3^2	0	80	15	0	1200
$4+2\sqrt{2}$	10	2^3	4	85	6	24	510
$13+9\sqrt{2}$	10	7	2	50	6	12	300
$8+5\sqrt{2}$	11	$2\cdot 7$	4	250	6	24	1500
$17+12\sqrt{2}$	11	1	1	1	6	6	6
$3+\sqrt{2}$	12	7	2	50	4	8	200
$12+8\sqrt{2}$	12	2^4	5	341	4	20	1364
$7+4\sqrt{2}$	13	17	0	288	7	0	2016
$16+11\sqrt{2}$	13	$2\cdot 7$	0	240	7	0	1680
2	14	2^2	3	21	2	6	42
$11+7\sqrt{2}$	14	23	2	530	2	4	1060
$20+14\sqrt{2}$	14	2^3	4	85	2	8	170
$6+3\sqrt{2}$	15	$2\cdot 3^2$	0	400	3	0	1200
$15+12\sqrt{2}$	15	5^2	0	624	3	0	1872
$10+6\sqrt{2}$	16	$2^2\cdot 7$	0	1008	1	0	1008
$19+13\sqrt{2}$	16	23	0	528	1	0	528
total						224	15362

Therefore

$$S_3(2, \chi, 1) = -(B_{2,\bar{\chi}}/4)\cdot A_{3,\chi}(1) - 15362$$
$$= -15376\,,$$

then by (3.6), $2^{-2}\cdot L(-2, \chi) = 15376/961 = 16$.

EXAMPLE 3. In the formula (2.19) of Theorem 2.1 and (2.21) of Theorem

2.2, we can take various λ satisfying $\lambda \equiv kg$ (mod. 2). So for each λ, we obtain another formula of $L(1-k, \chi)$. For example, we take $\boldsymbol{Q}$ for F and $\boldsymbol{Q}(\sqrt{-11})$ for K, then $\chi=\left(\frac{}{11}\right)$. Now we calculate h_K in the two special cases where $\lambda=3$ and $\lambda=5$. By Theorem 2.1, we obtain $r=1$,

$$R_1(3, \chi)=-(B_{3,\chi}/6)\cdot C_{4,0}+1-p^2,$$

$$R_1(5, \chi)=-(B_{5,\chi}/10)\cdot C_{6,0}+1-p^4,$$

$$S_1(3, \chi, 1)=-(B_{3,\chi}/6)-\sum_{j=1}^{p-1} b'_{3,\chi}(j)\cdot b'_{1,\chi}(p-j),$$

$$S_1(5, \chi, 1)=-(B_{5,\chi}/10)-\sum_{j=1}^{p-1} b'_{5,\chi}(j)\cdot b'_{1,\chi}(p-j),$$

then $h_K=-2\cdot S_1(\lambda, \chi, 1)/R_1(\lambda, \chi)$, for $\lambda=3, 5$. By Remark 2.1, $C_{4,0}=-240$, $C_{6,0}=504$, and $B_{3,\left(\frac{}{11}\right)}=18$, $B_{5,\left(\frac{}{11}\right)}=-12750/11$, then $R_1(3, \left(\frac{}{11}\right))=600$, $R_1(5, \left(\frac{}{11}\right))=481560/11$.

Table D.

j	$b'_{3,\left(\frac{}{11}\right)}(j)$	$b'_{5,\left(\frac{}{11}\right)}(j)$	$b'_{1,\left(\frac{}{11}\right)}(11-j)$	$b'_{3,\left(\frac{}{11}\right)}(j)\cdot b'_{1,\left(\frac{}{11}\right)}(11-j)$	$b'_{5,\left(\frac{}{11}\right)}(j)\cdot b'_{1,\left(\frac{}{11}\right)}(11-j)$
1			0	0	0
2	3	15	3	9	45
3			0	0	0
4			0	0	0
5			0	0	0
6	30	1230	2	60	2460
7	48	2400	1	48	2400
8	51	3855	2	102	7710
9			0	0	0
10	78	9390	1	78	9390
total				297	22005

Therefore by Table D, $S_1(3, \left(\frac{}{11}\right), 1)=-300$, $S_1(5, \left(\frac{}{11}\right), 1)=-240780/11$, then $h_K=2\cdot 300/600=2\cdot(11/481560)\cdot(240780/11)=1$.

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