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On a class of type I solvable Lie groups I

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1. Introduction.

Let G be a topological group, and U a unitary representation of G. If the von Neumann algebra generated by $\{U_x ; x \in G\}$ is of type I, we say that the representation is of type I. The group G is said to be of type I if all the strongly continuous unitary representations of G are of type I.

In this paper, we shall restrict ourselves to connected, simply connected solvable Lie groups. Then such a G is known to be of type I either if G is an exponential group by 0. Takenouchi [**10**], or if G is the universal covering group of the identity component of a suitable algebraic group in $GL(m, \boldsymbol{R})$, by results of J. Dixmier [**2**] and L. Pukanszky [**9**].

Let G be a connected Lie subgroup of $GL(m, \boldsymbol{R})$. In a previous paper, the author defined G to be *semi-algebraic* if a maximal compact connected subgroup of the algebraic hull of G is contained in G. The purpose of this paper is to prove the following theorem:

THEOREM. *Let G be a connected, simply connected solvable Lie group. If the adjoint group $Ad(G)$ of G is semi-algebraic, then G is of type I.*

Let G be a Lie group, and let g be the Lie algebra of G. Let g^* denote the vector space dual to g. We define a map $\mu : G \ni x \mapsto \mu(x) \in GL(g^*)$ by

$$\langle \mu(x)f, X\rangle = \langle f, Ad(x^{-1})X\rangle \quad \text{for} \quad X \in g \quad \text{and} \quad f \in g^*.$$

Then μ is a representation (the *coadjoint representation*) of G, and the corresponding representation $d\mu$ of the Lie algebra g is given by

$$\langle d\mu(X), Y\rangle = \langle f, [Y, X]\rangle,$$

for $X, Y \in g$ and $f \in g^*$. For $f \in g^*$, we put

$$G(f) = \{x \in G ; \mu(x)f = f\},$$

$$g(f) = \{X \in g ; d\mu(X)f = 0\}.$$

Then $G(f)$ is the isotropy group at f, and $g(f)$ is the Lie algebra of $G(f)$.

By the definition of $g(f)$, we have $\langle f, [g(f), g]\rangle=0$, and in particular $\langle f, [g(f), g(f)]\rangle=0$. Therefore

$$g(f)\ni X\mapsto i\langle f, X\rangle\in \boldsymbol{C} \quad (i=\sqrt{-1})$$

is a Lie algebra homomorphism. Suppose that G is connected, simply connected and solvable. Then $G(f)_e$ is closed and simply connected, and there corresponds a one-dimensional unitary representation (character) χ_f of $G(f)_e$ such that

$$\chi_f(\exp X)=e^{i\langle f, X\rangle} \qquad \text{for} \quad X\in g(f).$$

Our proof of the theorem is based on the following, see [**1**].

THEOREM (Auslander-Kostant). *Let G be a connected, simply connected, solvable Lie group. Then G is of type I if and only if the following two conditions are satisfied:*

(O) *Orbit condition. Any orbit $\mu(G)f$ of the coadjoint representation is a locally compact set.*

(I) *Integrability condition. The kernel $G(f)_0$ of the character χ_f contains the commutator subgroup of $G(f)$.*

REMARK. The above formulation of (I) is due to Pukanszky [**9**].

Let G be a connected, simply connected, solvable Lie group such that the adjoint group $Ad(G)$ is semi-algebraic. The orbit condition for G was established in [**6**] in a more general setting. In §2 we summarize the results on semi-algebraic groups, which are useful for us. In §3, the integrability condition will be discussed, and be obtained for G.

NOTATION AND TERMINOLOGY. The identity element of a group in question will be denoted always by e. For a Lie group L, let L_e denote the identity component, i.e. the connected component containing e. When the factor group L/L_e is finite, L is said to be *finitely connected.*

Let L be a (not necessarily connected) Lie group and l its Lie algebra. For x in L, the automorphism $L_e\ni y\rightarrow xyx^{-1}$ of L_e induces an automorphism of l. We denote it by $Ad(x)$. The group $Ad(L)=\{Ad(x);x\in L\}$ is called the *adjoint group* of L.

2. Semi-algebraic groups.

Let G be a subgroup of $GL(m, \boldsymbol{R})$. When G is an open subgroup of a suitable algebraic group, G is said to be *pre-algebraic.* For any subgroup H of $GL(m, \boldsymbol{R})$, there exists a smallest pre-algebraic group $\mathcal{A}(H)$ containing H. $\mathcal{A}(H)$ is called the pre-algebraic hull of H.

A closed subgroup S of $GL(m, \boldsymbol{R})$ is said to be *semi-algebraic* if the factor space $\mathcal{A}(S)/S$ is homeomorphic with a euclidean space $\boldsymbol{R}^k$, $k=0, 1, 2, \cdots$. A pre-algebraic group is semi-algebraic. We shall give results on semi-algebraic groups, for later use. The details can be found in [5] and [7].

(2. 1) A semi-algebraic group is finitely connected.

(2. 2) The intersection of any number of semi-algebraic groups in $GL(m, \boldsymbol{R})$ is semi-algebraic.

(2. 3) Let S be a semi-algebraic group in $GL(m, \boldsymbol{R})$. Then for any $v \in \boldsymbol{R}^n$, the orbit Sv is a locally compact set.

(2. 4) If S is a semi-algebraic group, then so is the adjoint group $Ad(S)$.

Let G be connected Lie group, and g the Lie algebra of G. The group G is said to be *adjoint semi-algebraic* if the adjoint group $Ad(G)$ is semi-algebraic in $GL(g)$.

(2. 5) If G is adjoint semi-algebraic, then there exists a semi-algebraic group G_1 in $GL(m, \boldsymbol{R})$ for a sufficiently large m, which is locally isomorphic with G.

PROPOSITION 1. *Let G be an adjoint semi-algebraic group, and let μ be the coadjoint representation of G. Then retaining the notation in* §1, *for $f \in g^*$ the orbit $\mu(G)f$ is a locally compact set.*

PROOF. For x in $GL(m, \boldsymbol{R})$, let x^* denote the transposed matrix of x^{-1}. Then $x \mapsto x^*$ is an automorphism of $GL(m, \boldsymbol{R})$. Furthermore, for any algebraic group H, $H^* = \{x^* ; x \in H\}$ is also algebraic. Hence if H is semi-algebraic, so is H^*. But $Ad(G)$ is semi-algebraic, and with respect to suitable bases in g and g^*, $\mu(G)$ can be identified with $\{x^* ; x \in Ad(G)\}$. Therefore, the proposition follows from (2. 3). Q. E. D.

REMARK. An exponential group with all roots real is adjoint semi-algebraic.

3. Integrability condition.

A Lie group L is called *faithfully representable* if there is a continuous one-one homomorphism $\lambda : L \rightarrow GL(m, \boldsymbol{R})$ for a sufficiently large m. A connected solvable Lie group L is faithfully representable if and only if the commutator subgroup L' of L is closed and simply connected; and in this case, there exists a closed, connected, simply connected normal subgroup N such that $L = TN$, $T \cap N = \{e\}$, where T is any maximal compact subgroup of L. The converse is also true; see [3].

PROPOSITION 2. *Let G be a connected, faithfully representable solvable Lie group, and let H be a finitely connected, closed subgroup of G. Then*

1) $H=KN, \quad K\cap N=\{e\}$,

where K is a compact abelian subgroup, and N is a closed, connected, simply connected normal subgroup, of H.

2) *Let $\tilde{G}$ be the universal covering group of G, and $\pi:\tilde{G}\to G$ the covering homomorphism. Then there exists a discrete subgroup $A\cong \boldsymbol{Z}^r$ for some $r=0, 1, 2, \cdots$ in $\tilde{G}$ such that*

$$\pi^{-1}H=A(\pi^{-1}H)_e, \quad A\cap(\pi^{-1}H)_e=\{e\}.$$

PROOF. Let h be the Lie algebra of H. Let H_e' be the commutator subgroup, and T a maximal compact subgroup, of H_e. Then H_e' is a closed characteristic subgroup of H_e, and the Lie algebra of H_e' is $[h, h]$. Since T is maximal compact in H_e, so is TH_e'/H_e' in H_e/H_e', by a theorem in Iwasawa [8]. However H_e/H_e' is a connected abelian Lie group and TH_e'/H_e' is the largest toral subgroup of H_e/H_e'. Hence TH_e' is a closed characteristic subgroup of H_e. We have $T\cap H_e'=\{e\}$ because H_e' is simply connected. Let t be the Lie algebra of T. Then $t\cap[h, h]=0$, and

$$Ad(H)(t+[h, h])=t+[h, h].$$

For x in H, let $\sigma(x)$ denote the restriction of $Ad(x)$ on $h/[h, h]$. We adopt the notation $(ad\,X)Y=[X, Y]$ for X, Y in a Lie algebra. Then $Ad(\exp X)Y-Y=\exp(ad\,X)Y-Y=\sum_{k=1}^{\infty}(ad\,X)^k Y/k!\in[h, h]$ for $X, Y\in h$. Hence $\sigma(\exp X)$ is the identity. Since $\exp h$ generates H_e, the kernel of σ contains H_e, and σ gives rise to a representation of the finite group H/H_e. Therefore, σ is completely reducible, and we can find a subspace p of h such that

$$h=p\oplus t\oplus[h, h] \quad \text{and}$$

$$Ad(H)(p+[h, h])=p+[h, h].$$

We put $p+[h, h]=n$. Then n is an ideal of h. Let N denote the connected Lie subgroup corresponding to n. Then N is closed, simply connected, and is a normal subgroup of H by $Ad(H)n=n$. Also $h=t\oplus n$ and

$$H_e=TN, \quad T\cap N=\{e\}.$$

Since H_e/N is compact and H/H_e is finite, the factor group H/N is compact. This fact, together with that N is a connected, simply connected solvable Lie group, implies that there exists a compact subgroup K of H such that

$H=KN$, $K\cap N=\{e\}$, by Iwasawa [8]. On the other hand, K is contained in some maximal compact subgroup C of G, which is a toral group. Hence K is abelian, and this proves 1).

Next, since C is a maximal compact subgroup of G, the underlying space of G is the direct product of C and a euclidean space. Hence $\pi^{-1}C=\tilde{C}$ is the universal covering group of C, and is a vector group, i. e. $\tilde{C}\cong\boldsymbol{R}^k$ for a suitable $k=0, 1, 2, \cdots$. Since $\pi^{-1}K$ is a closed subgroup of $\tilde{C}$,

$$\pi^{-1}K=A(\pi^{-1}K)_e \quad \text{and} \quad A\cap(\pi^{-1}K)_e=\{e\},$$

$$\text{where} \quad A\cong\boldsymbol{Z}^r \quad \text{and} \quad (\pi^{-1}K)_e\cong\boldsymbol{R}^s.$$

Next, let $\tilde{N}$ be the connected Lie subgroup of $\tilde{G}$ corresponding to the Lie algebra n. Then $\pi\tilde{N}=N$. In order to prove $\pi^{-1}K\cap\tilde{N}=\{e\}$, suppose that $x\in\pi^{-1}K\cap\tilde{N}$. Then $\pi(x)\in K\cap N=\{e\}$, and x is contained in the kernel of π. Recalling that N is simply connected, we have $\tilde{N}\cap\pi^{-1}\{e\}=\{e\}$. It follows that $x=e$, and hence we get $\pi^{-1}K\cap\tilde{N}=\{e\}$. Since $\pi^{-1}N=\pi^{-1}\{e\}\cdot\tilde{N}$,

$$\pi^{-1}H=\pi^{-1}K\cdot\pi^{-1}\{e\}\cdot\tilde{N}=\pi^{-1}K\cdot\tilde{N}.$$

Hence $\pi^{-1}H=A(\pi^{-1}H)_e$ and $A\cap(\pi^{-1}H)_e=\{e\}$. Q. E. D.

PROPOSITION 3. *Let L be a Lie group, l the Lie algebra of L, and suppose that there exists an abelian subgroup A of L such that $L=AL_e$. Then the commutator subgroup L' of L is a connected Lie subgroup of L_e, and the Lie algebra l' of L' is given by*

$$l'=\sum_{a\in A}(Ad(a)-1)l+[l, l].$$

PROOF. Let a and b be in A, and x and y in L_e. We shall prove that the commutator

$$c=(ax)(by)(ax)^{-1}(by)^{-1}$$

can be joined with e by an arc in L'. Let $\alpha(t)$ and $\beta(t)$ be arcs in L_e with

$$\alpha(0)=\beta(0)=e, \quad \alpha(1)=x \quad \text{and} \quad \beta(1)=y.$$

Then the arc

$$\gamma(t)=(a\alpha(t))(b\beta(t))(a\alpha(t))^{-1}(b\beta(t))^{-1}, \quad 0\leqq t\leqq 1,$$

is in L', $\gamma(0)=e$ and $\gamma(1)=c$. Since L' is generated by elements of the form c, we have that L' is arcwise connected. Hence L' is a connected Lie subgroup; see [4].

Let X be in l and a in A. Then the arc $a(\exp tX)a^{-1}(\exp(-tX))$ has the tangent vector $(Ad(a)-1)X$ at $t=0$. Since L' contains the commutator subgroup L_e' of L_e, and the Lie algebra of L_e' is $[l, l]$, we have

$$l' \supset l^{\#} = \sum_{a \in A} (Ad(a)-1)l + [l, l].$$

Let $L^{\#}$ be the connected Lie subgroup of L corresponding to the Lie algebra $l^{\#}$. Since $L^{\#} \supset L_e'$, the factor group $L_e/L^{\#}$ is abelian. Notice here that $L^{\#}$ may not be closed and we are considering only the group structure of $L_e/L^{\#}$. Next, for a, b in A

$$Ad(a)(Ad(b)-1)l = (Ad(b)-1)Ad(a)l = (Ad(b)-1)l,$$

and $\{Ad(a); a \in A\}$ leaves $l^{\#}$ invariant. Hence $L^{\#}$ is a normal subgroup of L.

Let us prove that $axa^{-1}x^{-1} \in L^{\#}$ for $a \in A$ and $x \in L_e$. Obviously, it suffices to prove this for x in any neighborhood of e in L_e. By the Campbell-Hausdorff formula, we have

$$a(\exp X)a^{-1}(\exp(-X)) = \exp((Ad(a)-1)X - \frac{1}{2}[Ad(a)X, X] + \cdots)$$
$$\in \exp l^{\#} \subset L^{\#},$$

for $X \in l$ sufficiently close to 0. Thus we have proved that $L/L^{\#}$ is abelian, that is $L^{\#} \supset L'$. Q. E. D.

PROPOSITION 4. *Let G be a connected, simply connected solvable Lie group. Let μ be the coadjoint representation of G. If for each $f \in g^*$, there exists an abelian subgroup $A(f)$ of G such that*

$$G(f) = A(f)G(f)_e,$$

then (I) *(the integrability condition) is satisfied for G.*

PROOF. By Proposition 3, the commutator subgroup $G(f)'$ is a connected Lie subgroup of $G(f)$ with the Lie algebra

$$g(f)' = \sum_{a \in A(f)} (Ad(a)-1)g(f) + [g(f), g(f)].$$

Let $g(f)_0$ denote the kernel of $f: g(f) \to \boldsymbol{R}$. Then $g(f)_0$ is the Lie algebra of the kernel $G(f)_0$ of the character $\chi_f: G(f)_e \to \boldsymbol{C}$. For $x \in G(f)$ and $X \in g$,

$$\langle f, X\rangle = \langle \mu(x^{-1})f, X\rangle = \langle f, Ad(x)X\rangle,$$

and so $\langle f, (Ad(x)-1)X\rangle = 0$, that is $(Ad(x)-1)g \subset g(f)_0$. Since $[g(f), g(f)] \subset g(f)_0$, we have that $g(f)' \subset g(f)_0$. Since $G(f)'$ is connected, we have $G(f)' \subset G(f)_0$. Q. E. D.

Directly from Proposition 2 and Proposition 4, we have

COROLLARY 1. *Let G be a connected, simply connected solvable Lie group. Let g be the Lie algebra of G, and g^* the dual vector space to g. Suppose there exists a locally faithful representation $\lambda: G \rightarrow GL(m, \boldsymbol{R})$, such that for any $f \in g^*$, the isotropy group $\lambda(G)(f)$ of the coadjoint representation of $\lambda(G)$ is finitely connected. Then* (I) *is satisfied for G.*

Now, the next corollary is what we wanted.

COROLLARY 2. *A connected, simply connected, solvable, adjoint semi-algebraic group G satisfies* (I) (*the integrability condition*).

PROOF. By (2.5), G has a locally faithful representation λ such that $\lambda(G)$, $\subset GL(m, \boldsymbol{R})$, is semi-algebraic. Let g be the Lie algebra of $\lambda(G)$ composed of m by m matrices, and let g^* be the dual vector space to g. Let N denote the normalizer of $\lambda(G)$. Then $N=\{x \in GL(m, \boldsymbol{R}); xgx^{-1}=g\}$ is an algebraic group. For $x \in N$, we put

$$\langle \mu(x)f, X\rangle = \langle f, x^{-1}Xx\rangle \qquad X \in g, \quad f \in g^*.$$

Then $\mu: N \rightarrow GL(g^*)$ is a rational homomorphism, and $\lambda(G) \ni x \mapsto \mu(x)$ coincides with the coadjoint representation of $\lambda(G)$. For $f \in g^*$, $N(f)=\{x \in N: \mu(x)f=f\}$ is an algebraic group, and $\lambda(G)(f)=\lambda(G) \cap N(f)$ is semi-algebraic by (2.2). Hence $\lambda(G)(f)$ is finitely connected, by (2.1), so Corollary 2 follows from Corollary 1. Q. E. D.

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