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On the invariant for a certain type of involutions on homology 3-spheres and its application

By Shinji FUKUHARA

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§ 1. Introduction.

The purpose of this paper is first to define a Z-invariant for involutions on homology 3-spheres which have circles as the sets of fixed points. Our definition of the invariant is derived from Hirzebruch's formula about the signature of ramified coverings [2]. For the second we shall present examples of involutions which are distinguished by our invariant. Finally, as an application of this invariant, we shall show a theorem on 4-dimensional homotopy smoothings. It will be proved that $\mathcal{HS}(P^2 \times D^2, \partial)$ has a nontrivial element.

§ 2. Definition of $\sigma(H, \tau)$.

We shall work in the smooth category. The following notations and conventions are used throughout the paper.

For an involution $T: X \to X$, X/T denotes its orbit space, and Fix$T$ denotes the set of fixed points of T. When $A \subset X$ is invariant under T, we write A/T instead of $A/(T|A)$. Let $i: X^n \to Y^m$ be an embedding of a compact oriented n-manifold X into an m-manifold Y such that $i^{-1}(\partial Y) = \partial X$. Then $[X, \partial X]$ denotes a homology class in $H_n(Y, \partial Y)$ represented by $(X, \partial X)$. As usual, $H_n(Y, \partial Y)$ means an n-dimensional integral homology group of $(Y, \partial Y)$. For a homology class x, $x^2 \in Z$ denotes its self-intersection number whenever it is defined. Suppose that a manifold X and its boundary A are oriented, we write $\partial X = A$ when {the orientation of A} $\times$ {the outward normal vector} coincides with the orientation of X.

Now we define $\sigma(H, \tau)$. Let H^3 be a homology 3-sphere, that is, a closed 3-manifold having an integral homology group isomorphic to that of a 3-sphere. Let τ be a smooth involution on H whose fixed points set Fixτ is diffeomorphic to a circle S^1. For (H, τ), we define the signature $\sigma(H, \tau) \in Z$ using the following two lemmas.

LEMMA 1. *H/τ is a homology 3-sphere.*

PROOF. Because $\mathrm{Fix}\tau$ is a codimension two submanifold in H, H/τ can be regarded as a manifold by the standard way. Let $p\in\mathrm{Fix}\tau$ and $\pi: H\to H/\tau$ be a canonical projection. For a given $x\in\pi_1(H/\tau, \pi(p))$, represent x by a smooth loop $l: [0, 1]\to H/\tau$ which intersects with $(\mathrm{Fix}\tau)/\tau$ only at $\pi(p)$ and transversely. Then clearly l can be "lifted" with respect to π, that is, there is a loop $l_1: [0, 1]\to H$ such that $\pi\circ l_1=l$. Thus $\pi_*: \pi_1(H, p)\to\pi_1(H/\tau, \pi(p))$ is epic. Therefore $\pi_*: H_1(H)\to H_1(H/\tau)$ is also epic. From the assumption that $H_1(H)=0$, $H_1(H/\tau)=0$ follows.

LEMMA 2. *For any* (H, τ), *there is a pair* (M, T) *of an oriented compact* 4-*manifold* M *and an orientation preserving involution* T *on* M *which satisfies the following conditions:*

(1) $\partial(M, T)=(H, \tau)$.

(2) $\{i_*^{-1}[(\mathrm{Fix}T)/T, \partial]\}^2=0$ *where* $i_*: H_2(M/T)\to H_2(M/T, \partial)$ *is an isomorphism induced from an inclusion map.*

(3) $w_2(M)=w_2(M/T)=0$ *where* w_2 *mean the second Stiefel-Whitney class.*

PROOF. It is well known that any orientable closed 3-manifold bounds a simply connected parallelisable 4-manifold (Refer to [4], p. 53). For H/τ, we supply such a manifold N. Because H/τ is a homology 3-sphere, $(\mathrm{Fix}\,\tau)/\tau\subset H/\tau$ bounds a connected Seifert surface, say S_1, in H/τ. Push the interior of S_1 into the interior of N. As the result, we obtain a surface S properly embedded in N such that $\partial S=(\mathrm{Fix}\tau)/\tau$ and its trace F of pushing is a 3-manifold embedded in N having $S\cup S_1$ as a boundary (See Figure 1.).

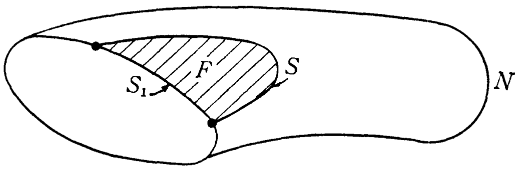

Figure 1.

Consider the following diagram (*).

$$\begin{array}{ccccc} H_2(H/\tau, H/\tau-(\mathrm{Fix}\tau)/\tau) & \to & H_1(H/\tau-(\mathrm{Fix}\tau)/\tau) & \to & H_1(H/\tau)\cong 0 \\ \downarrow & & \downarrow & & \downarrow \\ H_2(N, N-S) & \longrightarrow & H_1(N-S) & \longrightarrow & H_1(N)\cong 0 \end{array} \qquad (*)$$

From the fact that normal bundles of $(\mathrm{Fix}\tau)/\tau$ and S in H/τ and N are both trivial and the excision theorem, we obtain

$$H_2(H/\tau, H/\tau-(\mathrm{Fix}\,\tau)/\tau)\cong H_2((\mathrm{Fix}\tau)/\tau\times D^2, (\mathrm{Fix}\tau)/\tau\times S^1)$$

$$H_2(N, N-S)\cong H_2(S\times D^2, S\times S^1).$$

Since the homology groups above are generated by the image of the natural maps

$$H_2(D^2, S^1) \longrightarrow H_2((\mathrm{Fix}\tau)/\tau\times D^2, (\mathrm{Fix}\tau)/\tau\times S^1)$$

$$H_2(D^2, S^1) \longrightarrow H_2(S\times D^2, S\times S^1),$$

the map represented by the left vertical arrow in the diagram (*) is an isomorphism. Furthermore, since $N-F$ is a deformation retract of N, $H_2(N-F)\to H_2(N-S)\to H_2(N)$ is an isomorphism and $H_2(N, N-S)\to H_1(N-S)$ is a monomorphism. Hence $H_1(H/\tau-(\mathrm{Fix}\tau)/\tau)\to H_1(N-S)\cong Z$ induced from an inclusion is an isomorphism. Let $p: M\to N$ be a 2-fold ramified covering of N with branching locus S which corresponds to an epimorphism $\pi_1(N-S)\to H_1(N-S)\to Z_2$. More precisely M is constructed as follows. Let N_1 be a manifold obtained from M cut open along F. Since F is two-sided, we obtain cut ends F_1 and F_2. Let N_1', F_1' and F_2' be copies of N_1, F_1 and F_2. Then M is obtained from N_1 and N_1' by identifying F_1 with F_2', F_2 with F_1'. Since $\pi_1(N)=\pi_1(N_1)=\pi_1(N_1')=1$, $\pi_1(M)=1$ from van Kampen's theorem.

Let $T: M\to M$ be a covering transformation. We shall show that (M, T) satisfies (1)~(3). By the construction, we obtain

$$\partial(M, T)=(H, \tau) \quad\text{and}\quad \{i_*^{-1}[(\mathrm{Fix}T)/T, \partial]\}^2=\{i_*^{-1}[S, \partial]\}^2=\{i_*^{-1}[S_1, \partial]\}^2=0.$$

Let $\tilde{S}=p^{-1}(S)$, and $N(\tilde{S})$ be a tubular neighbourhood of $\tilde{S}$ in M. Let $\mathrm{int}\,N(\tilde{S})$ be interior of $N(\tilde{S})$ in M and $\delta N(\tilde{S})=N(\tilde{S})$-$\mathrm{int}\,N(\tilde{S})$. Consider the following sequence:

$$H^1(M; Z_2)\longrightarrow H^1(M\text{-int}N(\tilde{S}); Z_2)\oplus H^1(N(\tilde{S}); Z_2)\xrightarrow{j} H^1(\delta N(\tilde{S}); Z_2)$$

$$\longrightarrow H^2(M; Z_2)\xrightarrow{i} H^2(M\text{-int}N(\tilde{S}); Z_2)\oplus H^2(N(\tilde{S}); Z_2).$$

Taking $(H, \mathrm{Fix}\,\tau)$ and $(M, \tilde{S})$ instead of $(H/\tau, \mathrm{Fix}\,\tau/\tau)$ and (N, S), consider the similar diagram to (*). We obtain $H_1(M-\tilde{S})\cong Z$ and $H^1(M-\tilde{S}; Z_2)=H^1(M\text{-int}\,N(\tilde{S}); Z_2)=Z_2$. Since $H_1(\delta N(\tilde{S}); Z_2)=H^1(\tilde{S}\times S^1; Z_2)=H^1(\tilde{S}; Z_2)\oplus H^1(S^1; Z_2)$, $H^1(\delta N(\tilde{S}); Z_2)$ has the same rank as $H^1(M\text{-int}N(\tilde{S}); Z_2)\oplus H^1(N(\tilde{S}); Z_2)$ over Z_2. Furthermore j is a monomorphism, since $H^1(M; Z_2)=0$. Thus j is an isomorphism, and then i is a monomorphism. Note that $i=i_1\oplus(-i_2)$ where

$$i_1: H^2(M; Z_2)\longrightarrow H^2(M\text{-int}\,N(\tilde{S}); Z_2) \quad\text{and}$$

$$i_2: H^2(M; Z_2)\longrightarrow H^2(N(\tilde{S}); Z_2)\cong 0$$

are induced maps from inclusions.

Since $i_2(w_2(M))=w_2(M\text{-int}\,N(\tilde{S}))=p^*w_2(N-p(\mathrm{int}\,N(\tilde{S})))=0$, and $i_2(w_2(M))=w_2(N(\tilde{S}))=0$, we obtain $w_2(M)=0$. This completes the proof of Lemma 2.

We now define the signature $\sigma(H, \tau)$ for an involution τ on H which has a circle as the set of fixed points.

DEFINITION. For any (H, τ), take (M, T) satisfying the conditions (1) and (2) in Lemma 2. Then we define

$$\sigma(H, \tau)=\frac{1}{8}\{\operatorname{sign}(M)-2\operatorname{sign}(M/T)\}\in Q$$

where sign( ) means the ordinary signature defined by the intersection form.

Then we can prove the following lemma.

LEMMA 3. *$\sigma(H, \tau)$ is well defined and $\sigma(H, \tau)\in Z$.*

PROOF. Let (M, T) and (M', T') be pairs satisfying (1) and (2) in Lemma 2. Consider closed 4-manifold M_1 which consists of M and M' identified along their boundaries. Let T_1 be the involution on M_1 which is defined by $T_1|M=T$ and $T_1|M'=T'$.

Let

$$i_*: H_2(M/T) \longrightarrow H_2(M/T, \partial) \quad \text{and}$$

$$i'_*: H_2(M'/T') \longrightarrow H_2(M'/T', \partial)$$

be isomorphisms induced from inclusion maps. From the condition

$$(2) \qquad \{i_*^{-1}[(\mathrm{Fix}T)/T, \partial]\}^2=\{i'^{-1}_*[(\mathrm{Fix}T')/T', \partial]\}^2=0,$$

we obtain $[(\mathrm{Fix}T_1)/T_1]^2=0$ because, up to sign,

$$[(\mathrm{Fix}T_1)/T_1]^2 \quad \text{equals to}$$

$$\{i_*^{-1}[(\mathrm{Fix}T)/T, \partial]\}^2\pm\{i'^{-1}_*[(\mathrm{Fix}T')/T', \partial]\}^2.$$

Thus, applying Hirzebruch's theorem about the signature of ramified coverings [**2**], we obtain

$$\operatorname{sign}(M_1)-2\operatorname{sign}(M_1/T_1)=0.$$

Then we have

$$\operatorname{sign}(M)-2\operatorname{sign}(M/T)=\operatorname{sign}(M')-2\operatorname{sign}(M'/T')$$

and $\sigma(H, \tau)$ is well defined.

Let (M, T) be a pair satisfying not only (1) and (2) but also (3) in Lemma 2. Then sign (M) and sign (M/T) is divisible by 8 because intersection forms of M and M/T are even unimodular. This completes the proof.

Now we define the cobordism group Ω. Let (H_1, τ_1) and (H_2, τ_2) be involutions on homology 3-spheres which have circles as the sets of fixed point. Then we say that they are cobordant in our sense when there is an involution T on M satisfying:

(1) $\partial(M, T)=(H_1, \tau_1)\cup(-H_2, \tau_2)$,

(2) M is an H-cobordism (i.e. a homology cobordism) between H_1 and H_2,

(3) Fix$T$ is diffeomorphic to $S^1 \times I$.

To define the connected sum of (H_1, τ_1) and (H_2, τ_2), take an equivariant embedding

$$h_i : (D^3, D^1) \longrightarrow (H_i, \mathrm{Fix}\tau_i) \qquad (i=1, 2).$$

Then, attach $H_1 - h_1(\mathrm{int}D^3)$ and $H_2 - h_2(\mathrm{int}D^3)$ along their boundaries with an equivariant orientation reversing diffeomorphism. By the standard argument, we can show that the connected sum operation is compatible with the cobordism relation and the set of cobordism classes Ω obtains abelian group structure. It is clear that the function $\sigma : \Omega \rightarrow Z$ which assigns the signature $\sigma(H, \tau)$ to $[H, \tau] \in \Omega$, a class of (H, τ), is a well defined group homomorphism. In the following section, we shall show that σ is an epimorphism.

§ 3. Examples.

In this section we shall show the existence of (H, τ) with $\sigma(H, \tau)=k$ for any integer k.

Let V_k be the Brieskorn manifold defined by

$$V_k = \{(x, y, z) \in C^3 \mid x^2 + y^3 + z^{6k+1} = 0, \ |x|^2 + |y|^2 + |z|^2 = 1\}$$
$$\text{for} \quad k=0, 1, 2, \cdots.$$

Let $\tau_k : V_k \rightarrow V_k$ be a map defined by

$$\tau_k(x, y, z) = (-x, y, z).$$

Then it is well known that V_k is a homology 3-sphere (See e.g. [3]) and τ_k is a smooth involution with $\mathrm{Fix}\tau_k = V_k \cap \{x=0\} \cong S^1$. Let t_k denote the torus knot of type $(3, 6k+1)$ in S^3 and W_k be the 2-fold branched covering space with branching locus t_k. Denote by σ_k the covering transformation with respect to W_k. Then, as is well known [3], (V_k, τ_k) is equivariantly diffeomorphic to (W_k, σ_k). For (W_k, σ_k), $\sigma(W_k, \sigma_k)$ equals, up to sign, to 1/8{the signature of the torus knot $t_k\}=k$ (See [3].). Thus we have $|\sigma(V_k, \tau_k)| = k$ $(k=0, 1, 2, \cdots)$. Concluding these, we have the following proposition.

PROPOSITION 1. *$\sigma : \Omega \rightarrow Z$ is an epimorphism.*

§ 4. Some contractible 4-manifolds.

In this section we shall present an example of a homology 3-sphere which bounds a contractible 4-manifold. We shall use this example to construct a 4-dimensional nontrivial homotopy smoothing in the following section.

Now we describe the spinnable structure of V_k. Recall that

$$V_k=\{(x, y, z)\in C^3 \mid x^2+y^3+z^{6k+1}=0, \ |x|^2+|y|^2+|z|^2=1\}.$$

Define:

$$K_k=\{(x, y, z)\in V_k \mid z=0\},$$

$$F_k=\{(x, y, z)\in V_k \mid z\geqq 0\},$$

$$V=\{(x, y)\in C^2 \mid |x|^2+|y|^2=1\},$$

$$K=\{(x, y)\in V \mid x^2+y^3=0\},$$

$$F=\{(x, y)\in V \mid x^2+y^3\leqq 0\}.$$

Let $\psi_k: V_k\to V$ be a map defined by

$$\psi_k(x, y, z)=(s^{1/2}x, s^{1/3}y)$$

where s is the unique positive real number satisfying the equation $|x|^2s+|y|^2s^{2/3}=1$ (Refer to [**5**].).

Then, for $k\geqq 1$, ψ_k is a $(6k+1)$-fold cyclic branched covering map with $K=\psi_k(K_k)$ as the branching locus. Furthermore $\psi_k|F_k: F_k\to F$ is a diffeomorphism. Note that F is diffeomorphic to a punctured torus.

Now let E_k and E be mapping tori

$$F_k\times[0, 1]/((x, y, z), 0)\sim((-x, \omega^2y, z), 1)$$

and

$$F\times[0, 1]/((x, y), 0)\sim((-x, \omega^2y), 1)$$

where $\omega=\exp\frac{2}{3}\pi i$. Let consider the map $\varphi_k: E_k\to V_k$ defined by

$$\varphi_k((x, y, z), t)=\left(x\cdot\exp(6k+1)\pi it, \ y\cdot\exp\frac{2}{3}(6k+1)\pi it, \ z\cdot\exp 2\pi it\right).$$

Then we obtain the following proposition by the same method as used in [**1**].

PROPOSITION 2. *V_k is diffeomorphic to $E_k\cup\partial F_k\times D^2$ where an attaching diffeomorphism $h_k: \partial E_k\to\partial F_k\times S^1$ is defined by*

$$h_k((x, y, 0), t)=\left(\left(x\cdot\exp(6k+1)\pi it, \ y\cdot\exp\frac{2}{3}(6k+1)\pi it, \ 0\right), t\right).$$

Note that the diffeomorphism $\psi_k|F_k: F_k\to F$ induces the diffeomorphism from E_k to E. With the identification $F_k\approx F$ and $E_k\approx E$, we have the following immediately.

PROPOSITION 3. *$V_k\cong E\bigcup_{f_k}\partial F\times D^2$ where $f_k: \partial E\to\partial F\times S^1$ is defined by*

$$f_k((x, y), t)=\left(x\cdot\exp(6k+1)\pi it, \ y\cdot\exp\frac{2}{3}(6k+1)\pi it, \ t\right).$$

Now let B^2 be a 2-disk, and let $g: \partial B^2 \to \partial F$ be a diffeomorphism. Let $M^3 = E \bigcup_h B^2 \times S^1$ is defined by

$$h((x, y), t) = \left(g\left(x \cdot \exp \pi i t,\ y \cdot \exp \frac{2}{3} \pi i t\right),\ t\right).$$

We denote by T_1 a torus in M^3 which consists of $F \times \{0\} \subset E \subset M$ and $B^2 \times \{0\} \subset M$. From Mayor-Vietoris' sequence for $(M^3; E, B^2 \times S^1)$ and the fact that $H_*(E) = H_*(S^1)$, we can show M^3 is a homology $S^1 \times S^2$ and inclusion $j: T \to M$ induces an isomorphism $j^*: H_2(T) \to H_2(M)$. We can regard V_k as the manifold obtained from M by performing the surgery on $\{p\} \times S^1$ where $p \in \operatorname{int} B^2$. If we choose the standard framing decided by the product structure of $B^2 \times S^1$, then we obtain V_0. Furthermore, if we choose the k-times twisted one, we obtain V_k. With respect to the orientation of V_k, we adopt the following convention. Define the orientation of $D^4 = \{(x, y) \in C^2 \mid |x|^2 + |y|^2 \leqq 1\}$ as the induced one from the complex structure of C^2. We regard V as oriented with the relation $\partial D^4 = V$. Furthermore, as an orientation of V_k, we adopt the induced one from V with $\psi_k: V_k \to V$.

Next we construct a cobordism between V_k and V_0. Let W_k be a manifold obtained from $M \times I$ by attaching two 2-handles at $B^2 \times S^1 \times \{0\}$ and $B^2 \times S^1 \times \{1\}$ with untwisted and k-twisted framings respectively. Then, clearly, $\pi_1(W_k) = 1$ and components of ∂W_k are V_0 and V_k. Since V_0 is diffeomorphic to a 3-sphere, we can attach D^4 to W_k along V_0. By W'_k, we denote the resulting manifold. Now we survey 2-dimensional homology group of W'_k. Denote by S a 2-sphere in W'_k which consist of the core of the attached 2-handles and $\{p\} \times S^1 \times I \subset M \times I \subset W'_k$. Denote by T a torus in $M \times \{1/2\} \subset W'_k$ defined by $T = T_1 \times \{1/2\}$. We denote by $[S]$ and $[T]$ the elements of $H_2(W'_k)$ represented by 2-manifolds S and T with suitable orientations. By the easy computation we obtain that $[S]$ and $[T]$ are generator of $H_2(W'_k) \cong Z \oplus Z$. Furthermore, under the orientation of W'_k such that $\partial W_k = V_k$, the intersection matrix of W'_k is

$$\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}$$

when we take $[S]$ and $[T]$ as a base.

Next we shall try to kill the element of $\pi_2(W'_k) \cong H_2(W'_k)$ which corresponds to $[T]$ by Hurewicz homomorphism. But, in general, it is impossible because, by Rochlin's theorem, V_k does not bounds an acyclic 4-manifold for k odd. When $k=2$, we can do surgery to kill $\pi_2(W'_k)$. Its proof occupies the remainder of this section.

THEOREM 1. *V_2 bounds a contractible 4-manifold.*

PROOF. We shall show that we can represent a homology class $[T] \in$

$H_2(W_2')$ by an embedded sphere and, using this, we can kill $[T]$. For this reason, first we shall show that there is a menbrane P, that is, P is an embedded 2-disk in W_2' such that $T\cap P=\partial P$, with index $i(P)=0$ (See [6] for a precise definition and fundamental properties of menbranes and their indices.). Moreover P can be chosen so that $P\cap S=\emptyset$. The construction of P is as follows.

First we describe in Figure 2 how F_0 and K_0 take their positions in $V_0\cong S^3$.

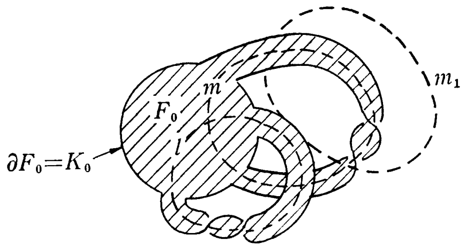

Figure 2.

Let l and m be loops on F_0 as in the figure above. Let push down the loop m such that the resulting loop m_1 does not intersect with l in V_0. Obviously l is unknotted in V_0 and bounds 2-disk D in V_0 such that $D\cap m_1=\emptyset$. We can regard $l\subset M$ because $l\subset E_0\approx E$ and $E\subset M$. Let $P_0=D\cup l\times[0, 1]\subset W_2'$. By smoothing P_0 near $l\times\{0\}$, we obtain a menbrane P_1. If we construct a new menbrane P_1' by moving P_1 with an isotopy to make P_1' intersect with P_1 transversely, then intersection number should be -1. This shows that $i(P_1)=-1$. According to Rochlin [5], from $S\subset W_2'$, we can make a menbrane Q with $i(Q)=[S]^2\pm1=2\pm1$ such that ∂Q is a trivial loop on T. For our use we choose the case of $i(Q)=2-1=1$. Then we connect P_1 and Q to make a new menbrane P with $i(P)=0$. By this menbrane P on T, we can construct an embedded 2-sphere, say S', which represents $[T]$. Since S' has a trivial normal bundle in W_2', we can perform surgery to kill $H_2(W_2')$. By W we denote the resulting manifold. Clearly W is acyclic.

The simply-connectivity of W is shown as follows. It will be sufficient if we show that a fiber of a normal S^1-bundle of $S'\subset W_2'$ is null homotopic in $W_2'-S'$. Let S_1 be an embedded 2-sphere which is isotopic to S. We can assume that S_1 intersects with T at a point transversely, and intersects with

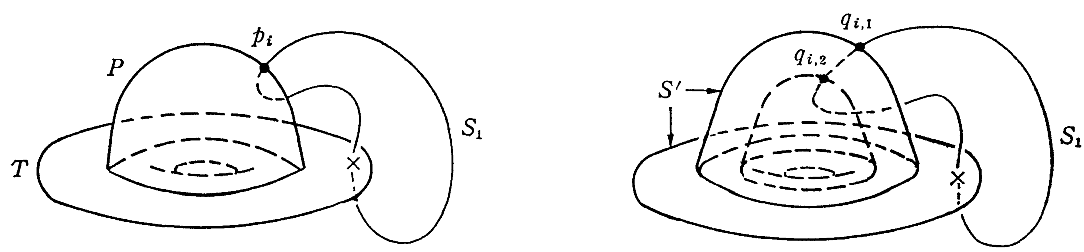

Figure 3.

P at $p_1, \cdots, p_i, \cdots, p_n$ transversely. Let $q_{i,1}$ and $q_{i,2}$ are intersection points of S and S_1 which correspond to p_i (See Figure 3.). Let l_i be a path in S' from $q_{i,1}$ to $q_{i,2}$. Let M_n be a manifold made from S_1 by removing D^2-fiber at $q_{i,1}$ and $q_{i,2}$ $(i=1, \cdots, n)$ of normal D^2-bundle of S' in W_2', and, in exchange, add normal S^1-bundle of S' in W_2' restricted on l_i $(i=1, \cdots, n)$ (See Figure 4). Then

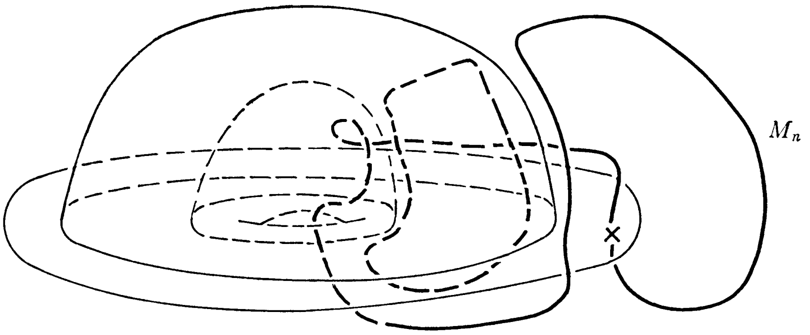

Figure 4.

the resulting manifold M_n is a surface with genus n which intersect with S' only at a point transversely.

Next we consider a loop $\bar{l}_i$ $(i=1, \cdots, n)$ which is constructed as follows. We choose a lifting l_i' of l_i on M_n and a path n_i $(i=1, \cdots, n)$ in M which joins an end point and a starting point of l_i' as in Figure 5. Let $\bar{l}_i = l_i' \cup n_i$ $(i=1, \cdots, n)$.

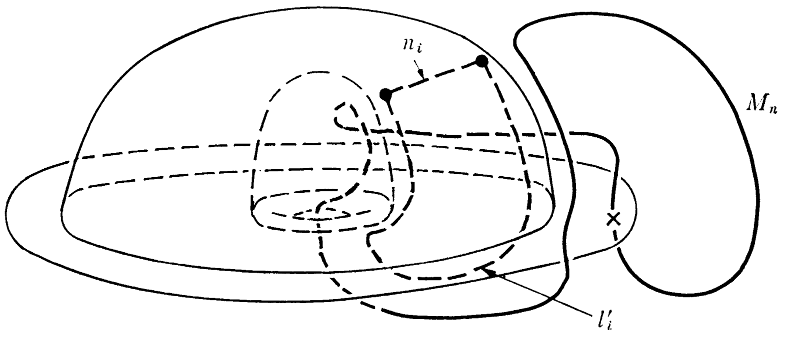

Figure 5.

By the suitable choice of n_i, we can assume that $\bar{l}_i$ is freely homotopic to the loop $m_1 \subset V_0$ in $W_2' - S'$ (note that the loop l and m in V_0 corresponds to the longitude and meridian of the torus T). But m_1 is null homotopic in $W_2' - S'$. Thus there is a 2-disk (not necessarily embedded) in $W_2' - S'$ whose boundary is $\bar{l}_i$ for $i=1, \cdots, n$. Hence we obtain a 2-sphere (not necessarily embedded) in W_2' which intersect S' at a point transversely. This shows that $\pi_1(W)=1$.

§ 5. A nontrivial element of $\mathcal{HS}(P^2\times D^2, \partial)$.

As usual we denote a real projective plane by P^2. In this section we shall present a nontrivial element of $\mathcal{HS}(P^2\times D^2, \partial)$. As far as the auther knows, very few are known about homotopy smoothings of 4-manifolds except for the case of $S^1\times S^3 \# k(S^2\times S^2)$ and $P^4 \# k(S^2\times S^2)$. We construct a nontrivial element of $\mathcal{HS}(P^2\times D^2, \partial)$ using the example presented in § 4.

The construction is as follows. First we consider a "suspension" of (V_2, τ_2). Let Σ be a 4-manifold which consists of two copies of W in the proof of Theorem 1, attached by $\tau_2: \partial W\to\partial W$ along their boundaries. Define an involution T on Σ so that if x belongs to the copy, then $T(x)$ may be a corresponding point in the other copy. Then we can easily show that Σ is a homotopy 4-sphere and T is well defined involution on Σ. Furthermore, when we restrict T on $V_2\subset\Sigma$, we obtain (V_2, τ_2). Let S^3 and S^4 be 3 and 4-spheres, and A_0 and A be standard involutions on S^3 and S^4 which have circles $S^1\subset S^3\subset S^4$ as sets of fixed points. We shall define an equivariant map $h: (\Sigma, T)\to(S^4, A)$ and show that h has some exotic property.

Let $h_0: (V_2, \tau_2)\to(S^3, A_0)$ be an equivariant map which satisfies the following condition (1) and (2):

(1) For equivariant tubular neighbourhoods N_0' and N_0 of Fixτ_2 and FixA_0 in V_2 and S^3 respectively, $h_0|N_0': N_0'\to N_0$ is a diffeomorphism.

(2) $h_0(V_2\text{-int}\, N_0')=S^3\text{-int}\, N_0$.

It is easy to show that such h_0 exists because we can construct a map from V_2/τ_2 to S^3/A_0 which satisfies the conditions similar to (1) and (2) above, then take the "lifting" of it as h_0.

We define $h: (\Sigma, T)\to(S^4, A)$ to be a "suspension" of h_0. More precisely, extend h_0 to a map from W to D^4, and extend further to a map from Σ to S^4 to be equivariant. With respect to h, we can assume the following (1) and (2).

(1) For equivariant tubular neighbourhoods N' and N of Fix$T$ and Fix$A$ in Σ and S^4, $h|N': N'\to N$ is a diffeomorphism.

(2) $h(\Sigma\text{-int}N)=S^4\text{-int}N$.

Let $Q'=(\Sigma\text{-int}N')/T$ and $Q=(S^4\text{-int}N)/A$. Let $\bar{h}: Q'\to Q$ be the map induced from $h|(\Sigma\text{-int}N')$. By the short computation we can show that h is a homotopy equivalent map of (Q', ∂) and (Q, ∂). Since Q is diffeomorphic to $P^2\times D^2$, we can consider $((Q', \partial), \bar{h})$ as a homotopy smoothing of $(P^2\times D^2, \partial)$. Now we can state our result on 4-dimensional homotopy smoothings.

THEOREM 2. *$\bar{h}: (Q', \partial)\to(Q, \partial)$ is not homotopic to a diffeomorphism fixing the boundary.*

Identifying (Q, ∂) with $(P^2\times D^2, \partial)$, we have the following corollary.

COROLLARY. *There is a nontrivial element in $\mathcal{HS}(P^2\times D^2, \partial)$.*

PROOF OF THEOREM 2. We shall prove by reduction to absurdity. Suppose $\bar{h}$ is homotopic to a diffeomorphism by a homotopy $H: Q'\times I\rightarrow Q$ where $H_0=\bar{h}$ and H_1 is a diffeomorphism and $H_t|\partial=\bar{h}|\partial$ for $0\leqq t\leqq 1$. Let $Q_0=(S^3\text{-int}\,N_0)/A_0$. Then Q_0 is a properly embedded submanifold in Q and diffeomorphic to $S^1\times D^2$. We can assume H is t-regular at Q_0. Let $E=H^{-1}(Q_0)$. Then we have:

LEMMA 4. $w_1(E)=w_2(E)=0$.

PROOF OF LEMMA 4. Let γ, η be normal bundles of E and Q_0 in $Q'\times I$ and Q respectively. Let $i: E\rightarrow Q'\times I$ and $j: Q_0\rightarrow Q$ be inclusion maps. We have a commutative diagram

$$\begin{array}{ccc} Q'\times I & \xrightarrow{H} & Q \\ i\big\uparrow\gamma & & j\big\uparrow\eta \\ E & \xrightarrow{H|E} & Q_0 \end{array} \qquad \text{and}$$

$$i^*W(Q'\times I)=W(E)W(\gamma)=W(E)(H|E)^*W(\eta) \qquad (**)$$

where $W(\ )$ means the total Stiefel-Whitney class.

Comparing 1-dimensional classes in (**), we obtain $i^*w_1(Q'\times I)=w_1(E)+(H|E)^*w_1(\eta)$. Let α be the generator of $H^1(Q;Z_2)\cong Z_2$. Since $w_1(\eta)=j^*\alpha$ and $w_1(Q'\times I)=H^*w_1(Q)=H^*\alpha$, we have $w_1(E)=i^*w_1(Q'\times I)+(H|E)^*w_1(\eta)=i^*H^*\alpha+(H|E)^*j^*\alpha=2i^*H^*\alpha=0$.

Comparing 2-dimensional classes in (**), we have $i^*w_2(Q'\times I)=w_2(E)$. From the homotopy invariance of w_2, $w_2(Q'\times I)=H^*w_2(Q)$. Hence $i^*w_2(Q'\times I)=i^*H^*w_2(Q)=(H|E)^*j^*w_2(Q)$. Since $H^2(Q_0;Z_2)=0$, $j^*w_2(Q)=0$, and we have $w_2(E)=0$. This completes the proof of Lemma 4.

We proceed the proof of Theorem 2. Let F be a subspace of $(\Sigma/T)\times I$ which is defined by $F=E\cup(N'/\tau_2)\times I$ (See Figure 6.).

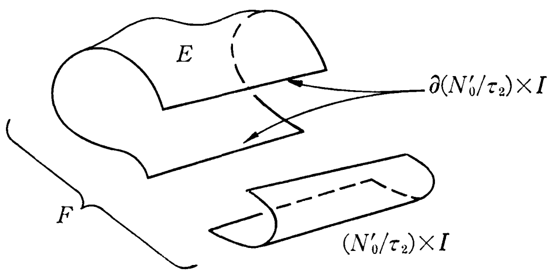

Figure 6.

Then F has the structure of a 4-manifold and possesses the following properties in Lemma 5.

LEMMA 5. (1) $\partial F\cong V_2/\tau_2\cup S^3/A_0$.

(2) $w_1(F)=w_2(F)=0$.

(3) $\operatorname{sign}(F)\equiv 0 \mod 16$.

PROOF OF LEMMA 5. From the construction, (1) and $w_1(F)=0$ hold. We shall prove $w_2(F)=0$. Consider the Mayor-Vietoris sequence and the diagram:

$$\begin{array}{ccc} H^1(E;Z_2)\oplus H^1((N_0'/\tau_2)\times I;Z_2) & \xrightarrow{j} & H^1(\partial(N_0'/\tau_2)\times I;Z_2) \longrightarrow \\ \uparrow & & \uparrow \\ H^1((S^3\text{-int}N_0)/A_0;Z_2)\oplus H^1(N_0/A_0;Z_2) & \longrightarrow & H^1(N_0/A_0;Z_2) \end{array}$$

$$H^2(F;Z_2) \xrightarrow{i} H^2(E;Z_2)\oplus H^2((N_0'/\tau_2)\times I;Z_2).$$

Observing the commutative diagram above, we obtain j is epic and therefore i is monic. Since $i_1^*w_2(F)=w_2(E)=0$ and $i_2^*w_2(F)=w_2(N_0'/\tau_2\times I)=0$ for inclusion maps $i_1: E\to F$ and $i_2: (N_0'/\tau_2)\times I\to F$, we have $w_2(F)=0$.

Since V_2/τ_2 and S^3/A_0 are both diffeomorphic to 3-spheres and $w_2(F)=0$, we can conclude that $\operatorname{sign}(F)\equiv 0 \mod 16$ by the well known theorem of Rochlin. This completes the proof.

We return to the proof of Theorem 2. Let $\pi: \Sigma\times I\to(\Sigma/T)\times I$ be a canonical projection and let $\tilde{F}=\pi^{-1}(F)$. Then $\tilde{F}$ is a branched covering space on F with $((\mathrm{Fix}\tau_2)\times I)\subset(N_0'/\tau_2)\times I\subset F$ as branching locus. $\tilde{F}$ separates $\Sigma\times I$ into two parts, say X and $T\times id_I(X)$, such that $X\cap T\times id_I(X)=\tilde{F}$ (See Figure 7.).

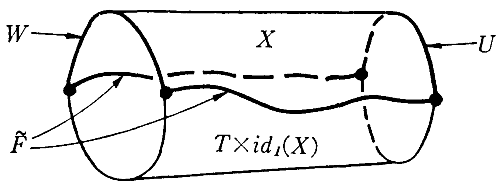

Figure 7.

Let $U=X\cap\Sigma\times\{1\}$. Note that U is diffeomorphic to a 4-disk. Then, the 4-manifold $W\cup\tilde{F}\cup U$ bounds a compact oriented 5-manifold X. Therefore $\operatorname{sign}(W\cup\tilde{F}\cup U)=0$. Hence $\operatorname{sign}(\tilde{F})=0$. Since $\partial(\tilde{F},T)\cong(V_2,\tau_2)\cup(S^3,A_0)$ and $\sigma(S^3,A_0)=0$, we have

$$\sigma(V_2,\tau_2)=\frac{1}{8}\{\operatorname{sign}(\tilde{F})-\operatorname{sign}(F)\}\equiv 0 \mod 4.$$

But, as we proved in § 3, $\sigma(V_2,\tau_2)=2$. This is a contradiction, and this completes the proof of Theorem 2.

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Shinji FUKUHARA
Department of Mathematics
Tsuda College
Kodaira
Tokyo, Japan