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Injective envelopes of C^*-algebras

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§ 1. Introduction.

For a Banach space, the existence and uniqueness of its injective envelope was proved by Cohen [**5**], and the present author [**9**] generalized this result to the case of Banach modules over a unital Banach algebra. In this paper we show a C^*-algebraic version of these results, i. e. that any unital C^*-algebra has a unique injective envelope (Theorem 4.1), where injectivity for C^*-algebras is understood as that considered by several authors, e. g. Hakeda and Tomiyama [**8**], Tomiyama [**16**], Choi and Effros [**4**], Loebl [**12**], et al. We also give two characterizations of injective C^*-algebras, one of which (Proposition 4.8) is similar to that of injective Banach modules (cf. [**9**; Lemma 3 (iv)]) and another (Proposition 4.11) is similar to that of von Neumann algebras whose commutant has property P of Schwartz ([**13**]; cf. also Remark 4.13). In the last section we give an example of an injective non W^*-, AW^*-factor of type III.

We recall the above-mentioned result of Cohen [**5**]. He considered the category whose objects are Banach spaces and whose morphisms are contractive linear maps, and defined "injectivity" and an "injective envelope" of a Banach space as follows: A Banach space Y is injective if any continuous linear map of a linear subspace of a Banach space Z into Y extends to a continuous linear map of the same norm on all of Z. An injective envelope of a Banach space X is a pair (Y, κ) of an injective Banach space Y and a linear isometry κ of X into Y such that Y itself is the only subspace of Y which is injective and contains $\kappa(X)$ [or equivalently, the identity map id_Y on Y ($\mathrm{id}_Y(y)=y$, $y\in Y$) is a unique contractive linear map of Y into itself which fixes each element of $\kappa(X)$ (cf. Isbell [**10**])]. This pair (Y, κ) is unique in the sense that if (Y_1, κ_1) is another injective envelope of X, there exists a linear isometry ι of Y onto Y_1 such that $\iota\circ\kappa=\kappa_1$.

In contrast to the case of Banach spaces, we consider the category whose objects are unital C^*-algebras and whose morphisms are unit-preserving completely positive linear maps. Hereafter, unless otherwise specified, C^*-algebras are unital, their C^*-subalgebras have the same units as the C^*-algebras containing them, and maps between C^*-algebras preserve units.

A C^*-algebra B is said to be injective if given any self-adjoint linear subspace S, containing the unit, of a C^*-algebra C, any completely positive linear map of S into B extends to a completely positive linear map of C into B (cf. Choi-Effros [4] and Loebl [12]). Let a C^*-algebra A be given. An extension of A is a pair (B, κ) of a C^*-algebra B and a *-monomorphism κ of A into B. The extension (B, κ) is called injective if B is injective, and it is called an injective envelope of A if it is an injective extension of A such that the identity map id_B on B is a unique completely positive linear map of B into itself which fixes each element of $\kappa(A)$. A result of Arveson [1; Theorem 1.2.3] says that the C^*-algebra $L(H)$ of all bounded linear operators on a Hilbert space H is injective, hence that each C^*-algebra, being represented faithfully on some Hilbert space, has an injective extension. The main result of this paper asserts that any C^*-algebra has a unique injective envelope (see Theorem 4.1).

For commutative C^*-algebras, their injective envelopes were studied by Gonshor ([6], [7]). His injective envelopes for commutative C^*-algebras coincide with their injective envelopes as Banach spaces (in fact, those become commutative AW^*-algebras which contain the original C^*-algebras as C^*-subalgebras) or those in the above sense.

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§ 2. Preliminaries.

This section is devoted to preparations for later use, most of which are known (cf. [3], [4]), but some of which are stated in a (possibly superficially) more general form (cf. Remark 2.5).

DEFINITION 2.1 (Choi-Effros [4] and Loebl [12]). A C^*-algebra B is injective if given any self-adjoint linear subspace S, containing the unit, of a C^*-algebra C, any completely positive linear map of S into B extends to a completely positive linear map of C into B.

DEFINITION 2.2. An extension of a C^*-algebra A is a pair (B, κ) of a C^*-algebra B and a *-monomorphism κ of A into B. The extension (B, κ) is injective if B is injective, and it is an injective envelope of A if it is an injective extension such that the identity map id_B on B is a unique completely positive linear map of B into itself which fixes each element of $\kappa(A)$.

Let B be a C^*-algebra and φ a unit-preserving contractive idempotent linear map of B into itself satisfying the Schwarz inequality:

$$\varphi(x)^*\varphi(x)\leqq\varphi(x^*x), \quad x\in B.$$

As in the proof of [4; Theorem 3.1], we define a multiplication "$\circ$" in $\operatorname{Im}\varphi=\varphi(B)$ by

$$x\circ y=\varphi(xy), \quad x, y\in\operatorname{Im}\varphi$$

and endow $\operatorname{Im}\varphi$ the involution and norm which are induced by those of B.

THEOREM 2.3. *In the above situation we have:*

(i) $\operatorname{Im}\varphi$ *is a unital* C^**-algebra.*

We denote this C^**-algebra by* $C^*(\varphi)$ *and the canonical map of* $\operatorname{Im}\varphi$ *onto* $C^*(\varphi)$ *by* j_φ.

(ii) *Let* $B_\varphi=\{x\in B : \varphi(x^*x)=\varphi(\varphi(x^*)\varphi(x)), \varphi(xx^*)=\varphi(\varphi(x)\varphi(x^*))\}$ *and* $I_\varphi=\{x\in B : \varphi(x^*x)=\varphi(xx^*)=0\}$. *Then* $B_\varphi=\operatorname{Im}\varphi+I_\varphi$, B_φ *is the largest* C^**-subalgebra of* B *restricted to which the map*

$$j_\varphi\circ\varphi : B \longrightarrow \operatorname{Im}\varphi \longrightarrow C^*(\varphi)$$

*becomes an onto *-homomorphism, and further* $\operatorname{Ker}(j_\varphi\circ\varphi|B_\varphi)=I_\varphi$. *Hence* $C^*(\varphi)$ *is *-isomorphic to the quotient* C^**-algebra* B_φ/I_φ.

PROOF. As in the proof of [4; Theorem 3.1], we have for $x, y\in\operatorname{Im}\varphi$

$$(x\circ y)^*=\varphi(xy)^*=\varphi(y^*x^*)=y^*\circ x^*,$$

$$\|x\circ y\|=\|\varphi(xy)\|\leqq\|xy\|\leqq\|x\|\,\|y\|,$$

and

$$\|x\|^2=\|x^*x\|\leqq\|\varphi(x^*x)\|=\|x^*\circ x\|\leqq\|x^*x\|=\|x\|^2$$

since φ is positive, contractive and

$$x^*x=\varphi(x)^*\varphi(x)\leqq\varphi(x^*x)=x^*\circ x$$

by the Schwarz inequality. Thus $\operatorname{Im}\varphi$ satisfies the axioms of C^*-algebras except for the associativity of the multiplication.

LEMMA 2.4. *Let* B *be a* C^**-algebra and* φ *a unit-preserving contractive idempotent linear map of* B *into itself. Then*

$$(*) \qquad \varphi(\varphi(x)^*\varphi(x))\leqq\varphi(x^*x) \qquad \textit{for all } x \textit{ in } B$$

if and only if

$$(**) \qquad \varphi(\varphi(x)\varphi(y))=\varphi(\varphi(x)y)=\varphi(x\varphi(y)) \qquad \textit{for all } x, y \textit{ in } B.$$

PROOF OF LEMMA 2.4. Let f be a state on B. Then $g=f\circ\varphi$ is also a state on B. Consider the cyclic representation $\{\pi_g, H_g\}$ of B induced by g and define a densely defined linear operator P_f on H_g by

$$P_f x_g=\varphi(x)_g, \quad x\in B,$$

where x_g stands for the canonical image of x in H_g; then $P_f^2=P_f$. Hence φ satisfies (*) if and only if

$$\|P_f x_g\|^2=f(\varphi(\varphi(x^*)\varphi(x)))\leqq f(\varphi(x^*x))=\|x_g\|^2$$

for all x in B and all state f on B if and only if P_f can be extended to a (self-adjoint) projection on H_g, i. e.

$$(P_f y_g, P_f(x^*)_g)=(y_g, P_f(x^*)_g)=(P_f y_g, (x^*)_g),$$
$$f(\varphi(\varphi(x)\varphi(y)))=f(\varphi(\varphi(x)y))=f(\varphi(x\varphi(y)))$$

for all state f on B if and only if φ satisfies (**). q. e. d.

The Schwarz inequality for φ: $\varphi(x)^*\varphi(x)\leqq\varphi(x^*x)$ implies $\varphi(\varphi(x)^*\varphi(x))\leqq\varphi(x^*x)$, hence φ satisfies (**) in Lemma 2. 4, so that we have for x, y, z in $\mathrm{Im}\,\varphi$,

$$x\circ(y\circ z)=\varphi(x\varphi(yz))=\varphi(\varphi(x)yz)=\varphi(xyz)$$
$$=\varphi(xy\varphi(z))=\varphi(\varphi(xy)z)=(x\circ y)\circ z.$$

(ii) First we show that $\mathrm{Im}\,\varphi+I_\varphi$ is a C^*-subalgebra of B. The iterated use of the equalities (**) in Lemma 2. 4 shows

(1) if $x, y\in I_\varphi$, $xy\in I_\varphi$;

(2) if $x\in\mathrm{Im}\,\varphi$ and $y\in I_\varphi$, $xy, yx\in I_\varphi$;

(3) if $x, y\in\mathrm{Im}\,\varphi$, $xy-\varphi(xy)\in I_\varphi$.

In fact (1) $\varphi((xy)^*xy)\leqq\|x\|^2\varphi(y^*y)=0$, $\varphi(xy(xy)^*)\leqq\|y\|^2\varphi(xx^*)=0$; hence $xy\in I_\varphi$.

(2) $\varphi((xy)^*xy)\leqq\|x\|^2\varphi(y^*y)=0$ and $\varphi(xy(xy)^*)=\varphi(\varphi(x)yy^*x^*)=\varphi(x\varphi(yy^*x^*))=\varphi(x\varphi(yy^*\varphi(x^*)))=\varphi(x\varphi(\varphi(yy^*)x^*))=0$. Similarly for yx.

(3) $\varphi((xy-\varphi(xy))^*(xy-\varphi(xy)))$

$$=\varphi((xy)^*xy)-\varphi((xy)^*\varphi(xy))-\varphi(\varphi(xy)^*xy)+\varphi(\varphi(xy)^*\varphi(xy))$$
$$=\varphi((xy)^*xy)-\varphi((xy)^*\varphi(xy)) \quad \text{and}$$
$$\varphi((xy)^*xy)=\varphi(\varphi(y^*)x^*xy)=\varphi(y^*\varphi(x^*xy))=\varphi(y^*\varphi(\varphi(x^*)xy))$$
$$=\varphi(y^*\varphi(x^*\varphi(xy)))=\varphi(\varphi(y^*)x^*\varphi(xy))=\varphi(y^*x^*\varphi(xy));$$

hence $\varphi((xy-\varphi(xy))^*(xy-\varphi(xy)))=0$.

Similarly

$$\varphi((xy-\varphi(xy))(xy-\varphi(xy))^*)=0.$$

We have for $x_i \in \operatorname{Im}\varphi$, $y_i \in I_\varphi$ $(i=1, 2)$,

$$(x_1+y_1)(x_2+y_2)=\varphi(x_1 x_2)+x_1 x_2-\varphi(x_1 x_2)+x_1 y_2+y_1 x_2+y_1 y_2$$
$$\in \operatorname{Im}\varphi+I_\varphi \quad \text{by } (1), (2), (3).$$

Further $\operatorname{Im}\varphi+I_\varphi$ is self-adjoint, closed since $I_\varphi \subset \operatorname{Ker}\varphi$ by the Schwarz inequality. Therefore $\operatorname{Im}\varphi+I_\varphi$ is a C^*-subalgebra of B and I_φ is its closed two-sided ideal by (1), (2).

Next we show that $B_\varphi \subset \operatorname{Im}\varphi+I_\varphi$. In fact, we have for x in B_φ,

$$\varphi((x-\varphi(x))^*(x-\varphi(x)))=\varphi(x^* x)-\varphi(x^*\varphi(x))-\varphi(\varphi(x^*)x)+\varphi(\varphi(x^*)\varphi(x))$$
$$=\varphi(x^* x)-\varphi(\varphi(x^*)\varphi(x))=0.$$

Similarly $\varphi((x-\varphi(x))(x-\varphi(x))^*)=0$; hence $x-\varphi(x)\in I_\varphi$, $x=\varphi(x)+x-\varphi(x) \in \operatorname{Im}\varphi+I_\varphi$.

Now, since the equalities defining the set B_φ are rewritten as

$$j_\varphi\circ\varphi(x^* x)=(j_\varphi\circ\varphi(x^*))\circ(j_\varphi\circ\varphi(x))$$

and

$$j_\varphi\circ\varphi(x x^*)=(j_\varphi\circ\varphi(x))\circ(j_\varphi\circ\varphi(x^*)),$$

it is clear that if C is a C^*-subalgebra of B such that $j_\varphi\circ\varphi|_C$ is a *-homomorphism, then $C \subset B_\varphi$. On the other hand, $j_\varphi\circ\varphi|_{\operatorname{Im}\varphi+I_\varphi}$ is a *-homomorphism because, for $x_i \in \operatorname{Im}\varphi$ and $y_i \in I_\varphi$ $(i=1, 2)$,

$$\varphi(y_1(x_2+y_2))=0=\varphi(\varphi(y_1)(x_2+y_2))$$

and so

$$\begin{aligned} & j_\varphi\circ\varphi((x_1+y_1)(x_2+y_2)) \\ &= j_\varphi\circ\varphi(x_1(x_2+y_2))+j_\varphi\circ\varphi(y_1(x_2+y_2)) \\ &= j_\varphi\circ\varphi(\varphi(x_1+y_1)(x_2+y_2)) \\ &= j_\varphi\circ\varphi(\varphi(x_1+y_1)\varphi(x_2+y_2)) \\ &= (j_\varphi\circ\varphi(x_1+y_1))\circ(j_\varphi\circ\varphi(x_2+y_2)). \end{aligned}$$

Thus $\operatorname{Im}\varphi+I_\varphi \subset B_\varphi$, so $\operatorname{Im}\varphi+I_\varphi=B_\varphi$.

Finally $\operatorname{Ker}(j_\varphi\circ\varphi|_{B_\varphi})=I_\varphi$ is immediate from $B_\varphi=\operatorname{Im}\varphi+I_\varphi$ and $I_\varphi \subset \operatorname{Ker}\varphi$.

q. e. d.

REMARK 2.5. In the proof of [**4**; Theorem 3.1], to conclude the equalities of the form (**) in Lemma 2.4, Choi and Effros used 2-positivity of φ. On the other hand, we used the Schwarz inequality for φ, which is implied by 2-positivity of φ (Choi [**3**; Corollary 2.8]). But the author does not know whether or not there is a unit-preserving contractive idempotent linear map on a C^*-algebra which satisfies the Schwarz inequality but is not 2-positive.

Following Arveson [**1**; Definition 1.2.1], we say that a linear map φ of a self-adjoint linear subspace S of a C^*-algebra B into another C^*-algebra C is completely isometric if, for each positive integer n, the map

$$\varphi \otimes 1 : S \otimes M_n \longrightarrow C \otimes M_n$$

is isometric, where M_n is the algebra of $n \times n$ matrices over $\boldsymbol{C}$ and 1 denotes the identity map on M_n. Obviously when φ is unit-preserving, φ is completely isometric if and only if φ is isometric and both φ and $\varphi^{-1} : \varphi(S) \to B$ are completely positive.

LEMMA 2.6 (cf. the proof of [**4**; Theorem 3.1]). *Let B, φ, $C^*(\varphi)$ and j_φ be as in Theorem 2.3. If B is injective and φ is completely positive, then $j_\varphi{}^{-1} : C^*(\varphi) \to \mathrm{Im}\,\varphi \subset B$ is completely isometric and $C^*(\varphi)$ is an injective C^*-algebra.*

LEMMA 2.7 (cf. the proof of [**4**; Theorem 3.1]). *A unit-preserving completely isometric linear map of a C^*-algebra onto another C^*-algebra is a $*$-isomorphism.*

LEMMA 2.8 (cf. Choi [**3**; Theorem 3.1]). *Let φ be a unit-preserving completely positive linear map of a C^*-algebra B into another C^*-algebra C. Then the set*

$$D = \{x \in B : \varphi(x^* x) = \varphi(x^*)\varphi(x),\ \varphi(xx^*) = \varphi(x)\varphi(x^*)\}$$

is the largest C^*-subalgebra of B restricted to which φ becomes a $*$-homomorphism, and moreover

$$\varphi(axb) = \varphi(a)\varphi(x)\varphi(b) \qquad \textit{for} \quad a, b \in D \quad \textit{and} \quad x \in B.$$

§ 3. Minimal projections on injective C^*-algebras.

Let B be a C^*-algebra and A its C^*-subalgebra.

DEFINITION 3.1. A linear map φ of B into itself is called a projection (resp. A-projection) on B if it is unit-preserving, completely positive and idempotent (resp. and further $\varphi(a) = a$ for all a in A).

DEFINITION 3.2. In the family of all A-projections on B we define a partial ordering $\prec$ by the rule $\varphi \prec \psi$ if $\varphi \circ \psi = \psi \circ \varphi = \varphi$. An A-projection on B which is minimal under this partial ordering is called a minimal A-projection.

DEFINITION 3.3. A seminorm p on B is called an A-seminorm if

$$p(x) \leqq \|x\|, \quad p(axb) \leqq \|a\| p(x) \|b\|$$

and

$$p(a) = \|a\| \qquad \text{for } a, b \text{ in } A \text{ and } x \text{ in } B.$$

In the family of all A-seminorms on B we define a partial ordering $\leqq$ by the rule $p\leqq q$ if $p(x)\leqq q(x)$ for all x in B.

Tomiyama's projection of norm one from a C^*-algebra B onto its C^*-subalgebra A [**14**] is an A-projection on B since it is completely positive ([**15**], [**18**]). Although the image of a projection on a C^*-algebra need not be a C^*-subalgebra, by Theorem 2.3, it is made into a C^*-algebra which is *-isomorphic to a quotient C^*-algebra of some C^*-subalgebra.

It is an immediate consequence of Zorn's lemma that there exists a minimal A-seminorm on B.

THEOREM 3.4. *Let B be an injective C^*-algebra and A its C^*-subalgebra. Then there exists a minimal A-projection on B.*

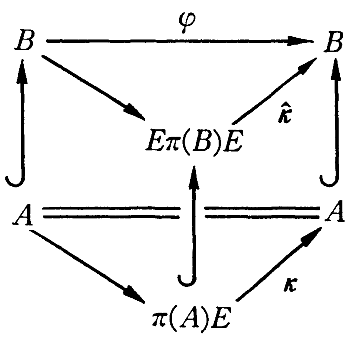
PROOF. Let p_0 be a minimal A-seminorm on B. Take a family $\{f_i\}_{i\in I}$ of pure states on A such that the direct sum $\sum_{i\in I}^{\oplus}\{\pi_{f_i}, H_{f_i}\}$ of the cyclic representations $\{\pi_{f_i}, H_{f_i}\}$ of A induced by f_i is faithful. By the Hahn-Banach theorem and the definition of A-seminorms, there exists a state extension g_i of each f_i to B such that

$$|g_i(x)|\leqq p_0(x) \qquad \text{for all } x \text{ in } B.$$

Let $\{\pi, H\}=\sum_{i\in I}^{\oplus}\{\pi_{g_i}, H_{g_i}\}$ be the direct sum of the cyclic representations $\{\pi_{g_i}, H_{g_i}\}$ of B induced by g_i and let E be the projection of H onto $\sum_{i\in I}^{\oplus}A_{g_i}$. Then $E\in\pi(A)'$, and by the choice of the family $\{f_i\}_{i\in I}$, the map

$$\kappa:\pi(A)E\longrightarrow A$$

given by $\kappa(\pi(a)E)=a$, $a\in A$, is a *-isomorphism and $\pi(A)$ acts irreducibly on each $A_{g_i}\subset H_{g_i}$. Since κ is completely positive and B is injective, there exists a completely positive map $\hat{\kappa}$ of $E\pi(B)E$ into B such that $\hat{\kappa}|_{\pi(A)E}=\kappa$.



Let $\varphi(x)=\hat{\kappa}(E\pi(x)E)$ for x in B. Then $\varphi|_A=\mathrm{id}_A$ and so φ is an A-module homomorphism, i.e. $\varphi(axb)=a\varphi(x)b$, $a, b\in A$, $x\in B$ by Lemma 2.8. We will show that $\|\varphi(x)\|\leqq p_0(x)$, $x\in B$. To this end we need only show that

$\|E\pi(x)E\| \leqq p_0(x)$, $x \in B$ since $\|\varphi(x)\| \leqq \|E\pi(x)E\|$. Take an $\varepsilon > 0$ and an x in B. Then there exist families $\{a_i\}_{i\in I}$, $\{b_i\}_{i\in I}$ $(a_i, b_i \in A)$ such that

$$\|\textstyle\sum_{i\in I}(a_i)_{g_i}\| = \|\sum_{i\in I}(b_i)_{g_i}\| = 1$$

and

$$|(\pi(x)\textstyle\sum_i(a_i)_{g_i}, \sum_j(b_j)_{g_j})| \geqq \|E\pi(x)E\| - \varepsilon.$$

Since $\pi(A)$ acts irreducibly on A_{g_i}, we may assume that

$$\|(a_i)_{g_i}\| = \|a_i\| \quad \text{and} \quad \|(b_j)_{g_j}\| = \|b_j\| \quad (i, j \in I).$$

We have then

$$\begin{aligned}
&|(\pi(x)\textstyle\sum_i(a_i)_{g_i}, \sum_j(b_j)_{g_j})| \\
&= |\textstyle\sum_i g_i(b_i^* x a_i)| \leqq \sum_i |g_i(b_i^* x a_i)| \leqq \sum_i p_0(b_i^* x a_i) \\
&\leqq p_0(x)\textstyle\sum_i \|b_i^*\| \, \|a_i\| = p_0(x)\sum_i \|(a_i)_{g_i}\| \, \|(b_i)_{g_i}\| \\
&\leqq p_0(x)(\textstyle\sum_i \|(a_i)_{g_i}\|^2)^{1/2}(\sum_i \|(b_i)_{g_i}\|^2)^{1/2} = p_0(x).
\end{aligned}$$

Hence $\|E\pi(x)E\| \leqq p_0(x)$ and so $\|\varphi(x)\| \leqq p_0(x)$.

The seminorms p_1, p_2 on B defined by

$$\begin{aligned}
&p_1(x) = \|\varphi(x)\| \\
&p_2(x) = \limsup_{n\to\infty} \|(\varphi + \varphi^2 + \cdots + \varphi^n)(x)/n\|
\end{aligned}$$

are A-seminorms $\leqq p_0$, so that the minimality of p_0 implies that $p_1 = p_2 = p_0$. Thus we have for each x in B,

$$\begin{aligned}
\|\varphi(x) - \varphi^2(x)\| &= p_1(x - \varphi(x)) = p_2(x - \varphi(x)) \\
&= \limsup_{n\to\infty} \|(\varphi(x) - \varphi^{n+1}(x))/n\| = 0,
\end{aligned}$$

i.e. $\varphi = \varphi^2$, so that φ is an A-projection on B.

To see the minimality of φ take an A-projection ψ on B with $\psi \prec \varphi$. Then, since $\|\psi(x)\| = \|\psi(\varphi(x))\| \leqq \|\varphi(x)\| = p_0(x)$, the minimality of p_0 implies that $\|\psi(x)\| = \|\varphi(x)\| = p_0(x)$, so that $\operatorname{Ker}\psi = p_0^{-1}(0) = \operatorname{Ker}\varphi$. On the other hand, $\varphi\circ\psi = \psi$ implies $\operatorname{Im}\psi \subset \operatorname{Im}\varphi$. Hence we have $\operatorname{Im}\psi = \operatorname{Im}\varphi$ and $\operatorname{Ker}\psi = \operatorname{Ker}\varphi$, i.e. $\psi = \varphi$. q.e.d.

REMARK 3.5. The above argument to conclude that $\varphi = \varphi^2$ is a modification of the one by Kaufman [**11**; the proof of Theorem 1].

REMARK 3.6. It follows from the argument analogous to the one in the proof of Theorem 3.4 that if we denote by p_φ the seminorm on B defined by $p_\varphi(x) = \|\varphi(x)\|$, then the map $\varphi \mapsto p_\varphi$ is a map of the set of all minimal A-projections on B onto the set of all minimal A-seminorms on B, and that if φ, ψ are minimal A-projections on B, then

$$\varphi\circ\psi\circ\varphi=\varphi$$

and $\varphi\circ\psi$ is a minimal A-projection on B such that

$$\operatorname{Im}\varphi\circ\psi=\operatorname{Im}\varphi \quad \text{and} \quad \operatorname{Ker}\varphi\circ\psi=\operatorname{Ker}\psi.$$

LEMMA 3.7. *Let B be an injective C^*-algebra, A its C^*-subalgebra and φ a minimal A-projection on B. Then the identity map $\mathrm{id}_{C^*(\varphi)}$ on the C^*-algebra $C^*(\varphi)$ is a unique completely positive linear map of $C^*(\varphi)$ into itself whose restriction to A coincides with id_A.*

PROOF. Let $\psi: C^*(\varphi)\rightarrow C^*(\varphi)$ be a completely positive linear map such that $\psi|_A=\mathrm{id}_A$. Since the seminorm p_φ is a minimal A-seminorm on B (Remark 3.6), the norm on $C^*(\varphi)$ is a unique A-seminorm on it, so that a reasoning similar to that of the proof of Theorem 3.4 shows that

$$\limsup_{n\to\infty}\|(\psi+\cdots+\psi^n)(x)/n\|=\|x\| \qquad \text{for } x \text{ in } C^*(\varphi).$$

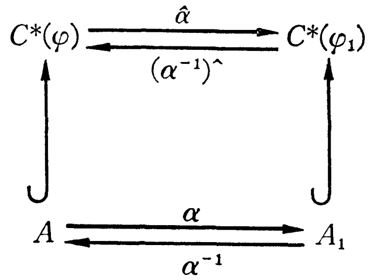
Hence we have for each x in $C^*(\varphi)$

$$\|x-\psi(x)\|=\limsup_{n\to\infty}\|(\psi+\cdots+\psi^n)(x-\psi(x))/n\|=0,$$

i.e. $\psi=\mathrm{id}_{C^*(\varphi)}$. q.e.d.

LEMMA 3.8. *Let A (resp. A_1) be a C^*-subalgebra of an injective C^*-algebra B (resp. B_1) and φ (resp. φ_1) a minimal A- (resp. A_1-) projection on B (resp. B_1). Suppose that there exists a *-isomorphism α of A onto A_1. Then α extends uniquely to a *-isomorphism $\hat{\alpha}$ of $C^*(\varphi)$ onto $C^*(\varphi_1)$.*

PROOF. Since $C^*(\varphi)$ [resp. $C^*(\varphi_1)$] is injective (Lemma 2.6), there exists a completely positive linear map $\hat{\alpha}$ [resp. $(\alpha^{-1})^\wedge$] of $C^*(\varphi)$ into $C^*(\varphi_1)$ [resp. $C^*(\varphi_1)$ into $C^*(\varphi)$] extending α (resp. α^{-1}).



Then Lemma 3.7 implies that $(\alpha^{-1})^\wedge\circ\hat{\alpha}=\mathrm{id}_{C^*(\varphi)}$ and $\hat{\alpha}\circ(\alpha^{-1})^\wedge=\mathrm{id}_{C^*(\varphi_1)}$, so that by Lemma 2.7 $\hat{\alpha}$ is a *-isomorphism of $C^*(\varphi)$ onto $C^*(\varphi_1)$. The uniqueness of $\hat{\alpha}$ follows again from Lemma 3.7. q.e.d.

§ 4. The main results.

With above preparations we can prove the following

THEOREM 4.1. *Any C^*-algebra A has an injective envelope (B, κ), which is unique in the sense that if another injective envelope (B_1, κ_1) is given, there exists a unique *-isomorphism ι of B onto B_1 such that $\iota\circ\kappa=\kappa_1$.*

PROOF. As stated before, there exists an injective C^*-algebra C containing A as a C^*-subalgebra. Let φ be a minimal A-projection on C (Theorem 3.4). Let $B=C^*(\varphi)$ and let κ be the canonical inclusion of A into B. Then, by Lemmas 2.6 and 3.7, (B, κ) is an injective envelope of A. If (B_1, κ_1) is another injective envelope of A, then id_{B_1} is a unique $\kappa_1(A)$-projection on B_1. Hence Lemma 3.8 implies the existence of a unique *-isomorphism ι of B onto B_1 such that $\iota\circ\kappa=\kappa_1$. q. e. d.

The next corollaries are immediate consequences of Theorem 4.1 and Lemma 3.8:

COROLLARY 4.2. *Let A be a C^*-algebra and (B, κ) its injective envelope. Then, for each *-automorphism α of A, there exists a unique *-automorphism $\hat{\alpha}$ of B such that $\kappa\circ\alpha=\hat{\alpha}\circ\kappa$. Hence the map $\alpha\mapsto\hat{\alpha}$ is a group-monomorphism of* Aut A *(=the group of all *-automorphisms of A) into* Aut B, *whose image consists of elements β such that $\beta(\kappa(A))=\kappa(A)$.*

COROLLARY 4.3. *With A, (B, κ) as in Corollary 4.2, the relative commutant $\kappa(A)'\cap B$ of $\kappa(A)$ in B coincides with the center of B.*

PROOF. Let u be a unitary element in $\kappa(A)'\cap B$. Then the map $x\mapsto uxu^*$ defines a *-automorphism of B which fixes each element of $\kappa(A)$, so it is the identity map on B. This shows that $\kappa(A)'\cap B \subset$ the center of B, and the converse inclusion is clear. q. e. d.

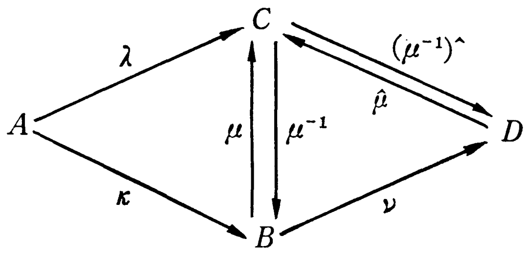
REMARK 4.4. By the construction it is obvious that a pair (B, κ) is the injective envelope of a C^*-algebra A if and only if B is an injective C^*-algebra and κ is a *-monomorphism of A into B such that the norm on B is a unique $\kappa(A)$-seminorm on B (cf. the proofs of Theorems 3.4 and 4.1).

We will give a characterization of the injective envelope of a C^*-algebra, which is similar to that of the injective envelope of a Banach module (cf. [**9**]).

DEFINITION 4.5. An extension (B, κ) of a C^*-algebra A is essential if for any completely positive linear map φ of B into a C^*-algebra C, φ is completely isometric whenever $\varphi\circ\kappa$ is.

LEMMA 4.6. *Let (C, λ) be an injective envelope of a C^*-algebra A. Then an extension (B, κ) of A is essential if and only if there exists a *-monomorphism μ of B into C such that $\mu\circ\kappa=\lambda$.*

PROOF. Necessity: Suppose that (B, κ) is essential. Since C is injective, we have a completely positive linear map μ of B into C such that $\mu\circ\kappa=\lambda$. Then, by hypothesis, μ is completely isometric. We will show that μ is a *-monomorphism. Let (D, ν) be the injective envelope of B. Since C and D are injective and μ is completely isometric, we have completely positive linear maps $\hat{\mu}: D\rightarrow C$ and $(\mu^{-1})^{\wedge}: C\rightarrow D$ such that $\hat{\mu}\circ\nu=\mu$ and $(\mu^{-1})^{\wedge}|_{\mu(B)}=\nu\circ\mu^{-1}$.



Hence $(\mu^{-1})^{\wedge}\circ\hat{\mu}: D\rightarrow D$ and $\hat{\mu}\circ(\mu^{-1})^{\wedge}: C\rightarrow C$ are completely positive linear maps such that

$$(\mu^{-1})^{\wedge}\circ\hat{\mu}|_{\nu(B)}=\mathrm{id}_{\nu(B)} \quad \text{and} \quad \hat{\mu}\circ(\mu^{-1})^{\wedge}|_{\lambda(A)}=\mathrm{id}_{\lambda(A)},$$

so that by the definition of the injective envelope,

$$(\mu^{-1})^{\wedge}\circ\hat{\mu}=\mathrm{id}_{D} \quad \text{and} \quad \hat{\mu}\circ(\mu^{-1})^{\wedge}=\mathrm{id}_{C}.$$

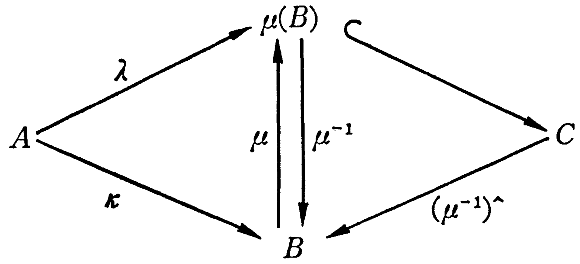
Thus $(\mu^{-1})^{\wedge}=\hat{\mu}^{-1}$ and by Lemma 2.7, $\hat{\mu}$ is a *-isomorphism of D onto C, so that $\mu=\hat{\mu}\circ\nu$ is a *-monomorphism.

Sufficiency: Suppose that there exists a *-monomorphism μ of B into C such that $\mu\circ\kappa=\lambda$ and let $\varphi: B\rightarrow E$ be a completely positive linear map of B into a C^*-algebra E such that $\varphi\circ\kappa$ is completely isometric. By replacing E by an injective C^*-algebra containing it as a C^*-subalgebra, we may assume that E itself is injective. Then an argument similar to above shows the existence of a completely isometric linear map $\psi: C\rightarrow E$ such that $\psi\circ\mu=\varphi$; hence φ is completely isometric. q. e. d.

PROPOSITION 4.7. *An extension (B, κ) of a C^*-algebra A is the injective envelope of A if and only if it is both injective and essential.*

PROOF. Necessity follows immediately from Lemma 4.6.

Sufficiency: Let (C, λ) be the injective envelope of A. Then Lemma 4.6 implies the existence of a *-monomorphism μ of B into C such that $\mu\circ\kappa=\lambda$. Since B is injective, we have a completely positive linear map $(\mu^{-1})^{\wedge}$ of C into B such that $(\mu^{-1})^{\wedge}|_{\mu(B)}=\mu^{-1}$.



Hence $\mu\circ(\mu^{-1})^{\wedge}: C \to C$ is a completely positive linear map such that $\mu\circ(\mu^{-1})^{\wedge}|_{\lambda(A)} = \mathrm{id}_{\lambda(A)}$, so that $\mu\circ(\mu^{-1})^{\wedge} = \mathrm{id}_C$ and consequently μ is a *-isomorphism of B onto C. q. e. d.

PROPOSITION 4.8. *A C^*-algebra B is injective if and only if it has no proper essential extension [i. e. if (C, λ) is an essential extension of B, then λ is a *-isomorphism of B onto C].*

PROOF. Necessity: Let (C, λ) be an essential extension of B. Since B is injective, there exists a completely positive linear map $(\lambda^{-1})^{\wedge}$ of C onto B such that $(\lambda^{-1})^{\wedge}|_{\lambda(B)} = \lambda^{-1}$, i. e. $(\lambda^{-1})^{\wedge}\circ\lambda = \mathrm{id}_B$. By hypothesis $(\lambda^{-1})^{\wedge}$ is completely isometric, and $(\lambda^{-1})^{\wedge}\circ(\lambda\circ(\lambda^{-1})^{\wedge} - \mathrm{id}_C) = 0$. Hence $\lambda\circ(\lambda^{-1})^{\wedge} = \mathrm{id}_C$, so λ is a *-isomorphism of B onto C.

Sufficiency: Let (C, λ) be an injective envelope of B. By Proposition 4.7, (C, λ) is an essential extension of B, so if B has no proper essential extension, then λ is a *-isomorphism of B onto C. Hence B is injective. q. e. d.

DEFINITION 4.9. A self-adjoint linear subspace S, containing the unit, of a C^*-algebra B is called a C^*-subspace of B if there exist a C^*-algebra A and a completely isometric linear map φ of A into B with $\mathrm{Im}\,\varphi = S$.

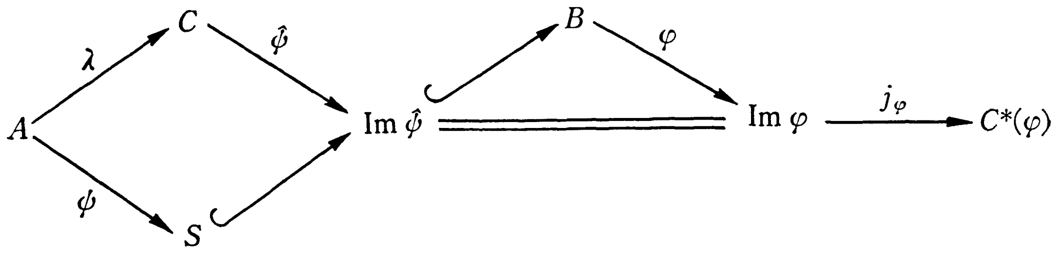
We note that if there exists another completely isometric linear map φ_1 of a C^*-algebra A_1 into B with $\mathrm{Im}\,\varphi_1 = S$, A and A_1 are *-isomorphic by Lemma 2.7.

PROPOSITION 4.10. *Let B be an injective C^*-algebra and S a closed self-adjoint linear subspace, containing the unit, of B. Then S is a C^*-subspace of B if and only if there exists a projection φ on B such that $\varphi(S^2) \subset S \subset \mathrm{Im}\,\varphi$.*

PROOF. Sufficiency: By Theorem 2.3 and Lemma 2.6, $j_{\varphi}^{-1}: C^*(\varphi) \to \mathrm{Im}\,\varphi \subset B$ is a completely isometric linear map of the C^*-algebra $C^*(\varphi)$ onto $\mathrm{Im}\,\varphi$. Noting the definition of the multiplication in $C^*(\varphi)$, we see that $\varphi(S^2) \subset S \subset \mathrm{Im}\,\varphi$ if and only if $j_{\varphi}(S)$ is a C^*-subalgebra of $C^*(\varphi)$. Hence $j_{\varphi}^{-1}|_{j_{\varphi}(S)}: j_{\varphi}(S) \to S \subset B$ is a completely isometric linear map of the C^*-algebra $j_{\varphi}(S)$ onto S, so that S is a C^*-subspace of B.

Necessity: Suppose that there exists a completely isometric linear map ϕ of a C^*-algebra A onto S and let (C, λ) be the injective envelope of A. Since

B is injective and (C, λ) is essential (Proposition 4.7), we have a completely isometric linear map $\hat{\phi}$ of C into B such that $\hat{\phi}\circ\lambda=\phi$. Then there exists a projection φ on B such that $\operatorname{Im}\varphi=\operatorname{Im}\hat{\phi}$.



Since $j_\varphi\circ\hat{\phi}$ is a *-isomorphism of C onto $C^*(\varphi)$ (Lemma 2.7), $j_\varphi(S)=(j_\varphi\circ\hat{\phi})\circ\lambda(A)$ is a C^*-subalgebra of $C^*(\varphi)$, so that the condition in the statement of this proposition is satisfied. q. e. d.

We will give a necessary and sufficient condition that a C^*-subalgebra of an injective C^*-algebra be injective: Let $A\subset B$ be C^*-algebras with B injective. For each x in B, set

$$C_A(x)=\{y\in B : \|a+\textstyle\sum_{i=1}^n b_i y c_i\|\leqq\|a+\sum_{i=1}^n b_i x c_i\| \text{ for all } a, b_i, c_i \text{ in } A,\ n=1, 2, \cdots\}.$$

PROPOSITION 4.11. *With notations as above A is injective if and only if $C_A(x)\cap A\neq\emptyset$ for all x in B.*

PROOF. Take a minimal A-projection φ on B (Theorem 3.4). Then obviously A is injective if and only if $\operatorname{Im}\varphi=A$.

Necessity: If $\operatorname{Im}\varphi=A$, then φ is a contractive A-module homomorphism (Lemma 2.8), so that $\varphi(x)\in C_A(x)\cap A$ for all x in B.

Sufficiency: Suppose that the above condition is satisfied, but A is not injective. Then there exist an $x_0\in\operatorname{Im}\varphi\setminus A$ and an $a_0\in C_A(x_0)\cap A$. Let X be the Banach A-bimodule generated by A and x_0, i.e. X is the norm closure of the subset

$$\{a+\textstyle\sum_{i=1}^n b_i x_0 c_i :\ a, b_i, c_i\in A,\ n=1, 2, \cdots\}$$

of B. Define a seminorm p on X by

$$p(a+\textstyle\sum_{i=1}^n b_i x_0 c_i)=\|a+\sum_{i=1}^n b_i a_0 c_i\|.$$

Then

$$p(axb)\leqq\|a\|p(x)\|b\| \quad \text{for } a, b\in A \text{ and } x\in X,$$

$$
\begin{aligned}
p(a+\textstyle\sum_{i=1}^{n} b_i x_0 c_i) &= \|a+\textstyle\sum_{i=1}^{n} b_i a_0 c_i\| \\
&\leqq \|a+\textstyle\sum_{i=1}^{n} b_i x_0 c_i\| \\
&= \|\varphi(a+\textstyle\sum_{i=1}^{n} b_i x_0 c_i)\| \\
&= p_\varphi(a+\textstyle\sum_{i=1}^{n} b_i x_0 c_i)
\end{aligned}
$$

(p_φ denotes the seminorm on B defined in Remark 3.6)
for all $a+\sum_{i=1}^{n} b_i x_0 c_i \in X$, $a, b_i, c_i \in A$,
and

$$p(a)=\|a\| \quad \text{for} \quad a\in A.$$

On the other hand,

$$p(-a_0+x_0)=\|-a_0+a_0\|=0<\|-a_0+x_0\|=p_\varphi(-a_0+x_0).$$

Since p_φ is a minimal A-seminorm (Remark 3.6), this inequality and the following lemma would yield a contradiction:

LEMMA 4.12. *There exists an A-seminorm p_1 on B such that*

$$p_1|_X=p \quad \text{and} \quad p_1\leqq p_\varphi.$$

PROOF OF LEMMA 4.12. Let $U=\{x\in X: p(x)\leqq 1\}$, $V=\{y\in B: p_\varphi(y)\leqq 1\}$ and W the convex hull of $U\cup V$ in B. Then the Minkowski functional p_1 of W:

$$p_1(y)=\inf\{\lambda>0: y\in\lambda W\}, \quad y\in B$$

is the desired seminorm. In fact $p_1|_X=p$ follows from $V\cap X\subset U$, and the remainder of the proof is immediate. q. e. d.

REMARK 4.13. In the above proposition, let A be a von Neumann algebra on a Hilbert space H and let $B=L(H)$. Then Schwartz's property P [**13**] for the commutant A' of A implies the above condition for A, hence the existence of a projection of norm one from B onto A (cf. [**13**; Lemma 5]).

Let A be a C^*-algebra, B an injective C^*-algebra containing A as a C^*-subalgebra, and (C, λ) an injective envelope of A. We know that C can be embedded in B as a C^*-subspace of B, i.e. there exists a completely isometric linear map φ of C into B such that $\varphi\circ\lambda=\mathrm{id}_A$ (cf. Proposition 4.7). But the author does not know whether or not C can be embedded in B as a C^*-subalgebra of B, i.e. the above φ can be chosen as a $*$-monomorphism. (Added March 1978: This is not the case for a general C^*-algebra A.) A necessary condition for this is stated as follows:

PROPOSITION 4.14. *Let A, B and C be as above and let K be the set of all completely isometric linear maps φ of C into B such that $\varphi\circ\lambda=\mathrm{id}_A$. Then K is* *a convex subset of $L(C, B)$, the Banach space of all continuous linear maps of C into B, and if $\varphi_0 \in K$ is a *-monomorphism, then φ_0 is an extreme point of K.*

PROOF. Let $\varphi = \mu\varphi_1 + (1-\mu)\varphi_2$, where $\varphi_1, \varphi_2 \in K$ and $0 < \mu < 1$. Then $\varphi : C \to B$ is completely positive and $\varphi \circ \lambda = \mathrm{id}_A$, so by Proposition 4.7, φ is completely isometric; hence $\varphi \in K$.

Suppose that $\varphi_0 \in K$ is a *-monomorphism and that $\varphi_0 = (\varphi_1 + \varphi_2)/2$, $\varphi_1, \varphi_2 \in K$. Then the Schwarz inequality shows that for each x in C,

$$\left\{\frac{1}{2}(\varphi_1(x)+\varphi_2(x))\right\}^* \left\{\frac{1}{2}(\varphi_1(x)+\varphi_2(x))\right\} = \varphi_0(x)^* \varphi_0(x)$$

$$= \varphi_0(x^* x) = \frac{1}{2}(\varphi_1(x^* x) + \varphi_2(x^* x)) \geqq \frac{1}{2}(\varphi_1(x)^* \varphi_1(x) + \varphi_2(x)^* \varphi_2(x)),$$

$$0 \geqq (\varphi_1(x) - \varphi_2(x))^* (\varphi_1(x) - \varphi_2(x));$$

hence

$$\varphi_1(x) = \varphi_2(x) = \varphi_0(x), \quad \varphi_1 = \varphi_2 = \varphi_0.$$

q. e. d.

PROPOSITION 4.15. *Let A be a unital C^*-algebra and (B, κ) its injective envelope. Then if A is simple, so is B too. Hence, in particular, B is an AW^*-factor.*

PROOF. Let I be a proper closed two-sided ideal of B. Since A is unital and simple, $A \cap I = \{0\}$. Hence the map $\pi \circ \kappa : A \to B \to B/I$, where $\pi : B \to B/I$ is the quotient map, is a *-monomorphism, so that the seminorm $x \mapsto \|\pi(x)\|$ on B defines a $\kappa(A)$-seminorm. Thus Remark 4.4 implies that $I = \mathrm{Ker}\,\pi = \{0\}$, hence that B is simple. An injective C^*-algebra is monotone closed (Tomiyama [**16**; Theorem 7.1]); in particular, it is an AW^*-algebra. Hence the simple AW^*-algebra B is an AW^*-factor. q. e. d.

§ 5. An example.

We give an example of an injective non W^*-, AW^*-factor of type III.

EXAMPLE 5.1. Let $A = L(H)/LC(H)$ be the Calkin algebra, where H is a separable infinite dimensional Hilbert space, and let (B, κ) be the injective envelope of A. Then B is an injective non W^*-, AW^*-factor of type III.

PROOF. Since A is simple, Proposition 4.15 implies that B is a simple AW^*-factor. Hence B must be of type I_n $(n < \infty)$ or II_1 or III. The first two cases are excluded since A is infinite dimensional and contains an infinite projection; so B is of type III. To see that B is non W^*, we follow the argument of Birrell [2; Example (c)]: If B were W^*, since it is simple, it must be a countably decomposable W^*-factor of type III. But there exists an uncountable orthogonal family of non-zero projections in A, hence in B, a contradiction. q. e. d.

REMARK 5.2. Professor Sakai kindly pointed out to the author that a result of Voiculescu can be applied to show that the Calkin algebra A is not AW^*, hence that the injective envelope B of A, being AW^*, contains $\kappa(A)$ properly: In fact, let C be the C^*-subalgebra of $A=L(H)/LC(H)$ generated by $S+LC(H)$, where S is the simple unilateral shift on H. Then Voiculescu [**17**; Corollary 1.9] implies that C, being separable, is equal to its bicommutant. Hence if A were AW^*, then C also would be so. But this is absurd since $C\cong C(T)$, the C^*-algebra of continuous functions on the 1 dimensional torus T.

References

[1] W. Arveson, Subalgebras of C^*-algebras, Acta Math., **123** (1969), 141-224.

[2] I.D. Birrell, Maximal simple C^*-algebras I, Bull. London Math. Soc., **6** (1974), 141-144.

[3] M.-D. Choi, A Schwarz inequality for positive linear maps on C^*-algebras, Illinois J. Math., **18** (1974), 565-574.

[4] M.-D. Choi and E.G. Effros, Injectivity and operator spaces, J. Functional Analysis, **24** (1977), 156-209.

[5] H.B. Cohen, Injective envelopes of Banach spaces, Bull. Amer. Math. Soc., **70** (1964), 723-726.

[6] H. Gonshor, Injective hulls of C^*-algebras, Trans. Amer. Math. Soc., **131** (1968), 315-322.

[7] H. Gonshor, Injective hulls of C^*-algebras II, Proc. Amer. Math. Soc., **24** (1970), 486-491.

[8] J. Hakeda and J. Tomiyama, On some extension property of von Neumann algebras, Tôhoku Math. J., **19** (1967), 315-323.

[9] M. Hamana, Injective envelopes of Banach modules, to appear in Tôhoku Math. J.

[10] J.R. Isbell, Three remarks on injective envelopes of Banach spaces, J. Math. Anal. Appl., **27** (1969), 516-518.

[11] R. Kaufman, A type of extension of Banach spaces, Acta Sci. Math. (Szeged), **27** (1966), 163-166.

[12] R.I. Loebl, Injective von Neumann algebras, Proc. Amer. Math. Soc., **44** (1974), 46-48.

[13] J. Schwartz, Two finite, non-hyperfinite, non-isomorphic factors, Comm. Pure Appl. Math., **16** (1963), 19-26.

[14] J. Tomiyama, On the projection of norm one in W^*-algebras, Proc. Japan Acad., **32** (1957), 608-612.

[15] J. Tomiyama, On the product projection of norm one in the direct product of operator algebras, Tôhoku Math. J., **11** (1959), 303-313.

[16] J. Tomiyama, Tensor products and projections of norm one in von Neumann algebras, Lecture notes for seminar given at University of Copenhagen, 1970.

[17] D. Voiculescu, A non-commutative Weyl-von Neumann theorem, Rev. Roumaine Math. Pures Appl., 21 (1976), 97-113.
[18] M. Nakamura, M. Takesaki and H. Umegaki, A remark on the expectation of operator algebras, Kōdai Math. Sem. Rep., 12 (1960), 82-90.

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