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A generalized Lüroth Theorem for curves

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Let k be a field. The famous "Lüroth Theorem" asserts that if R is a field with $k\subset R\subseteqq k(X)$, then $R=k(Y)$, a simple transcendental extension of k. [**5**, p. 198]. As was proved by Igusa [**2**], [**3**], Lüroth's Theorem can be generalized to say that if $X_1, \cdots, X_n$ are algebraically independent over k and R is a field of transcendence degree one over k such that $k\subset R\subseteqq k(X_1, \cdots, X_n)$, then $R=k(Y)$, a simple transcendental extension of k. Related results for the case when R has transcendence degree >1 over k are given by Zariski [**6**], Swan [**4**], and Clemens-Griffiths [**1**].

These striking results naturally motivate the search for similar phenomena or generalization. For this purpose we use the following notation. If R is a function field of one variable over k, then the degree of irrationality of R over k, $\mathrm{irr}(R)=\min\{[R:k(x)]:x\in R\}$. The classical Lüroth Theorem can then be stated: if $R\subseteqq S$ are function fields of one variable over k and $\mathrm{irr}(S)=1$, then $\mathrm{irr}(R)=1$. In this form, Lüroth's Theorem naturally calls for the study of the pair of numbers $(\mathrm{irr}(S), \mathrm{irr}(R))$ for the case $\mathrm{irr}(S)>1$. Our result is the following.

THEOREM. *Let $R\subseteqq S$ be function fields of one variable over a field k. For any $x\in S$, let y denote the norm of x with respect to R. If y is not algebraic over k, then $[S:k(x)]\geqq[R:k(y)]$. In particular, if k is an infinite field, then the degree of irrationality of R, $\mathrm{irr}(R)\leqq\mathrm{irr}(S)$, the degree of irrationality of S.*

PROOF. We first consider the case when S is separable over R. Let T be a normal closure of S over R, and let G be the Galois group of T over R. We recall that if H is the subgroup of G fixing S and $G=g_1H\cup\cdots\cup g_mH$ is a coset decomposition of G with respect to H, then $y=\prod_{i=1}^{m}g_i(x)$ is the norm of x [**7**, p. 91]. Note that $m=[S:R]$. Since $[T:k(x)]=[T:k(g_i(x))]$ is equal to the degree of the polar divisor of x or $g_i(x)$ in T, and since the

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degree of the polar divisors of a product is less than or equal to the sum of the degrees of the polar divisors of the factors, we have

$m[T:k(x)]\geqq[T:k(y)]$. Thus

$m[T:S][S:k(x)]\geqq[T:R][R:k(y)]$, and since

$m[T:S]=[T:R]$, we have $[S:k(x)]\geqq[R:k(y)]$

in the separable case.

In the general case, let S' be the separable closure of R in S, and let $p^e=[S:S']$. Then for $x\in S$, $x^{p^e}=x'$ is the norm of x in S' and $[S:k(x)]=[S':k(x')]$. If y is the norm of x with respect to S/R, then y is also the norm of x' with respect to S'/R. It follows from the separable case that $[S:k(x)]\geqq[R:k(y)]$.

For k infinite, and $x\in S$ such that $[S:k(x)]=\text{irr}(S)$, we show the existence of an element c of k such that the norm of $x-c$ with respect to R is transcendental over k; thus establishing $\text{irr}(S)\geqq\text{irr}(R)$. Let $f(x)$ be the field equation for x with respect to S over R, then the norm of $x-c$ for $c\in k$ is $\pm f(c)$. By the Lagrange interpolation formula $f(c)$ can not be algebraic over k for more than $[S:R]$ elements c, for otherwise x would be algebraic over k.

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