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Markov partitions of hyperbolic sets

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§ 1. Introduction and preliminaries.

In this paper we prove that pseudo-orbits near a hyperbolic set are shadowed, and for a hyperbolic set there is its arbitrarily slight extension which has Markov partitions.

Bowen proved similar results for a basic set of an Axiom A diffeomorphism ([1], [3]).

Suppose $f: M \to M$ is a diffeomorphism of a Riemannian manifold M with some Riemannian metric $\|\cdot\|$. A compact f-invariant set $\Lambda \subset M$ is said to be hyperbolic if $T_\Lambda M$ (the tangent bundle of M over Λ) splits into a Whitney sum of Tf-invariant subbundles

$$T_\Lambda M = E^s \oplus E^u,$$

and if there are $c > 0$ and $0 < \lambda < 1$ such that

$$\|Tf^n v\| \leqq c\lambda^n \|v\| \qquad \text{if} \quad v \in E^s$$

$$\|Tf^{-n} v\| \leqq c\lambda^n \|v\| \qquad \text{if} \quad v \in E^u$$

for $n > 0$.

The following was proved in [7].

THEOREM 0. *Suppose Λ is a hyperbolic set for a diffeomorphism $f: M \to M$, and U is a neighbourhood of Λ. Then there is a hyperbolic set Λ' with $\Lambda \subset \Lambda' \subset U$ which is a quotient of a subshift of finite type. More precisely there are a subshift of finite type Σ on symbols $\mathcal{B} = \{B_1, \cdots, B_N\}$ determined by an $N \times N$ 0-1 matrix $T = (t_{ij})$, and a map $\pi: \Sigma \to \Lambda'$ satisfying the followings.*

(1) *$\pi(\Sigma) = \Lambda'$ and $f\pi = \pi\sigma$, where σ is the shift transformation.*

(2) *B_i is a topologically embedded m-disk in M for $i = 1, \cdots, N$.*

(3) *The map π is given by*

$$\pi((a_i)_{i \in \boldsymbol{Z}}) = \bigcap_{i \in \boldsymbol{Z}} f^{-i}(a_i) \qquad \textit{for} \quad (a_i)_{i \in \boldsymbol{Z}} \in \Sigma.$$

Here $\boldsymbol{Z}$ denotes the integers.

(4) *For any* $\varepsilon>0$, *we can choose* $\mathscr{B}$ *and* T *so that*

$$\operatorname{diam} B_i<\varepsilon$$

for $i=1, \cdots, N$ ($\operatorname{diam} B_i$ *is the diameter of* B_i).

(5) *There is an associated covering* $\{U_i\}_{i=1,\cdots,N}$ *of* Λ *such that*

$$U_i \subset B_i$$

and

$$f(U_i) \cap U_j \neq \emptyset \qquad \textit{implies} \quad t_{ij}=1$$

for $i, j=1, \cdots, N$.

In the above the subshift Σ is defined by

$$\Sigma=\{(a_i)_{i\in Z}\in\mathscr{B}^{Z} \mid t_{n_i n_{i+1}}=1 \text{ where } a_i=B_{n_i}\}$$

where $\mathscr{B}^{Z}$ is the space of maps from $\boldsymbol{Z}$ into $\mathscr{B}$ with the compact-open topology ($\mathscr{B}$ and $\boldsymbol{Z}$ have the discrete topologies). The shift transformation $\sigma : \Sigma \to \Sigma$ is given by

$$\sigma((a_i)_{i\in Z})=(a'_i)_{i\in Z} \qquad \text{where} \quad a'_i=a_{i+1}.$$

§ 2. Pseudo-orbits in neighbourhoods of hyperbolic sets.

A sequence $\{x_n\}_{n=j,\cdots,k}$ ($j=-\infty$ or $k=+\infty$ is permitted and $x_n\in M$ for $n=j, \cdots, k$) is called an ε-pseudo-orbit for $f: M\to M$, if x_n satisfy

$$d(f(x_n), x_{n+1})<\varepsilon$$

for $n=j, \cdots, k-1$. Here $d(,)$ denotes the metric on M. A pseudo-orbit $\{x_n\}_{n=j,\cdots,k}$ is δ-shadowed by $x\in M$ if

$$d(f^n(x), x_n)\leqq\delta$$

for $n=j, \cdots, k$.

Bowen proved that if Ω is the non-wandering set for an Axiom A diffeomorphism f, for each $\delta>0$ there is an $\varepsilon>0$ so that every ε-pseudo-orbit $\{x_n\}_{n=j,\cdots,k}$ of $f|\Omega$ is δ-shadowed by some $x\in\Omega$ ([3], [4]).

We show the following.

LEMMA 2.1. *Let* $\Lambda\subset M$ *be a hyperbolic set for a diffeomorphism* f. *Then for any* $\delta>0$, *there are* $\varepsilon>0$ *and a neighbourhood* W *of* Λ *such that every* ε*-pseudo-orbit* $\{x_n\}_{n=j,\cdots,k}$ *with* $x_n\in W$ *is* δ*-shadowed by some* $x\in M$. *Moreover we can choose* x *so that*

$$d(f^n(x), \Lambda)<\delta \qquad \textit{for} \quad n\in\boldsymbol{Z}.$$

PROOF. By Theorem 0, there is a subshift of finite type Σ on $\mathcal{B}=\{B_1, \cdots, B_N\}$ determined by a matrix $T=(t_{ij})$ such that

$$\operatorname{diam} B_i<\frac{\delta}{2} \quad \text{for} \quad i=1, \cdots, N. \tag{2.1}$$

There is an associated covering $\{U_n\}_{n=1,\cdots,N}$ of Λ such that

$$U_i \subset B_i$$

and

$$f(U_i) \cap U_j \neq \emptyset \quad \text{implies} \quad t_{ij}=1$$

for $i, j=1, \cdots, N$. Let $\{U'_n\}_{n=1,\cdots,N}$ be a covering of Λ such that $\bar{U}'_n \subset U_n$, where $\bar{U}'_n$ is the closure of U'_n. Let $\varepsilon_1>0$ satisfy

$$\varepsilon_1<d(U'_n, M-U_n) \tag{2.2}$$

for $n=1, \cdots, N$. Let V be a neighbourhood of Λ with $\bar{V}$ compact. Then there is $\varepsilon_2>0$ such that

$$d(f(x), f(y))<\frac{\varepsilon_1}{3} \quad \text{when} \quad d(x, y)<\varepsilon_2$$

for $x, y \in V$.

Define

$$\varepsilon=\min\left\{\frac{\delta}{2}, \frac{\varepsilon_1}{3}, \varepsilon_2\right\},$$

and

$$W=\{x \in V \mid d(x, \Lambda)<\varepsilon\}.$$

For any ε-pseudo-orbits $\{x_n\}_{n=j,\cdots,k}$ in W, there is a sequence $\{y_n\}_{n=j,\cdots,k}$ such that $y_n \in \Lambda$ and $d(x_n, y_n)<\varepsilon$ for $n=j, \cdots, k$. Then for $n=j, \cdots, k-1$,

$$\begin{aligned} d(f(y_n), y_{n+1}) &\leqq d(f(y_n), f(x_n))+d(f(x_n), x_{n+1})+d(x_{n+1}, y_{n+1}) \\ &\leqq \frac{\varepsilon_1}{3}+\varepsilon+\varepsilon \\ &\leqq \varepsilon_1. \end{aligned} \tag{2.3}$$

In the case when $j>-\infty$ or $k<+\infty$, we extend $\{y_n\}_{n=j,\cdots,k}$ to the two-sided sequence $\{y_n\}_{n\in Z}$ as follows.

Set for $i>0$

$$y_{k+i}=f^i(y_k) \quad \text{if} \quad k<+\infty$$

and

$$y_{j-i}=f^{-i}(y_j) \quad \text{if} \quad j>-\infty.$$

Choose $U'_{i_n}(n\in \boldsymbol{Z})$ such that

$$y_n\in U'_{i_n}.$$

From (2.2) and (2.3), it follows that

$$f(U_{i_n})\cap U_{i_{n+1}}\neq 0.$$

This implies $t_{i_n i_{n+1}}=1$ for $n\in \boldsymbol{Z}$. Then there is $(a_n)_{n\in Z}\in\Sigma$ such that $a_n=B_{i_n}$ for $n\in \boldsymbol{Z}$. Put

$$x=\pi((a_n)_{n\in Z}).$$

Then

$$f^n(x)\in B_{i_n}. \tag{2.4}$$

From

$$y_n\in U_{i_n}\subset B_{i_n}, \tag{2.5}$$

it follows that

$$\begin{aligned} d(f^n(x), x_n) &\leqq d(f^n(x), y_n)+d(y_n, x_n) \\ &<\delta \end{aligned}$$

for $n=j, \cdots, k$. By (2.1), (2.4) and (2.5),

$$d(f^n(x), \Lambda)<\delta \qquad \text{for} \quad n\in \boldsymbol{Z}.$$

This completes the proof.

COROLLARY 2.2. *Given any $\delta>0$, there are an $\varepsilon>0$ and a neighbourhood W of Λ such that for any $x\in W$ with*

$$d(f^n(x), x)<\varepsilon$$

and

$$f^k(x)\in W \qquad \textit{for} \quad k=1, \cdots, n$$

there is a periodic point $p\in M$ of period n with

$$d(f^k(x), f^k(p))\leqq\delta \qquad \textit{for} \quad k=1, \cdots, n$$

and

$$d(p, \Lambda)\leqq\delta.$$

PROOF. In Lemma 2.1, suppose δ is sufficiently small. Then

$$\Lambda'=\bigcap_{n\in Z} f^n(\{y\in M \mid d(y, \Lambda)\leqq\delta\})$$

is a hyperbolic set and a point which δ-shadows an ε-pseudo-orbit in W is contained in Λ'. Letting δ be small, we can choose Λ' such that its expansive constant is arbitrarily close to that of Λ. Hence the corollary follows as in [**4**, p. 75-76].

§ 3. Markov partitions of hyperbolic sets.

Suppose Λ is a hyperbolic set for a diffeomorphism $f: M \to M$. Let $\varepsilon>0$ be small so that $W_\varepsilon^s(x)$ intersects $W_\varepsilon^u(y)$ transversally and $W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ is at most one point for $x, y \in \Lambda$, where $W_\varepsilon^s(x)$ (resp. $W_\varepsilon^u(x)$) denotes the local stable manifold (resp. the local unstable manifold) of size ε through x ([**6**]).

DEFINITION. A subset $R \subset \Lambda$ is called a rectangle if the diameter of R is less than ε and $W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \in R$ for $x, y \in R$, where ε is so small that $W_\varepsilon^s(x) \cap W_\varepsilon^u(y) \neq 0$. A rectangle R is proper if $R=\overline{\operatorname{int} R}$, where $\operatorname{int} R$ denotes the interior of R in Λ.

DEFINITION. A finite covering $\mathcal{R}$ of Λ is a Markov partition of Λ if

(1) $\mathcal{R}=\{R_1, \cdots, R_n\}$ where R_i is a proper rectangle in Λ for $i=1, \cdots, n$.

(2) $R_i \cap R_j=\partial R_i \cap \partial R_j$, where $\partial R_i=R_i-\operatorname{int} R_i$.

(3) $fW^s(x, R_i) \subset W^s(f(x), R_j)$, and $f^{-1}W^u(f(x), R_j) \subset W^u(x, R_i)$ for $x \in \operatorname{int} R_i \cap \operatorname{int} f^{-1}R_j$, where $W^s(x, R_i)=W_\varepsilon^s(x) \cap R_i$ and $W^u(x, R_i)=W_\varepsilon^u(x) \cap R_i$.

Sinai proved that an Anosov diffeomorphism has Markov partitions ([**9**], [**10**]). Bowen constructed Markov partitions of a basic set for an Axiom A diffeomorphism ([**1**]). We will prove that for a hyperbolic set there is its arbitrarily slight extension which has Markov partitions.

THEOREM 1. *Suppose Λ is a hyperbolic set for a diffeomorphism $f: M \to M$, and U is a neighbourhood of Λ. Then there is a hyperbolic set Λ' such that $\Lambda \subset \Lambda' \subset U$ and Λ' has a Markov partition.*

PROOF. Let $\delta>0$ be small. By Lemma 2.1, there is $\varepsilon>0$ such that an ε-pseudo-orbit $\{x_n\}_{n \in \boldsymbol{Z}}$ in Λ is δ-shadowed by some $x \in M$ with $f^n(x) \in U$ for $n \in \boldsymbol{Z}$. Let $\gamma>0$ satisfy

$$\gamma<\frac{\varepsilon}{2}$$

and

$$d(f(x), f(y))<\frac{\varepsilon}{2} \quad \text{if} \quad d(x, y)<\gamma$$

for $x, y \in U$ (we may assume that $\bar{U}$ is compact).

Let $P=\{p_1, \cdots, p_r\}$ be a finite set of points in Λ such that

$$U_\gamma(x)\cap P\neq\emptyset \quad \text{for any} \quad x\in\Lambda,$$

where $U_\gamma(x)=\{y\in\Lambda|d(x, y)<\gamma\}$. Then the subset $\Sigma(P)$ of $P^{\mathbf{Z}}$ defined by

$$\Sigma(P)=\{(q_i)_{i\in\mathbf{Z}}\in P^{\mathbf{Z}}|d(f(q_i), q_{i+1})<\varepsilon \text{ for } i\in\mathbf{Z}\}$$

is a subshift of finite type. Let the map

$$\theta:\Sigma(P)\longrightarrow U$$

be given by $\theta((q_i)_{i\in\mathbf{Z}})=x$, where x δ-shadows $(q_i)_{i\in\mathbf{Z}}$. Then θ is continuous, $\theta(\Sigma(P))\subset U$ is f-invariant and $\Lambda\subset\theta(\Sigma(P))$. Define

$$\Lambda'=\theta(\Sigma(P)).$$

Then Λ' is compact because $\Sigma(P)$ is compact. If U is sufficiently small, Λ' is a hyperbolic set ([**6**, p. 161]). We will construct a Markov partition of Λ' from the subshift $\Sigma(P)$ along the line of [**4**].

For $s=1, \cdots, r$, define

$$T_s=\{\theta(q_i)_{i\in\mathbf{Z}}|(q_i)_{i\in\mathbf{Z}}\in\Sigma(P),\ q_0=p_s\}.$$

Then

$$T_s \text{ is closed,} \tag{3.1}$$

$$T_s \text{ is a rectangle,} \tag{3.2}$$

for $s=1, \cdots, r$, and

$$\{T_s\}_{s=1,\cdots,r} \text{ is a finite covering of } \Lambda'. \tag{3.3}$$

For T_j and T_k $(1\leqq j, k\leqq r)$ with $T_j\cap T_k\neq\emptyset$, define

$$T_{jk}{}^1=T_j\cap T_k,$$

$$T_{jk}{}^2=\{x\in T_j|W^u(x, T_j)\cap T_k\neq\emptyset,\ W^s(x, T_j)\cap T_k=\emptyset\},$$

$$T_{jk}{}^3=\{x\in T_j|W^u(x, T_j)\cap T_k=\emptyset,\ W^s(x, T_j)\cap T_k\neq\emptyset\},$$

and

$$T_{jk}{}^4=\{x\in T_j|W^u(x, T_j)\cap T_k=\emptyset,\ W^s(x, T_j)\cap T_k=\emptyset\}.$$

Then each $T_{jk}{}^n$ $(1\leqq n\leqq 4)$ is a rectangle. Let K_x $(x\in\Lambda')$ be given by

$$K_x=\bigcap\{T_{jk}{}^n|x\in T_{jk}{}^n\},$$

and a finite covering $\mathcal{K}$ of Λ' be given by

$$\mathcal{K}=\{K_x\}_{x\in\Lambda'}=\{K_1, \cdots, K_m\},$$

where we omit the empty set and suppose $K_i\neq\emptyset$ $(1\leqq i\leqq m)$.

In general $\operatorname{int} T_j$ and $\operatorname{int} T_{jk}{}^n$ are not rectangles open in Λ', because Λ' does not have the local product structure. Nevertheless we can prove that $\operatorname{int} K_i$ is a rectangle open in Λ' for $i=1, \cdots, m$. To do this, we need Lemma 3.2 ~ Lemma 3.4 as follows.

LEMMA 3.2.

(1) *$x \in \partial T_j$ if and only if $x \notin \operatorname{Int} W^s(x, T_j)$ or $x \notin \operatorname{Int} W^u(x, T_j)$.*

(2) *$x \in \partial K_j$ if and only if $x \notin \operatorname{Int} W^s(x, K_j)$ or $x \notin \operatorname{Int} W^u(x, K_j)$.*

Here $\operatorname{Int} W^s(x, T_j)$ *(resp.* $\operatorname{Int} W^u(x, T_j)$*) denotes the interior of* $W^s(x, T_j)$ *(resp.* $W^u(x, T_j)$*) in* $W^s_\varepsilon(x) \cap \Lambda'$ *(resp.* $W^u_\varepsilon(x) \cap \Lambda'$*).* $\operatorname{Int} W^s(x, K_j)$ *and* $\operatorname{Int} W^u(x, K_j)$ *are defined similarly.*

PROOF OF (2). Suppose $x \in \partial K_j$. Then there is a sequence $\{z_n\}_{n \in Z}$ in Λ' such that $\{z_n\}_{n \in Z}$ converges to x and

$$z_n \notin K_j \quad \text{for} \quad n \in \boldsymbol{Z}. \tag{3.5}$$

By (3.3) there are a rectangle T_l and a subsequence $\{z_{i_n}\}_{n \in Z} \subset \{z_n\}_{n \in Z}$ with

$$z_{i_n} \in T_l \quad \text{for} \quad n \in \boldsymbol{Z}. \tag{3.6}$$

By (3.1), we have

$$x \in T_l. \tag{3.7}$$

From (3.2), (3.6) and (3.7), it follows that

$$[z_{i_n}, x] \in T_l \subset \Lambda' \tag{3.8}$$

and

$$[x, z_{i_n}] \in T_l \subset \Lambda', \tag{3.9}$$

where $[z_{i_n}, x] = W^s_\varepsilon(z_{i_n}) \cap W^u_\varepsilon(x)$. We have that $[z_{i_n}, x] \notin W^u(x, K_j)$ or $[x, z_{i_n}] \notin W^s(x, K_j)$ for $n \in \boldsymbol{Z}$, because

$$[z_{i_n}, x] \in W^u(x, K_j) \quad \text{and} \quad [x, z_{i_n}] \in W^s(x, K_j)$$

implies

$$z_{i_n} = [[z_{i_n}, x], [x, z_{i_n}]] \in K_j.$$

Hence there is a subsequence $\{z_{j_n}\}_{n \in Z} \subset \{z_{i_n}\}_{n \in Z}$ such that

$$[z_{j_n}, x] \notin W^u(x, K_j) \quad \text{for} \quad n \in \boldsymbol{Z} \tag{3.10}$$

or

$$[x, z_{j_n}] \notin W^s(x, K_j) \quad \text{for} \quad n \in \boldsymbol{Z}. \tag{3.11}$$

By (3.8), (3.10) implies

$$[z_{j_n}, x] \in W^n_\varepsilon(x) \cap \Lambda' - W^u(x, K_j)$$

and

$$[z_{j_n}, x] \longrightarrow x \quad \text{as} \quad n \to \infty.$$

By (3. 9), (3. 11) implies

$$[x, z_{j_n}] \in W^s_\varepsilon(x) \cap \Lambda' - W^s(x, K_j)$$

and

$$[x, z_{j_n}] \longrightarrow x \quad \text{as} \quad n \to \infty.$$

Thus $x \notin \mathrm{Int}\, W^u(x, K_j)$ or $x \notin \mathrm{Int}\, W^s(x, K_j)$. The converse is obvious. The proof of (1) is similar.

LEMMA 3. 3. (1) *Suppose* $x \in \mathrm{Int}\, W^u(x, T_i)$, $y \notin \mathrm{Int}\, W^u(y, T_i)$ *and* $y \in W^s(x, T_i)$. *Then there is* T_j *such that* $x \notin T_j$ *and* $y \in T_j$.

(2) *Suppose* $x \in \mathrm{Int}\, W^s(x, T_i)$, $y \notin \mathrm{Int}\, W^s(y, T_i)$ *and* $y \in W^u(x, T_i)$. *Then there is* T_j *such that* $x \notin T_j$ *and* $y \in T_j$.

PROOF OF (1). Let x, y and T_i satisfy

$$x \in \mathrm{Int}\, W^u(x, T_i), \tag{3. 12}$$

$$y \notin \mathrm{Int}\, W^u(y, T_i) \tag{3. 13}$$

and

$$y \in W^s(x, T_i). \tag{3. 14}$$

By (3. 13) there is a sequence $\{z_n\}_{n \in Z}$ such that

$$z_n \in W^u_\varepsilon(y) \cap \Lambda' - W^u(y, T_i) \tag{3. 15}$$

and

$$z_n \longrightarrow y \quad \text{as} \quad n \to \infty. \tag{3. 16}$$

By (3. 3) there are a subsequence $\{z_{i_n}\}_{n \in Z} \subset \{z_n\}_{n \in Z}$ and a rectangle T_j with

$$z_{i_n} \in T_j. \tag{3. 17}$$

By (3. 1) and (3. 16), this implies

$$y \in T_j. \tag{3. 18}$$

We will show

$$x \notin T_j. \tag{3. 19}$$

If we assume $x \in T_j$, then

$$[z_{i_n}, x] \in T_j \subset \Lambda'. \tag{3. 20}$$

On the other hand we have

$$[z_{i_n}, x] \notin T_i$$

by (3. 2), (3. 14) and (3. 15). Combining with (3. 20) this implies

$$[z_{i_n}, x] \in W^u_\varepsilon(x) \cap \Lambda' - W^u(x, T_i),$$

and from (3.16) and (3.14) it follows that

$$[z_{i_n}, x] \longrightarrow x \quad \text{as} \quad n \to \infty.$$

This contradicts (3.12). Thus we proved (3.19).

Proof of (2) is similar.

LEMMA 3.4. (1) *If* $x \in \partial K_i$ *and* $x \notin \operatorname{Int} W^s(x, K_i)$. *Then* $y \notin \operatorname{Int} W^s(y, K_i)$ *for any* $y \in W^u(x, K_i)$.

(2) *If* $x \in \partial K_i$ *and* $x \notin \operatorname{Int} W^u(x, K_i)$. *Then* $y \notin \operatorname{Int} W^u(y, K_i)$ *for any* $y \in W^s(x, K_i)$.

PROOF OF (1). Let x, y, and K_i satisfy

$$x \in \partial K_i, \tag{3.21}$$

$$x \notin \operatorname{Int} W^s(x, K_i) \tag{3.22}$$

and

$$y \in W^u(x, K_i). \tag{3.23}$$

Let

$$\mathcal{T}_i \subset \{T_{kl}^m \mid T_{kl}^m \neq \emptyset,\ 1 \leq k,\ l \leq r,\ 1 \leqq m \leqq 4\}$$

satisfy

$$K_i = \bigcap \{T_{kl}^m \mid T_{kl}^m \in \mathcal{T}_i\}$$

By (3.22), there is a sequence $\{z_n\}_{n \in Z}$ such that

$$z_n \longrightarrow x \quad \text{as} \quad n \to \infty \tag{3.24}$$

and

$$\begin{aligned} z_n &\in W_\varepsilon^s(x) \cap \Lambda' - K_i \\ &= W_\varepsilon^s(x) \cap \Lambda' - \bigcap \{T_{kl}^m \mid T_{kl}^m \in \mathcal{T}_i\}. \end{aligned} \tag{3.25}$$

Then there are a subsequence $\{z_{i_n}\}_{n \in Z} \subset \{z_n\}_{n \in Z}$ and $T_{\alpha\beta}^\gamma \in \mathcal{T}_i$ so that

$$z_{i_n} \in W_\varepsilon^s(x) \cap \Lambda' - T_{\alpha\beta}^\gamma \quad \text{for} \quad n \in \boldsymbol{Z}. \tag{3.26}$$

We have

Case 1. $x \in \operatorname{Int} W^s(x, T_k)$ for any T_k with $T_{kl}^m \in \mathcal{T}_i$ (for some l, m),

or

Case 2. $x \notin \operatorname{Int} W^s(x, T_k)$ for some T_k with $T_{kl}^m \in \mathcal{T}_i$ (for some l, m).

In case 1, by (3.24) and (3.25) there is an integer $N > 0$ such that

$$z_{i_n} \in W^s(x, T_\alpha) \quad \text{for} \quad n \geqq N.$$

Then

$$[y, z_{i_n}] \notin T_{\alpha\beta}^\gamma, \tag{3.27}$$

because if $[y, z_{i_n}]\in T_{\alpha\beta}^{\gamma}$ we have

$$z_{i_n}=[x, [y, z_{i_n}]]\in T_{\alpha\beta}^{\gamma}$$

which contradicts (3.26). From (3.27) and

$$[y, z_{i_n}] \longrightarrow y \quad \text{as} \quad n\to\infty,$$

it follows that

$$y\notin \operatorname{Int} W^s(y, K_i).$$

In case 2, assume $y\in \operatorname{Int} W^s(y, T_k)$. By Lemma 3.3 (2), (3.22) and (3.23), this implies that there is a rectangle T_m such that $x\in T_m$ and $y\notin T_m$. So x and y can not belong to the same rectangle of $\mathcal{K}$. This is a contradiction. Therefore

$$y\notin \operatorname{Int} W^s(y, T_k).$$

Then

$$y\notin \operatorname{Int} W^s(y, K_i),$$

because $T_{kl}^m\in\mathcal{T}_i$ for some l, m. Thus we proved Lemma 3.4 (1). The proof of Lemma 3.4 (2) is similar.

Now we will prove that $\operatorname{int} K_i$ is a rectangle open in Λ' if $K_i\in\mathcal{K}$. Let $\operatorname{int} K_i\neq\emptyset$ and $x, y\in \operatorname{int} K_i$. Assume

$$z=[x, y]\in\partial K_i.$$

From Lemma 3.2 (2), it follows that

$$z\notin \operatorname{Int} W^s(z, K_i) \tag{3.28}$$

or

$$z\notin \operatorname{Int} W^u(z, K_i). \tag{3.29}$$

Because $y\in \operatorname{int} K_i$ and $z\in W^u(y, K_i)$, (3.28) contradicts Lemma 3.4 (1). On the other hand (3.29) contradicts Lemma 3.4 (2), because $x\in \operatorname{int} K_i$ and $z\in W^s(x, K_i)$. Thus $[x, y]\in \operatorname{int} K_i$ if $x, y\in \operatorname{int} K_i$, i.e. $\operatorname{int} K_i$ is a rectangle. Because $\operatorname{int} K_i$ is contained in a closed rectangle T_m for some m, $\overline{\operatorname{int} K_i}$ is a closed rectangle.

Set

$$\mathcal{R}=\{\overline{\operatorname{int} K_i}\mid K_i\in\mathcal{K},\ \operatorname{int} K_i\neq\emptyset\}.$$

Then $\mathcal{R}$ is a Markov partition of Λ' as in [**4**, p. 78-83]. This completes the proof of Theorem 1.

REMARK 3.1. A Markov partition of Λ' induces a subshift Σ of finite type and a surjective map $\pi:\Sigma\to\Lambda'$ such that $f\pi=\pi\sigma$ where σ is the shift transformation ([**1**]). And π is a finite to one map, i.e. there is a positive integer N such that the cardinal number of $\pi^{-1}(x)$ is less than N for any $x \in \Lambda'$ ([2]).

DEFINITION. Let $g: X \rightarrow X$ be a homeomorphism of a compact space X. A subset $A \subset X$ is minimal for g if A is a closed g-invariant non-empty set which is minimal, i.e. when $B \subset A$ is a closed g-invariant non-empty set, then $B = A$.

Bowen proved that the dimension of a minimal set in a basic set for an Axiom A diffeomorphism is zero ([2]). We have the following.

COROLLARY 3.2. *Suppose Λ is a hyperbolic set for a diffeomorphism $f: M \rightarrow M$. If $B \subset \Lambda$ is a minimal set for f, the dimension of B is zero.*

PROOF. Let $\Lambda' \supset \Lambda$ be a hyperbolic set which has a Markov partition. Then B is also minimal in Λ'. As in [2], it follows that the dimension of B is zero. Q.E.D.

DEFINITION. Let $f: M \rightarrow M$ be a diffeomorphism with $N_m < \infty$ for $m > 0$, where N_m is the number of the fixed points of f^m. The zeta function ζ_f of f is the formal power series defined by

$$\zeta_f(z) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} z^m\right).$$

Manning proved that Axiom A diffeomorphisms have rational zeta functions ([8]).

COROLLARY 3.3. *Suppose Λ is a hyperbolic set for a diffeomorphism $f: M \rightarrow M$, and U is a neighbourhood of Λ. Then there is a hyperbolic set Λ' such that $\Lambda \subset \Lambda' \subset U$ and the zeta function of $f|\Lambda'$ is rational.*

PROOF. By Theorem 1, there is a hyperbolic set Λ' such that $\Lambda \subset \Lambda' \subset U$ and Λ' has a Markov partition. From Remark 3.1 and [8], it follows that the zeta function of $f|\Lambda'$ is rational.

§ 4. Hyperbolic ω-limit sets.

DEFINITION. A homeomorphism $f: X \rightarrow X$ of a topological space X is called an abstract ω-limit set if there are a compact metric space Y, a homeomorphism $g: Y \rightarrow Y$ and a point $x \in Y$ such that there is a homeomorphism $h: X \rightarrow \omega(x)$ with $gh = hf$, where $\omega(x)$ is the ω-limit set of x in Y.

The following is an extension of [3] for hyperbolic sets.

THEOREM 2. *Suppose $\Lambda \subset M$ is a hyperbolic set for a diffeomorphism $f: M \rightarrow M$. If Λ is an abstract ω-limit set, Λ is an ω-limit set of a point in M,* i.e. *there is $x \in M$ with $\Lambda = \omega(x)$. Moreover for any $\delta > 0$, we can choose $x \in M$ so that $d(f^n(x), \Lambda) < \delta$ for $n \in \boldsymbol{Z}$.*

PROOF. By the proof of Theorem 1 of [3], for any $\varepsilon>0$ there is an ε-pseudo-orbit $\{x_n\}_{n\in Z}$ in Λ so that

$$d(f(x_n), x_{n+1}) \longrightarrow 0 \quad (n \to +\infty)$$

and

$$\{x_n\}_{n\in Z} \text{ is dense in } \Lambda.$$

If ε is sufficiently small, Lemma 2.1 implies that there is $x\in M$ such that x δ-shadows $\{x_n\}_{n\in Z}$ and $d(f^n(x), \Lambda)<\delta$ for $n\in \boldsymbol{Z}$. As in [3, Theorem 2], $d(f^n(x), x_n)\to 0$ $(n\to+\infty)$. Therefore $\omega(x)=\Lambda$. Q.E.D.

DEFINITION. A homeomorphism $g: X\to X$ is topologically transitive if there is a point whose orbit is dense in X.

PROPOSITION 4.2. *Suppose Λ is a hyperbolic set for a diffeomorphism $f: M \to M$, and Λ is an ω-limit set of some $x\in M$. Then for any neighbourhood U of Λ there is a hyperbolic set Λ' such that $\Lambda\subset\Lambda'\subset U$ and $f|\Lambda'$ is topologically transitive and has dense periodic points.*

PROOF. By Theorem 0, there are a subshift Σ of finite type on symbols $\mathcal{B}=\{B_1, \cdots, B_N\}$ determined by a matrix $T=(t_{ij})$, a hyperbolic set Λ' with $\Lambda\subset\Lambda'\subset U$ and a surjective map $\pi: \Sigma\to\Lambda'$ with $f\pi=\pi\sigma$. Moreover there is an associated covering $\{U_1, \cdots, U_N\}$ of Λ such that $t_{ij}=1$ when $f(U_i)\cap U_j\neq\emptyset$. Because Λ is an ω-limit set of some $x\in M$, for any i, j $(1\leqq i, j\leqq N)$ there is a finite sequence $n_1, \cdots, n_m$ $(1\leqq n_k\leqq N)$ satisfying $n_1=i$, $n_m=j$ and $f(U_{n_i})\cap U_{n_{i+1}}\neq\emptyset$ for $i=1, \cdots, m-1$. This implies $t_{n_i n_{i+1}}=1$ for $i=1, \cdots, m-1$. Then Σ is topologically transitive and has dense periodic points. Therefore $f|\Lambda'$ is topologically transitive and has dense periodic points. This completes the proof.

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